Weakly solutions to the complex Monge-Ampère equation on bounded plurifinely hyperconvex domains

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Abstract

Let μ be a non-negative measure defined on bounded $\mathcal{F}$-hyperconvex domain Ω. We are interested in giving sufficient conditions on μ such that we can find a plurifinely plurisubharmonic function satisfying $NP(dd^c u)^n = \mu$ in $QB(\Omega)$.

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1. Introduction

The plurifine topology $\mathcal{F}$ on a Euclidean open set D is the smallest topology that makes all plurisubharmonic functions on D continuous. Notions pertaining to the plurifine topology are indicated with the prefix $\mathcal{F}$ to distinguish them from notions pertaining to the Euclidean topology on $\mathbb{C}^n$. For a set $A \subset \mathbb{C}^n$ we write $\overline{A}$ for the closure of A in the one point compactification of $\mathbb{C}^n$, $\overline{A}^{\mathcal{F}}$ for the $\mathcal{F}$-closure of A and $\partial_{\mathcal{F}} A$ for the $\mathcal{F}$-boundary of A.

In 2003, El Kadiri [19] defined the notion of $\mathcal{F}$-plurisubharmonic function in an $\mathcal{F}$-open subset of $\mathbb{C}^n$ and studied properties of such functions. Later, El Marzguioui and Wiegerinck [23] proved the continuity properties of the plurifinely plurisubharmonic functions. Next, El Kadiri, Fuglede and Wiegerinck [20] proved the most important properties of the plurifinely plurisubharmonic functions. El Kadiri and Wiegerinck [22] defined the Monge-Ampère operator on finite plurifinely plurisubharmonic functions in $\mathcal{F}$-open sets. They showed that it defines a non-negative measure which vanishes on all pluripolar sets. Note that the measure is in general not a Radon measure, i.e. not Euclidean locally finite. They also defined the non-polar part $NP(dd^c u)^n$ of $\mathcal{F}$-plurisubharmonic function u by

$$\int_A NP(dd^c u)^n = \lim_{j\to+\infty} \int_A (dd^c \max(u, -j))^n, \ A \in QB(\Omega).$$

El Kadiri and Smit [21] introduced the notion of $\mathcal{F}$-maximal $\mathcal{F}$-plurisubharmonic functions and studied properties of such functions. Hong, Hai and Viet [15] proved that $\mathcal{F}$-maximality is an $\mathcal{F}$-local notion for bounded $\mathcal{F}$-plurisubharmonic functions. Trao, Viet and Hong [24] studied the approximation of a negative $\mathcal{F}$-plurisubharmonic function by an increasing sequence of plurisubharmonic functions. They defined the notion of bounded $\mathcal{F}$-hyperconvex domain which extends the notion of bounded hyperconvex domain of $\mathbb{C}^n$ in a natural way. Recently, Hong [13] studied the Dirichlet problem in $\mathcal{F}$-domain. He proved that under the suitable conditions, the complex Monge-Ampère equation can be solved.

The aim of this paper is to study the complex Monge-Ampère equations in bounded $\mathcal{F}$-hyperconvex domains. Namely, we prove the following theorem.

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Theorem 1.1. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let μ be a non-negative measure on $QB(\Omega)$ which vanishes on all pluripolar subsets of Ω such that*

$$\int_\Omega (-\psi) d\mu < +\infty \text{ , for some } \psi \in \mathcal{F}\text{-}PSH^-(\Omega). \tag{1.1}$$

Then, there exists $u \in \mathcal{F}\text{-}PSH^-(\Omega)$ such that $NP(dd^c u)^n = \mu$ on $QB(\Omega)$.

The paper is organized as follows. In Section 2, we recall some notions of plurifine pluripotential theory. Section 3 is devoted to prove Theorem 1.1. In Section 4, we prove a result to show that the condition (1.1) in Theorem 1.1 is sharp.

2. Preliminaries

In this section, we recall some elements of pluripotential theory (plurifine potential theory) that will be used throughout the paper. All those results can be found in [1]-[25].

2.1. The Cegrell's classes

Firstly, we recall the following definitions (see [6], [7]).

Definition 2.1. Let D be a bounded hyperconvex domain in $\mathbb{C}^n$. We say that a bounded, negative plurisubharmonic function φ in D belongs to $\mathcal{E}_0(D)$ if $\{\varphi < -\varepsilon\} \Subset D$ for all $\varepsilon > 0$ and

$$\int_D (dd^c \varphi)^n < +\infty.$$

Let $\mathcal{F}(D)$ be the family of plurisubharmonic functions φ defined on D such that there exists a decreasing sequence $\{\varphi_j\} \subset \mathcal{E}_0(D)$ that converges pointwise to φ on D as $j \to +\infty$ and

$$\sup_j \int_D (dd^c \varphi_j)^n < +\infty.$$

Proposition 2.2. *If $u, v \in \mathcal{F}(D)$ such that $u \leq v$ and $(dd^c u)^n \leq (dd^c v)^n$ then $u = v$ in D.*

Proof. See Theorem 3.6 in [1]. □

Proposition 2.3. *If $\mu$is a non-negative measure in D such that $\mu \leq (dd^c w)^n$ for some $w \in \mathcal{F}(D)$ then there exists $u \in \mathcal{F}(D)$ such that $(dd^c u)^n = \mu$ in D.*

Proof. See Theorem 4.14 in [1]. □

Proposition 2.4. *If $D_1 \Subset \mathbb{C}^{n_1}$, $D_2 \Subset \mathbb{C}^{n_2}$ are two hyperconvex domains and $u_1 \in \mathcal{E}_0(D_1)$, $u_2 \in \mathcal{E}_0(D_2)$ then*

$$\int_{D_1 \times D_2} h(\max(u_1, u_2))(dd^c \max(u_1, u_2))^{n_1+n_2} = \int_{D_1 \times D_2} h(\max(u_1, u_2))(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$$

for all upper semicontinuous functions $h : (-\infty, 0] \to \mathbb{R}$.

Proof. See [2]. □

2.2. The $\mathcal{F}$-plurisubharmonic functions and the Monge-Ampère operator

Next, we recall the following definitions (see [19], [20], [23], [25]).

Definition 2.5. Let Ω be an $\mathcal{F}$-open subset of $\mathbb{C}^n$. A function $u : \Omega \to [-\infty, +\infty)$ is said to be $\mathcal{F}$-plurisubharmonic if u is $\mathcal{F}$-upper semicontinuous and for every complex line l in $\mathbb{C}^n$, the restriction of u to any $\mathcal{F}$-component of the finely open subset $l \cap \Omega$ of l is either finely subharmonic or $\equiv -\infty$.

The set of all negative $\mathcal{F}$-plurisubharmonic functions defined in $\mathcal{F}$-open set Ω is denoted by $\mathcal{F}\text{-}PSH^-(\Omega)$.

Definition 2.6. Let $\Omega \subset \mathbb{C}^n$ be an $\mathcal{F}$-open set and let $u \in \mathcal{F}\text{-}PSH^-(\Omega)$. Denote by $QB(\mathbb{C}^n)$ the measurable space on $\mathbb{C}^n$ generated by the Borel sets and the pluripolar subsets of $\mathbb{C}^n$ and $QB(\Omega)$ is the trace of $QB(\mathbb{C}^n)$ on Ω.

(i) If u is finite then there exist a pluripolar set $E \subset \Omega$, a sequence of $\mathcal{F}$-open subsets $\{O_j\}$ and plurisubharmonic functions f_j, g_j defined in Euclidean neighborhoods of $\overline{O}_j$ such that $\Omega = E \cup \bigcup_{j=1}^{\infty} O_j$ and $u = f_j - g_j$ on O_j. The Monge-Ampère measure $(dd^c u)^n$ on $QB(\Omega)$ is defined by

$$\int_A (dd^c u)^n := \sum_{j=1}^{\infty} \int_{A \cap (O_j \setminus \bigcup_{k=1}^{j-1} O_k)} (dd^c(f_j - g_j))^n, \ \ A \in QB(\Omega).$$

(ii) The non-polar part $NP(dd^c u)^n$ is defined by

$$\int_A NP(dd^c u)^n = \lim_{j \to +\infty} \int_A (dd^c \max(u, -j))^n, \ A \in QB(\Omega).$$

Proposition 2.7. *Let Ω be an $\mathcal{F}$-domain in $\mathbb{C}^n$ and let $v, w \in \mathcal{F}\text{-}PSH(\Omega)$ be finite with $w \leq -v$. Then, for every measure μ on the $QB(\Omega)$ with $0 \leq \mu \leq (dd^c w)^n$, there exists a finite plurifinely plurisubharmonic function u defined on Ω such that $w \leq u \leq -v$ and*

$$(dd^c u)^n = \mu \text{ in } QB(\Omega).$$

Proof. See [13]. □

Definition 2.8. Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $u \in \mathcal{F}\text{-}PSH(\Omega)$. We say that u is $\mathcal{F}$-maximal in Ω if for every bounded $\mathcal{F}$-open set G of $\mathbb{C}^n$ with $\overline{G} \subset \Omega$, and for every function $v \in \mathcal{F}\text{-}PSH(G)$ that is bounded from above on G and extends $\mathcal{F}$-upper semicontinuously to $\overline{G}^{\mathcal{F}}$ with $v \leq u$ on $\partial_{\mathcal{F}} G$ implies $v \leq u$ on G.

Proposition 2.9. *If u is a finite $\mathcal{F}$-maximal $\mathcal{F}$-plurisubharmonic function defined on an $\mathcal{F}$-domain then $(dd^c u)^n = 0$.*

Proof. See Theorem 4.8 in [21]. □

Proposition 2.10. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and assume that $u \in \mathcal{F}\text{-}PSH(\Omega)$ is bounded. Then, u is $\mathcal{F}$-maximal in Ω if and only if $(dd^c u)^n = 0$ on $QB(\Omega)$.*

Proof. See [15]. □

2.3. The Cegrell's classes for $\mathcal{F}$-plurisubharmonic functions

Finally, we recall the definition of bounded $\mathcal{F}$-hyperconvex domain Ω and the class $\mathcal{F}_p(\Omega)$ (see [24]).

Definition 2.11. (i) A bounded $\mathcal{F}$-domain Ω in $\mathbb{C}^n$ is called $\mathcal{F}$-hyperconvex if there exist a negative bounded plurisubharmonic function γ_Ω defined in a bounded hyperconvex domain Ω' such that $\Omega = \Omega' \cap \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega$ is $\mathcal{F}$-plurisubharmonic in Ω.

(ii) We say that a bounded negative $\mathcal{F}$-plurisubharmonic function u defined on a bounded $\mathcal{F}$-hyperconvex domain Ω belongs to $\mathcal{E}_0(\Omega)$ if $\int_\Omega (dd^c u)^n < +\infty$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\overline{\Omega \cap \{u < -\varepsilon\}} \subset \Omega' \cap \{\gamma_\Omega > -1 + \delta\}.$$

(iii) Denote by $\mathcal{F}_p(\Omega)$, $p > 0$ the family of negative $\mathcal{F}$-plurisubharmonic functions u defined on Ω such that there exist a decreasing sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to u on Ω and

$$\sup_{j \geq 1} \int_\Omega (1 + (-u_j)^p)(dd^c u_j)^n < +\infty.$$

Proposition 2.12. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded $\mathcal{F}$-hyperconvex domain. Assume that $u \in \mathcal{F}_1(\Omega)$ is bounded and $v \in \mathcal{F}\text{-}PSH^-(\Omega)$ such that $(dd^c u)^n \leq (dd^c v)^n$ in $\Omega \cap \{v > -\infty\}$. Then, $u \geq v$ in Ω.*

Proof. Without loss of generality we can assume that $-1 \leq u \leq 0$ on Ω. Let $j \in \mathbb{N}^*$ and define

$$v_j := (1 + \frac{1}{j})(v - \frac{1}{j}) \text{ in } \Omega.$$

Choose $p > 0$ such that $j^p < 1 + \frac{1}{j}$. It is easy to see that

$$(1 + (-u)^p)(dd^c u)^n \leq 2(dd^c u)^n \leq 2(dd^c v)^n \leq (1 + (-v_j)^p)(dd^c v_j)^n \text{ on } \Omega \cap \{v_j > -\infty\}.$$

Since u is bounded, so $u \in \mathcal{F}_p(\Omega)$, and hence, Proposition 4.4 in [24] implies that $u \geq v_j$ in Ω. Letting $j \to +\infty$ we conclude that $u \geq v$ in Ω. The proof is complete. □

3. The complex Monge-Ampère equations

We need the following auxiliary lemma.

Lemma 3.1. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $\Omega \Subset D$ be a bounded $\mathcal{F}$-hyperconvex domain. Assume that $u \in \mathcal{E}_0(\Omega)$ and define*

$$w := \sup\{\varphi \in PSH^-(D) : \varphi \leq u \text{ on } \Omega\}.$$

Then, $w \in \mathcal{E}_0(D)$ and $(dd^c w)^n \leq 1_\Omega (dd^c u)^n$ in D.

Proof. It is easy to see that $w \in \mathcal{E}_0(D)$. Without loss of generality we can assume that $-\frac{1}{2} \leq u < 0$ in Ω, and hence, $-\frac{1}{2} \leq w < 0$ in D. First, we claim that

$$(dd^c w)^n \leq (dd^c u)^n \text{ on } \Omega \cap \{w = u\}. \tag{3.1}$$

Indeed, let Ω' be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $\gamma_\Omega \in PSH^-(\Omega') \cap L^\infty(\Omega')$ such that $\Omega = \Omega' \cap \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega \in \mathcal{F}\text{-}PSH(\Omega)$. Choose $\varepsilon, \delta > 0$ such that $\sup_\Omega w < -2\varepsilon$ and

$$\overline{\Omega \cap \{u < -\varepsilon\}} \subset \Omega' \cap \{\gamma_\Omega > -1 + 2\delta\}.$$

Let j be an integer number with $j\varepsilon > 1$. Proposition 2.3 in [21] states that the functions

$$f := \begin{cases} \max(-\frac{1}{\delta}, u + \frac{1}{\delta}\gamma_\Omega) & \text{in } \Omega \\ -\frac{1}{\delta} & \text{in } \Omega' \setminus \Omega \end{cases} \quad \text{and } f_j := \begin{cases} \max(-\frac{1}{\delta}, \max(u, w + \frac{1}{2j}) + \frac{1}{\delta}\gamma_\Omega) & \text{in } \Omega \\ -\frac{1}{\delta} & \text{in } \Omega' \setminus \Omega \end{cases}$$

are $\mathcal{F}$-plurisubharmonic and Proposition 2.14 in [20] states that $f, f_j \in PSH(\Omega')$ because Ω' is a Euclidean open set. Since $u = f - \frac{1}{\delta}\gamma_\Omega$, $\max(u, w + \frac{1}{2j}) = f_j - \frac{1}{\delta}\gamma_\Omega$ in $\{\gamma_\Omega > -1 + \delta\}$, by [4] we have

$$\lim_{j \to +\infty} \int_\Omega \chi (dd^c \max(u, w + \frac{1}{j}))^n = \int_\Omega \chi (dd^c u)^n \tag{3.2}$$

for every bounded $\mathcal{F}$-continuous function χ with compact support on $\{\gamma_\Omega > -1 + \delta\}$. Let $K \subset \Omega \cap \{w = u\}$ be a compact set. Since $\Omega \cap \{w = u\} \subset \Omega \cap \{u < -\varepsilon\} \subset \{\gamma_\Omega > -1 + 2\delta\}$, there exists a decreasing sequence of bounded $\mathcal{F}$-continuous functions $\{\chi_k\}$ with compact support on $\{\gamma_\Omega > -1 + \delta\}$ such that $\chi_k \searrow 1_K$ as $k \nearrow +\infty$. By Theorem 4.8 in [22] we conclude by (3.2) that

$$\int_K (dd^c w)^n \leq \lim_{j \to +\infty} \int_\Omega \chi_k (dd^c \max(u, w + \frac{1}{j}))^n = \int_\Omega \chi_k (dd^c u)^n, \ \forall k \geq 1.$$

Letting $k \to +\infty$, we obtain that

$$\int_K (dd^c w)^n \leq \int_K (dd^c u)^n.$$

Therefore, $(dd^c w)^n \le (dd^c u)^n$ on $\Omega \cap \{w = u\}$. This proves the claim. Now, since u is $\mathcal{F}$-continuous on Ω, it follows that the function

$$h := \begin{cases} u & \text{on } \Omega \\ 0 & \text{in } D \backslash \Omega \end{cases}$$

is $\mathcal{F}$-continuous on D, and hence,

$$U := D \cap \{w < h\} \text{ is } \mathcal{F}\text{-open set.}$$

Let $z \in U$ and let $a \in \mathbb{R}$ be such that $w(z) < a < h(z)$. Let V be a connected component of the $\mathcal{F}$-open set $D \cap \{w < a\} \cap \{h > a\}$ which contains the point z. We claim that w is $\mathcal{F}$-maximal in V. Indeed, let G be a bounded $\mathcal{F}$-open set in $\mathbb{C}^n$ with $\overline{G} \subset V$ and let $v \in \mathcal{F}\text{-}PSH(G)$ such that v is bounded from above on G, extends $\mathcal{F}$-upper semicontinuously to $\overline{G}^{\mathcal{F}}$ and $v \le w$ on $\partial_{\mathcal{F}} G$. Since D is a Euclidean open set, Proposition 2.3 in [21] and Proposition 2.14 in [20] imply that the function

$$\varphi := \begin{cases} \max(w, v) & \text{on } G \\ w & \text{on } D \backslash G \end{cases}$$

is plurisubharmonic in D. Because $\overline{G} \subset V \subset D \cap \{w < a\}$, we infer that $\varphi < a$ on $\overline{G}$, and therefore, $\varphi \le h$ in D. It follows that $\varphi = w$ in D. Thus, $v \le w$ in G, and hence, w is $\mathcal{F}$-maximal in V. This proves the claim. Therefore, w is $\mathcal{F}$-locally $\mathcal{F}$-maximal in U. Theorem 1 in [15] implies that

$$(dd^c w)^n = 0 \text{ on } U.$$

Combining this with (3.1) we arrive at $(dd^c w)^n \le 1_\Omega (dd^c u)^n$ in D. The proof is complete. □

We now able to give the proof of theorem 1.1.

Proof of Theorem 1.1. Without loss of generality we can assume that $\psi \in \mathcal{E}_0(\Omega)$ and $-1 \le \psi < 0$ in Ω. Let $r > 0$ be such that $\Omega \Subset \mathbb{B}(0, r)$. Let $j \ge 1$ be an integer number. Since

$$\int_{\mathbb{B}(0,r)} 1_{\Omega \cap \{-\frac{1}{j} \le \psi < -\frac{1}{j+1}\}} d\mu \le (j+1) \int_\Omega (-\psi) d\mu < +\infty,$$

Theorem 6.2 in [7] implies that there exist $\psi_j \in \mathcal{E}_0(\mathbb{B}(0, r))$ and $0 \le f_j \in L^1((dd^c \psi_j)^n)$ such that

$$1_{\Omega \cap \{-\frac{1}{j} \le \psi < -\frac{1}{j+1}\}} \mu = f_j (dd^c \psi_j)^n \text{ in } \mathbb{B}(0, r).$$

Let $a_j > 0$ be such that

$$\sum_{k=1}^{j} \psi_k > -a_j \text{ in } \mathbb{B}(0, r).$$

Thanks to Theorem 4.8 in [22] we have

$$\begin{aligned} (dd^c \max(\sum_{k=1}^{j} \psi_k, a_j(j+1)\psi))^n &\ge 1_{\Omega \cap \{\psi < -\frac{1}{j+1}\}} (dd^c (\sum_{k=1}^{j} \psi_k))^n \\ &\ge \sum_{k=1}^{j} 1_{\Omega \cap \{-\frac{1}{k} \le \psi < -\frac{1}{k+1}\}} (dd^c \psi_k)^n \ge \sum_{k=1}^{j} \min(f_k, j)(dd^c \psi_k)^n \text{ on } QB(\Omega). \end{aligned}$$

By Theorem 1.1 in [13] we can find $u_j \in \mathcal{F}\text{-}PSH(\Omega)$ such that $\max(\sum_{k=1}^{j} \psi_k, a_j(j+1)\psi) \le u \le 0$ on Ω and

$$(dd^c u_j)^n = \sum_{k=1}^{j} \min(f_k, j)(dd^c \psi_k)^n \text{ on } QB(\Omega).$$

Since $\max(\sum_{k=1}^{j}\psi_k, a_j(j+1)\psi) \in \mathcal{E}_0(\Omega)$, Proposition 3.4 in [24] implies that $u_j \in \mathcal{E}_0(\Omega)$, and hence, Proposition 2.12 states that $u_j \geq u_{j+1}$ in Ω because $(dd^c u_j)^n \leq (dd^c u_{j+1})^n$ on $QB(\Omega)$. Put

$$u := \lim_{j\to+\infty} u_j \text{ on } \Omega.$$

We claim that $u \not\equiv -\infty$ in Ω. Indeed, without loss of generality we can assume that $G := \{\psi < -\frac{1}{2}\} \neq \emptyset$. We set

$$v_j := \sup\{\varphi \in \mathcal{F}\text{-}PSH^-(\Omega) : \varphi \leq u_j \text{ on } G\}.$$

Then, $v_j \in \mathcal{E}_0(\Omega)$, $u_j \leq v_j < 0$ in Ω and $v_j = u_j$ on G. By Proposition 3.1 in [21] we have u_j is $\mathcal{F}$-maximal on $\Omega \cap \{\psi > -\frac{1}{2}\}$, and hence, $(dd^c v_j)^n = 0$ on $\Omega \cap \{\psi > -\frac{1}{2}\}$. Therefore, using Proposition 3.4 in [24] we infer that

$$\begin{aligned}
\int_\Omega (dd^c v_j)^n &= \int_{\Omega\cap\{\psi\leq -\frac{1}{2}\}} (dd^c v_j)^n \leq 2\int_\Omega (-\psi)(dd^c v_j)^n \\
&\leq 2\int_\Omega (-\psi)(dd^c u_j)^n = 2\sum_{k=1}^{j}\int_\Omega (-\psi)\min(f_k, j)(dd^c\psi_k)^n \\
&\leq 2\sum_{k=1}^{j}\int_\Omega (-\psi)1_{\Omega\cap\{-\frac{1}{k}\leq\psi<-\frac{1}{k+1}\}}d\mu \leq 2\int_\Omega(-\psi)d\mu.
\end{aligned} \tag{3.3}$$

Lemma 3.1 states that the function

$$w_j := \sup\{\varphi \in PSH^-(\mathbb{B}(0,r)) : \varphi \leq v_j \text{ on } \Omega\}$$

belongs to $\mathcal{E}_0(\mathbb{B}(0,r))$ and

$$(dd^c w_j)^n \leq 1_\Omega (dd^c v_j)^n \text{ in } \mathbb{B}(0,r).$$

Hence, by the hypotheses and using (3.3) we get

$$\sup_j \int_{\mathbb{B}(0,r)} (dd^c w_j)^n \leq \sup_j \int_\Omega (dd^c v_j)^n \leq 2\int_\Omega (-\psi)d\mu < +\infty.$$

This implies that $w := \lim_{j\to+\infty} w_j \in \mathcal{E}_0(\mathbb{B}(0,r))$. Thanks to Theorem 2.3 in [20] we arrive at $\mathbb{B}(0,r)\cap\{w = -\infty\}$ has no $\mathcal{F}$-interior point. Moreover, since $w \leq v = u$ on G so $u \not\equiv -\infty$ on G. Thus, $u \not\equiv -\infty$ on Ω. This proves the claim, and therefore, $u \in \mathcal{F}\text{-}PSH(\Omega)$. Since $u_j \searrow u$ on $\Omega\cap\{u > -\infty\}$ as $j \nearrow +\infty$, by Theorem 4.5 in [21] we have the sequence of measures $(dd^c u_j)^n$ converges $\mathcal{F}$-locally vaguely to $(dd^c u)^n$ on $\Omega\cap\{u > -\infty\}$. Moreover, since

$$(dd^c u_j)^n = \sum_{k=1}^{j}\min(f_k, j)(dd^c\psi_k)^n \nearrow \sum_{k=1}^{\infty} f_k(dd^c\psi_k)^n = \mu \text{ on } \Omega,$$

it follows that

$$(dd^c u)^n = \mu \text{ on } QB(\Omega\cap\{u > -\infty\}).$$

Thus, $NP(dd^c u)^n = \mu$ on $QB(\Omega)$. The proof is complete. □

4. Measures without solutions

First we prove the following.

Proposition 4.1. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}$ that has no Euclidean interior points. Then, there exists a non-negative measure μ on $QB(\Omega)$ such that*

(i) μ vanishes on all pluripolar subsets of Ω;

(ii) $\int_\Omega(-\psi)d\mu < +\infty$, for some negative finely subharmonic function ψ in Ω;

(iii) There is no subharmonic function w defined on Euclidean open neighborhood of Ω satisfying

$$NP(dd^c w) = \mu \text{ on } QB(\Omega).$$

Proof. Let γ_Ω be a negative bounded subharmonic function defined in a bounded hyperconvex domain Ω' such that $\Omega = \Omega' \cap \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega$ is finely subharmonic in Ω. Since Ω has no Euclidean interior points, we can find $\{a_j\} \subset \Omega$ such that $a_j \to a \in \partial_{\mathcal{F}}\Omega \cap \Omega'$ and

$$-1 < \gamma_\Omega(a_{j+1}) < \gamma_\Omega(a_j) < -1 + \frac{1}{2^j}.$$

Theorem 4.14 in [1] implies that there exist $u_j \in \mathcal{F}(\Omega')$ be such that

$$dd^c u_j = \delta_{a_j} \ \text{ in } \Omega',$$

where δ_{a_j} denotes the Dirac measure at a_j. Let $r_j \in (0, \frac{1}{2^j})$ and let $\varphi_j \in SH^-(\mathbb{B}(a_j, 2r_j))$ be such that $\varphi_j(a_j) > -1$ and

$$\mathbb{B}(a_j, 2r_j) \cap \{\varphi_j > -2\} \subset \Omega \cap \{\gamma_\Omega(a_{j+1}) < \gamma_\Omega < -1 + \frac{1}{2^j}\}.$$

We set

$$u_{j,k} := \sup\{\varphi \in SH^-(\Omega') : \varphi \le \max(u_j, -k) \text{ on } \mathbb{B}(a_j, r_j) \cap \{\varphi_j > -1\}\}.$$

Since $\mathbb{B}(a_j, r_j) \cap \{\varphi_j > -1\}$ is $\mathcal{F}$-open set, by Corollary 3.10 in [20] we infer that $u_{j,k} \in SH^-(\Omega')$. Proposition 3.2 in [21] implies that u_{j,k_j} is $\mathcal{F}$-maximal on the $\mathcal{F}$-interior of $\Omega' \backslash (\mathbb{B}(a_j, r_j) \cap \{\varphi_j > -1\})$, and hence, by Theorem 4.8 in [21] (also see [15]) we get

$$dd^c u_{j,k_j} = 0 \ \text{ on } \Omega' \backslash (\overline{\mathbb{B}}(a_j, r_j) \cap \{\varphi_j \ge -1\}). \tag{4.1}$$

We now claim that $u_{j,k} \searrow u_j$ in Ω' as $k \nearrow +\infty$. Indeed, since $u_j \le u_{j,k+1} \le u_{j,k}$ in Ω' so $v_j := \lim_k u_{j,k} \in SH^-(\Omega')$ and $v_j \ge u_j$. Because $u_j = v_j$ on $\mathbb{B}(a_j, r_j) \cap \{\varphi_j > -1\}$, by Lemma 4.1 in [1] and Theorem 1.1 in [9] we get

$$dd^c v_j \le dd^c u_j = 1_{\{a_j\}} dd^c u_j = 1_{\{a_j\}} dd^c v_j \le dd^c v_j \ \text{ in } \Omega'.$$

This implies that $dd^c v_j = dd^c u_j = \delta_{a_j}$ in Ω'. According to Theorem 3.6 in [1] we have $v_j = u_j$ in Ω', and hence, $u_{j,k} \searrow u_j$ on Ω' as $k \nearrow +\infty$. This proves the claim. Therefore, Corollary 3.4 in [1] implies that

$$\lim_{k \to +\infty} \int_{\Omega'} dd^c u_{j,k} = \int_{\Omega'} dd^c u_j = 1.$$

Let $k_j \ge 1$ be such that

$$\frac{1}{2} \le \int_{\Omega'} dd^c u_{j,k_j} \le 1. \tag{4.2}$$

We set $\psi := -1 - \gamma_\Omega$ and

$$\mu := \sum_{j \ge 1} dd^c u_{j,k_j} \text{ on } QB(\Omega).$$

Then, ψ is negative finely subharmonic function on Ω and μ vanishes on all pluripolar subsets of Ω. Since

$$\overline{\mathbb{B}}(a_j, r_j) \cap \{\varphi_j \ge -1\} \subset \mathbb{B}(a_j, 2r_j) \cap \{\varphi_j > -2\} \subset \Omega \cap \{\gamma_\Omega < -1 + \frac{1}{2^j}\},$$

by (4.1) and (4.2) we arrive at

$$\begin{aligned}
\int_\Omega (-\psi) d\mu &= \sum_{j \ge 1} \int_\Omega (1 + \gamma_\Omega) dd^c u_{j,k_j} \\
&= \sum_{j \ge 1} \int_{\Omega \cap \{\gamma_\Omega < -1 + \frac{1}{2^j}\}} (1 + \gamma_\Omega) dd^c u_{j,k_j} \\
&\le \sum_{j \ge 1} \frac{1}{2^j} \int_{\Omega'} dd^c u_{j,k_j} \le 1.
\end{aligned}$$

We now assume that there exists a subharmonic function w defined on Euclidean open neighborhood O of Ω such that

$$NP(dd^c w) = \mu \text{ on } QB(\Omega).$$

Let $r > 0$ and let $j_0 \in \mathbb{N}$ be such that $\mathbb{B}(a_j, 2r_j) \Subset \mathbb{B}(a, r) \Subset O$ for all $j \geq j_0$. Again by (4.1) and (4.2) we get

$$\begin{aligned}+\infty > \int_{\mathbb{B}(a,r)} dd^c w &\geq \int_{\mathbb{B}(a,r)\cap\Omega} d\mu \\ &\geq \sum_{j\geq j_0} \int_{\overline{\mathbb{B}}(a_j,r_j)\cap\{\varphi_j\geq -1\}} dd^c u_{j,k_j} \\ &= \sum_{j\geq j_0} \int_{\Omega'} dd^c u_{j,k_j} \geq \sum_{j\geq j_0} \frac{1}{2} = +\infty.\end{aligned}$$

This is impossible. The proof is complete. □

We now show that the condition (1.1) in Theorem 1.1 is sharp.

Proposition 4.2. *Let n be an integer number with $n \geq 2$. Then, there exist a bounded $\mathcal{F}$-hyperconvex domain $\Omega \subset \mathbb{C}^n$ and a non-negative measure μ on $QB(\Omega)$ such that*

(i) Ω has no Euclidean interior points;

(ii) μ vanishes on all pluripolar subsets of Ω;

(iii) $\int_\Omega (-\psi)^p d\mu < +\infty$ for all $p > 1$, for some $\psi \in \mathcal{F}\text{-}PSH^-(\Omega)$;

(iv) There is no function $w \in \mathcal{F}\text{-}PSH^-(\Omega)$ satisfying $NP(dd^c w)^n = \mu$ on $QB(\Omega)$.

Proof. Let Δ be a unit disc in $\mathbb{C}$. By Example 3.3 in [24] we can find a bounded $\mathcal{F}$-hyperconvex domain $D \subset \Delta$ and an increasing sequence of negative subharmonic functions ρ_j defined on bounded hyperconvex domains D_j such that $D \subset D_{j+1} \subset D_j$, D has no Euclidean interior points and $\rho_j \nearrow \rho \in \mathcal{E}_0(D)$ a.e. on D. Let $a \in D$. Thanks to Theorem 4.14 in [1] we can find an increasing sequence of negative subharmonic functions $u_j \in \mathcal{F}(D_j)$ such that $u_j \leq u_{j+1}$ on D_{j+1} and

$$dd^c u_j = \delta_a \text{ on } D_j,$$

where δ_a denotes the Dirac measure at a. Let u be the least $\mathcal{F}$-upper semicontinuous majorant of $(\sup_{j\geq 1} u_j)$ in D. Then, $u_j \nearrow u$ a.e. on D as $j \nearrow +\infty$. Let $v \in \mathcal{F}(\Delta^{n-1})$ be such that $\lim_{z\ni\Delta^{n-1}\to\partial\Delta^{n-1}} v(z) = 0$ and

$$(dd^c v)^{n-1} = \delta_o \text{ on } \Delta^{n-1}, \text{ where } o \text{ is the origin of } \mathbb{C}^{n-1}.$$

We set $\Omega := D \times \Delta^{n-1}$ and $\mu := \sum_{k\geq 1} (dd^c w_k)^n$ on Ω, where

$$w_k(t, z) := \max(2^k u(t), kv(z), -1), \quad (t, z) \in D \times \Delta^{n-1}.$$

It is easy to see that Ω is bounded $\mathcal{F}$-hyperconvex domain that has no Euclidean interior point exists. Using Theorem 4.6 in [20] we obtain that $w_k \in \mathcal{F}\text{-}PSH^-(\Omega) \cap L^\infty(\Omega)$, and therefore, μ vanishes on all pluripolar subsets of Ω. We now claim that

$$\int_{D\times\Delta^{n-1}} (-\max(u, v))^p d\mu < +\infty, \ \forall p > 1.$$

Indeed, since $dd^c \max(u_j, -\frac{1}{2^k}) = 0$ on $D_j \cap \{u_j \neq -\frac{1}{2^k}\}$, by Corollary 2.1 in [2] and Corollary 4.2 in [2] we get

$$\begin{aligned}&\int_{D_j\times\Delta^{n-1}} (-\max(u_j, v))^p (dd^c \max(2^k u_j, kv, -1))^n \\ &= \int_{D_j\times\Delta^{n-1}} (-\max(u_j, v))^p dd^c \max(2^k u_j, -1) \wedge (dd^c \max(kv, -1))^{n-1}\end{aligned}$$

$$
\begin{aligned}
&= 2^k k^{n-1} \int\limits_{D_j\times\Delta^{n-1}} (-\max(u_j, v))^p dd^c \max(u_j, -\frac{1}{2^k}) \wedge (dd^c \max(v, -\frac{1}{k}))^{n-1} \\
&\le 2^{-k(p-1)} k^{n-1} \int\limits_{D_j\times\Delta^{n-1}} dd^c \max(u_j, -\frac{1}{2^k}) \wedge (dd^c \max(v, -\frac{1}{k}))^{n-1} \\
&= 2^{-k(p-1)} k^{n-1} \int\limits_{D_j} dd^c \max(u_j, -\frac{1}{2^k}) \int\limits_{\Delta^{n-1}} (dd^c \max(v, -\frac{1}{k}))^{n-1} \\
&= 2^{-k(p-1)} k^{n-1}.
\end{aligned}
$$

Proposition 2.7 in [24] implies that

$$
\begin{aligned}
\int\limits_{D\times\Delta^{n-1}} (-\max(u, v))^p d\mu &\le \liminf_{j\to+\infty} \int\limits_{D_j\times\Delta^{n-1}} (-\max(u_j, v))^p \sum_{k\ge 1} (dd^c \max(2^k u_j, kv, -1))^n \\
&\le \sum_{k\ge 1} 2^{-k(p-1)} k^{n-1} < +\infty.
\end{aligned}
$$

This proves the claim. Now, assume that $\mu = NP(dd^c w)^n$ for some $w \in \mathcal{F}\text{-}PSH^-(\Omega)$. We claim that $w \le w_k$ in Ω for any $k \ge 1$. Indeed, let $h \ge 1$ be an integer number and define

$$
\varphi_h(t, z) := \max(h\rho(t), 2^k u(t), kv(z), -1), \quad (t, z) \in D \times \Delta^{n-1}.
$$

By Proposition 2.7 in [24] and Corollary 2.1 in [2] we have

$$
\begin{aligned}
\int_\Omega (dd^c \varphi_h)^n &\le \liminf_{j\to+\infty} \int_{D_j\times\Delta^{n-1}} (dd^c \max(h\rho_j, 2^k u_j, kv, -1))^n \\
&= \liminf_{j\to+\infty} \int_{D_j} dd^c \max(h\rho, 2^k u, -1) \int_{\Delta^{n-1}} (dd^c \max(kv, -1))^{n-1} \\
&\le \liminf_{j\to+\infty} \int_{D_j} dd^c \max(2^k u, -1) \int_{\Delta^{n-1}} (dd^c \max(kv, -1))^{n-1} \\
&= 2^k k^{n-1}.
\end{aligned}
$$

This implies that $\varphi_h \in \mathcal{E}_0(\Omega)$ and

$$
\sup_{h\ge 1} \int_\Omega (dd^c \varphi_h)^n \le 2^k k^{n-1}.
$$

Since w_k is bounded and $\varphi_h \searrow w_k$ on Ω as $h \nearrow +\infty$, we infer that $w_k \in \mathcal{F}_1(\Omega)$. Proposition 2.12 implies that $w \le w_k$ in Ω because $(dd^c w)^n \le (dd^c w_k)^n$ on $\{w > -\infty\}$. This proves the claim. Letting $k \to \infty$ we arrive that $w \le -1$ on Ω, and hence, $w + 1 \in \mathcal{F}\text{-}PSH^-(\Omega)$. Replace w by $w + 1$ and using above argument we obtain that $w + 1 \le -1$ on Ω. Therefore, $w \le -2$ on Ω. By induction we obtain that

$$
w \equiv -\infty \text{ in } \Omega.
$$

This is impossible. Thus, there is no function $w \in \mathcal{F}\text{-}PSH^-(\Omega)$ satisfying $NP(dd^c w)^n = \mu$ on $QB(\Omega)$. The proof is complete. □

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