Robust finite-time stability and stabilization of a class of fractional-order switched nonlinear systems

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Abstract

The problem of finite-time boundedness and finite-time stabilization boundedness of fractional-order switched nonlinear systems with exogenous inputs is considered in this paper. By constructing a simple Lyapunov-like functional and using some properties of Caputo derivative, we obtain some new sufficient conditions for the problem via linear matrix inequalities, which can be efficiently solved by using existing convex algorithms. A constructive geometric is used to design switching laws amongst the subsystems. Two numerical examples are provided to demonstrate the validity of our method.

Keywords: Fractional-order switched systems, Finite-time boundedness, Linear matrix inequalities

1. Introduction

In recent years, switched systems have attracted significant research attention in the literature (see, for example, [1, 2]). The common characteristic of switched systems is that they are constructed from a subsystem family being modeled in differential or difference equations and associated with a switching law. For the last decade, Lyapunov stability and its variations including H_∞ control, passivity analysis and reachable sets bounding, which are mainly defined in an infinite time interval, have been intensively investigated for switched systems (see, [3–6] and the references therein). However, in many practical applications, it is possible that, over a finite-time interval, the states of a dynamical system do not exceed a certain threshold if a specific bound is given on the initial condition. For that, the concept of finite-time stability, in which the transient behavior of a system over a finite-time is taken into account rather than the asymptotic behavior of the system response [7–9]. Roughly speaking, a switched system is said to be finite-time stable if its state does not exceed a certain threshold during a specified time interval when the initial conditions are within a specific bound. As noted by many researchers, the concept of finite-time stability and Lyapunov asymptotic stability (LAS) are independent concepts; indeed a system can be finite-time stability but not LAS, and vice versa [10, 11]. Therefore, there are some interesting results on stability of integer-order switched systems in the literature (see, for example, [12–15]).

Fractional-order systems have recently arisen in interdisciplinary areas as a consequence of their wide applications to physics, engineering, and economics [16]. Many important results on Lyapunov stability of fractional-order systems have been obtained in the literature, see, books [17, 18], and papers [19–21]. Furthermore, regarding switched systems, fractional-order state space models appear naturally and can effectively describe dynamical systems associated with the dynamic parameters being switched [22]. Hence, some interesting works accounted for Lyapunov stability and stabilization of fractional-

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order switched systems have been presented in the literature [23–27]. For example, by employing a well known Lyapunov function scheme, the authors of [23, 24] solved the problem of Lyapunov stability of fractional-order switched linear systems. In the work of [25], based on the convex analysis and using linear matrix inequalities (LMIs) technique, the authors investigated the stabilization of linear time invariant fractional-order switched systems. Meanwhile, the authors in [26] utilized the Mittag–Leffler function and fractional-order multiple Lyapunov functions to derive some sufficient conditions ensuring the Lyapunov stability of nonlinear fractional-order impulsive switched systems. Very recently, by introducing a couple of new concepts namely Mittag–Leffler increment and average Mittag–Leffler increment, the authors in [27] investigated the stability of a class of fractional-order switched systems with nonlinear perturbations.

With regard to finite-time stability of fractional-order switched systems, there are a few results were reported in the literature. In particular, by employing Mittag-Leffler function, fractional-order Lyapunov function and Gronwall-Bellman lemma, the authors in [28] considered the problem of finite-time stability for fractional-order impulsive switched systems. It should be mentioned here that this result was derived based on constructing abstract Lyapunov functions. In fact, Lyapunov direct method is a very effective tool for analyzing the finite-time stability of switched systems. However, for fractional-order switched systems, it is difficult to construct a Lyapunov function and calculate its fractional derivative since the well-known Leibniz rule does not hold for fractional derivatives [29]. In [28], some stable conditions were derived, however, they were not associated with a practical algebraic criterion which offers a solution to the problem. Recently, the problem of finite-time stability of fractional-order positive switched systems was addressed in the work of [30] where the linear copositive Lyapunov function integrated with average dwell time switching technique was employed. However, in this work, the concept of finite-time boundedness, which is applied to fractional-order positive systems, is different from the concept of finite-time boundedness used in general systems (non-positive systems). Therefore, the scheme in [30] cannot be extended to fractional-order switched nonlinear systems. The aforementioned discussion inspires us to the present study.

In this paper, we study the problem of finite-time boundedness and finite-time stabilization boundedness of fractional-order switched nonlinear systems. The significance of this work can be summarized as follows. Firstly, by constructing a simple Lyapunov-like functional and employing some properties of Caputo derivative, we derive some new sufficient conditions dealing with the problem. The conditions are with the form of LMIs, which can be effectively solved by utilizing existing convex algorithms that can be found in [31]. Secondly, the switching rule amongst the subsystems of the systems is designed. Lastly, two numerical examples are presented to demonstrate the effectiveness and applicability of the proposed method.

The remainder of this paper is organized as follows. Some necessary definitions and lemmas are given in the Section 2. Sufficient conditions ensuring the finite-time boundedness of fractional-order switched nonlinear systems are derived in Section 3. Two numerical examples are given in Section 4 and a conclusion of the paper is presented in Section 5.

Notation: The following notations will be used throughout this paper: $\mathbb{R}^n$ is the $n-$dimensional real Euclidean space with the Euclidean norm $\|.\|$ given by $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}, x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$; $\mathbb{R}^{n\times m}$ is the set of all real $n \times m$ matrices. For a real matrix $A, \lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. A matrix P is positive definite $(P > 0)$ if $x^T P x > 0, \forall x \neq 0; P > Q$ means $P - Q > 0$. $\mathbb{S}_n^+$ is the set of symmetric positive definite matrices in $\mathbb{R}^{n\times n}$.

2. Problem statement and preliminaries

We shall begin with recalling some basic notations and properties for fractional calculus. For further details, we refer the reader to [18]. The fractional integral of order $\alpha > 0$ on $[t_0, t]$ of an arbitrary integrable function $x(t)$ is defined as follows

$$ {}_{t_0}I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} x(s) ds, $$

where $\Gamma(.)$ represents the gamma function. The Caputo fractional-order derivative of order $\alpha > 0$ for a function $x(t) \in C^{n+1}([t_0, +\infty), \mathbb{R})$ is defined as follows

$$ {}_{t_0}^{C}D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t \geq t_0 \geq 0,\ n-1 < \alpha < n, $$

where n is a positive integer. In particular, when $0 < \alpha < 1$, we have

$$ {}_{t_0}^{C}D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\dot{x}(s)}{(t-s)^{\alpha}} ds, \quad t \geq t_0 \geq 0. $$

Especially, as in [18], we have ${}_{t_0}^{C}D_t^{0} x(t) = x(t)$ and ${}_{t_0}^{C}D_t^{1} x(t) = \dot{x}(t)$.

We now consider the following Caputo fractional-order switched nonlinear system:

$$ (\Sigma_\sigma) \begin{cases} {}_{0}^{C}D_t^{\alpha} x(t) = [A_\sigma + \Delta A_\sigma(t)] x(t) + f_\sigma(x(t)) + W_\sigma d(t) + B_\sigma u(t), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1} $$

where $\alpha \in (0,1), x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the exogenous input vector, $u(t) \in \mathbb{R}^q$ is the control input vector, $\sigma(.) : \mathbb{R}^n \longrightarrow \mathcal{N} = \{1, 2, \ldots, N\}$ is the switching rule, which is a piece-wise constant function depending on the state in each time. $\sigma(x(t)) = i,\ i = 1, 2, \ldots, N$ implies that the system realization is chosen as Σ_i. Matrices A_i, W_i, B_i are constant and of appropriate dimensions.

For $i = 1, 2, \ldots, N$, the following assumptions are made:

Assumption 1. The uncertainties $\Delta A_i(t)$ satisfy the following conditions:

$$ \Delta A_i(t) = \mathscr{G}_i \mathscr{F}_i(t) \mathscr{H}_i, $$

where $\mathscr{G}_i, \mathscr{H}_i$ are given real constant matrices; $\mathscr{F}_i(t)$ are time-varying, real matrices satisfying

$$ \mathscr{F}_i^T(t) \mathscr{F}_i(t) \leq I. $$

Assumption 2. The nonlinear perturbations $f_i(x(t))$ satisfy the following condition:

$$ f_i^T(x(t)) f_i(x(t)) \leq a_i x^T(t) x(t), $$

where a_i are given positive numbers.

Assumption 3. The exogenous input $d(t)$ is time-varying and satisfying the following condition:

$$ d^T(t) d(t) \leq d, \quad t \in [0, T_f], $$

where d is a given positive constant.

The unforced system of (1) is expressed as follows:

$$\begin{cases} {}_{0}^{C}D_t^{\alpha}x(t) = [A_{\sigma}+\Delta A_{\sigma}(t)]x(t)+f_{\sigma}(x(t))+W_{\sigma}d(t), \quad t\geq 0, \\ x(0)=x_0\in\mathbb{R}^n. \end{cases} \tag{2}$$

We will use the following definitions and well-known auxiliary lemmas to derive the main results of the paper.

Definition 1. ([9]) Given positive constants c_1, c_2, d, T_f with $c_1 < c_2$ and a symmetric positive definite matrix R, the fractional-order switched nonlinear system (2) is said to be finite-time boundedness with respect to (c_1, c_2, d, T_f, R) if there exists switching rule $\sigma(.)$ such that

$$x^T(0)Rx(0)\leq c_1 \Rightarrow x^T(t)Rx(t) < c_2, \quad \forall t\in[0,T_f],$$

for all the disturbances $d(t)\in\mathbb{R}^m$ satisfy Assumption 3.

Definition 2. The set of matrices $\{L_i\}$ is said to be strictly complete if for every $x\in\mathbb{R}^n\backslash\{0\}$ there is $i\in\mathcal{N}=\{1,2,\ldots,N\}$ such that $x^TL_ix<0$.

Setting

$$\Lambda_i=\{x\in\mathbb{R}^n : x^TL_ix<0\}.$$

It is not hard to verify that the set of matrices $\{L_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N}\Lambda_i=\mathbb{R}^n\backslash\{0\}.$$

Remark 1. As noted in [32], if there exist numbers $\beta_i\geq 0$, $\sum\limits_{i=1}^{N}\beta_i>0$ such that

$$\sum_{i=1}^{N}\beta_iL_i<0$$

then the set of matrices $\{L_i\}$ is strictly complete.

Lemma 1. *([33]) If $x(t)\in C^n([0,+\infty),\mathbb{R})$ and $n-1<\alpha<n, (n\geq 1, n\in\mathbb{Z}^+)$, then*

$${}_0I_t^{\alpha}\left({}_0^{C}D_t^{\alpha}x(t)\right)=x(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}x^{(k)}(0).$$

In particular, when $0<\alpha<1$, we have

$${}_0I_t^{\alpha}\left({}_0^{C}D_t^{\alpha}x(t)\right)=x(t)-x(0).$$

Lemma 2. *([34]) Let $x(t)\in\mathbb{R}^n$ be a vector of differentiable function. Then, for any time instant $t\geq t_0$, the following relationship holds*

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}\left(x^T(t)Px(t)\right)\leq x^T(t)P\,{}_{t_0}^{C}D_t^{\alpha}x(t), \quad \forall\alpha\in(0,1), \forall t\geq t_0\geq 0,$$

where $P\in\mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

3. Main results

3.1. Finite-time boundedness

Our main objective of this subsection is to derive a simple procedure for constructing the switching rule to guarantee that the unforced fractional-order switched nonlinear system (2) is finite-time boundedness.

We denote for $i = 1, 2, \ldots, N$,

$$\begin{aligned}
&\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},\ \kappa_1 = \lambda_{\min}(\overline{P}),\ \kappa_2 = \lambda_{\max}(\overline{P}),\ \gamma = \max_{1\le i\le N} \gamma_i,\\
&L_i(P) = PA_i + A_i^T P + \varepsilon_i^{-1} P\mathscr{G}_i\mathscr{G}_i^T P + \mu_i^{-1} PP + \gamma_i^{-1} PW_iW_i^T P + \varepsilon_i \mathscr{H}_i^T \mathscr{H}_i + \mu_i a_i I,\\
&\mathscr{S}_i = \{x \in \mathbb{R}^n : x^T L_i(P) x < 0\},
\end{aligned}$$

and

$$\overline{\mathscr{S}}_1 = \mathscr{S}_1, \overline{\mathscr{S}}_2 = \mathscr{S}_2 \backslash \left(\mathscr{S}_2 \cap \overline{\mathscr{S}}_1\right), \ldots, \overline{\mathscr{S}}_p = \mathscr{S}_p \backslash \left(\mathscr{S}_p \cap \left(\cup_{j=1}^{p-1} \overline{\mathscr{S}}_j\right)\right), \ldots, \overline{\mathscr{S}}_N = \mathscr{S}_N \backslash \left(\mathscr{S}_N \cap \left(\cup_{k=1}^{N-1} \overline{\mathscr{S}}_k\right)\right). \tag{3}$$

Theorem 1. *Suppose that Assumptions 1, 2, 3 are satisfied. For given positive numbers c_1, c_2, d, T_f $(c_1 < c_2)$ and matrix $R \in \mathbb{S}_n^+$, the system (2) is finite-time boundedness with respect to (c_1, c_2, d, T_f, R) if there exist a matrix $P \in \mathbb{S}_n^+$, positive scalars $\varepsilon_i, \mu_i, \gamma_i (i = 1, 2, \ldots, N)$ such that the following conditions are satisfied:*

(i) The set of matrices $L_i(P)$ is strictly complete,

(ii) $\kappa_2 c_1 + \frac{\gamma d}{\Gamma(\alpha+1)} T_f^{\alpha} < \kappa_1 c_2$.

Moreover, the switching laws amongst the subsystems is chosen $\sigma(x(t)) = i \in \mathscr{N}$ whenever $x(t) \in \overline{\mathscr{S}}_i$.

Proof. Let us consider the following non-negative quadratic function

$$V(x(t)) = x^T(t) P x(t).$$

It follows from Lemma 2 that we obtain the $\alpha-$order $(0 < \alpha < 1)$ Caputo derivative of $V(x(t))$ along the trajectories of any subsystem ith $(i = 1, \ldots, N)$ as follows:

$$\begin{aligned}
{}_0^C D_t^{\alpha} V(x(t)) &\le 2x^T(t) P\, {}_0^C D_t^{\alpha} x(t)\\
&= x^T(t) \left[PA_i + A_i^T P\right] x(t) + 2x^T(t) P\mathscr{G}_i \mathscr{F}_i(t) \mathscr{H}_i x(t)\\
&\quad + 2x^T(t) P f_i(x(t)) + 2x^T(t) PW_i d(t).
\end{aligned} \tag{4}$$

By using the Cauchy matrix inequality, we have

$$\begin{aligned}
&2x^T(t) P\mathscr{G}_i \mathscr{F}_i(t) \mathscr{H}_i x(t) \le \varepsilon_i^{-1} x^T(t) P\mathscr{G}_i\mathscr{G}_i^T P x(t) + \varepsilon_i x^T(t) \mathscr{H}_i^T \mathscr{H}_i x(t),\\
&2x^T(t) P f_i(x(t)) \le \mu_i^{-1} x^T(t) PPx(t) + \mu_i f^T(x(t)) f(x(t))\\
&\qquad\qquad\qquad\quad \le \mu_i^{-1} x^T(t) PPx(t) + \mu_i a_i x^T(t) x(t),\\
&2x^T(t) PW_i d(t) \le \gamma_i^{-1} x^T(t) PW_iW_i^T P x(t) + \gamma_i d^T(t) d(t).
\end{aligned} \tag{5}$$

From (4) and (5), we obtain

$${}_0^C D_t^{\alpha} V(x(t)) \le x^T(t) L_i(P) x(t) + \gamma_i d^T(t) d(t), \forall t \in [0, T_f]. \tag{6}$$

From the condition (i) and Remark 1, we have

$$\bigcup_{i=1}^{N} \mathscr{S}_i = \mathbb{R}^n \backslash \{0\}.$$

Based on the sets $\mathscr{S}_i, (i = 1, \ldots, N)$, we construct the sets $\overline{\mathscr{S}}_i$ by (3) and we can verify that

$$\overline{\mathscr{S}}_i \bigcap \overline{\mathscr{S}}_j = \emptyset, (i \neq j), \quad \bigcup_{i=1}^{N} \overline{\mathscr{S}}_i = \mathbb{R}^n \backslash \{0\}. \tag{7}$$

From condition (7), we construct the switching rule $\sigma(x(t)) = i \in \mathscr{N}$ whenever $x(t) \in \overline{\mathscr{S}}_i$. Thus, from (6) and the fact that $\gamma = \max_{1 \leq i \leq N} \gamma_i$, we obtain

$${}_0^C D_t^\alpha V(x(t)) \leq \gamma d^T(t) d(t), \forall t \in [0, T_f]. \tag{8}$$

Integrating with order α both sides of (8) from 0 to $t(0 < t < T_f)$ and using Lemma 1, we have

$$\begin{aligned} x^T(t)Px(t) &\leq x^T(0)Px(0) + {}_0I_t^\alpha \left(\gamma d^T(t) d(t)\right) \\ &= x^T(0)Px(0) + \frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d^T(s) d(s)\, ds \\ &\leq x^T(0)Px(0) + \frac{\gamma d}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq x^T(0)Px(0) + \frac{\gamma d}{\Gamma(\alpha+1)} T_f^\alpha. \end{aligned} \tag{9}$$

On the other hand, we have

$$x^T(t)Px(t) = x^T(t) R^{\frac{1}{2}} \overline{P} R^{\frac{1}{2}} x(t) \geq \lambda_{\min}(\overline{P}) x^T(t) R x(t) = \kappa_1 x^T(t) R x(t) \tag{10}$$

and

$$x^T(0)Px(0) = x^T(0) R^{\frac{1}{2}} \overline{P} R^{\frac{1}{2}} x(0) \leq \lambda_{\max}(\overline{P}) x^T(0) R x(0) = \kappa_2 x^T(0) R x(0) \leq \kappa_2 c_1. \tag{11}$$

From (9) to (11), we have

$$\kappa_1 x^T(t) R x(t) \leq V(x(t)) = x^T(t) P x(t) \leq \kappa_2 c_1 + \frac{\gamma d}{\Gamma(\alpha+1)} T_f^\alpha.$$

From condition (ii), we obtain $x^T(t)Rx(t) < c_2$, which completes the proof of the theorem. □

Remark 2. From the Remark 1, the condition (i) in Theorem 1 is satisfied if there exist numbers $\beta_i \geq 0, \sum_{i=1}^{N} \beta_i > 0$ such that

$$\sum_{i=1}^{N} \beta_i L_i(P) < 0. \tag{12}$$

By using Schur Complement Lemma, we have the condition (12) is equivalent to the following condition

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & -\Xi_{22} \end{bmatrix} < 0, \tag{13}$$

where

$$\Xi_{11} = \sum_{i=1}^{N} \left(\beta_i(PA_i + A_i^T P) + \beta_i\varepsilon_i \mathscr{H}_i^T \mathscr{H}_i + a_i\beta_i\mu_i I\right),$$
$$\Xi_{12} = \begin{bmatrix}\beta_1 P\mathscr{G}_1 & \dots & \beta_N P\mathscr{G}_N & \beta_1 P & \dots & \beta_N P & \beta_1 PW_1 & \dots & \beta_N PW_N\end{bmatrix},$$
$$\Xi_{22} = \operatorname{diag}\{\beta_1\varepsilon_1 I, \dots, \beta_N\varepsilon_N I, \beta_1\mu_1 I, \dots, \beta_N\mu_N I, \beta_1\gamma_1 I, \dots, \beta_N\gamma_N I\}.$$

Note that, matrix inequality (13) can be represented into LMI with N scalars $\beta_1, \dots, \beta_N$. To solve matrix inequality (13), we combine a N-dimensional search method with a convex optimisation algorithm in [31].

Remark 3. From Theorem 1 and Remark 2, we have the following procedure to to solve the problem of switching design for the finite-time boundedness of fractional-order switched nonlinear system (2):
Step 1. Solve matrix inequality (13) and condition (ii) to find a matrix $P \in \mathbb{S}_n^+$ and positive scalars $\varepsilon_i, \mu_i, \gamma_i,\ i = 1, \dots, N$;
Step 2. Construct the sets $\mathscr{S}_i$, and then $\overline{\mathscr{S}}_i$;
Step 3. Choose the switching laws amongst the subsystems as $\sigma(x(t)) = i \in \mathscr{N}$ whenever $x(t) \in \overline{\mathscr{S}}_i$.

3.2. Finite-time stabilization boundedness

We now consider the problem of finite-time stabilization boundedness of the fractional-order switched nonlinear system (1). Our main objective is to design a state feedback controller $u(t) = K_\sigma x(t)$ such that the following closed-loop system

$$(\Sigma_\sigma)\begin{cases} {}_0^C D_t^\alpha x(t) = [A_\sigma + B_\sigma K_\sigma + \Delta A_\sigma(t)]\, x(t) + f_\sigma(x(t)) + W_\sigma d(t), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{14}$$

is finite-time boundedness with respect to (c_1, c_2, d, T_f, R).

For the simplicity of matrix representation, we denote for $i = 1, 2, \dots, N$,

$$\hat{P} = R^{-\frac{1}{2}} P^{-1} R^{-\frac{1}{2}},\ \nu_1 = \lambda_{\min}(\hat{P}),\ \nu_2 = \lambda_{\max}(\hat{P}),\ \theta = \max_{1\leq i\leq \mathscr{N}} \gamma_i^{-1},$$
$$\mathscr{L}_i(P) = A_i P + PA_i^T - B_i B_i^T + \varepsilon_i \mathscr{G}_i \mathscr{G}_i^T + \mu_i I + \gamma_i W_i W_i^T + \varepsilon_i^{-1} P \mathscr{H}_i^T \mathscr{H}_i P + \mu_i^{-1} a_i PP,$$
$$\mathbf{S}_i = \{x \in \mathbb{R}^n : x^T \mathscr{L}_i(P) x < 0\},$$
$$\Lambda_i = \{Px : x \in \mathbf{S}_i\},$$

and

$$\overline{\Lambda}_1 = \Lambda_1, \overline{\Lambda}_2 = \Lambda_2 \backslash \left(\Lambda_2 \cap \overline{\Lambda}_1\right), \dots, \overline{\Lambda}_p = \Lambda_p \backslash \left(\Lambda_p \cap \left(\cup_{j=1}^{p-1} \overline{\Lambda}_j\right)\right), \dots, \overline{\Lambda}_N = \Lambda_N \backslash \left(\Lambda_N \cap \left(\cup_{k=1}^{N-1} \overline{\Lambda}_k\right)\right). \tag{15}$$

Theorem 2. *Assume that Assumption 1, 2, 3 are satisfied. For given positive numbers $c_1, c_2, d, T_f (c_1 < c_2)$ and a matrix $R \in \mathbb{S}_n^+$, the system (1) is finite-time stabilization boundedness with respect to (c_1, c_2, d, T_f, R) if there exist a matrix $P \in \mathbb{S}_n^+$, positive scalars $\varepsilon_i, \mu_i, \gamma_i\ (i = 1, 2, \dots, N)$ such that following conditions hold:*
(i) The set of matrices $\mathscr{L}_i(P)$ is strictly complete,
(ii) $\nu_2 c_1 + \frac{\theta d}{\Gamma(\alpha+1)} T_f^\alpha < \nu_1 c_2$.
Moreover, the switching rule is chosen $\sigma(x(t)) = i \in \mathscr{N}$ whenever $x(t) \in \overline{\Lambda}_i$ and the feedback controller is given by

$$u(t) = -\frac{1}{2} B_\sigma^T P^{-1} x(t), \quad t \in [0, T_f].$$

Proof. Since $\{\mathscr{L}_i(P)\}$ is strictly complete, so we have $\mathbf{S}_i \cap \mathbf{S}_j = \emptyset\,(i \neq j)$ and $\bigcup_{i=1}^{N} \mathbf{S}_i = \mathbb{R}^n \backslash \{0\}$. Based on the set $\mathbf{S}_i$, we construct the sets Λ_i and we will show that

$$\Lambda_i \cap \Lambda_j = \emptyset \quad (i \neq j), \quad \bigcup_{i=1}^{N} \Lambda_i = \mathbb{R}^n \backslash \{0\}. \tag{16}$$

Clearly, $\Lambda_i \cap \Lambda_j = \emptyset\,(i \neq j)$. For any $x \in \mathbb{R}^n \backslash \{0\}$, there is $i \in \mathscr{N}$ such that $y = P^{-1}x \in \mathbf{S}_i$. Hence $x = PP^{-1}x = Py \in \Lambda_i$. Therefore, $\bigcup_{i=1}^{N} \Lambda_i = \mathbb{R}^n \backslash \{0\}$. From (15), we obtain

$$\overline{\Lambda}_i \cap \overline{\Lambda}_j = \emptyset \quad (i \neq j), \quad \bigcup_{i=1}^{N} \overline{\Lambda}_i = \mathbb{R}^n \backslash \{0\}. \tag{17}$$

The switching rule is chosen $\sigma(x(t)) = i \in \mathscr{N}$ whenever $x(t) \in \overline{\Lambda}_i$. So when $x(t) \in \overline{\Lambda}_i$, the ith subsystem is activated and then we have the following subsystem

$$(\Sigma_i) \begin{cases} {}_0^C D_t^\alpha x(t) = [A_i + B_i K_i + \Delta A_i(t)]\, x(t) + f_i(x(t)) + W_i d(t), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{18}$$

Setting $K_i = -0.5 B_i P^{-1}$. Consider the following non-negative quadratic function

$$V(x(t)) = x^T(t) P^{-1} x(t)$$

and based on the similar lines of work as in Theorem 1, we obtain

$${}_0^C D_t^\alpha V(x(t)) \leq \eta^T(t) \mathscr{L}_i(P) \eta(t) + \gamma_i^{-1} d^T(t) d(t), \forall t \in [0, T_f], \tag{19}$$

where

$$\eta(t) = P^{-1} x(t).$$

Noting that $x(t) \in \Lambda_i$ implies $\eta(t) = P^{-1}x(t) \in \mathbf{S}_i$ and $\eta^T(t)\mathscr{L}_i(P)\eta(t) < 0$. From (19) and the fact that $\theta = \max_{1 \leq i \leq N} \gamma_i^{-1}$, we obtain

$${}_0^C D_t^\alpha V(x(t)) \leq \theta d^T(t) d(t), \forall t \in [0, T_f]. \tag{20}$$

Integrating with order α both sides of (20) from 0 to $t\,(0 < t < T_f)$ and using Lemma 1, we have

$$\begin{aligned} x^T(t) P^{-1} x(t) &\leq x^T(0) P^{-1} x(0) + {}_0 I_t^\alpha \left(\theta d^T(t) d(t)\right) \\ &= x^T(0) P^{-1} x(0) + \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d^T(s) d(s)\, ds \\ &\leq x^T(0) P^{-1} x(0) + \frac{\theta d}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq x^T(0) P^{-1} x(0) + \frac{\theta d}{\Gamma(\alpha+1)} T_f^\alpha. \end{aligned} \tag{21}$$

On the other hand, we have

$$x^T(t) P^{-1} x(t) = x^T(t) R^{\frac{1}{2}} \hat{P} R^{\frac{1}{2}} x(t) \geq \lambda_{\min}(\hat{P}) x^T(t) R x(t) = \nu_1 x^T(t) R x(t) \tag{22}$$

and

$$x^T(0)P^{-1}x(0) = x^T(0)R^{\frac{1}{2}}\hat{P}R^{\frac{1}{2}}x(0) \leq \lambda_{\max}(\hat{P})x^T(0)Rx(0) = \nu_2 x^T(0)Rx(0) \leq \nu_2 c_1. \tag{23}$$

From (21) to (23), we have

$$\nu_1 x^T(t)Rx(t) \leq V(x(t)) = x^T(t)P^{-1}x(t) \leq \nu_2 c_1 + \frac{\theta d}{\Gamma(\alpha+1)}T_f^{\alpha}.$$

From condition (ii), we obtain $x^T(t)Rx(t) < c_2$, which completes the proof of the theorem. □

Remark 4. From the Remark 1, the condition (i) in Theorem 2 is satisfied if there exist numbers $\tau_i \geq 0$, $\sum_{i=1}^{N} \tau_i > 0$ such that

$$\mathscr{M} = \sum_{i=1}^{N} \tau_i \mathscr{L}_i(P) < 0. \tag{24}$$

By using Schur Complement Lemma, we have the condition (24) is equivalent to the following condition

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T & -\Psi_{22} \end{bmatrix} < 0, \tag{25}$$

where

$$\begin{aligned}
\Psi_{11} &= \sum_{i=1}^{N} \tau_i \left(A_iP + PA_i^T - B_iB_i^T + \varepsilon_i \mathscr{G}_i\mathscr{G}_i^T + \mu_i I + \gamma_i W_iW_i^T\right), \\
\Psi_{12} &= \begin{bmatrix} \tau_1 P\mathscr{H}_1^T & \dots & \tau_N P\mathscr{H}_N^T & \tau_1 a_1 P & \dots & \tau_N a_N P \end{bmatrix}, \\
\Psi_{22} &= \mathrm{diag}\{\tau_1\varepsilon_1, \dots, \tau_N\varepsilon_N, \tau_1\mu_1, \dots, \tau_N\mu_N\}.
\end{aligned}$$

Note that, matrix inequality (25) can be represented into LMI with N scalars $\tau_1, \dots, \tau_N$. Therefore, it can be efficiently solved by using existing convex algorithms.

Remark 5. From Theorem 2 and Remark 4, we have the following algorithm to solve the problem of switching design for the finite-time stabilization boundedness of fractional-order switched nonlinear system (1).

Step 1. Solve matrix inequality (25) and condition (ii) to find a matrix $P \in \mathbb{S}_n^+$ and positive scalars $\varepsilon_i, \mu_i, \gamma_i,\ i = 1, \dots, N$;

Step 2. Constructing the sets Λ_i, and then $\overline{\Lambda}_i$;

Step 3. Choose the switching laws amongst the subsystems as $\sigma(x(t)) = i \in \mathscr{N}$, whenever $x(t) \in \overline{\Lambda}_i$ and the feedback controller is given by

$$u(t) = -\frac{1}{2}B_\sigma^T P^{-1}x(t), \quad t \in [0, T_f].$$

4. Numerical examples

Example 1. Let us consider the fractional-order switched nonlinear system (2) which consists of two subsystems and the following parameters:

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \mathscr{G}_1 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \mathscr{H}_1 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}, W_1 = \begin{bmatrix} 1.0 \\ 0.9 \end{bmatrix},$$

$$f_1(x(t)) = \begin{bmatrix} \sqrt{0.5}x_1(t) \\ \sqrt{0.5}x_2(t) \end{bmatrix}, \mathscr{F}_1(t) = \sin 0.1t,$$
$$A_2 = \begin{bmatrix} -0.5 & 0 \\ -0.9 & -2 \end{bmatrix}, \mathscr{G}_2 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, \mathscr{H}_2 = \begin{bmatrix} 0.3 & 0.9 \end{bmatrix}, W_2 = \begin{bmatrix} -1.0 \\ 0.5 \end{bmatrix},$$
$$f_2(x(t)) = \begin{bmatrix} \sqrt{0.3}x_1(t) \\ \sqrt{0.3}x_2(t) \end{bmatrix}, \mathscr{F}_2(t) = \cos 0.1t,$$

and $\alpha = 0.6, d(t) = 0.2\sin 0.1t$. Given $T_f = 10, c_1 = 1, c_2 = 2, d = 0.04$, and $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The conditions in Theorem 1 and Remark 2 are satisfied with $\beta_1 = 0.4, \beta_2 = 0.6, \varepsilon_1 = 1.4743, \varepsilon_2 = 1.3495, \mu_1 = 1.5828, \mu_2 = 1.5075, \gamma_1 = 1.4827, \gamma_2 = 1.4453$, and

$$P = \begin{bmatrix} 0.6667 & 0.0404 \\ 0.0404 & 0.7289 \end{bmatrix}.$$

By using Remark 3, the sets $\overline{\mathscr{S}}_1, \overline{\mathscr{S}}_2$ is constructed as follows

$$\overline{\mathscr{S}}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : -1.2402x_1^2 + 1.9544x_1x_2 + 0.3031x_2^2 < 0\},$$
$$\overline{\mathscr{S}}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : -1.2402x_1^2 + 1.9544x_1x_2 + 0.3031x_2^2 > 0\}.$$

The switching law between two subsystems is chosen as:

$$\sigma(x(t)) = \begin{cases} 1, & \text{if} \quad x(t) \in \overline{\mathscr{S}}_1 \\ 2, & \text{if} \quad x(t) \in \overline{\mathscr{S}}_2 \end{cases}.$$

By Theorem 1, the system is finite-time boundedness with respect to $(1, 2, 0.04, 10, R)$.

Example 2. We now consider the controlled fractional-order switched nonlinear system (1) which consists of two subsystems and the following parameters:

$$A_1 = \begin{bmatrix} -0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix}, \mathscr{G}_1 = \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix}, \mathscr{H}_1 = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}, W_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix},$$
$$f_1(x(t)) = \begin{bmatrix} \sqrt{0.2}x_1(t) \\ \sqrt{0.2}x_2(t) \end{bmatrix}, \mathscr{F}_1(t) = \cos t,$$
$$A_2 = \begin{bmatrix} 5 & 0 \\ -0.9 & -1 \end{bmatrix}, \mathscr{G}_2 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \mathscr{H}_2 = \begin{bmatrix} 0.3 & 0.6 \end{bmatrix}, W_2 = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix} B_2 = \begin{bmatrix} 4 \\ 0.5 \end{bmatrix},$$
$$f_2(x(t)) = \begin{bmatrix} \sqrt{0.9}x_1(t) \\ \sqrt{0.9}x_2(t) \end{bmatrix}, \mathscr{F}_2(t) = \cos t,$$

and $\alpha = 0.8, d(t) = 0.1\sin t$. Given $T_f = 10, c_1 = 1, c_2 = 3, d = 0.01$, and $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The conditions in Theorem 2 and Remark 4 are satisfied with $\tau_1 = 0.2, \tau_2 = 0.8, \varepsilon_1 = 3.9620, \varepsilon_2 = 3.7714, \mu_1 = 4.4101, \mu_2 = 3.8321, \gamma_1 = 3.3896, \gamma_2 = 3.4341$, and

$$P = \begin{bmatrix} 0.7191 & -0.0286 \\ -0.0286 & 1.3705 \end{bmatrix}.$$

Then the set of matrices $\{\mathscr{L}_1(P), \mathscr{L}_2(P)\}$, where

$$\mathscr{L}_1(P) = \begin{bmatrix} 3.2272 & 13.5360 \\ 13.5360 & -25.3291 \end{bmatrix}, \mathscr{L}_2(P) = \begin{bmatrix} -4.7981 & -2.7561 \\ -2.7561 & 4.1242 \end{bmatrix}$$

are strictly complete. Using Remark 5, the switching regions are constructed by

$$
\begin{aligned}
&\mathbf{S}_1 = \{(x_1,x_2) \in \mathbb{R}^2 : 3.2272x_1^2 + 27.0720x_1x_2 - 25.3291x_2^2 < 0\},\\
&\mathbf{S}_2 = \{(x_1,x_2) \in \mathbb{R}^2 : -4.7981x_1^2 - 5.5122x_1x_2 + 4.1242x_2^2 < 0\},\\
&\Lambda_1 = \{Px : x \in \mathbf{S}_1\},\\
&\Lambda_2 = \{Px : x \in \mathbf{S}_2\},\\
&\overline{\Lambda}_1 = \Lambda_1, \overline{\Lambda}_2 = \Lambda_2 \setminus \left(\Lambda_2 \cap \overline{\Lambda}_1\right).
\end{aligned}
$$

The switching law between two subsystems is chosen as:

$$
\sigma(x(t)) = \begin{cases} 1, & \text{if} \quad x(t) \in \overline{\Lambda}_1 \\ 2, & \text{if} \quad x(t) \in \overline{\Lambda}_2 \end{cases}.
$$

By Theorem 2, the closed-loop system is finite-time boundedness with respect to $(1,3,0.01,10,R)$. Moreover, the feedback controller is given by $u(t) = K_i x(t), (i = 1,2), t \in [0,10]$, where

$$
K_1 = -\frac{1}{2}B_1^T P^{-1} = \begin{bmatrix}1.3048 & -2.1618\end{bmatrix}, K_2 = -\frac{1}{2}B_2^T P^{-1} = \begin{bmatrix}-2.7909 & -0.2406\end{bmatrix}.
$$

5. Conclusion

This paper has considered the problem of finite-time boundedness and finite-time stabilization boundedness of fractional-order switched nonlinear systems. Based on constructing a simple Lyapunov-like functional and using some properties of Caputo fractional derivative, some sufficient conditions for the problem are derived in the form of linear matrix inequalities. A constructive geometric design of switching laws has also been presented. Two numerical examples have been provided to demonstrate effectiveness of the obtained results. Our future works will focus on finite-time stability and boundedness for fractional-order switched nonlinear systems with time-varying delays since time delay is one of the most common phenomena in real-life systems.

Acknowledgments

This paper was completed when the authors were working as researchers at Vietnam Institute for Advance Study in Mathematics (VIASM). Authors would like to thank the VIASM for providing a fruitful research environment and extending support and hospitality during their visit. The research of Mai Viet Thuan and Dinh Cong Huong was funded by the Vietnam National Foundation for Science and Technology Development under grants 101.01-2017.300 and 101.02-2017.310, respectively.

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