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Existence of infinitely many solutions for semilinear degenerate Schrödinger equations

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ABSTRACT

In this paper, we study the existence of infinitely many nontrivial solutions of to the semilinear Δ_γ differential equations in $\mathbb{R}^N$

$$\begin{cases} -\Delta_\gamma u + b(x)u = f(x,u) \text{ in } \mathbb{R}^N, \\ u \in S^2_\gamma(\mathbb{R}^N), \end{cases}$$

where Δ_γ is a subelliptic operator, the potential $b(x)$ and nonlinearity $f(x,u)$ are not assumed to be continuous. Multiplicity of nontrivial solutions for semilinear Laplace equations in $\mathbb{R}^N$ with continuous potential and nonlinearity was considered in many works, such as [4,15,18,24].

1. Introduction

In the last years, the semilinear elliptic partial differential equation

$$-\Delta u + b(x)u = f(x,u), \quad x \in \mathbb{R}^N, \qquad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

has been studied by many authors. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for problem (1.1) have been extensively investigated in the literature over the past several decades. Many papers deal with the autonomous case where the potential $b(x)$ and the nonlinearity f are independent of x, or with the radially symmetric case where $b(x)$ and f depend on $|x|$. We quote here [5,6,16,17], where the autonomous case is studied, and [2,3,12], where the radial nonautonomous case is considered. If the radial symmetry is lost, the problem becomes very different because of the lack of

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compactness. Ever since the work of W.Y. Ding and W.M. Ni [7], Y. Li [11] and P.H. Rabinowitz [15], this situation has been treated in a great number of papers under various growth conditions on $b(x)$ and f.

In this paper, we study the existence and multiplicity of nontrivial weak solutions to the following problem

$$\begin{cases} -\Delta_\gamma u + b(x)u = f(x,u) \text{ in } \mathbb{R}^N, \\ u \in S^2_\gamma(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where Δ_γ is a subelliptic operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right),\ \gamma = (\gamma_1, \gamma_2, \dots, \gamma_N) : \mathbb{R}^N \to \mathbb{R}^N.$$

The Δ_γ-operator was considered by A.E. Kogoj and E. Lanconelli in [10]. This operator has the same form as in [8], however the functions $\gamma(x)$ in [10] are more generalized than those considered in [8]. The Δ_γ-operator contains many degenerate elliptic operators such as the Grushin-type operator

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha \geq 0,$$

where x denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ (see [9,20,21]), and the operator of the form

$$P_{\alpha,\beta} := \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z, \quad (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where α, β are nonnegative real numbers (see [19,22]).

To study the problem (1.2), we make the following assumptions:

(A_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,\xi)| \leq f_1(x)\,|\xi| + f_2(x)\,|\xi|^{p-1} \text{ for almost every } (x,\xi) \in \mathbb{R}^N \times \mathbb{R},$$

where $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ are nonnegative and $f_1(x) \in L^{p_1}(\mathbb{R}^N) \cap L^{p_3}(\mathbb{R}^N) \cap L^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}(\mathbb{R}^N)$, $f_2(x) \in L^{p_2}(\mathbb{R}^N) \cap L^{p_3}(\mathbb{R}^N), 2p_1/(p_1-1) < 2^*_\gamma := \frac{2\widetilde{N}}{\widetilde{N}-2}$ (where $\widetilde{N}$ is defined by formula (2.1)) $pp_2/(p_2-1) < 2^*_\gamma, p \in (2, 2^*_\gamma), p_1, p_2 > 1, p_3 \geq \frac{2^*_\gamma}{2^*_\gamma - p}, p_3(2^*_\gamma - 2p + 2) \leq 2.2^*_\gamma$;

(A_2) $\lim\limits_{|\xi|\to\infty} \frac{|F(x,\xi)|}{\xi^2} = \infty$, for almost every $x \in \mathbb{R}^N$, and there exists $r_0 \geq 0$ such that

$$F(x,\xi) \equiv \int_0^\xi f(x,\tau)\mathrm{d}\tau \geq 0 \quad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}, |\xi| \geq r_0;$$

(A_3) There are constants $\mu > 2$ and $r_1 > 0$ such that

$$\mu F(x,\xi) \leq \xi f(x,\xi) \quad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}, |\xi| \geq r_1;$$

(A_4) $f(x,-\xi) = -f(x,\xi)$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}$;

(B_1) $b : \mathbb{R}^N \to \mathbb{R}$ such that $b \in L^1_{loc}(\mathbb{R}^N)$ and

$$\mu_0 = \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} b(x) := \sup\{\mu \in \mathbb{R} : Vol(\{x \in \mathbb{R}^N, b(x) < \mu\}) = 0\} > 0;$$

(B_2) For any $M > 0$

$$Vol\left(\left\{x \in \mathbb{R}^N, b(x) \leq M\right\}\right) < \infty,$$

where $Vol(\cdot)$ denotes the Lebesgue measure of a set in $\mathbb{R}^N$.

We would like to mention the results obtained in the past for the case $\Delta_\gamma \equiv \Delta$.

In [15], P.H. Rabinowitz assumed that $b(x), f(x, \xi)$ satisfy the following conditions:

(Rab_1) $b \in C^1(\mathbb{R}^N, \mathbb{R})$ and there is a $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \mathbb{R}^N$;

(Rab_2) $b(x) \to \infty$ as $|x| \to \infty$;

(Raf_1) $f \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}), f(x, 0) = f_\xi(x, 0) = 0$;

(Raf_2) There are constants $C_1, C_2 > 0$ and $p \in (1, (N+2)/(N-2))$ such that for all $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}$

$$|f_\xi(x, \xi)| \leq C_1 + C_2 |\xi|^{p-1};$$

(Raf_3) There is a constant $\mu > 2$ such that

$$0 < \mu F(x, \xi) \equiv \mu \int_0^\xi f(x, \tau) \mathrm{d}\tau \leq \xi f(x, \xi),$$

for all $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R} \backslash \{0\}$.

Then he proved that the problem (1.2) (for $\Delta_\gamma = \Delta$) has a nontrivial solution.

In [4], T. Bartsh and Z.Q. Wang assumed that $b(x), f(x, \xi)$ satisfy the following conditions:

(Bab_1) $b \in C(\mathbb{R}^N, \mathbb{R})$ satisfies

$$\mu_0 = \inf_{x \in \mathbb{R}^N} b(x) > 0;$$

(Bab_2) For any $M > 0$

$$Vol\left(\left\{x \in \mathbb{R}^N, b(x) \leq M\right\}\right) < \infty;$$

(Baf_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$; $f(x, \xi) = o(|\xi|)$ as $|\xi| \to 0$ uniformly in x;

(Baf_2) There are constants $C_1, C_2 > 0$ and $p \in (1, (N+2)/(N-2))$ such that for all $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}$

$$|f(x, \xi)| \leq C_1 + C_2 |\xi|^p;$$

(Baf_3) There is a constant $\mu > 2$ such that

$$0 < \mu F(x, \xi) \equiv \mu \int_0^\xi f(x, \tau) \mathrm{d}\tau \leq \xi f(x, \xi),$$

for all $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R} \backslash \{0\}$;

(Baf_4) f is odd in ξ, that is,

$$f(x, -\xi) = -f(x, \xi) \text{ for } x \in \mathbb{R}^N, \xi \in \mathbb{R}.$$

Then they proved that the problem (1.2) (for $\Delta_\gamma = \Delta$) possesses infinitely many nontrivial solutions.

Next, we can state the main theorem of the paper.

Theorem 1.1. *Assume that b and f satisfy $(A_1), (A_2), (A_3), (A_4), (B_1)$ and (B_2). Then the problem (1.2) possesses infinitely many nontrivial solutions.*

Remark 1.2. Even in the case $\Delta_\gamma \equiv \Delta$, that is $\gamma_1 = \gamma_2 = \cdots = \gamma_N \equiv 1$, our result is not covered by those in [4]. For example, when $b(x) = n+1$ for $n \leq |x| < n+1$, for every $n \in \mathbb{N}$ and $f(x,\xi) = f_1(x)\xi^3 - f_2(x)\xi$ where $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are positive $f_1 \in (L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3) \cap L^{\frac{9}{5}}(\mathbb{R}^3)) \backslash (C(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3))$, $f_2 \in L^3(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $f_1(x) > 2f_2(x)$ for all $x \in \mathbb{R}^3$, it is easy to check that $f(x,\xi), b(x)$ satisfy $(A_1), (A_2), (A_3), (A_4), (B_1), (B_2)$ but do not satisfy the conditions $(Bab_1), (Baf_1), (Baf_2)$, and function $b(x), f(x,\xi)$ do not satisfy the conditions $(V_1), (S_1)$ in [18,24] ($(V_1) : b \in C(\mathbb{R}^N, \mathbb{R})$ is bounded from below; $(S_1) : f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists constants $c_1, c_2 > 0$ and $p \in (2, \frac{2N}{N-2})$ such that $|f(x,\xi)| \leq c_1 |\xi| + c_2 |\xi|^{p-1}$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}$).

Obviously, condition (B_1) is weaker than condition (Bab_1) in [4].

The paper is organized as follows. In Section 2 for convenience of the readers, we recall some function spaces, embedding theorems and associated functional settings. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary results

2.1. Function spaces and embedding theorems

We recall the functional setting in [10,13]. We consider the operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j} \left(\gamma_j^2 \partial_{x_j}\right), \ \partial_{x_j} = \frac{\partial}{\partial x_j}, j = 1, 2, \ldots, N.$$

Here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \backslash \Pi$, where

$$\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, we assume the following properties:

i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that

$$\delta_t : \mathbb{R}^N \to \mathbb{R}, \delta_t (x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N), 1 = \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N,$$

such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.,

$$\gamma_j (\delta_t (x)) = t^{\varepsilon_j - 1} \gamma_j (x), \forall x \in \mathbb{R}^N, \forall t > 0, \ j = 1, \ldots, N.$$

The number

$$\widetilde{N} := \sum_{j=1}^{N} \varepsilon_j \tag{2.1}$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

ii)

$$\gamma_1 = 1, \gamma_j (x) = \gamma_j (x_1, x_2, \ldots, x_{j-1}), \; j = 2, \ldots, N.$$

iii) There exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \gamma_j (x) \leq \rho \gamma_j (x), \forall k \in \{1, 2, \ldots, j-1\}, \forall j = 2, \ldots, N,$$

and for every $x \in \overline{\mathbb{R}}_+^N := \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \geq 0, \forall j = 1, 2, \ldots, N\right\}$.

iv) Equalities $\gamma_j (x) = \gamma_j (x^*)$ $(j = 1, 2, \ldots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = (|x_1|, \ldots, |x_N|) \text{ if } x = (x_1, x_2, \ldots, x_N).$$

Definition 2.1. By $S^p_\gamma(\mathbb{R}^N)$ $(1 \leq p < +\infty)$ we will denote the set of all functions $u \in L^p(\mathbb{R}^N)$ such that $\gamma_j \partial_{x_j} u \in L^p(\mathbb{R}^N)$ for all $j = 1, \ldots, N$. We define the norm in this space as follows

$$\|u\|_{S^p_\gamma(\mathbb{R}^N)} = \left(\int\limits_{\mathbb{R}^N} (|u|^p + |\nabla_\gamma u|^p) \, \mathrm{d}x \right)^{\frac{1}{p}},$$

where $\nabla_\gamma u = (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u)$.

If $p = 2$ we can also define the scalar product in $S^2_\gamma(\mathbb{R}^N)$ as follows

$$(u, v)_{S^2_\gamma(\mathbb{R}^N)} = (u, v)_{L^2(\mathbb{R}^N)} + (\nabla_\gamma u, \nabla_\gamma v)_{L^2(\mathbb{R}^N)}.$$

Define

$$S^2_{\gamma,b(x)}(\mathbb{R}^N) = \left\{ u \in S^2_\gamma(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} \left(|\nabla_\gamma u|^2 + b(x) u^2 \right) \mathrm{d}x < +\infty \right\}$$

with $b(x)$ satisfying conditions $(B_1), (B_2)$, then $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ is a Hilbert space with the norm

$$\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \left(\int\limits_{\mathbb{R}^N} \left(|\nabla_\gamma u|^2 + b(x) u^2 \right) \mathrm{d}x \right)^{\frac{1}{2}}.$$

By (B_1) the embedding $S^2_{\gamma,b(x)}(\mathbb{R}^N) \hookrightarrow S^2_\gamma(\mathbb{R}^N)$ is continuous. From an embedding inequality in [1] and Hölder's inequality, we have $S^2_{\gamma,b(x)}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $2 \leq q \leq 2^*_\gamma$. Moreover, we have

Lemma 2.2. *Let $(B_1), (B_2)$ be satisfied. Then the embedding map from $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is compact for $2 \leq q < 2^*_\gamma$.*

Proof. Let $\{u_n\}_{n=1}^\infty \subset S^2_{\gamma,b(x)}(\mathbb{R}^N)$ be a bounded sequence of $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. Then, by the Sobolev embedding theorem, $u_n \to u$ strongly in $L^p_{loc}(\mathbb{R}^N)$ for $2 \leq q < 2^*_\gamma$. We claim that

$$u_n \to u \text{ strongly in } L^2(\mathbb{R}^N). \tag{2.2}$$

To prove (2.2), we only need to prove that $\alpha_n := \|u_n\|^2_{L^2(\mathbb{R}^N)} \to \|u\|^2_{L^2(\mathbb{R}^N)}$ since the space $L^2(\mathbb{R}^N)$ is uniformly convex. Assume, up to a subsequence, that $\alpha_n \to \alpha$.

Set

$$\begin{aligned} B_R &= \{x \in \mathbb{R}^N : |x| < R\}, \\ \mathbb{R}^N_{M,b(x),R} &= \{x \in \mathbb{R}^N \backslash B_R : b(x) \geq M\}, \\ \complement\mathbb{R}^N_{M,b(x),R} &= \{x \in \mathbb{R}^N \backslash B_R : b(x) < M\}, \end{aligned}$$

then

$$\begin{aligned} \int\limits_{\mathbb{R}^N_{M,b(x),R}} |u_n|^2 \,\mathrm{d}x &\leq \int\limits_{\mathbb{R}^N_{M,b(x),R}} \frac{b(x)}{M} |u_n|^2 \,\mathrm{d}x \\ &\leq \frac{1}{M} \int\limits_{\mathbb{R}^N} \left(|\nabla_\gamma u_n|^2 + b(x) u_n^2 \right) \mathrm{d}x \leq \frac{\|u_n\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}{M}. \end{aligned}$$

Choose $s \in \left(1, \frac{\widetilde{N}}{\widetilde{N}-2}\right)$ and s' such that $\frac{1}{s} + \frac{1}{s'} = 1$, then applying Hölder's inequality we have

$$\begin{aligned} \int\limits_{\complement\mathbb{R}^N_{M,b(x),R}} |u_n|^2 \,\mathrm{d}x &\leq \left(\int\limits_{\complement\mathbb{R}^N_{M,b(x),R}} |u_n|^{2s} \right)^{\frac{1}{s}} \left(Vol(\complement\mathbb{R}^N_{M,b(x),R}) \right)^{\frac{1}{s'}} \\ &\leq C \|u_n\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \left(Vol(\complement\mathbb{R}^N_{M,b(x),R}) \right)^{\frac{1}{s'}}. \end{aligned}$$

Since $\left\{ \|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \right\}_{n=1}^\infty$ is bounded and conditions $(B_1), (B_2)$ hold, we may choose R, M large enough such that the quantities $\frac{\|u_n\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}{M}$ and $\left(Vol(\complement\mathbb{R}^N_{M,b(x),R}) \right)^{\frac{1}{s'}}$ are small enough. Hence, for all $\varepsilon > 0$, we have

$$\int\limits_{\mathbb{R}^N \backslash B_R} |u_n|^2 \,\mathrm{d}x = \int\limits_{\mathbb{R}^N_{M,b(x),R}} |u_n|^2 \,\mathrm{d}x + \int\limits_{\complement\mathbb{R}^N_{M,b(x),R}} |u_n|^2 \,\mathrm{d}x < \varepsilon.$$

Thus,

$$\begin{aligned} \|u\|^2_{L^2(\mathbb{R}^N)} &= \|u\|^2_{L^2(B_R)} + \|u\|^2_{L^2(\mathbb{R}^N \backslash B_R)} \\ &\geq \lim_{n\to\infty} \|u_n\|^2_{L^2(B_R)} = \lim_{n\to\infty} \left(\|u_n\|^2_{L^2(\mathbb{R}^N)} - \|u\|^2_{L^2(\mathbb{R}^N \backslash B_R)} \right) \geq \alpha^2 - \varepsilon. \end{aligned}$$

On the other hand, let Ω be an arbitrary domain in $\mathbb{R}^N$, then

$$\int\limits_{\Omega} |u_n|^2 \,\mathrm{d}x \leq \int\limits_{\mathbb{R}^N} |u_n|^2 \,\mathrm{d}x \to \alpha^2,$$

hence $\|u\|_{L^2(\mathbb{R}^N)} \leq \alpha$. By the arbitrariness of ε, we have $\alpha = \|u\|_{L^2(\mathbb{R}^N)}$. So (2.2) is proved.

Finally, we prove that $u_n \to u$ in $L^q(\mathbb{R}^N)$ for $2 \leq q < 2^*_\gamma$. In fact, if $q \in (2, 2^*_\gamma)$, there is a number $\theta \in (0,1)$ such that $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{2^*_\gamma}$. Then by Hölder's inequality,

$$\|u_n - u\|_{L^q(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} |u_n - u|^{\theta p} |u_n - u|^{(1-\theta)q} \, \mathrm{d}x \le \|u_n - u\|_{L^2(\mathbb{R}^N)}^{\theta q} \|u_n - u\|_{L^{2_\gamma^*}(\mathbb{R}^N)}^{(1-\theta)q}.$$

Since u_n is bounded in $L^{2_\gamma^*}(\mathbb{R}^N)$ and $\|u_n - u\|_{L^2(\mathbb{R}^N)} \to 0$, we have $u_n \to u$ in $L^q(\mathbb{R}^N)$. □

Lemma 2.3. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying*

$$|f(x,\xi)| \le f_1(x) |\xi| + f_2(x) |\xi|^{p-1} \quad \textit{almost everywhere} \quad (x,\xi) \in \mathbb{R}^N \times \mathbb{R},$$

where $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ are nonnegative and $f_1(x) \in L^{p_1}(\mathbb{R}^N) \cap L^{p_3}(\mathbb{R}^N) \cap L^{\frac{2_\gamma^* p_3}{p_3(p-1)+2_\gamma^*}}(\mathbb{R}^N)$, $f_2(x) \in L^{p_2}(\mathbb{R}^N) \cap L^{p_3}(\mathbb{R}^N), 2p_1/(p_1-1) \le 2_\gamma^*, pp_2/(p_2-1) \le 2_\gamma^*, p \in (2, 2_\gamma^*), p_1, p_2 > 1, p_3 \ge \frac{2_\gamma^*}{2_\gamma^* - p}, p_3(2_\gamma^* - 2p + 2) \le 2.2_\gamma^*$. Then $\Phi_1(u) \in C^1(S_\gamma^2(\mathbb{R}^N), \mathbb{R})$ and

$$\Phi_1'(u)(v) = \int_{\mathbb{R}^N} f(x,u) v \mathrm{d}x$$

for all $v \in S_\gamma^2(\mathbb{R}^N)$, where

$$\Phi_1(u) = \int_{\mathbb{R}^N} F(x,u) \mathrm{d}x.$$

Proof. First, we prove the existence of the Gâteaux derivative of $\Phi_1(u)$. Let $u, v \in S_\gamma^2(\mathbb{R}^N)$. Given $x \in \mathbb{R}^N$, $t \in \mathbb{R}$ and $0 < |t| < 1$ by the Mean Value Theorem

$$\begin{aligned} &\left| \frac{F(x, u(x) + tv(x)) - F(x, u(x))}{t} \right| \\ &\le \left[f_1(x)(|u(x)| + |v(x)|) + 2^{p-1} f_2(x) \left(|u(x)|^{p-1} + |v(x)|^{p-1} \right) \right] |v(x)| := \mathcal{F}(x). \end{aligned}$$

Applying Hölder's inequality, we conclude that

$$\int_{\mathbb{R}^N} f_1(x) |v(x)|^2 \, \mathrm{d}x \le \|f_1\|_{L^{p_1}(\mathbb{R}^N)} \|v\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)}^2,$$

$$\int_{\mathbb{R}^N} f_2(x) |v(x)|^p \, \mathrm{d}x \le \|f_2\|_{L^{p_2}(\mathbb{R}^N)} \|v\|_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}^p,$$

$$\int_{\mathbb{R}^N} f_1(x) |u(x)| |v(x)| \, \mathrm{d}x \le \|f_1\|_{L^{p_1}(\mathbb{R}^N)} \|u\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} \|v\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)},$$

$$\int_{\mathbb{R}^N} f_2(x) |u(x)|^{p-1} |v(x)| \, \mathrm{d}x \le \|f_2\|_{L^{p_2}(\mathbb{R}^N)} \|u\|_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}^{p-1} \|v\|_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}.$$

Hence

$$\left| \frac{F(x, u(x) + tv(x)) - F(x, u(x))}{t} \right| \le \mathcal{F}(x) \in L^1(\mathbb{R}^N).$$

Therefore, for any $u, v \in S^2_\gamma(\mathbb{R}^N)$ by the Mean Value Theorem and Lebesgue's Theorem, we have

$$\lim_{t\to 0}\frac{\Phi_1(u+tv)-\Phi_1(u)}{t} = \int\limits_{\mathbb{R}^N}\lim_{t\to 0}\frac{F(x,u(x)+tv(x))-F(x,u(x))}{t}\mathrm{d}x$$
$$= \int\limits_{\mathbb{R}^N} f(x,u(x))v(x)\mathrm{d}x := F_0(u,v).$$

Obviously, $F_0(u,v)$ is linear in v. Further, applying Hölder's and Sobolev's inequalities, we have

$$|F_0(u,v)| \le \int\limits_{\mathbb{R}^N}\left(f_1(x)|u(x)||v(x)| + f_2(x)|u(x)|^{p-1}|v(x)|\right)\mathrm{d}x$$
$$\le \|f_1\|_{L^{p_1}(\mathbb{R}^N)}\|u\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)}\|v\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} + \|f_2\|_{L^{p_2}(\mathbb{R}^N)}\|u\|^{p-1}_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}\|v\|_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}$$
$$\le C\left(\|f_1\|_{L^{p_1}(\mathbb{R}^N)}\|u\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} + \|f_2\|_{L^{p_2}(\mathbb{R}^N)}\|u\|^{p-1}_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}\right)\|v\|_{S^2_\gamma(\mathbb{R}^N)}.$$

It follows that $F_0(u,v)$ is linear and bounded in v on $S^2_\gamma(\mathbb{R}^N)$. Therefore, $D\Phi_1(u) = F_0(u,\cdot)$ is the Gâteaux derivative of Φ_1 at u.

Next, we establish that the Gâteaux derivative of $\Phi_1(u)$ is continuous in u in the uniform $(S^2_\gamma(\mathbb{R}^N))^*$-topology. Assume that $u_n \to u$ in $S^2_\gamma(\mathbb{R}^N)$ by a Sobolev embedding theorem, hence

$$\begin{aligned} &u_n \to u \text{ in } L^q(\mathbb{R}^N), 2\le q\le 2^*_\gamma \text{ as } n\to\infty,\\ &u_n \to u \text{ a.e. in } \mathbb{R}^N \text{ as } n\to\infty. \end{aligned} \tag{2.3}$$

Since f is a Carathéodory function

$$f(x,u_n) \to f(x,u) \text{ a.e. in } \mathbb{R}^N \text{ as } n\to\infty. \tag{2.4}$$

Let

$$\begin{aligned} \varphi_n(x) &:= f_1(x) + (f_1(x)+f_2(x))\,|u_n(x)|^{p-1}, \text{ for any } n = 1,2,\dots,\\ \varphi(x) &:= f_1(x) + (f_1(x)+f_2(x))\,|u(x)|^{p-1}. \end{aligned} \tag{2.5}$$

Then by (2.3), (2.5) and (A_1), we have

$$|f(x,u_n)| \le \varphi_n(x) \text{ for almost every } x\in\mathbb{R}^N,\ \varphi_n(x)\to\varphi(x) \text{ a.e. in } \mathbb{R}^N. \tag{2.6}$$

For any $v \in S^2_\gamma(\mathbb{R}^N)$ applying Hölder's and Sobolev's inequalities, we have

$$|\langle D\Phi_1(u_n) - D\Phi_1(u), v\rangle| = \left|\int\limits_{\mathbb{R}^N}(f(x,u_n)-f(x,u))v\mathrm{d}x\right|$$
$$\le \|f(x,u_n)-f(x,u)\|_{L^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}(\mathbb{R}^N)}\|v\|_{L^{\frac{2^*_\gamma p_3}{2^*_\gamma p_3 - p_3(p-1)-2^*_\gamma}}(\mathbb{R}^N)}$$
$$\le C\|f(x,u_n)-f(x,u)\|_{L^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}(\mathbb{R}^N)}\|v\|_{S^2_\gamma(\mathbb{R}^N)}.$$

This implies that

$$\|D\Phi_1(u_n) - D\Phi_1(u)\|_{(S^2_\gamma(\mathbb{R}^N))^*} \le C\,\|f(x,u_n) - f(x,u)\|_{L^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}(\mathbb{R}^N)}. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} |\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^N} |f_1(x) + f_2(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \left||u_n(x)|^{p-1} - |u(x)|^{p-1}\right|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}\,\mathrm{d}x \\
&\le \left(\int\limits_{\mathbb{R}^N} |f_1(x) + f_2(x)|^{p_3}\,\mathrm{d}x\right)^{\frac{2^*_\gamma}{p_3(p-1)+2^*_\gamma}} \left(\int\limits_{\mathbb{R}^N} \left||u_n|^{p-1} - |u|^{p-1}\right|^{\frac{2^*_\gamma}{p-1}}\,\mathrm{d}x\right)^{\frac{p_3(p-1)}{p_3(p-1)+2^*_\gamma}}.
\end{aligned}$$

Since $u_n \to u$ in $L^q(\mathbb{R}^N)$ for all $2 \le q \le 2^*_\gamma$, we have $\int\limits_{\mathbb{R}^N} \left||u_n|^{p-1} - |u|^{p-1}\right|^{\frac{2^*_\gamma}{p-1}}\,\mathrm{d}x \to 0$. Therefore

$$\int\limits_{\mathbb{R}^N} |\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}\,\mathrm{d}x \to 0,\ \text{as } n \to \infty. \tag{2.8}$$

Since there are constants $C > 0, \overline{C} > 0$ such that

$$\begin{aligned}
&|f(x,u_n) - f(x,u)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \\
&\le C\,|f(x,u_n)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + C\,|f(x,u)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \\
&\le C\,|\varphi_n(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + C\,|\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \\
&\le \overline{C}\,|\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + \overline{C}\,|\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}, \quad \text{for almost every } x \in \mathbb{R}^N,
\end{aligned}$$

by Fatou's Lemma

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \liminf_{n\to\infty} \Big(\overline{C}\,|\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + \overline{C}\,|\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \\
&\qquad\qquad - |f(x,u_n) - f(x,u)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}\Big)\mathrm{d}x \\
&\le \liminf_{n\to\infty} \int\limits_{\mathbb{R}^N} \Big(\overline{C}\,|\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + \overline{C}\,|\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \\
&\qquad\qquad - |f(x,u_n) - f(x,u)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}\Big)\mathrm{d}x
\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&\leq \limsup_{n\to\infty} \int\limits_{\mathbb{R}^N} \Big(\overline{C}\, |\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + \overline{C}\, |\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \\
&\qquad\qquad - |f(x,u_n) - f(x,u)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \Big) \mathrm{d}x \\
&\leq \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} \Big(\overline{C}\, |\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + \overline{C}\, |\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \Big) \mathrm{d}x.
\end{aligned}
$$

Moreover from (2.4), (2.6) and (2.8), we have

$$
\begin{aligned}
&\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} \Big(\overline{C}\, |\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + \overline{C}\, |\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \Big) \mathrm{d}x = \overline{C} \int\limits_{\mathbb{R}^N} |\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \,\mathrm{d}x, \\
&\int\limits_{\mathbb{R}^N} \liminf_{n\to\infty} \Big(\overline{C}\, |\varphi_n(x) - \varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} + \overline{C}\, |\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \\
&\qquad - |f(x,u_n) - f(x,u)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \Big) \mathrm{d}x = \overline{C} \int\limits_{\mathbb{R}^N} |\varphi(x)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \,\mathrm{d}x.
\end{aligned}
\tag{2.10}
$$

Combining (2.9) and (2.10) we derive that

$$
\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |f(x,u_n) - f(x,u)|^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}} \,\mathrm{d}x = 0.
$$

Therefore, $D\Phi_1(u_n) \to D\Phi_1(u)$. This means that $D\Phi_1(u)$ is continuous in u. Hence, $\Phi_1'(u) = D\Phi_1(u)$, i.e., $\Phi_1 \in C^1(S^2_\gamma(\mathbb{R}^N), \mathbb{R})$. This proves the theorem. $\square$

Remark 2.4. Lemma 2.3 is a generalization of Lemma 3.10 in [23].

Define the Euler–Lagrange functional associated with the problem (1.2) as follows

$$
\Phi(u) = \frac{1}{2} \int\limits_{\mathbb{R}^N} \left(|\nabla_\gamma u|^2 + b(x)u^2 \right) \mathrm{d}x - \int\limits_{\mathbb{R}^N} F(x,u)\mathrm{d}x.
$$

From Lemma 2.3 and f satisfies (A_1), $b(x)$ satisfies (B_1), we have Φ is well defined on $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ and $\Phi \in C^1(S^2_{\gamma,b(x)}(\mathbb{R}^N), \mathbb{R})$ with

$$
\Phi'(u)(v) = \int\limits_{\mathbb{R}^N} (\nabla_\gamma u \cdot \nabla_\gamma v + b(x)uv)\, \mathrm{d}x - \int\limits_{\mathbb{R}^N} f(x,u)\, v\mathrm{d}x
$$

for all $v \in S^2_{\gamma,b(x)}(\mathbb{R}^N)$. One can also check that the critical points of Φ are weak solutions of problem (1.2).

2.2. Mountain Pass Theorem

Definition 2.5. Let $\mathbb{X}$ be a real Banach space with its dual space $\mathbb{X}^*$ and $J \in C^1(\mathbb{X}, \mathbb{R})$. For $c \in \mathbb{R}$ we say that J satisfies the $(C)_c$ condition if for any sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{X}$ with

$$
J(x_n) \to c \text{ and } (1 + \|x_n\|_{\mathbb{X}})\, \|J'(x_n)\|_{\mathbb{X}^*} \to 0,
$$

then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges strongly in $\mathbb{X}$. If J satisfies the $(C)_c$ condition for all $c > 0$ then we say that J satisfies the Cerami condition.

We will use the following version of the Mountain Pass Theorem.

Lemma 2.6 *(see [14]). Let $\mathbb{X}$ be an infinite dimensional Banach space, $\mathbb{X} = \mathbb{Y} \bigoplus \mathbb{Z}$, where $\mathbb{Y}$ is finite dimensional and let $J \in C^1(\mathbb{X}, \mathbb{R})$ satisfy the $(C)_c$ condition for all $c > 0, J(0) = 0, J(-u) = J(u)$ for all $u \in \mathbb{X}$, and*

(i) There are constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in \mathbb{Z}$ such that $\|u\|_{\mathbb{X}} = \rho$;
(ii) For any finite dimensional subspace $\widehat{\mathbb{X}} \subset \mathbb{X}$, there is $R = R(\widehat{\mathbb{X}}) > 0$ such that $J(u) \leq 0$ on $\widehat{\mathbb{X}} \backslash B_R$.

Then J possesses an unbounded sequence of critical values.

3. Proof of Theorem 1.1

We prove Theorem 1.1 by verifying that all conditions of Lemma 2.6 are satisfied. First, we check the Cerami condition in this lemma:

Lemma 3.1. *Let $(A_1), (A_3), (B_1)$ and (B_2) be satisfied. Then Φ satisfies the $(C)_c$ condition for all $c > 0$ on $S^2_{\gamma,b(x)}(\mathbb{R}^N)$.*

Proof. Let $\{u_m\}_{m=1}^{\infty}$ be a sequence in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ such that

$$\left(1 + \|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right) \|\Phi'(u_m)\|_{(S^2_{\gamma,b(x)}(\mathbb{R}^N))^*} \to 0 \text{ and } \Phi(u_m) \to c > 0 \text{ as } m \to \infty, \tag{3.1}$$

hence

$$\Phi'(u_m)(u_m) \to 0 \quad \text{and} \quad \frac{1}{2} \|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N} F(x, u_m)\mathrm{d}x \to c \text{ as } m \to \infty. \tag{3.2}$$

When m is large enough, we have

$$\begin{aligned} c + 1 &\geq \Phi(u_m) - \frac{1}{\mu}\Phi'(u_m)(u_m) \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} \left(\frac{1}{\mu} f(x, u_m)u_m - F(x, u_m)\right) \mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} + \int\limits_{\Omega_m(0,r_1)} \left(\frac{1}{\mu} f(x, u_m)u_m - F(x, u_m)\right) \mathrm{d}x \\ &+ \int\limits_{\Omega_m(r_1,\infty)} \left(\frac{1}{\mu} f(x, u_m)u_m - F(x, u_m)\right) \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} + \int\limits_{\Omega_m(0,r_1)} \left(\frac{1}{\mu} f(x, u_m)u_m - F(x, u_m)\right) \mathrm{d}x, \end{aligned} \tag{3.3}$$

where $\Omega_m(a, b) = \{x \in \mathbb{R}^N : a \leq |u_m(x)| < b\}$ for $0 \leq a < b$.

We first show that $\{u_m\}_{m=1}^{\infty}$ is bounded in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ by a contradiction argument. Indeed, suppose that

$$\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to \infty \text{ as } m \to \infty. \tag{3.4}$$

Setting

$$v_m = \frac{u_m}{\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}},$$

then $\|v_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$. Passing to a subsequence, we may assume that $v_m \rightharpoonup v$ weakly in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$, then by Lemma 2.2, $v_m \to v$ strongly in $L^q(\mathbb{R}^N), 2 \le q < 2^*_\gamma$, and $v_m \to v$ a.e. on $\mathbb{R}^N$.

From (3.3) and (3.4), we obtain

$$\limsup_{m\to\infty} \frac{1}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} \int_{\Omega_m(0,r_1)} \left(\frac{1}{\mu} f(x,u_m)u_m - F(x,u_m)\right) \mathrm{d}x \le \frac{1}{\mu} - \frac{1}{2} < 0. \tag{3.5}$$

If $v \equiv 0$, then $v_m \to 0$ strongly in $L^q(\mathbb{R}^N), 2 \le q < 2^*_\gamma$, and $v_m \to 0$ a.e. on $\mathbb{R}^N$. Hence, it follows from (A_1) that

$$\begin{aligned}
\left| \int_{\Omega_m(0,r_1)} \frac{f(x,u_m)u_m - \mu F(x,u_m)}{\mu \|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} \mathrm{d}x \right| &\le \int_{\Omega_m(0,r_1)} \frac{|f(x,u_m)u_m - \mu F(x,u_m)|}{\mu |u_m|^2} |v_m|^2 \mathrm{d}x \\
&\le C \int_{\Omega_m(0,r_1)} (|f_1(x)| + |f_2(x)|) |v_m|^2 \mathrm{d}x \le C \int_{\mathbb{R}^N} (|f_1(x)| + |f_2(x)|) |v_m|^2 \mathrm{d}x \\
\le C \left(\|f_1\|_{L^{p_1}(\mathbb{R}^N)} \|v_m\|^2_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} \right. &\left. + \|f_2\|_{L^{p_2}(\mathbb{R}^N)} \|v_m\|^2_{L^{\frac{2p_2}{p_2-1}}(\mathbb{R}^N)} \right) \to 0 \text{ as } m \to \infty,
\end{aligned}$$

which contradicts (3.5).

Set $\Omega^* = \{x \in \mathbb{R}^N : v(x) \ne 0\}$ then $Vol(\Omega^*) > 0$. For almost every $x \in \Omega^*$, we have $\lim\limits_{m\to\infty} |u_m(x)| = \infty$. Hence $\Omega^* = \Omega_1^* \bigcup \Omega_2^*$ where $\Omega_2^* \subset \Omega_m(r_0,\infty), Vol(\Omega_1^*) = 0$ for large $m \in \mathbb{N}$. It follows from (A_1), (3.2) and Fatou's Lemma that

$$\begin{aligned}
0 &= \lim_{m\to\infty} \frac{c+1}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} = \lim_{m\to\infty} \frac{\Phi(u_m)}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} \\
&= \lim_{m\to\infty} \left[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u_m)}{u_m^2} v_m^2 \mathrm{d}x \right] \\
&= \lim_{m\to\infty} \left[\frac{1}{2} - \int_{\Omega_m(0,r_0)} \frac{F(x,u_m)}{u_m^2} v_m^2 \mathrm{d}x - \int_{\Omega_m(r_0,\infty)} \frac{F(x,u_m)}{u_m^2} v_m^2 \mathrm{d}x \right] \\
&\le \limsup_{m\to\infty} \left[\frac{1}{2} + \|f_1\|_{L^{p_1}(\mathbb{R}^N)} \|v_m\|^2_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} + \|f_2\|_{L^{p_2}(\mathbb{R}^N)} \|v_m\|^2_{L^{\frac{2p_2}{p_2-1}}(\mathbb{R}^N)} \right. \\
&\quad \left. - \int_{\Omega_m(r_0,\infty)} \frac{F(x,u_m)}{u_m^2} v_m^2 \mathrm{d}x \right] \le C_1 - \liminf_{m\to\infty} \int_{\Omega_m(r_0,\infty)} \frac{|F(x,u_m)|}{u_m^2} v_m^2 \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq C_1 - \liminf_{m\to\infty} \int\limits_{\mathbb{R}^N} \frac{|F(x,u_m)|}{u_m^2} \chi_{\Omega_m(r_0,\infty)}(x) v_m^2 \mathrm{d}x \\
&\leq C_1 - \int\limits_{\mathbb{R}^N} \liminf_{m\to\infty} \frac{|F(x,u_m)|}{u_m^2} \chi_{\Omega_m(r_0,\infty)}(x) v_m^2 \mathrm{d}x = -\infty
\end{aligned} \tag{3.6}
$$

which is a contradiction, where χ_I denotes the characteristic function associated to the mensurable subset $I \subset \mathbb{R}$. Thus $\{u_m\}_{m=1}^{\infty}$ is bounded in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$.

Because of the above result, without loss of generality, we can suppose that

$$
\begin{aligned}
&u_m \rightharpoonup u \text{ in } S^2_{\gamma,b(x)}(\mathbb{R}^N) \quad \text{as } m\to\infty \\
&u_m \to u \text{ in } L^q(\mathbb{R}^N) \quad \text{as } m\to\infty, 2\leq q < 2^*_\alpha.
\end{aligned} \tag{3.7}
$$

By (A_1), we obtain

$$
\begin{aligned}
\left| \int\limits_{\mathbb{R}^N} f(x,u_m)(u_m-u)\mathrm{d}x \right| &\leq \int\limits_{\mathbb{R}^N} |f_1(x)|\,|u_m|\,|u_m-u|\,\mathrm{d}x + \int\limits_{\mathbb{R}^N} |u_m-u|\,|u_m|^{p-1}\,|f_2(x)|\,\mathrm{d}x \\
&\leq \|u_m-u\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} \|u_m\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} \|f_1\|_{L^{p_1}(\mathbb{R}^N)} \\
&\quad + \|u_m-u\|_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)} \|u_m\|^{p-1}_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)} \|f_2\|_{L^{p_2}(\mathbb{R}^N)}.
\end{aligned}
$$

Since (3.7), we can conclude that

$$
\int\limits_{\mathbb{R}^N} f(x,u_m)(u_m-u)\mathrm{d}x \to 0 \text{ as } m\to\infty.
$$

Therefore

$$
\int\limits_{\mathbb{R}^N} \left[f(x,u_m)-f(x,u)\right](u_m-u)\mathrm{d}x \to 0 \text{ as } m\to\infty. \tag{3.8}
$$

Observe that

$$
\|u_m-u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \langle \Phi'(u_m)-\Phi'(u), u_m-u\rangle + \int\limits_{\mathbb{R}^N} \left[f(x,u_m)-f(x,u)\right](u_m-u)\mathrm{d}x.
$$

It is clear that

$$
\langle \Phi'(u_m)-\Phi'(u), u_m-u\rangle \to 0 \text{ as } m\to\infty. \tag{3.9}
$$

From (3.8)–(3.9), we deduce that

$$
\|u_m-u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to 0 \text{ as } m\to\infty.
$$

Therefore, we conclude that $u_m \to u$ strongly in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. The proof of Lemma 3.1 is complete. $\square$

Lemma 3.2. *Let* (A_1)*–*$(A_3), (B_1)$ *and* (B_2) *be satisfied. Then for any finite dimensional subspace* $\widehat{\mathbb{X}} \subset S^2_{\gamma,b(x)}(\mathbb{R}^N)$*, there is* $R = R(\widehat{\mathbb{X}}) > 0$ *such that*

$$\Phi(u) \leq 0, \quad \forall u \in \widehat{\mathbb{X}}, \ \ \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \geq R.$$

Proof. Arguing by contradiction, suppose that for some sequence $\{u_n\}_{n=1}^{\infty} \subset \widehat{\mathbb{X}}$ with $\|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to \infty$, there is $M > 0$ such that $\Phi(u_n) \geq -M$ for all $n \in \mathbb{N}$. Set

$$v_n(x) = \frac{u_n}{\|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}},$$

then $\|v_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$. Therefore we can (by passing to a subsequence if necessary) suppose that

$$\begin{aligned}
&v_n \rightharpoonup v \text{ weakly in } S^2_{\gamma,b(x)}(\mathbb{R}^N) \ \text{ as } n \to \infty,\\
&v_n \to v \text{ a.e. in } \mathbb{R}^N \ \text{ as } n \to \infty,\\
&v_n \to v \text{ strongly in } L^q(\mathbb{R}^N) \ \text{ as } n \to \infty, 2 \leq q < 2^*_\gamma.
\end{aligned}$$

Since $\widehat{\mathbb{X}}$ is finite dimensional, then

$$v_n \to v \text{ strongly in } \widehat{\mathbb{X}} \ \text{ as } n \to \infty$$

and $v \in \widehat{\mathbb{X}}, \|v\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$. Therefore, it follows from (3.6) that

$$\begin{aligned}
0 = \lim_{m\to\infty} \frac{-M}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} &\leq \lim_{m\to\infty} \frac{\Phi(u_m)}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\\
&\leq C_1 - \int\limits_{\mathbb{R}^N} \liminf_{m\to\infty} \frac{|F(x,u_m)|}{u_m^2} \chi_{\Omega_m(r_0,\infty)}(x) v_m^2 \mathrm{d}x = -\infty.
\end{aligned}$$

Hence we arrive at a contradiction. So, there is $R = R(\widehat{\mathbb{X}}) > 0$ such that $\Phi(u) \leq 0$ for $u \in \widehat{\mathbb{X}}$ and $\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \geq R$. $\square$

Let $\{e_j\}_{j=1}^{\infty}$ be a total orthonormal basis of $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ and define $\mathbb{X}_j = \mathbb{R}e_j$,

$$\mathbb{Y}_k = \bigoplus_{j=1}^{k} \mathbb{X}_j, \quad \mathbb{Z}_k = \bigoplus_{j=k+1}^{\infty} \mathbb{X}_j, \quad k \in \mathbb{N}.$$

Let

$$\beta_k = \sup_{\substack{u\in\mathbb{Z}_k\\ \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}=1}} \|u\|_{L^q(\mathbb{R}^N)}, 2 \leq q < 2^*_\alpha \tag{3.10}$$

then $\beta_k \to 0$ as $k \to \infty$. Indeed, suppose that this is not the case. Then there is an $\varepsilon_0 > 0$ and $\{u_j\}_{j=1}^{\infty} \subset S^2_{\gamma,b(x)}(\mathbb{R}^N), \|u_j\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$, with $u_j \perp \mathbb{Y}_{k_j}, \|u_j\|_{L^q(\mathbb{R}^N)} \geq \varepsilon_0$ where $k_j \to \infty$ as $j \to \infty$. For any $v \in S^2_{\gamma,b(x)}(\mathbb{R}^N)$, we may find a $w_j \in \mathbb{Y}_{k_j}$ such that $w_j \to v$ as $j \to \infty$. Therefore

$$\left|(u_j, v)_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right| = \left|(u_j, w_j - v)_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right| \leq \|w_j - v\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}$$

as $j \to \infty$, i.e., $u_j \rightharpoonup 0$ weakly in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. Hence, $u_j \to 0$ in $L^p(\mathbb{R}^N)$, a contradiction.

Lemma 3.3. *Let $(A_1), (B_1)$ and (B_2) be satisfied. Then there exists constants $\rho, \alpha, k > 0$ such that $\Phi(u) \geq \alpha$ for all $u \in \mathbb{Z}_k$ such that $\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \rho$.*

Proof. For any $u \in \mathbb{Z}_k$, using Hölder's inequality, we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \|f_1\|_{L^{p_1}(\mathbb{R}^N)}\|u\|^2_{L^{\frac{p_1}{p_1-1}}(\mathbb{R}^N)} - \|f_2\|_{L^{p_2}(\mathbb{R}^N)}\|u\|^p_{L^{\frac{pp_1}{p_1-1}}(\mathbb{R}^N)} \\ &= \frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \|f_1\|_{L^{p_1}(\mathbb{R}^N)}\left\|\frac{u}{\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\right\|^2_{L^{\frac{p_1}{p_1-1}}(\mathbb{R}^N)}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \\ &\quad - \|f_2\|_{L^{p_2}(\mathbb{R}^N)}\left\|\frac{u}{\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\right\|^p_{L^{\frac{pp_1}{p_1-1}}(\mathbb{R}^N)}\|u\|^p_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}.\end{aligned}$$

Because $2 \leq 2p_1/(p_1-1) < 2^*_\gamma, 2 \leq pp_2/(p_2-1) < 2^*_\gamma$, we have

$$\Phi(u) \geq \frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \|f_1\|_{L^{p_1}(\mathbb{R}^N)}\beta_k^2\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \|f_2\|_{L^{p_2}(\mathbb{R}^N)}\beta_k^p\|u\|^p_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}.$$

By (3.10), we can choose k large enough, and $\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \frac{1}{2}$ such that

$$\frac{1}{8} - \frac{1}{4}\|f_1\|_{L^{p_1}(\mathbb{R}^N)}\beta_k^2 - \|f_2\|_{L^{p_2}(\mathbb{R}^N)}\beta_k^p\frac{1}{2^p} = \alpha > 0. \quad \square$$

Proof of Theorem 1.1. Let $\mathbb{X} \equiv S^2_{\gamma,b(x)}(\mathbb{R}^N), \mathbb{Y} \equiv \mathbb{Y}_k, \mathbb{Z} \equiv \mathbb{Z}_k$. By Lemmas 3.1, 3.2 and 3.3 all conditions of Lemma 2.6 are satisfied. Thus, problem (1.2) possesses infinitely many nontrivial solutions. $\square$

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References

[1] C.T. Anh, Global attractor for a semilinear strongly degenerate parabolic equation on $\mathbb{R}^N$, NoDEA Nonlinear Differential Equations Appl. 21 (5) (2014) 663–678.
[2] T. Bartsch, M. Willem, Infinitely many nonradial solutions of a Euclidean scalar field equation, J. Funct. Anal. 117 (2) (1993) 447–460.
[3] T. Bartsch, M. Willem, Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 124 (3) (1993) 261–276.
[4] T. Bartsh, Z.Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Comm. Partial Differential Equations 20 (9–10) (1995) 1725–1741.
[5] H. Berestycki, P.L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (4) (1983) 313–345.
[6] H. Berestycki, P.L. Lions, Nonlinear scalar field equations. II. Existence of infinitely many solutions, Arch. Ration. Mech. Anal. 82 (4) (1983) 347–375.
[7] W.Y. Ding, W.M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation, Arch. Ration. Mech. Anal. 91 (4) (1986) 283–308.
[8] B. Franchi, E. Lanconelli, An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality, Comm. Partial Differential Equations 9 (13) (1984) 1237–1264.
[9] V.V. Grushin, A certain class of hypoelliptic operators, Mat. Sb. (N.S.) 83 (1970) 456–473 (in Russian).
[10] A.E. Kogoj, E. Lanconelli, On semilinear Δ_λ-Laplace equation, Nonlinear Anal. 75 (12) (2012) 4637–4649.
[11] Y. Li, Remarks on a semilinear elliptic equation on $\mathbb{R}^N$, J. Differential Equations 74 (1) (1988) 34–49.

[12] Y. Li, Nonautonomous nonlinear scalar field equations, Indiana Univ. Math. J. 39 (2) (1990) 283–301.
[13] D.T. Luyen, N.M. Tri, Existence of solutions to boundary value problems for semilinear Δ_γ differential equations, Math. Notes 97 (1) (2015) 73–84.
[14] P.H. Rabinowitz, Multiple critical points of perturbed symmetric functionals, Trans. Amer. Math. Soc. 272 (2) (1982) 753–769.
[15] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (2) (1992) 270–291.
[16] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (2) (1977) 149–162.
[17] M. Struwe, Multiple solutions of differential equations without the Palais–Smale condition, Math. Ann. 261 (3) (1982) 399–412.
[18] X.H. Tang, Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity, J. Math. Anal. Appl. 401 (1) (2013) 407–415.
[19] P.T. Thuy, N.M. Tri, Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations, NoDEA Nonlinear Differential Equations Appl. 19 (3) (2012) 279–298.
[20] N.T.C. Thuy, N.M. Tri, Some existence and nonexistence results for boundary value problems for semilinear elliptic degenerate operators, Russ. J. Math. Phys. 9 (3) (2002) 365–370.
[21] N.M. Tri, On the Grushin equation, Mat. Zametki 63 (1) (1998) 95–105.
[22] N.M. Tri, Recent Progress in the Theory of Semilinear Equations Involving Degenerate Elliptic Differential Operators, Publishing House for Science and Technology of the Vietnam Academy of Science and Technology, 2014, 380 pp.
[23] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996, x+162 pp.
[24] Q. Zhang, B. Xu, Multiplicity of solutions for a class of semilinear Schrödinger equations with sign-changing potential, J. Math. Anal. Appl. 377 (2) (2011) 834–840.