Switching law design for finite-time stability of singular fractional-order systems with delay

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Abstract

In this paper, we present an approach based on the Laplace transform and "inf-sup" method is proposed for studying finite-time stability of singular fractional-order switched systems with delay. A constructive geometric design for switching laws based on the construction of a partition of the stability state regions in convex cones is proposed. Using the proposed approach, new delay-dependent sufficient conditions such that the system is regular, impulse-free and finite-time stable are developed in terms of tractable matrix inequalities and Mittag-Leffler functions. An example is provided to illustrate the effectiveness of the proposed results.

Keywords: Fractional derivative, switching law, singularity, finite-time stability, Mittag-Leffler functions, time delay, matrix inequalities.

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1. Introduction

Switched systems, which contain a family of subsystems with a rule switching among them, play an important role in hybrid dynamical systems due to their great flexibility in modeling of control, engineering, and social sciences [1, 2]. In recent years, much attention has been focused on stability and control problems for singular systems with or without delay [3-6]. Many effective methods have been presented to find conditions for the stability of switched singular systems, such as the Lyapunov function method, the state-space decomposition approach, and average dwell-time scheme [7-9]. On the other hand, stability analysis and applications of fractional-order systems (FOS) in different areas have been studied by many authors and many basic and interesting results have been obtained; see for example [10-13] for FOS with or without delays, [14-16] for singular FOS. Up to now, there are some research studies on the stability of fractional-order switched systems (FOSS) in literature. Existing attempts for stability analysis of FOSS are mere extensions of the methods for normal linear switched systems. In [17, 18] using frequency domain approach and convex analysis the authors proposed stability conditions for linear FOSS in term of common Lyapunov function. The authors of [19] proposed MittagLeffler increment approach to study asymptotic stablity of nonlinear FOSS. By using a method of minimum dwell-time with fractional order multiple Lyapunov functions, a switching law is designed [20] to ensure the asymptotic

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stability of nonlinear fractional-order impulsive switched systems. It should be noticed that the stability results of these papers were obtained for FOSS systems without singularity constraint. For singular FOS, it is well known that stability analysis is more complicated than that of FOS, because fractional derivatives are nonlocal and have weakly singular kernels. For singular integer-oder switched systems, by using Lyapunovlike Krasovskii approach, the relationship between the average dwell-time of the switched nonlinear singular time-delay system and the exponential decay rate of differential and algebraic states is given in [21]. Recently, based on the state-space singular value decomposition approach, a constructive geometric design of switching laws is proposed in [22] for the finite-time stability of singular nonlinear integer-order switched systems with delay. However, to the best of our knowledge, the stability analysis of singular fractional-order switched systems with delay has not been yet investigated in the literature and it motivates the main purpose of our paper.

In this paper, we develop some results on finite-time stability for singular FOSS with delay. The main contributions of this paper are given as follows. Using an approach based on Laplace transform and "inf-sup" method, we propose a constructive geometric design of state-dependent switching laws to guarantee the regularity, impulse-absence and finite-time stability for such systems. The switching rule for the system is constructed based on the construction of a partition of the state space in convex cones such that each system mode is active in one particular conic zone and each subregion is defined to make particular quadratic form negative. The advantage of the proposed method is that the obtained conditions are presented in terms of a set of tractable matrix inequalities and Mittag-Leffler functions.

The paper is organized as follows. In Section 2, we provide some preliminaries on fractional derivatives, Laplace transforms, finite-time stability problem and some auxiliary lemmas needed in next section. Main reulst on deriving new delay-dependent sufficient conditions for finite-time stability of SFDE with time-varying delay are presented in Section 3. The effectiveness of the theoretical result is illustrated by a numerical example.

2. Preliminaries

$\mathbb{N}$ denotes the set of all non-negative integers, $\overline{1,p} = \{1,2,\ldots,p\}$, $\mathbb{C}$ denotes the complex space; $R^{n\times r}$ denotes the space of all $(n\times r)-$ matrices; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{max}(A) = max\{Re(\lambda) : \lambda \in \lambda(A)\}$; $\lambda_{min}(A) = min\{Re(\lambda) : \lambda \in \lambda(A)\}$; $\|A\|$ denotes the spectral norm defined by $\sqrt{\lambda_{\max}(A^\top A)}$; $C([h,0],R^n)$ denotes the set of all R^n-valued continuously functions on $[h,0]$; $AC([a,b],R^n)$ denotes the set of all R^n-valued absolutely continuous functions on $[a,b]$; $[a]$ denotes the integer part of number a.

We first give some basic concepts of fractional calculus used in the paper.

Definition 1. ([10]) The Caputo fractional-order derivative of order $0<\alpha<1$ for continuous and derivable function $f(t)$ is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{f}(s)}{(t-s)^\alpha}ds, \quad t\geq 0.$$

Lemma 1. [10]. *For $\alpha\in(0,1)$, and $z\in\mathbb{C}$, $Re(z)>0$, we have*

(i) The Gamma function converges.

(ii) $\Gamma(z+1) = z\Gamma(z)$. *In particular,*

$$\Gamma(n+1) = n!,\ n = 1,2,\ldots, \quad \Gamma(1) = 1.$$

The Mittag-Lefller function with two parameters is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)},$$

where $\alpha > 0,\ \beta > 0$, and $z \in \mathbb{C}$. For $\beta = 1$, we denote $E_\alpha(z) := E_{\alpha,1}(z)$.

Lemma 2. [11]. *Given* $\alpha > 0$, *we have*
(i) $E_\alpha(z) \geq 1,\ \forall z \in R^+$,
(ii) $E_\alpha(z)$ *is a increasing function on* R^+.

The Laplace transform $\mathbb{L}[f(t)](s)$ of the integrable function $f(.)$, which is defined by

$$F(s) = \mathbb{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)dt.$$

Lemma 3. [11]. *Let* $f(.) : R^+ \to R$ *be a integrable function,* $\alpha \in (0,1), \beta > 0,\ h > 0$, *we have*
(i) $\mathbb{L}[D^\alpha f(t)](s) = s^\alpha \mathbb{L}[f(t)](s) - s^{\alpha-1} f(0)$,
(ii) For $k \in \mathbb{N}, Re(s) > h^{1/\alpha}$, *we have*

$$\mathbb{L}[t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(ht^\alpha)](s) = \frac{k! s^{\alpha-\beta}}{(s^\alpha - h)^{k+1}},$$

(iii) $\mathbb{L}[f * g(t)](s) = \mathbb{L}[f(t)](s) \cdot \mathbb{L}[g(t)](s)$,
where $f(t),\ g(t)$ *are integrable functions on* R^+, *the convolution of* $f(t)$ *and* $g(t)$ *is defined by* $f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

We now consider the following singular fractional-order switched system with delay:

$$\begin{cases} ED^\alpha x(t) = A_\sigma x(t) + D_\sigma x(t-h), & t \geq 0, \\ x(t) = \phi(t), & t \in [-h,0], \end{cases} \tag{1}$$

where $\alpha \in (0,1), x(t) \in \mathbb{R}^n$ is the state vector, $E \in R^{n\times n}$ is a singular matrix, rank$E = r < n$; $\sigma : R^n \to \{1,...,p\}$ is a switching rule depending on the system state at each time and takes its values in the finite set of modes $\{1,...,p\}$; the system matrices (A_σ, D_σ) take values in the finite set of $(A_i,\ D_i),\ i \in \overline{1,p}$, where $A_i, D_i, B_i \in R^{n\times n}$ are given constant matrices.

Corresponding to the switching law $\sigma(x(t))$, we assume that the system is activated by the $l-$th switching mode, which means that $\sigma(x(t)) = l$.

Definition 2. ([1]) For the switching law $\sigma(.)$, the system (1) is said to be (i) regular if the polynomial $det(sE-A_l)$ is not identically zero for each $\sigma(x(t))=l$; (ii) impulse-free if the deg(det$(sE-A_l)$) = rank E for each $\sigma(x(t))=l$.

Similar to singular delay systems, system (2.1) may have an impulsive solution, however, the regularity and the absence of impulses of the pair (E,A) ensure the existence and uniqueness of an impulse-free solution to the system, which is shown in following lemma.

Lemma 4. *([23,24]) Assume that the system (1) is regular and impulse-free. Then for every continuous initial condition $\phi(t)$ system (1) has a unique solution $x(t,\phi)\in AC[0,+\infty)$.*

Definition 3. ([25]) For given positive numbers T,c_1,c_2, the system (1) is finite-time stable w.r.t. (c_1,c_2,T) under switching law $\sigma(\cdot)$ if it is regular, impulse-free and every solution $x_\sigma(t,\varphi)$ of the system satisfies the condition:

$$\sup_{s\in[-h,0]}\{\varphi(s)^\top\varphi(s)\}\le c_1 \;\Rightarrow\; x_\sigma(t,\varphi)^\top x_\sigma(t,\varphi)\le c_2,\ \forall t\in[0,T].$$

Definition 4. ([26]) The system of matrices $\{L_i\}_{i=1}^{p}$ is strictly complete if for every $x\in R^n\setminus\{0\}$ there is $i\in\{1,2,\dots,p\}$ such that $x^\top L_i x<0$.

It is easy to see that system $\{L_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{p}\Omega_l = R^n\setminus\{0\},\ \text{where}\ \Omega_i=\{x\in R^n:\ x^\top L_i x<0\},\ i=\overline{1,p}.$$

Remark 1. Note that (see [26]) the system $\{L_i\}_{i=1}^{p}$ is strictly complete if there exist numbers $\xi_l\ge 0,\ i=\overline{1,p},\ \sum\limits_{i=1}^{p}\xi_l>0$, such that

$$\sum_{i=1}^{p}\xi_i L_i<0.$$

Definition 5. For given positive numbers c_1,c_2,T, the system (1) is finite-time stable w.r.t. (c_1,c_2,T), if

$$\sup_{s\in[-h,0]}\varphi(s)^\top\varphi(s)\le c_1\Rightarrow x(t)^\top x(t)\le c_2,\ \forall t\in[0,T].$$

Now, we recall the following auxiliary lemmas needed for the proof of the main result in next section.

Lemma 5. *([27]) For any continuous and derivable function $x(t)\in\mathbb{R}^n$, and symmetric positive definite matrix $P\in\mathbb{R}^{n\times n}$, we have*

$$\frac{1}{2}D^\alpha(x^\top(t)Px(t))\le x^\top(t)PD^\alpha x(t),\quad \forall\alpha\in(0,1),\forall t\ge 0.$$

3. Main result

The purpose of this section is to study finite-time stability of singular FOSS (1). We first establish delay-dependent conditions to check the regularity and impulse-absence of the systems based on singular value decomposition method. Then, we prove the finite-time stability based on the Laplace transform and "inf-sup" method. Consider the system (1), where *rank* $E < n$. Then there are two nonsingular matrices M, G such that

$$MEG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Let us set

$$MA_lG = \begin{bmatrix} A^l_{11} & A^l_{12} \\ A^l_{21} & A^l_{22} \end{bmatrix}, \quad MD_lG = \begin{bmatrix} D^l_{11} & D^l_{12} \\ D^l_{21} & D^l_{22} \end{bmatrix}.$$

Under the state transformation $y(t) = G^{-1}x(t), y(t)^\top = \left(y_1(t)^\top,\ y_2(t)^\top\right), y_1(t) \in R^r,\ y_2(t) \in R^{n-r}$, the system (1) takes the following form

$$\begin{cases} D^\alpha y_1(t) & = A^l_{11}y_1(t) + A^l_{12}y_2(t) + D^l_{11}y_1(t-h) + D^l_{12}y_2(t-h), \\ 0 & = A^l_{21}y_1(t) + A^l_{22}y_2(t) + D^l_{21}y_1(t-h) + D^l_{22}y_2(t-h), \\ y(t) & = G^{-1}\varphi(t),\ t \in [-h, 0]. \end{cases} \tag{2}$$

Before introducing the main result, the following notations of several matrix variables are defined for simplicity.

$$G^\top PEG = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad p = \frac{\lambda_{max}(PE)}{\lambda_{min}(P_1)}, \quad g^2 = \lambda_{max}([G^{-1}]^\top [G^{-1}],$$
$$\beta = \lambda_{max}(G^\top G), \quad \gamma = \max\{\|(A^l_{22})^{-1}A^l_{21}\|, \|[A^l_{22}]^{-1}D^l_{21}\|, \|[A^l_{22}]^{-1}D^l_{22}\|\},$$
$$\eta = \Big[\max_{i=0,1,2,\ldots,[\frac{T}{h_1}]} 2\sum_{k=0}^{i} \gamma^{k+1} + 2g\gamma^{i+2}\Big]^2,$$
$$L_i = PA_l + A_l^\top P^\top - hPE + K_lA_l + [K_lA_l]^\top,$$
$$\Omega_i = \{x \in R^n:\ x^\top L_i x < 0\},\ i = \overline{1,p},$$
$$\overline{\Omega}_1 = \Omega_1 \cup \{0\},\ \overline{\Omega}_i = \Omega_i \setminus \bigcup_{k=1}^{i-1} \overline{\Omega}_k,\ l = 2, 3, \ldots, p.$$

Theorem 6. *For given positive numbers T, c_1, c_2, the system (1) is finite-time stable w.r.t. (c_1, c_2, T) if there exist a nonsingular matrix $P \in R^{n\times n}$, any matrices $K_l \in R^{n\times n}$, $l \in \overline{1,p}$, scalars $\xi_l \geq 0$, $l = \overline{1,p}$, $\sum_{l=1}^{p} \xi_l > 0$, such that the following conditions hold:*

$$PE = E^\top P^\top, \tag{3}$$

$$\sum_{i=1}^{p} \xi_i L_i < 0, \tag{4}$$

$$\begin{bmatrix} PA_l + A_l^\top P^\top - hPE & 2PD_l + [K_lA_l]^\top + K_lD_l & [K_lA_l]^\top - K_l \\ * & -2hPE + K_lD_l + [K_lD_l]^\top & -K_l + [K_lD_l]^\top \\ * & * & -K_l - K_l^\top \end{bmatrix} < 0, \ l = \overline{1,p}, \tag{5}$$

$$\beta\Big[p(1+g)\sum_{j=0}^{[T/h_1]+1}(E_\alpha(h_2T^\alpha)-1)^jE_\alpha(h_2T^\alpha)+g^2\gamma_1\Big] \le \frac{c_2}{c_1}. \tag{6}$$

The switching rule is chosen as $\sigma(x(t)) = i$ whenever $x(t) \in \overline{\Omega}_i$.

Proof. The proof is divided into two steps. The first step is to prove the regularity and the impulse-free of the singular system (1). The second step will focus on getting conditions for design the state-dependent switching laws for finite-time stability by using Lyapunov-like function method and LMI technique.

Step 1. Regularity and impulse-free of the system.

Let us set

$$G^\top = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \ G^\top PM^{-1} = \begin{bmatrix} P_1 & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

From the condition (3), $PE = E^\top P^\top \ge 0$, it is easily seen that

$$G^\top PEG = G^\top PM^{-1}MEG = G^\top PM^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ P_{21} & 0 \end{bmatrix} \ge 0,$$

$$G^\top E^\top P^\top G = \begin{bmatrix} P_1^\top & P_{21}^\top \\ 0 & 0 \end{bmatrix} \ge 0,$$

and hence

$$P_{21} = 0, \ P_1 = P_1^\top \ge 0, \ G^\top PEG = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7}$$

Since matrix P is nonsingular, it follows the non-singularity of $G^\top PM^{-1} = \begin{bmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{bmatrix}$ such that from (7) it follows that $det(P_1) \ne 0$, and hence $P_1 > 0$. Next, note that the LMI (5) implies the following inequality

$$G^\top[PA_l + A_l^\top P^\top - hPE]G < 0. \tag{8}$$

On the other hand, we rewrite the expression $G^\top PA_lG$ as follows

$$\begin{aligned} G^\top PA_lG = G^\top PM^{-1}MA_lG &= \begin{bmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{bmatrix}\begin{bmatrix} A_{11}^l & A_{12}^l \\ A_{21}^l & A_{22}^l \end{bmatrix} \\ &= \begin{bmatrix} P_1A_{11}^l + P_{12}A_{21}^l & P_1A_{12}^l + P_{12}A_{22}^l \\ P_{22}A_{21}^l & P_{22}A_{22}^l \end{bmatrix}, \end{aligned}$$

Therefore, taking (7)-(8) into account we have

$$P_{22}A_{22}^l + [A_{22}^l]^\top P_{22}^\top < 0,$$

which gives $det(A_{22}^l) \neq 0$ and then the system is regular and impulse-free (see [17]).

Step 2. Finite-time stability.

Consider the following non-negative quadratic function:

$$V(t,x_t) = x(t)^\top 2PEx(t).$$

Assuming that the system is activated by the $l-$th switching mode, which means that $\sigma(x(t)) = l$, we take the derivative of $V(t,x_t)$ in t along the solution of the system:

$$\begin{aligned} D^\alpha(V(t,x(t))) &\leq 4x(t)^\top PED^\alpha(x(t)) = 4x(t)^\top P\Big(A_l x(t) + D_l x(t-h)\Big) \\ &\leq 4x(t)^\top P\Big(A_l x(t) + D_l x(t-h)\Big) - 2hx(t-h)^\top PEx(t-h) + 2hx(t-h)^\top PEx(t-h) \\ &\qquad - 2hx(t)^\top PEx(t) + hx(t)^\top 2PEx(t). \end{aligned} \tag{9}$$

To estimate the derivative of $V(t,x_t)$, we need the following inequalities. Firstly, multiplying the both side of the following identity by $2x(t)^\top K_l$, $2x(t-h)^\top K_l$, $2[ED^\alpha(x(t))]^\top K_l$, respectively, from the left hand side of (1), we have

$$\begin{aligned} 0 &= -2x(t)^\top K_l ED^\alpha(x(t)) + 2x(t)^\top K_l\Big[A_l x(t) + D_l x(t-h)\Big], \\ 0 &= -2x(t-h)^\top K_l ED^\alpha(x(t)) + 2x(t-h)^\top K_l\Big[A_l x(t) + D_l x(t-h)\Big], \\ 0 &= -2[ED^\alpha(x(t))]^\top K_l ED^\alpha(x(t)) + 2[ED^\alpha(x(t))]^\top K_l\Big[A_l x(t) + D_l x(t-h)\Big] \end{aligned} \tag{10}$$

Hence, we obtain from (9)-(10) that

$$\dot V(\cdot) - hV(\cdot) \leq \xi(t)^\top C_l \xi(t) + x(t)^\top L_l x(t) + 2hx(t-h)^\top PEx(t-h), \tag{11}$$

where $\xi(t)^\top = [x(t)^\top,\ x(t-h)^\top,\ [ED^\alpha(x(t))]^\top]$, and

$$C_l = \begin{bmatrix} PA_l + A_l^\top P^\top - hPE & 2PD_l + [K_l A_l]^\top + K_l D_l & [K_l A_l]^\top - K_l \\ * & -2hPE + K_l D_l + [K_l D_l]^\top & -K_l + [K_l D_l]^\top \\ * & * & -K_l - K_l^\top \end{bmatrix},$$

$$L_l = PA_l + A_l^\top P^\top - hPE + K_l A_l + [K_l A_l]^\top.$$

Since the system of matrices $\{L_l:\ l = \overline{1,p}\}$ is strictly complete due to (4), we get

$$\bigcup_{l=1}^{p} \Omega_l = R^n \setminus \{0\},$$

and hence by constructing the sets $\overline{\Omega}_l$, we have

$$\bigcup_{l=1}^{p} \overline{\Omega}_l = R^n \ \text{ and } \ \overline{\Omega}_{l_1} \cap \overline{\Omega}_{l_1} = \emptyset, \ \text{ for all } \ l_1 \neq l_2.$$

Therefore, for $x(t) \in R^n$, there exists a unique $l \in \{1,2,...,p\}$ such that $x(t) \in \overline{\Omega}_l$ and

$$x(t)^\top L_l x(t) \leq 0. \tag{12}$$

Choosing the switching rule as $\sigma(x(t)) = l$ whenever $x(t) \in \overline{\Omega}_l$. From the condition (5) , $C_l \leq 0,\ l = \overline{1,p}$, and from the inequality (11)-(12), it follows that

$$D^{\alpha}(V(t,x(t))) - hV(t,x(t)) \leq hx(t-h)^{\top}2PEx(t-h).$$

Let us set

$$M(t) = D^{\alpha}(V(t,x(t))) - hV(t,x(t)), \tag{13}$$

we have

$$M(t) \leq hx(t-h)^{\top}2PEx(t-h).$$

Applying the Laplace transform to both side of (13), by Lemma 3 (i), we have

$$s^{\alpha}\mathbb{V}(s) - V(0,x(0))s^{\alpha-1} = h\mathbb{V}(s) + \mathbb{M}(s),$$

where $\mathbb{V}(s) = \mathbb{L}[V(t,x(t))](s),\ \mathbb{M}(s) = \mathbb{L}[M(t)](s)$, and hence

$$\mathbb{V}(s) = (s^{\alpha} - h)^{-1}(V(0,x(0))s^{\alpha-1} + \mathbb{M}(s)). \tag{14}$$

On the other hand, we can verify the correctness of following relations:

$$(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h(t-\tau)^{\alpha}) \geq 0, \forall t \geq 0,\ \tau \in [0,t],\ h > 0,$$

$$\sup_{0\leq\tau\leq t} M(\tau) \leq h \sup_{0\leq\tau\leq t} x(\tau-h)^{\top}2PEx(\tau-h),$$

$$\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h(t-\tau)^{\alpha})d\tau = \frac{1}{h}[E_{\alpha}(ht^{\alpha}) - 1].$$

Taking the inverse Laplace transform to both sides of equation (14), by Lemma 3 (ii)-(iii), we have

$$\begin{aligned}
V(t,x(t)) =& V(0,x(0))E_{\alpha}(ht^{\alpha}) + \int_0^t M(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h(t-\tau)^{\alpha})d\tau \\
\leq& V(0,x(0))E_{\alpha}(ht^{\alpha}) + \int_0^t M(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h(t-\tau)^{\alpha})d\tau \\
\leq& V(0,x(0))E_{\alpha}(ht^{\alpha}) + \sup_{0\leq\tau\leq t} M(\tau)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h(t-\tau)^{\alpha})d\tau \\
\leq& V(0,x(0))E_{\alpha}(ht^{\alpha}) + \Big(E_{\alpha}(ht^{\alpha}) - 1\Big) \sup_{0\leq\tau\leq t} x(\tau-h)^{\top}2PEx(\tau-h) \\
\leq& V(0,x(0))E_{\alpha}(ht^{\alpha}) + \Big(E_{\alpha}(ht^{\alpha}) - 1\Big) \sup_{-h\leq\theta\leq t-h} x(\theta)^{\top}2PEx(\theta),
\end{aligned}$$

and we obtain that

$$x(t)^{\top}2PEx(t) \leq \varphi(0)^{\top}2PE\varphi(0)E_{\alpha}(ht^{\alpha}) + \Big(E_{\alpha}(ht^{\alpha}) - 1\Big) \sup_{-h\leq\theta\leq t-h} x(\theta)^{\top}2PEx(\theta). \tag{15}$$

We now estimate the value $x(\tau)^\top 2PEx(\tau)$ on $\tau \in [-h,t]$. Firstly, note that by Lemma 2 the function $E_\alpha(ht^\alpha) \geq 1$, we have for $\tau \in [-h,0]$:

$$x(\tau)^\top 2PEx(\tau) \leq \sup_{\theta\in[-h,0]} \varphi(\theta)^\top 2PE\varphi(\theta)E_\alpha(ht^\alpha).$$

Since $E_\alpha(.)$ is non-decreasing, applying the derived condition (15) for $0 \leq \tau \leq t \leq T$, we get

$$\begin{aligned}
x(\tau)^\top 2PEx(\tau) \leq & \varphi(0)^\top 2PE\varphi(0)E_\alpha(h\tau^\alpha) + \Big(E_\alpha(h\tau^\alpha)-1\Big) \sup_{-h\leq\theta\leq\tau-h} x(\theta)^\top 2PEx(\theta) \\
\leq & \varphi(0)^\top 2PE\varphi(0)E_\alpha(ht^\alpha) + \Big(E_\alpha(ht^\alpha)-1\Big) \sup_{-h\leq\theta\leq t-h} x(\theta)^\top 2PEx(\theta) \\
\leq & \sup_{\theta\in[-h,0]} \varphi(\theta)^\top 2PE\varphi(\theta)E_\alpha(hT^\alpha) \\
& + \Big(E_\alpha(hT^\alpha)-1\Big) \sup_{-h\leq\theta\leq t-h} x(\theta)^\top 2PEx(\theta),
\end{aligned}$$

which implies

$$\begin{aligned}
\sup_{-h\leq\theta\leq t} x(\theta)^\top 2PEx(\theta) \leq & \sup_{\theta\in[-h,0]} \varphi(\theta)^\top 2PE\varphi(\theta)E_\alpha(hT^\alpha) \\
& + \Big(E_\alpha(hT^\alpha)-1\Big) \sup_{-h\leq\theta\leq t-h} x(\theta)^\top 2PEx(\theta).
\end{aligned}$$

Let us denote $H(t) = \sup_{-h\leq\theta\leq t} x(\theta)^\top 2PEx(\theta),\ a = E_\alpha(hT^\alpha),\ b = \Big(E_\alpha(hT^\alpha)-1\Big)$, we have

$$H(t) \leq aH(0) + bH(t-h),\ \forall t \in [0,T].$$

Next, we prove that

$$H(t) \leq H(0)q, \forall t \in [0,T],$$

where $q = E_\alpha(hT^\alpha) \sum_{j=0}^{[T/h]+1} (E_\alpha(hT^\alpha)-1)^j$.

In fact, for each $t \in [0,T]$, there is $m \in \mathbb{N}$ such that $mh \leq t < (m+1)h$. By induction, we have

$$H(t) \leq \begin{cases} \Big[a+ba+\cdots+b^m a\Big]H(0) + b^{m+1}H(t-(m+1)h) & m \geq 1, \\ aH(0) + bH(t-(m+1)h) & m = 0. \end{cases}$$

From $-h \leq t-(m+1)h < 0$, and the funtion $H(t)$ is nondecreasing, we get

$$H(t-(m+1)h) \leq H(0).$$

Hence, by $a \geq 1$,

$$H(t) \leq \begin{cases} \Big[a+ba+\cdots+b^m a+b^{m+1}a\Big]H(0) & m \geq 1, \\ (a+ba)H(0) & m = 0. \end{cases}$$

$$= a\sum_{j=0}^{m+1} b^j H(0).$$

Besides, $t \leq T$ leads to $m \leq [T/h]$ and $H(t) \leq qH(0)$.

Consequently,

$$x(t)^\top 2PEx(t) \leq H(t) \leq H(0)q \leq \sup_{\theta\in[-h,0]} \varphi(\theta)^\top 2PE\varphi(\theta)q, \quad \forall t \in [0,T].$$

Besides, it is easy to see that

$$x(t)^\top 2PEx(t) = y(t)^\top G(2PE)Gy(t) = y(t)^\top \begin{bmatrix} 2P_1 & 0 \\ 0 & 0 \end{bmatrix} y(t) \geq 2\lambda_{min}(P_1)\|y_1(t)\|^2$$

$$\varphi(\theta)^\top 2PE\varphi(\theta) \leq 2\lambda_{max}(PE)\varphi(\theta)^\top\varphi(\theta) \leq 2\lambda_{max}(PE)c_1.$$

Hence, we obtain

$$\|y_1(t)\|^2 \leq \frac{\lambda_{max}(PE)}{\lambda_{min}(P_1)}c_1 q = pqc_1, \quad t \in [0,T]. \tag{16}$$

Next, we estimate the second state $\|y_2(t\|$ as follows. Consider the second equation of (2)

$$y_2(t) = -[A_{22}^l]^{-1}\Big[A_{21}^l y_1(t) + D_{21}^l y_1(t-h(t)) + D_{22}^l y_2(t-h)\Big].$$

Applying estimation (16) for $t \in [0,T]$ gives

$$\|y_1(t-h(t))\|^2 \leq \lambda_{max}([G^{-1}]^\top[G^{-1}])c_1 + \frac{\lambda_{max}(PE)}{\lambda_{min}(P_1)}qc_1 = (g^2+pq)c_1,$$

and hence

$$\|y_2(t)\| \leq \zeta\sqrt{c_1} + \gamma\|y_2(t-h(t))\|, \quad t \in [0,T], \tag{17}$$

where $\zeta = 2\gamma\sqrt{pq+g^2}$. On the other hand, using inequality (17) for $t \in [0,h_1]$, we obtain that

$$\|y_2(t)\| \leq (\zeta + \gamma g)\sqrt{c_1},$$

because of $t - h_2 \leq t - h(t) \leq t - h_1 \leq 0$, and

$$\|y_2(t-h(t))\|^2 \leq \|y(t-h(t))\|^2 = \lambda_{max}([G^{-1}]^\top[G^{-1}])c_1 = g^2 c_1.$$

By induction, for $t \in [ih_1,(i+1)h_1] \cap [0,T],\ ih_1 \leq T,\ i = 0,1,...,$ we have

$$\|y_2(t)\| \leq \zeta[\sum_{k=0}^{i}\gamma^k + g\gamma^{i+1}]\sqrt{c_1},$$

and hence

$$\|y_2(t)\| \leq \sqrt{(g^2+pq)\eta c_1}, \tag{18}$$

where $\eta = \Big[\max_{i=0,1,2,\dots,[\frac{T}{h_1}]} 2\sum_{k=0}^{i}\gamma^{k+1}+2g\gamma^{i+2}\Big]^2$. Finally, combining conditions (16), (18) with condition (6), we obtain that

$$\begin{aligned}\|x(t)\|^2 =& x(t)^\top x(t) = y(t)^\top G^\top G y(t) \le \lambda_{max}(G^\top G)\|y(t)\|^2\\ =&\beta(\|y_1(t)\|^2+\|y_2(t)\|^2)\\ \le&\beta(pq+(g^2+pq)\eta)c_1 \le c_2, \quad t\in[0,T].\end{aligned}$$

The proof of the theorem is completed. □

Remark 2. In Theorem 1 the conditions (3),(5) guarantee the regularity and impulse-absence of the system. Moreover, the condition(5) is an LMI, and since the given number c_1, c_2 do not include in LMIs (5) and in the right-hand side of (6) such that we can easily determine solutions P, K_l from LMI(5) and then verify the condition (6).

Remark 3. It is worth noting that the condition (4) in Theorem 1 is a bilinear matrix inequality (BMI) with respect to ξ_i and P. To solve this BMI we can use the branch and bound methods proposed in [28] or the homotopy-based algorithm in [29].

Remark 4. It is worth noting that the method used in [30, 31] can not be applied to the system (1), since the constructed Lyapunov-Krasovskii function $V(x_t)$ associated with the Riemann-Liouville fractional integral in these papers can not guarantee the positive definiteness of the function $V(x_t)$, and hence the use of the fractional Lyapuov stability theorem for system (1) was impossible. To overcome this drawback, we construct a delay-independent Lyapunov functional $V(x(t))$ as for systems without delay, which allows us to overcome the positive definiteness of $V(x_t)$.

In the sequel, a numerical example is given to demonstrate the effectiveness of the method proposed in this paper

Example 1. Consider system (1), where

$$E=\begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix},\ M=G=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \alpha=0.5,\ h=1,$$

$$A_1=\begin{bmatrix}0 & 0\\ 0 & 0.001\end{bmatrix},\ A_2=\begin{bmatrix}0 & 0\\ 0 & -0.002\end{bmatrix},$$

$$D_1=\begin{bmatrix}0 & 0\\ 0 & 0.001\end{bmatrix},\ D_2=\begin{bmatrix}0 & 0\\ 0 & 0.001\end{bmatrix},$$

By using LMI Toolbox in Matlab, LMI (5) is feasible with

$$P=\begin{bmatrix}1205.7314 & 0\\ 0 & 0.0105\end{bmatrix},$$

$$K_1=\begin{bmatrix}0 & 0\\ 0 & -1570063.6709\end{bmatrix},\ K_2=\begin{bmatrix}0 & 0\\ 0 & -404055.3796\end{bmatrix}.$$

In this case, it can be computed that

$$L_1 = \begin{bmatrix} -12057.3140 & 0 \\ 0 & 0.2097e-12 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -12057.3140 & 0 \\ 0 & -0.4195e-12 \end{bmatrix},$$

$$L_1 + L_2 = \begin{bmatrix} -24114.6280 & 0 \\ 0 & -0.2097e-12 \end{bmatrix} < 0.$$

Thus, the system of matrices $\{L_1, L_2\}$ is strictly complete. The sets $\overline{\Omega}_l$ are given as (see Fig. 1)

$$\overline{\Omega}_1 = \Big\{ x = (x_1, x_2)^\top : \ (-x_1 + 4.1711e-09x_2)(x_1 + 4.1711e-09x_2) \leq 0 \Big\},$$
$$\overline{\Omega}_2 = \Big\{ x = (x_1, x_2)^\top : \ (-x_1 + 4.1711e-09x_2)(x_1 + 4.1711e-09x_2) > 0 \Big\}.$$

Moreover, we see that the condition (3) satisfies

$$PE = E^\top P^\top = \begin{bmatrix} 12057.3140 & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

and the condition (6) holds with

$$c_1 = 0.01, c_2 = 60, T = 2.5.$$

In fact, we can estimate $[T/h] = 2$,

$$q = E_{0.5}(\sqrt{2.5}) \sum_{j=0}^{3} (E_{0.5}(\sqrt{2.5}) - 1)^j = 590263.3106,$$

$$\lambda_{max}(G^\top G) = 1, \ \lambda_{max}(PE) = \lambda_{min}(P_1) = 12057.3140.$$

Hence, the condition (6) holds, and by Theorem 1, the system (1) with the switching rule $\sigma(x(t)) = l$ whenever $x(t) \in \overline{\Omega}_l$, is finite-time stable w.r.t. $(0.01, \ 60, \ 2.5)$.

4. Conclusion

We have studied the finite-time stability of singular fractional-order switched systems with delay. The approach based on the Laplace transform and "inf-sup" method was proposed to derive new delay-dependent sufficient conditions for the regularity, impulse-absence and finite-time stability. The obtained conditions for constructing geometric switching laws in in terms of the Mittag-Leffler function and matrix inequalities. An illustrative example is given to show the validity of the obtained result.

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