Constructing 3-level Foldover Screening Designs using Cyclic Generators

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Abstract

Most factors in chemical science and engineering are quantitative. Therefore, chemists and engineers are more familiar with the notion that factors should necessarily have three levels. Jones & Nachtsheim (2011) introduced a new class of 3-level screening designs which allows the assessment of curvature in the factor-response

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relationship. They call these designs definitive screening designs (DSDs). These DSDs are (i) saturated for estimating the intercept, the m main effects and the m quadratic effects; (ii) unlike resolution III designs, all main effects are orthogonal to 2-factor interactions; (iii) unlike resolution IV designs, 2-factor interactions are not fully aliased with one another; and (iv) unlike resolution III, IV and IV designs, the quadratic effects can be estimated and are orthogonal to main effects and not fully aliased with 2-factor interactions; (v) when the design is sufficiently large, it allows efficient estimation of the full quadratic model in any three factors. This paper introduces a new class of DSD-like designs generated by cyclic generators. This new class can be used to study the presence of the second-order effects more efficiently.

Keywords: Conference matrix; Definitive screening designs; D-efficiency; Interchange algorithm; Response Surface Designs.

1 Introduction

Screening designs are used in the early stage of experimentation to single out a subset of factors from a list of potential factors for further studies. Traditional screening designs are 2-level, such as resolution III or IV fractional factorial designs (FFDs) and Plackett-Burman designs (Plackett & Burman, 1946). See Mee (2009) for reviews and examples of using these designs. However, factors of many experiments in chemical science and engineering are often quantitative and as such should be set at three levels.

Consider the experiment reported by Cheng, et al. (2012) in which 13 variables (chitosan, glucose, peptone, $(NH_4)_2SO_4$, urea, NH_4Cl, $MgSO_4$, KH_2PO_4 , $FeSO_47H_2O$, $ZnSO_47H_2O$, $CaCl_26H_2O$, $MnSO_4H_2O$ and NaCl) suspected to significantly influence the

chitosanase production were screened using 13 columns of a Plackett-Burman design for 20 runs. These 13 independent medium compositions were evaluated at two levels, namely high and low (designated as level $+1$ and level -1 respectively). Converting a 3-level factor to a 2-level one for the sake of using a 2-level design as in this example is not always recommended as 2-level designs could not study the existence of the second-order effects. DSDs introduced by Jones & Nachtsheim (2011) seem to be more suitable for this experiment. The design matrix for a DSD can be written as:

$$\begin{pmatrix} \mathbf{C} \\ \mathbf{0}' \\ -\mathbf{C} \end{pmatrix}. \tag{1}$$

Here, $\mathbf{C}$ is an $m \times m$ $(0, \pm 1)$ matrix with zero diagonal and $\mathbf{0}$ is a column vector of 0's. Stylianou (2011) and Xiao et al. (2012) pointed out that if we use a *conference* matrix of order m for $\mathbf{C}$, i.e., if $\mathbf{C}'\mathbf{C} = (m-1)\,\mathbf{I}_{m\times m}$ then the DSD is also orthogonal for main effects, i.e., all main effects are orthogonal to one another (see Stylianou, 2011 and Xiao et al., 2012). For even $m \leq 50$, with the exception of $m = 22$ and $m = 34$, the $\mathbf{C}$ matrices which are also conference matrices, are given in Xiao et al. (2012) and Nguyen & Stylianou (2013). For odd m, the $\mathbf{C}$ matrices, constructed such that $|\mathbf{C}'\mathbf{C}|$ is maximized, are given in Jones & Nachtsheim (2011) and Nguyen & Stylianou (2013).

DSDs, with $n = 2m + 1$ runs where m is the number of factors, have the following desirable properties:

(i) they are mean orthogonal, i.e., each factor has the same number of ± 1;

(ii) they are saturated for estimating the intercept, the m main effects and the m

quadratic effects;

(iii) all main effects are orthogonal to 2-factor interactions. This is not true with resolution III designs;

(iv) 2-factor interactions are not fully aliased with one another. This is not true with resolution IV designs;

(v) quadratic effects can be estimated, orthogonal to the main effects and are not fully aliased with 2-factor interactions. This is not true with resolution III, and V designs; and

(vi) when the design is sufficiently large, say with six or more factors, it possesses the capability to estimate all possible full quadratic models involving three or fewer factors. This feature is important as they may render follow-up experiments unnecessary (Jones & Nachtsheim, 2011).

Note that properties (iii) and (iv) are at our disposal due to the foldover structure of a DSD. An example of a 7-factor DSD is given in Figure 1 (a). In spite of the usefulness and popularity of DSDs, there has been criticism against this type of designs. The main criticism is that the number of middle/zero levels (considered as "desired" or "best" level) of each factor is always fixed, which is three regardless of the size of the DSD. Thus, for a 30-factor DSD (in 61 runs), the proportion of middle/zero levels is about 5% ($\approx 3/61$). As such, while the DSDs seem to be reasonably good designs for the first-order model, they might not be very suitable for studying second-order effects particularly when m is large. In addition, the correlation between two pure quadratic-effects columns of a DSD is increasing in m and approaches $+1/3$ as $m \to \infty$ (see p. 6 of Jones & Nachtsheim, 2011).

This paper introduces a new class of DSD-like designs generated by cyclic generators called cyclic DSDs or CDSDs. Like DSDs, these CDSDs are constructed by (1). Unlike the

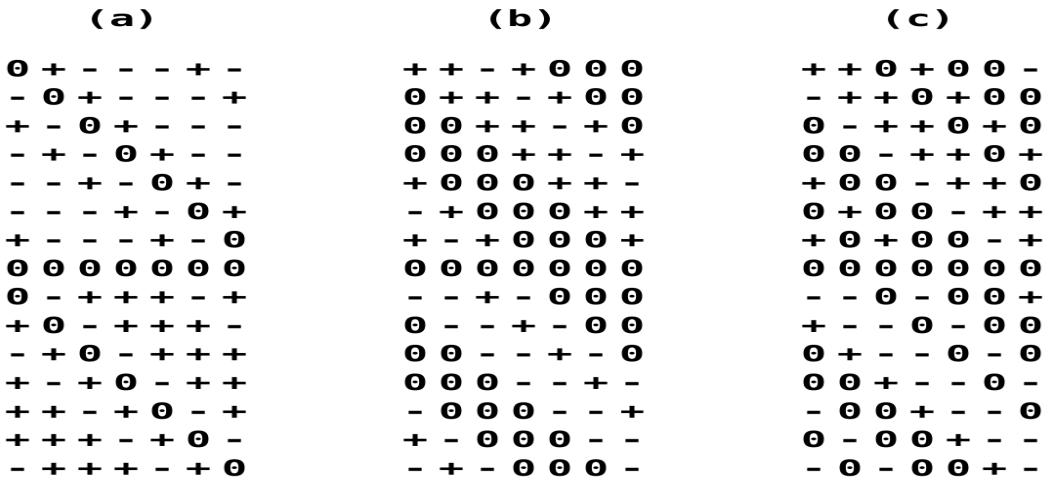


Figure 1: Two designs for seven factors in seven runs: (a) a DSD and (b) a CDSD.

C matrices of DSDs, the ones of CDSDs are circulant generated by the generating vectors (i.e. their first columns) which might include any number of zeros. Three CDSDs for seven factors are given in Figures 1 (a), (b) and (c). Although CDSDs such as the one in Figure 1 (a) are also DSDs, in this paper we call them DSDs instead of CDSDs.

2 Brief introduction to the circulant matrices

Cyclic generators have been used to generate the Plackett-Burman designs, the 2-level supersaturated designs (Nguyen, 1996), the 3-level saturated and supersaturated designs (Nguyen & Pham, 2017), the conference matrices constructing DSDs (Nguyen & Stylianou, 2013) and various type of cyclic incomplete block designs (John & Williams, 1995). Each cyclic generator is associated with a circulant matrix. In the following, we will give a brief introduction to the circulant matrices.

An $m \times m$ circulant matrix $\mathbf{C} = (c_{ij})$, generated by a generating vector $\mathbf{c} = (c_0, c_1, \cdots,$

c_{m-1}) (or its first column) of length m, is the following matrix :

$$\begin{pmatrix} c_0 & c_{m-1} & \cdots & c_1 \\ c_1 & c_0 & \cdots & c_2 \\ \ddots & \ddots & \ddots & \ddots \\ c_{m-1} & c_{m-2} & \cdots & c_0 \end{pmatrix}. \tag{2}$$

The first seven rows of Figures 1 (a), (b) and (c) are three circulant matrices generated by the generating vector $(0, -1,\ 1, -1, -1, -1,\ 1)$, $(1,\ 0,\ 0,\ 0,\ 1, -1,\ 1)$ and $(1, -1,\ 0,\ 0,\ 1,\ 0,\ 1)$ respectively.

Note that the matrix $\mathbf{A} = (a_{ij}) = \mathbf{C}'\mathbf{C}$ is circulant with the generating vector $\mathbf{a} = (a_0, a_1, \cdots, a_{m-1})$, where

$$a_k = \sum_{j=0}^{m-1} c_j c_{(j+k) \bmod m}. \tag{3}$$

The generating vector $\mathbf{a}$'s correspond to the generating vectors in the previous paragraphs are $(6, -1, -1,\ 1,\ 1, -1-1)$, $(4, -1,\ 0,\ 1,\ 1,\ 0, -1)$ and $(4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)$.

Now if we square each element of c_{ij} in (2), the matrix $\mathbf{A}$ will become $\mathbf{A}^* = (a_{ij}^2)$. $\mathbf{A}^*$ is also circulant with the generating vector $\mathbf{a}^* = (a_0^*, a_1^*, \cdots, a_{m-1}^*)$, where

$$a_k^* = \sum_{j=0}^{m-1} (c_j c_{(j+k) \bmod m})^2. \tag{4}$$

The corresponding generating vector $\mathbf{a}^*$'s are $(6,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5)$, $(4,\ 3,\ 2,\ 1,\ 1,\ 2,\ 3)$, and $(4,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2)$.

3 Criteria for comparing new designs

Consider the following pure-quadratic model for a DSD or CDSD with m 3-level factors in $n = 2m + 1$ runs:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{5}$$

where $\mathbf{Y}_n$ is the column vector of responses, $\mathbf{X}_{n\times p}$ is the expanded design matrix for the pure-quadratic model and $p = 2m+1$ is the number of parameters in (5), $\boldsymbol{\beta}_p$ is the column vector containing p coefficients for fixed effects, $\boldsymbol{\epsilon}_n$ is the column vector of random errors with zero mean and variance-covariance matrix $\sigma^2_\epsilon \mathbf{I}_n$, and $\mathbf{I}_n$ is the identity matrix of order n.

Let the u-th row of $\mathbf{X}$ in (5) be written as $(1, x^2_{u1}, \ldots, x^2_{um_3}, x_{u1}, \ldots, x_{um})$ and let the information matrix $\mathbf{M} = \mathbf{X}'\mathbf{X}$ be partitioned as:

$$2\begin{pmatrix} \mathbf{A} & \mathbf{0}' \\ \mathbf{0} & \mathbf{D} \end{pmatrix}, \tag{6}$$

where $\mathbf{A}$ and $\mathbf{D}$ are the $(1 + m) \times (1 + m)$ and $m \times m$ sub-matrices respectively. $\mathbf{0}$ is the $m \times (m + 1)$ sub-matrix of 0's.

Again, the matrix $\mathbf{A}$ in (6) can be partitioned as:

$$\begin{pmatrix} a & b\mathbf{1}' \\ b\mathbf{1} & \mathbf{A}^* \end{pmatrix}, \tag{7}$$

where $\mathbf{A}^*$ is the core of $\mathbf{A}$ (i.e. matrix $\mathbf{A}$ with the first row and column removed), $a = \frac{1}{2}n$

and b is the number of ± 1's in each column of $\mathbf{C}$.

The determinant of $\mathbf{X}'\mathbf{X}$, i.e. $|\mathbf{X}'\mathbf{X}|$ can thus be computed as $2^n\ |\mathbf{A}|\ |\mathbf{D}|$ or

$$|\mathbf{X}'\mathbf{X}| = 2^n\ a\ |\mathbf{A}^* - \frac{b^2}{a}\mathbf{J}|\ |\mathbf{D}|, \tag{8}$$

where $\mathbf{J}$ is the matrix of 1's. As both $\mathbf{D}$ and $\mathbf{A}^* - \frac{b^2}{a}\mathbf{J}$ are circulant (since $\mathbf{A}^*$ is circulant), their determinants can be computed from their generating vectors (See the Appendix).

The $\mathbf{X}$ matrix and p, the number of parameters given in the previous paragraph, are for the pure-quadratic model. For the first-order model, columns 2 to $m+1$ of $\mathbf{X}$ (and columns/rows 2 to $m+1$ of $\mathbf{X}'\mathbf{X}$ which correspond to the squared terms) are dropped and p becomes $1+m$. Then

$$|\mathbf{X}'\mathbf{X}| = 2^{1+m}\ a\ |\mathbf{D}|. \tag{9}$$

4 An algorithm for constructing CDSD

In this paper we use (m, m_0) CDSD to denote a CDSD constructed by a circulant matrix $\mathbf{C}$ in (1), generated by the generating vector $\mathbf{c}$ of length m with m_0 elements having value 0's and the remaining $b = m - m_0$ elements having values ± 1's. The following are the steps for constructing this cyclic generator $\mathbf{c}$:

1. Construct a vector $\mathbf{c}$ with m_0 elements having value 0 and b elements having values ± 1's. Randomize their positions. Calculate the first column of $\mathbf{D}$ in (6) using (3), the first

column of $\mathbf{A}^* - \frac{b^2}{a}\mathbf{J}$ in (8) using (4) and $|\mathbf{D}|$, $|\mathbf{A}^* - \frac{b^2}{a}\mathbf{J}|$ using (13) in the Appendix and then the objective function $f = |\mathbf{A}^* - \frac{b^2}{a}\mathbf{J}|\ |\mathbf{D}|$.

2. Repeat searching for a pair of elements of $\mathbf{c}$ which have different values such that swapping their positions will result in the biggest increase in f. If the search is successful, update f and $\mathbf{c}$.

Step 2 is repeated until f cannot be improved further.

Remarks

1. The number of $+1$'s and -1's of each generating vector $\mathbf{c}$ need not be the same.

2. Steps 1-2 make up one try. Each try corresponds to a candidate design with a particular f. Among several candidate designs, the one with the largest f will be chosen.

3. Our algorithm is an example of the interchange algorithm. This type of algorithm has been used in Nguyen (1996) to construct the 2-level supersaturated designs and in Nguyen & Pham (2017) to construct the 3-level saturated and supersaturated designs.

Figures 1 (b) and (c) show two steps of the algorithm for constructing a generating vector for a (7, 3) CDSD. In Step 1, the generating vector $(1,\ 0,\ 0,\ 0,\ 1, -1,\ 1)$ was constructed. In Step 2, the two elements at the second and sixth positions swap their positions and this vector becomes $(1, -1,\ 0,\ 0,\ 1,\ 0,\ 1)$. The objective function f has increased from 3588.27 to 1,118,481.07. After this swap, f cannot be increased further and the algorithm stops.

(a)

```
+ - 0 - 0 + + + - + + 0 0
0 + - 0 - 0 + + + - + + 0
0 0 + - 0 - 0 + + + - + +
+ 0 0 + - 0 - 0 + + + - +
+ + 0 0 + - 0 - 0 + + + -
- + + 0 0 + - 0 - 0 + + +
+ - + + 0 0 + - 0 - 0 + +
+ + - + + 0 0 + - 0 - 0 +
+ + + - + + 0 0 + - 0 - 0
0 + + + - + + 0 0 + - 0 -
- 0 + + + - + + 0 0 + - 0
0 - 0 + + + - + + 0 0 + -
- 0 - 0 + + + - + + 0 0 +
```

(b)

```
0 + - - - - - + - + + - -
- 0 + - - - - - + - + + -
- - 0 + - - - - - + - + +
+ - - 0 + - - - - - + - +
+ + - - 0 + - - - - - + -
- + + - - 0 + - - - - - +
+ - + + - - 0 + - - - - -
- + - + + - - 0 + - - - -
- - + - + + - - 0 + - - -
- - - + - + + - - 0 + - -
- - - - + - + + - - 0 + -
- - - - - + - + + - - 0 +
+ - - - - - + - + + - - 0
```

Figure 2: Two **C** matrices of two candidate designs for 13 3-level factors in 27 runs: (a) a (13,4) CDSD and (b) a 13-factor DSD.

(a)

```
0 + + + + - 0 - + 0 - 0 + - 0
0 0 + + + + - 0 - + 0 - 0 + -
- 0 0 + + + + - 0 - + 0 - 0 +
+ - 0 0 + + + + - 0 - + 0 - 0
0 + - 0 0 + + + + - 0 - + 0 -
- 0 + - 0 0 + + + + - 0 - + 0
0 - 0 + - 0 0 + + + + - 0 - +
+ 0 - 0 + - 0 0 + + + + - 0 -
- + 0 - 0 + - 0 0 + + + + - 0
0 - + 0 - 0 + - 0 0 + + + + -
- 0 - + 0 - 0 + - 0 0 + + + +
+ - 0 - + 0 - 0 + - 0 0 + + +
+ + - 0 - + 0 - 0 + - 0 0 + +
+ + + - 0 - + 0 - 0 + - 0 0 +
+ + + + - 0 - + 0 - 0 + - 0 0
```

(b)

```
0 - - - + - - + - + + + - - -
- 0 - - - + - - + - + + + - -
- - 0 - - - + - - + - + + + -
- - - 0 - - - + - - + - + + +
+ - - - 0 - - - + - - + - + +
+ + - - - 0 - - - + - - + - +
+ + + - - - 0 - - - + - - + -
- + + + - - - 0 - - - + - - +
+ - + + + - - - 0 - - - + - -
- + - + + + - - - 0 - - - + -
- - + - + + + - - - 0 - - - +
+ - - + - + + + - - - 0 - - -
- + - - + - + + + - - - 0 - -
- - + - - + - + + + - - - 0 -
- - - + - - + - + + + - - - 0
```

Figure 3: Two **C** matrices of two candidate designs for 15 3-level factors in 31 runs: (a) a (15, 5) CDSD and (b) a 15-factor DSD.

5 Results and Discussion

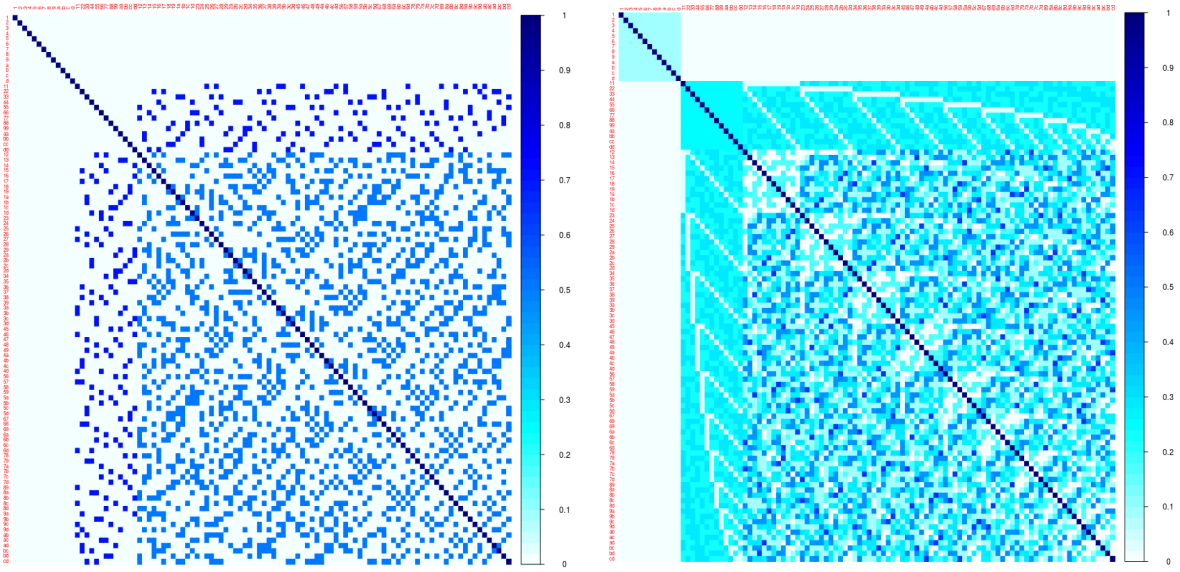
The D-efficiencies of a design is calculated as:

$$|\mathbf{X}'\mathbf{X}|^{1/p}/n. \tag{10}$$

Here $\mathbf{X}$ and p are the model matrix and number of parameters in the model respectively. Let $(d_1, d_2, r_{\max})$ triplet be the first-order D-efficiency, the pure-quadratic D-efficiency and the maximum correlation coefficient (in terms of the absolute value) among the $2m$ columns of the model matrix $\mathbf{X}$ respectively. A design is considered good if it has high d values and small $r_{\max}$.

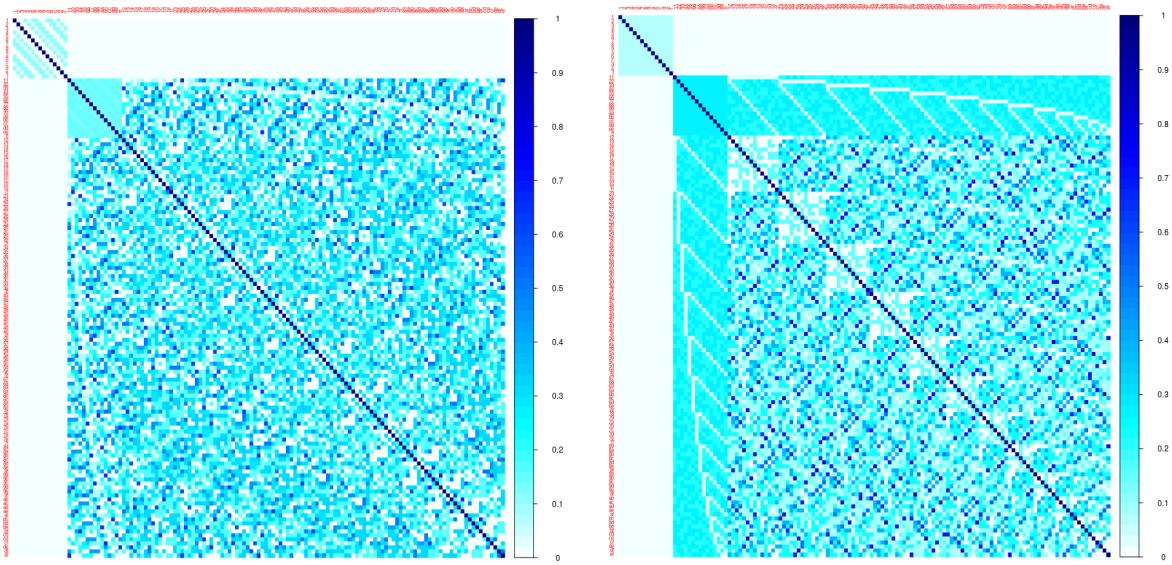
Table 1 compares 167 CDSDs for m from 7 to 30 and m_0 from 3 to $\lfloor \frac{m}{2} \rfloor$ with 24 DSDs for $m = 7$ to 30 in terms of the values of the $(d_1, d_2, r_{\max})$ triplets. These DSDs are from Nguyen & Stylianou (2013). It can be seen that all CDSDs have less d_1 values than DSDs. At the same time, most CDSDs have higher d_2 values than DSDs. Comparing to DSDs, many CDSDs have higher d_2 values and smaller $r_{\max}$'s.

The Introduction Section briefly mentioned the experiment of Cheng, et al. (2012) to optimize the chitosanase production in which 13 quantitative factors were studied. The authors used the Plackett-Burman design for 20 runs. The follow-up experiment to study the three significant factors, glucose, peptone and $MgSO_4$, was a Box-Behnken design for three factors. Figure 2 display the $\mathbf{C}$ matrices of two candidate screening designs for 13 3-level factors in 27 runs using a (13, 4) CDSD and a 13-factor DSD. The $(d_1, d_2, r_{\max})$ values of CDSD are (0.686, 0.399, 0.0) and the ones of the DSD are (0.861, 0.281, 0.25). This means CDSD has increased the DSD's d_2 or the pure-quadratic D-efficiency 42%.



(a) (b)

Figure 4: CCPs of two screening designs 13 factors in 27 runs: (a) a CDSD and (b) a DSD.



(a) (b)

Figure 5: CCPs of two screening designs 15 factors in 31 runs: (a) a CDSD and (b) a DSD.

Poorna & Kulkarni (1995) investigated four carbon sources (inulin, fructose, glucose,

sucrose), four organic nitrogen sources (corn steep liquor peptone, urea, yeast extract), four organic nitrogen sources (corn steep liquor, peptone, urea, yeast extract) and three others parameters (trace element solution, inoculum level, pH) suspected to affect the insulinase production. The design for this experiment, a regular 2^{15-11} FFD and its analysis were summarized in Example 6.4 of Mee (2009). Figure 2 displays the $\mathbf{C}$ matrices of two alternative candidate screening designs for 15 3-level factors in 31 runs using a (15, 5) CDSD and a 15-factor DSD. The $(d_1, d_2, r_{\max})$ values of CDSD are (0.639, 0.365, 0.155) and the ones of the DSD are (0.882, 0.264, 0.262). This means the CDSD has increased the DSD's d_2 or pure-quadratic D-efficiency 38%.

Figures 4 and 5 show two correlation cell plots (CCPs) of the designs whose $\mathbf{C}$ matrices are in Figures 2 and 3 respectively. These CCPs, advocated by Jones & Nachtsheim (2011), show the pairwise (absolute) correlation between two terms under study as a coloured square. The color of each cell goes from white to dark. The white cells mean no correlation while the dark ones mean a correlation of 1 or close to 1 (see the vertical bar on the right side of each graph). Each CCP in Figure 4 has 105^2 coloured cells (105 is the number of 13 main effects, 13 quadratic effects and $\binom{13}{2}$ 2-factor interactions). Each CCP in Figure 5 has 136^2 coloured cells. All four CCPs in Figures 4 and 5 show that the main effects are orthogonal to the quadratic effects and interactions. They also show that none of the quadratic effect is fully aliased with the 2-factor interactions. Figure 4 (a) shows the orthogonality among the main effects and among the quadratic effects. Figure 5 shows that the magnitude of correlation among quadratic effects is more visible in the CCP of the DSD than the one of the corresponding CDSD.

To study the performance of the algorithm in Section 4, we generate a (9, 4) CDSD and

a (15, 5) CDSD each with 10,000 tries on an HP laptop with Intel®Core™ i7 CPU Q720 @ 1.60GHz. 10,000 tries take about 20 seconds for the (9, 4) CDSD and about 32 seconds for the (15, 5) CDSD. Out of 10,000 tries for the parameter set (9, 4) CDSD, 1,669 results in designs having the same $(d_1, d_2, r_{\max})$ as the CDSD whose $\mathbf{C}$ matrix appear in Figure 2 (a). Out of 10,000 tries for the (15, 5) CDSD, only 407 results in designs having the same $(d_1, d_2, r_{\max})$ as the CDSD whose $\mathbf{C}$ matrix appear in Figure 3 (a). In general, larger designs, require more tries to obtain good designs, and each try consumes more time.

6 Conclusion

Most factors in chemical science and engineering are quantitative and are necessarily set at 3-level. This paper describes an algorithm for constructing the cyclic generators used for constructing 3-level foldover screening designs. We use CDSD to denote these designs. Unlike DSD, we can specify m_0, the number of zeros for each column of $\mathbf{C}$ in (1) and in turns the number of zeros/middle levels for each columns of the CDSD. To many experimenters, the zero/middle levels are considered "desired" ones and are expected to be replicated more often than the extreme levels ± 1's. Most constructed CDSDs have higher pure-quadratic D-efficiencies and smaller $r_{\max}$'s than the corresponding DSDs. This is important as the experiments with quantitative factors, the experimenters require the more accurate assessment of the curvature in the factor-response relationship.

DSDs and CDSDs are only suitable for screening experiments in which all factors quantitative. When factors are of mixed type, i.e. some are at 3-level and some at 2-level or categorical, readers are referred to Jones & Nachtsheim (2013) and Nguyen & Pham (2016).

Data obtained from experiments using DSDs or CDSDs can be analyzed by standard multiple-regression when the multiple-regression model only include the main and quadratic effects such as (5). When the 2-factor interactions are also suspected to be active and a part of the model, Jones & Nachtsheim (2011) recommended the use of forward stepwise regression where the set of model terms include all first- and second-order effects with provisions to ensure models with strong heredity.

The text files contains the DSDs and CDSDs in Table 1 can be found at `http://designcomputing.net/Cmatrices/DSD.html` and `http://designcomputing.net/Cmatrices/CDSD.html` and are in the supplemental material. The Java implementation of the algorithm for constructing CDSD in Section 4 is also available in the supplemental material.

Appendix: Calculating the determinant of a circulant matrix

Let $\mathbf{C}$ be a circulant matrix $\mathbf{C}$ generated by the generating vector $\mathbf{c} = (c_0, c_1, \cdots, c_{m-1})'$. Assuming $\mathbf{C}$ is nonsingular, its determinant can then be defined as $|\mathbf{C}| = \prod_{j=0}^{m-1} \lambda_j$ where $\lambda_j,\ j = 0, \cdots, m-1$ are the eigenvalues of $\mathbf{C}$ and are given by

$$\lambda_j = \sum_{k=0}^{m-1} c_k \omega_j^{m-k}, \tag{11}$$

where $\omega_j = e^{i\frac{2\pi j}{m}} = \cos\frac{2\pi j}{m} + i\sin\frac{2\pi j}{m},\ j = 0, \cdots, m-1$ are the solutions of equation $\omega^m = 1$ and i is the imaginary unit (i.e. unit satisfying $i^2 = -1$).

Hence,

$$|\mathbf{C}| = \prod_{j=0}^{m-1} \Big(\sum_{k=0}^{m-1} c_k \cos\frac{2\pi j(m-k)}{m} + i \sum_{k=0}^{m-1} c_k \sin\frac{2\pi j(m-k)}{m} \Big)$$
$$= \prod_{j=0}^{m-1} \Big(\sum_{k=0}^{m-1} c_k \cos\frac{2\pi jk}{m} - i \sum_{k=0}^{m-1} c_k \sin\frac{2\pi jk}{m} \Big). \quad (12)$$

Multiply two conjugate terms in the product and use $i^2 = -1$, we can get the explicit formula of the determinant (which is more suitable for programming purpose) as follows:

$$|\mathbf{C}| = S\ T \prod_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \Big\{ \Big(\sum_{k=0}^{m-1} c_k \cos\frac{2\pi jk}{m} \Big)^2 + \Big(\sum_{k=0}^{m-1} c_k \sin\frac{2\pi jk}{m} \Big)^2 \Big\} \quad (13)$$

where $S = \sum_{k=0}^{m-1} c_k$ and $T = 1$ if m is odd and $T = \sum_{k=0}^{m-1} (-1)^k c_k$ otherwise.

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Table 1. $(d_1, d_2, r_{\max})$ values of DSD (first column) and CDSD for m from 7 to 30 and m_0 from 3 to $\lfloor \frac{m}{2} \rfloor$

m/m_0	1	3	4	5	6	7	8	9	10	11	12	13	14	15
7	0.76	0.58												
	0.36	**0.39**												
	0.17	**0.07**												
8	0.84	0.51												
	0.36	0.34												
	0.19	0.46												
9	0.81	0.64	0.53											
	0.33	**0.36**	**0.34**											
	0.21	0.36	0.27											
10	0.87	0.65	0.56	0.44										
	0.32	**0.34**	**0.33**	0.31										
	0.22	0.29	0.22	0.40										
11	0.84	0.70	0.57	0.51										
	0.30	**0.35**	**0.36**	**0.36**										
	0.23	**0.23**	0.29	**0.17**										
12	0.89	0.66	0.59	0.51	0.44									
	0.30	**0.34**	**0.37**	**0.34**	**0.31**									
	0.24	**0.22**	0.31	0.30	0.33									
13	0.86	0.73	0.69	0.58	0.51									
	0.28	**0.34**	**0.40**	**0.36**	**0.34**									
	0.25	**0.23**	**0.00**	**0.23**	**0.19**									
14	0.90	0.72	0.62	0.61	0.53	0.47								
	0.28	**0.35**	**0.36**	**0.36**	**0.35**	**0.31**								
	0.26	**0.25**	0.30	**0.22**	**0.25**	0.38								
15	0.88	0.76	0.72	0.64	0.57	0.45								
	0.26	**0.32**	**0.37**	**0.37**	**0.37**	**0.34**								
	0.26	0.26	**0.25**	**0.16**	**0.21**	**0.25**								
16	0.91	0.75	0.71	0.64	0.58	0.51	0.46							
	0.26	**0.33**	**0.35**	**0.36**	**0.35**	**0.34**	**0.33**							
	0.27	0.28	**0.22**	**0.18**	0.27	**0.22**	**0.25**							
17	0.88	0.79	0.73	0.67	0.62	0.57	0.51							
	0.25	**0.32**	**0.35**	**0.36**	**0.36**	**0.36**	**0.34**							
	0.27	0.29	**0.20**	**0.21**	**0.27**	**0.17**	**0.14**							
18	0.92	0.78	0.74	0.68	0.63	0.59	0.52	0.45						
	0.25	**0.31**	**0.34**	**0.36**	**0.37**	**0.36**	**0.34**	**0.33**						
	0.28	0.30	**0.18**	**0.22**	**0.19**	**0.18**	**0.20**	**0.22**						
19	0.90	0.81	0.76	0.69	0.66	0.59	0.54	0.47						
	0.24	**0.30**	**0.34**	**0.37**	**0.36**	**0.36**	**0.36**	**0.34**						
	0.28	0.30	**0.16**	**0.24**	**0.15**	**0.17**	**0.18**	**0.10**						
20	0.93	0.81	0.75	0.71	0.67	0.63	0.57	0.51	0.46					
	0.23	**0.30**	**0.33**	**0.36**	**0.37**	**0.37**	**0.35**	**0.35**	**0.32**					
	0.28	0.31	**0.15**	**0.26**	**0.14**	**0.16**	**0.21**	**0.18**	**0.17**					
21	0.90	0.82	0.78	0.75	0.68	0.63	0.59	0.55	0.50					
	0.22	**0.31**	**0.33**	**0.38**	**0.36**	**0.38**	**0.37**	**0.35**	**0.34**					
	0.28	0.32	**0.16**	**0.02**	**0.13**	**0.18**	**0.17**	**0.13**	**0.12**					
22	0.93	0.82	0.77	0.73	0.69	0.65	0.60	0.55	0.51	0.47				
	0.22	**0.29**	**0.33**	**0.35**	**0.35**	**0.37**	**0.36**	**0.35**	**0.35**	**0.33**				
	0.29	0.32	**0.17**	**0.28**	**0.14**	**0.20**	**0.13**	**0.15**	**0.17**	**0.18**				
23	0.92	0.84	0.80	0.75	0.72	0.67	0.64	0.58	0.54	0.46				
	0.22	**0.28**	**0.32**	**0.36**	**0.36**	**0.37**	**0.37**	**0.36**	**0.35**	**0.34**				
	0.29	0.33	**0.18**	**0.19**	**0.15**	**0.22**	**0.11**	**0.14**	**0.21**	**0.17**				
24	0.94	0.82	0.79	0.74	0.73	0.67	0.63	0.58	0.56	0.51	0.47			
	0.21	**0.28**	**0.32**	**0.35**	**0.36**	**0.37**	**0.37**	**0.37**	**0.37**	**0.35**	**0.33**			
	0.29	0.33	**0.18**	**0.17**	**0.16**	**0.15**	**0.13**	**0.14**	**0.17**	**0.15**	**0.17**			
25	0.91	0.85	0.81	0.77	0.74	0.70	0.65	0.61	0.57	0.54	0.50			
	0.20	**0.27**	**0.31**	**0.34**	**0.36**	**0.37**	**0.37**	**0.37**	**0.37**	**0.35**	**0.34**			
	0.29	0.34	**0.19**	**0.16**	**0.17**	**0.13**	**0.12**	**0.17**	**0.13**	**0.14**	**0.10**			
26	0.95	0.84	0.79	0.77	0.73	0.70	0.65	0.62	0.58	0.55	0.51	0.48		
	0.20	**0.27**	**0.32**	**0.33**	**0.35**	**0.37**	**0.37**	**0.36**	**0.36**	**0.36**	**0.35**	**0.34**		
	0.29	0.34	**0.20**	**0.15**	**0.19**	**0.12**	**0.13**	**0.18**	**0.13**	**0.13**	**0.09**	**0.15**		
27	0.92	0.85	0.83	0.79	0.74	0.71	0.67	0.63	0.61	0.57	0.53	0.51		
	0.20	0.27	**0.30**	**0.34**	**0.35**	**0.36**	**0.37**	**0.37**	**0.37**	**0.36**	**0.35**	**0.34**		
	0.30	0.35	**0.20**	**0.14**	**0.19**	**0.10**	**0.15**	**0.13**	**0.12**	**0.13**	**0.17**	**0.16**		
28	0.95	0.85	0.81	0.79	0.75	0.70	0.68	0.65	0.62	0.58	0.54	0.50	0.47	
	0.20	**0.28**	**0.31**	**0.33**	**0.36**	**0.36**	**0.36**	**0.37**	**0.37**	**0.36**	**0.35**	**0.35**	**0.33**	
	0.30	0.35	**0.21**	**0.13**	**0.20**	**0.10**	**0.17**	**0.11**	**0.11**	**0.17**	**0.15**	**0.13**	**0.14**	
29	0.94	0.87	0.83	0.80	0.76	0.72	0.69	0.67	0.61	0.60	0.56	0.52	0.50	
	0.19	**0.26**	**0.29**	**0.32**	**0.34**	**0.35**	**0.37**	**0.38**	**0.37**	**0.37**	**0.36**	**0.35**	**0.34**	
	0.30	0.35	**0.21**	**0.12**	**0.21**	**0.11**	**0.17**	**0.10**	**0.11**	**0.15**	**0.12**	**0.13**	**0.13**	
30	0.95	0.86	0.83	0.80	0.75	0.72	0.69	0.66	0.63	0.59	0.58	0.54	0.49	0.47
	0.19	**0.25**	**0.30**	**0.32**	**0.35**	**0.36**	**0.36**	**0.36**	**0.36**	**0.36**	**0.36**	**0.35**	**0.34**	**0.33**
	0.30	0.35	**0.22**	**0.11**	**0.22**	**0.12**	**0.18**	**0.10**	**0.16**	**0.12**	**0.11**	**0.12**	**0.13**	**0.13**