On the approximation of weakly plurifinely plurisubharmonic functions

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Abstract

In this note, we study the approximation of singular plurifine plurisubharmonic function u defined on a plurifine domain Ω. Under some condition we prove that u can be approximated by an increasing sequence of plurisubharmonic functions defined on Euclidean neighborhoods of Ω.

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1. Notation and main result

The plurifine topology $\mathcal{F}$ on a Euclidean open set D is the smallest topology that makes all plurisubharmonic functions on D continuous. Notions pertaining to the plurifine topology are indicated with the prefix $\mathcal{F}$ to distinguish them from notions pertaining to the Euclidean topology on $\mathbb{C}^n$. For a set $A \subset \mathbb{C}^n$ we write $\overline{A}$ for the closure of A in the one point compactification of $\mathbb{C}^n$, $\overline{A}^{\mathcal{F}}$ for the $\mathcal{F}$-closure of A and $\partial_{\mathcal{F}} A$ for the $\mathcal{F}$-boundary of A.

Let Ω be a bounded $\mathcal{F}$-domain in $\mathbb{C}^n$. A function $u : \Omega \to [-\infty, +\infty)$ is said to be $\mathcal{F}$-plurisubharmonic if u is $\mathcal{F}$-upper semicontinuous and for every complex line l in $\mathbb{C}^n$, the restriction of u to any $\mathcal{F}$-component of the finely open subset $l \cap \Omega$ of l is either finely subharmonic or $\equiv -\infty$. El Kadiri, Fuglede and Wiegerinck [16] proved the most important properties of the $\mathcal{F}$-plurisubharmonic functions. El Kadiri and Wiegerinck [18] defined the complex Monge-Ampère operator for finite $\mathcal{F}$-plurisubharmonic functions on an $\mathcal{F}$-domain Ω. Recently, Hong and coauthors have been successfully pushing the theory of $\mathcal{F}$-plurisubharmonic functions (see [12], [13], [14], [19]). The aim of this note is to study the conditions on u and Ω such that u can be approximated by an increasing sequence of plurisubharmonic functions defined on Euclidean neighborhoods of Ω.

When Ω is bounded Euclidean domain with C^1-boundary, Fornæss and Wiegerinck [9] proved that if u is continuous on $\overline{\Omega}$ then u can be approximated uniformly on $\overline{\Omega}$ by a sequence of smooth plurisubharmonic functions defined on Euclidean neighborhoods of Ω.

When Ω is bounded hyperconvex domain, according to the results by [4], [5], [8], [10] and other authors, the approximation is possible if the domain Ω has the $\mathcal{F}$-approximation property and u belongs to one of the Cegrell's classes in Ω.

When Ω is bounded $\mathcal{F}$-domain, the authors gave in [19] the kind of Ω and u that are in line with the $\mathcal{F}$-set up to make the approximation possible.

The purpose of this note is to extend the result of [19]. In analogy with the set up of the hyperconvex domain to make the approximation possible, we introduce the following:

Definition 1.1. Let Ω be a bounded $\mathcal{F}$-hyperconvex domain, i.e., it is a bounded, connected, and $\mathcal{F}$-open set such that there exists a negative bounded plurisubharmonic function γ_Ω defined in a bounded hyperconvex domain $\Omega' \supset \Omega$ such that $\Omega = \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega$ is $\mathcal{F}$-plurisubharmonic in Ω. We say that Ω has the $\mathcal{F}$-approximation property if there

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exists an increasing sequence of negative plurisubharmonic functions ρ_j defined on bounded hyperconvex domains Ω_j such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and $\rho_j \nearrow \rho \in \mathcal{E}_0(\Omega)$ a.e. on Ω as $j \nearrow +\infty$. Here

$$\mathcal{E}_0(\Omega) := \{u \in \mathcal{F}\text{-}PSH^-(\Omega) \cap L^\infty(\Omega) : \int_\Omega (dd^c u)^n < +\infty$$
$$\text{and } \forall \varepsilon > 0,\ \exists \delta > 0,\ \overline{\{u < -\varepsilon\}} \subset \{\gamma_\Omega > -1 + \delta\}\}.$$

Every bounded hyperconvex domain is $\mathcal{F}$-hyperconvex. Example 3.3 in [19] showed that there exists a bounded $\mathcal{F}$-hyperconvex domain Ω that has the $\mathcal{F}$-approximation property, moreover, it has no Euclidean interior point. For the precise definition and properties of the class $\mathcal{F}(\Omega)$ we refer the reader to the next section. Our main result is the following theorem.

Theorem 1.2. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain and let $u \in \mathcal{F}(\Omega)$. Assume that Ω has the $\mathcal{F}$-approximation property. Then there exists an increasing sequence of plurisubharmonic functions u_j defined on Euclidean neighborhoods of Ω such that $u_j \nearrow u$ a.e. on Ω as $j \nearrow +\infty$.*

The note is organized as follows. In Section 2, we introduce and investigate the class $\mathcal{F}(\Omega)$. Section 3 is devoted to prove Theorem 1.2.

2. The class $\mathcal{F}(\Omega)$

Some elements of pluripotential theory (plurifine potential theory) that will be used throughout the paper can be found in [1]-[22]. We denote by $\mathcal{F}\text{-}PSH^-(\Omega)$ the set of negative $\mathcal{F}$-plurisubharmonic functions defined in an $\mathcal{F}$-open set Ω. First, we recall the definition of the complex Monge-Ampère measure for finite $\mathcal{F}$-plurisubharmonic functions.

Definition 2.1. Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $QB(\Omega)$ be the trace of $QB(\mathbb{C}^n)$ on Ω, where $QB(\mathbb{C}^n)$ denotes the σ-algebra on $\mathbb{C}^n$ generated by the Borel sets and the pluripolar subsets of $\mathbb{C}^n$. Assume that $u_1, \ldots, u_n$ are finite $\mathcal{F}-$plurisubharmonic functions in Ω. Using the quasi-Lindelöf property of the plurifine topology and Theorem 2.17 in [18], there exist a pluripolar set $E \subset \Omega$, a sequence of $\mathcal{F}$-open subsets $\{O_k\}$ and plurisubharmonic functions $f_{j,k}, g_{j,k}$ defined in Euclidean neighborhoods of $\overline{O}_k$ such that $\Omega = E \cup \bigcup_{k=1}^\infty O_k$ and $u_j = f_{j,k} - g_{j,k}$ on O_k. We define $O_0 := \emptyset$ and

$$\int_A dd^c u_1 \wedge \ldots \wedge dd^c u_n := \sum_{k=1}^\infty \int_{A \cap (O_k \setminus \bigcup_{h=0}^{k-1} O_h)} dd^c(f_{1,k} - g_{1,k}) \wedge \ldots \wedge dd^c(f_{n,k} - g_{n,k}),\ A \in QB(\Omega). \tag{2.1}$$

Theorem 3.6 in [18] implies that the measure defined by (2.1) is independent on E, $\{O_k\}$, $\{f_{j,k}\}$ and $\{g_{j,k}\}$. This measure is called the complex Monge-Ampère measure.

Note that from Theorem 2.17 in [18] and Lemma 4.1 in [18] we infer that $dd^c u_1 \wedge \ldots \wedge dd^c u_n$ is a non-negative measure on $QB(\Omega)$. We now give the following definition which is an extension of the class $\mathcal{F}(\Omega)$ introduced and investigated by Cegrell [6] when Ω is a bounded hyperconvex domain in $\mathbb{C}^n$.

Definition 2.2. Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$. We denote by $\mathcal{F}(\Omega)$ the family of negative $\mathcal{F}$-plurisubharmonic functions u defined on Ω such that there exist a decreasing sequence $\{\varphi_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to u on Ω and

$$\sup_{j \geq 1} \int_\Omega (dd^c \varphi_j)^n < +\infty.$$

Furthermore, if $p > 0$ satisfies

$$\sup_{j \geq 1} \int_\Omega (1 + (-\varphi_j)^p)(dd^c \varphi_j)^n < +\infty$$

then we say that $u \in \mathcal{F}_p(\Omega)$.

Note that $\mathcal{E}_0(\Omega) \subset \mathcal{F}(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}_p(\Omega) \subset \mathcal{F}(\Omega)$ for all $p > 0$. When Ω is a bounded Euclidean hyperconvex domain, the classes $\mathcal{E}_0(\Omega)$, $\mathcal{F}(\Omega)$ are the same as the classical Cegrell's classes.

Proposition 2.3. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $u \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$. Then the following statements hold:*

(i) If $\{\varphi_j\} \subset \mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow u$ on Ω and $\sup_{j\geq 1} \int_\Omega (dd^c\varphi_j)^n < +\infty$ then

$$\int_\Omega (-\rho)(dd^c u)^n = \sup_{j\geq 1} \int_\Omega (-\rho)(dd^c\varphi_j)^n, \quad \forall \rho \in \mathcal{F}\text{-}PSH^-(\Omega) \cap L^\infty(\Omega).$$

(ii) If $v \in \mathcal{F}\text{-}PSH(\Omega)$ with $u \leq v < 0$ then $v \in \mathcal{F}(\Omega)$ and $\int_\Omega (dd^c v)^n \leq \int_\Omega (dd^c u)^n$.

Proof. The statements follow from Proposition 4.2 in [19] and Proposition 4.3 in [19]. □

Proposition 2.4. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $u \in \mathcal{F}(\Omega)$. If $\{u_j\} \subset \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ such that $u_j \searrow u$ in Ω as $j \nearrow +\infty$ then*

$$\sup_{j\geq 1} \int_\Omega (dd^c u_j)^n < +\infty.$$

and

$$\int_\Omega (dd^c \max(u,\rho))^n = \sup_{j\geq 1} \int_\Omega (dd^c u_j)^n$$

for every $\rho \in \mathcal{F}\text{-}PSH^-(\Omega) \cap L^\infty(\Omega)$ with $\sup_\Omega \rho < 0$. In particular,

$$\int_\Omega (dd^c \max(u,-1))^n < +\infty.$$

Proof. Let $\{\varphi_k\} \subset \mathcal{E}_0(\Omega)$ such that $\varphi_k \searrow u$ in Ω as $k \nearrow +\infty$ and

$$\sup_{k\geq 1} \int_\Omega (dd^c\varphi_k)^n < +\infty.$$

Since $\max(u_j, \rho_k) \searrow u_j$ in Ω as $k \nearrow +\infty$, by Proposition 3.4 in [19] and Proposition 4.2 in [19] we infer that

$$\int_\Omega (dd^c u_j)^n = \sup_{k\geq 1} \int_\Omega (dd^c \max(u_j,\varphi_k))^n \leq \sup_{k\geq 1} \int_\Omega (dd^c\varphi_k)^n.$$

This implies that

$$\sup_{j\geq 1} \int_\Omega (dd^c u_j)^n \leq \sup_{k\geq 1} \int_\Omega (dd^c\varphi_k)^n < +\infty.$$

Now, assume that $\rho \in \mathcal{F}\text{-}PSH^-(\Omega) \cap L^\infty(\Omega)$, $\sup_\Omega \rho < 0$. Thanks to Proposition 3.4 in [19] and Proposition 2.3 we have

$$\begin{aligned}\int_\Omega (dd^c \max(u,\rho))^n &= \sup_{k\geq 1} \int_\Omega (dd^c \max(\varphi_k,\rho))^n \\ &= \sup_{k\geq 1} \int_\Omega (dd^c\varphi_k)^n \\ &= \sup_{k\geq 1} \sup_{j\geq 1} \int_\Omega (dd^c \max(u_j,\varphi_k))^n = \sup_{j\geq 1} \int_\Omega (dd^c u_j)^n.\end{aligned}$$

The proof is complete. □

Proposition 2.5. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$. Assume that $u \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ and $v \in \mathcal{F}\text{-}PSH^-(\Omega)$ such that $(dd^c u)^n \leq (dd^c v)^n$ in $\{v > -\infty\}$. Then, $u \geq v$ in Ω.*

Proof. Without loss of generality we can assume that $-1 \le u \le 0$ on Ω. Let $j \in \mathbb{N}^*$ and define

$$v_j := (1 + \frac{1}{j})(v - \frac{1}{j}) \text{ in } \Omega.$$

Choose $p > 0$ such that $j^p < 1 + \frac{1}{j}$. It is easy to see that

$$(1 + (-u)^p)(dd^c u)^n \le 2(dd^c u)^n \le 2(dd^c v)^n \le (1 + (-v_j)^p)(dd^c v_j)^n \text{ on } \{v_j > -\infty\}.$$

Proposition 4.4 in [19] implies that $u \ge v_j$ in Ω. Letting $j \to +\infty$ we conclude that $u \ge v$ in Ω which is what we wanted to prove. □

3. Proof of Theorem 1.2

We need the following.

Lemma 3.1. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $u, v \in \mathcal{F}(\Omega)$ be such that*
(i) $u \ge v$ in Ω;
(ii) $(dd^c u)^n \le (dd^c v)^n$ on $\{v > -\infty\}$;
(iii) $\int_\Omega (dd^c \max(u, -1))^n \ge \int_\Omega (dd^c \max(v, -1))^n$.
Then, $u = v$ in Ω.

Proof. Let $R > 0$ be such that $\Omega \Subset \mathbb{B}(0, R)$ and define $\rho(z) := |z|^2 - R^2$, $z \in \mathbb{C}^n$. Let $\varepsilon, \delta \in (0, 1)$. We set

$$u_{\varepsilon,\delta} := \max(u, -2\delta R^2 \varepsilon^{-1}) \text{ and } v_{\varepsilon,\delta} := \max((1 - \varepsilon)u_{\varepsilon,\delta} + \delta\rho, v).$$

Since $\sup_\Omega \rho < 0$ and $u \ge v$ in Ω, by Proposition 2.3 and Proposition 2.4 we conclude by (iii) that

$$\begin{aligned} \int_\Omega (dd^c v_{\varepsilon,\delta})^n &= \int_\Omega (dd^c \max(v, -1))^n \\ &\le \int_\Omega (dd^c \max(u, -1))^n = \int_\Omega (dd^c u_{\varepsilon,\delta})^n. \end{aligned} \tag{3.1}$$

Since $u_{\varepsilon,\delta} = u$ on $\{v > -2\delta R^2 \varepsilon^{-1}\}$, by Theorem 4.8 in [18] and using (ii), we get

$$(dd^c v)^n \ge (dd^c u)^n = (dd^c u_{\varepsilon,\delta})^n \text{ on } \{v > -2\delta R^2 \varepsilon^{-1}\}.$$

Hence, Proposition 2.6 in [19] implies that

$$(dd^c v_{\varepsilon,\delta})^n \ge (1 - \varepsilon)^n (dd^c u_{\varepsilon,\delta})^n \text{ on } \{v > -2\delta R^2 \varepsilon^{-1}\}. \tag{3.2}$$

Because $v_{\varepsilon,\delta} = (1 - \varepsilon)u_{\varepsilon,\delta} + \delta\rho$ in $\{v < -\delta R^2 \varepsilon^{-1}\} \cup \{v < u - \delta R^2\}$, by Theorem 4.8 in [18] we infer that

$$(dd^c v_{\varepsilon,\delta})^n \ge (1 - \varepsilon)^n (dd^c u_{\varepsilon,\delta})^n + \delta^n (dd^c \rho)^n \text{ on } \{v < -\delta R^2 \varepsilon^{-1}\} \cup \{v < u - \delta R^2\}.$$

Combining this with (3.2) we arrive at

$$(dd^c v_{\varepsilon,\delta})^n \ge (1 - \varepsilon)^n (dd^c u_{\varepsilon,\delta})^n + \delta^n 1_{\{v<u-\delta R^2\}} (dd^c \rho)^n \text{ on } \Omega.$$

It follows that

$$\int_\Omega (dd^c v_{\varepsilon,\delta})^n \ge (1 - \varepsilon)^n \int_\Omega (dd^c u_{\varepsilon,\delta})^n + \delta^n \int_{\{v<u-\delta R^2\}} (dd^c \rho)^n.$$

Letting $\varepsilon \to 0$ we conclude by (3.1) that

$$\int_{\{v<u-\delta R^2\}} (dd^c \rho)^n = 0, \ \forall \delta > 0.$$

Therefore, by Proposition 2.3 in [19] we infer that $v \ge u$ in Ω, and hence, $u = v$ in Ω. The proof is complete. □

We now are able to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Since Ω has the $\mathcal{F}$-approximation property, there exists an increasing sequence of negative plurisubharmonic functions ρ_j defined on bounded hyperconvex domains Ω_j such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and $\rho_j \nearrow \rho \in \mathcal{E}_0(\Omega)$ a.e. on Ω. Let $k \in \mathbb{N}$ be such that $k \geq 1$. Proposition 2.4 implies that

$$\int_\Omega (dd^c \max(u, -k))^n = \int_\Omega (dd^c \max(u, -1))^n < +\infty.$$

Since the measure $1_\Omega (dd^c \max(u, -k))^n$ vanishes on all pluripolar subsets of Ω_j, by Lemma 5.14 in [6] there exists $u_{j,k} \in \mathcal{F}(\Omega_j)$ such that

$$(dd^c u_{j,k})^n = 1_\Omega (dd^c \max(u, -k))^n \text{ in } \Omega_j.$$

Theorem 3.7 in [20] states that the function $u_j := (\limsup_{k\to+\infty} u_{j,k})^*$ belongs to $\mathcal{F}(\Omega_j)$, where * denotes the upper semi-continuous regularization. By Theorem 5.5 in [6] and Proposition 2.5 we infer that $u_{j,k} \leq u_{j+1,k} \leq \max(u, -k)$ on Ω, and hence,

$$u_j \leq u_{j+1} \leq u \text{ on } \Omega.$$

We now claim that

$$(dd^c u_j)^n \geq (dd^c u)^n \text{ on } \Omega \cap \{u > -\infty\} \tag{3.3}$$

and

$$\int_{\Omega_j} (dd^c u_j)^n \leq \int_\Omega (dd^c \max(u, -1))^n. \tag{3.4}$$

Indeed, fix $a > 0$ and let $k \in \mathbb{N}^*$ be such that $k \geq a$. Since

$$(dd^c u_{j,k+s})^n \geq 1_{\Omega\cap\{u>-a\}} (dd^c u)^n \text{ in } \Omega_j,\ \forall s \geq 0,$$

Proposition 4.3 in [20] implies that

$$(dd^c \max(u_{j,k}, \ldots, u_{j,k+s}))^n \geq 1_{\Omega\cap\{u>-a\}} (dd^c u)^n \text{ in } \Omega_j,\ \forall s \geq 0.$$

Main Theorem in [7] states that

$$(dd^c (\sup_{l\geq 0} u_{j,k+l})^*)^n \geq 1_{\Omega\cap\{u>-a\}} (dd^c u)^n \text{ in } \Omega_j$$

because $\max(u_{j,k}, \ldots, u_{j,k+s}) \nearrow (\sup_{l\geq 0} u_{j,k+l})^*$ a.e. in Ω_j as $s \nearrow +\infty$. Moreover, since $(\sup_{l\geq 0} u_{j,k+l})^* \searrow u_j$ a.e. in Ω_j as $k \nearrow +\infty$, again by Main Theorem in [7] we infer that

$$(dd^c u_j)^n \geq 1_{\Omega\cap\{u>-a\}} (dd^c u)^n \text{ in } \Omega_j.$$

Letting $a \to +\infty$, we get

$$(dd^c u_j)^n \geq (dd^c u)^n \text{ on } \Omega \cap \{u > -\infty\}.$$

Now, by Lemma 3.3 in [1] and Corollary 3.4 in [1] we have

$$\begin{aligned}\int_{\Omega_j} (dd^c u_j)^n &= \lim_{k\to+\infty} \int_{\Omega_j} (dd^c (\sup_{l\geq 0} u_{j,k+l})^*)^n \\ &\leq \sup_{k\geq 1} \int_{\Omega_j} (dd^c u_{j,k})^n = \int_\Omega (dd^c \max(u, -1))^n.\end{aligned}$$

This proves the claim. Let v be the least $\mathcal{F}$-upper semicontinuous majorant of $\sup_{j\geq 1} u_j$ in Ω. Then, $v \in \mathcal{F}\text{-}PSH^-(\Omega)$ and $v \leq u$ on Ω. By Theorem 4.5 in [17] and using (3.3) we infer that

$$(dd^c v)^n \geq (dd^c u)^n \text{ on } \{v > -\infty\}. \tag{3.5}$$

We claim that $v \in \mathcal{F}(\Omega)$. Indeed, put $v_k := \max(v, k\rho)$, where $k \in \mathbb{N}^*$. Proposition 3.4 in [19] implies that $v_k \in \mathcal{E}_0(\Omega)$. Since $\max(u_j, k\rho_j) \nearrow v_k$ a.e. in Ω as $j \nearrow +\infty$, by Proposition 2.7 in [19] and Lemma 3.3 in [1] we obtain by (3.4) that

$$\begin{aligned}\int_\Omega (dd^c v_k)^n &\leq \liminf_{j\to+\infty} \int_\Omega (dd^c \max(u_j, k\rho_j))^n \\ &\leq \liminf_{j\to+\infty} \int_{\Omega_j} (dd^c u_j)^n \leq \int_\Omega (dd^c \max(u, -1))^n.\end{aligned}$$

Since $v_k \searrow v$, by Proposition 2.4 we obtain $v \in \mathcal{F}(\Omega)$. This proves the claim. Now, again by Proposition 2.7 in [19] and Proposition 3.4 in [19] we have

$$\begin{aligned}\int_\Omega (dd^c \max(v, -1))^n &\leq \liminf_{k\to+\infty} \int_\Omega (dd^c \max(v_k, -1))^n \\ &\leq \liminf_{k\to+\infty} \int_\Omega (dd^c v_k)^n \leq \int_\Omega (dd^c \max(u, -1))^n.\end{aligned}$$

Combining this with (3.5) and using Lemma 3.1 we conclude that $v = u$ in Ω. Thus, $u_j \nearrow u$ a.e. in Ω as $j \nearrow +\infty$. This completes the proof of Theorem 1.2. □

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