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State and unknown disturbance simultaneous estimation for one-sided Lipschitz fractional-order systems with time-delay

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Abstract This paper addresses the problem of estimating simultaneously the state and unknown disturbance of one-sided Lipschitz fractional-order systems with time-delay. The nominal models of nonlinearities are assumed to satisfy both the one-sided Lipschitz condition and the quadratically inner-bounded condition. We employ the Razumikhin stability theorem and a recent result on the Caputo fractional derivative of a quadratic function to derive a sufficient condition for the asymptotic stability of the observer error dynamic system. The stability condition is obtained in terms of linear matrix inequalities, which can be effectively solved by using existing convex algorithms. Two examples are provided to show the effectiveness of the proposed design approach.

Keywords One-sided Lipschitz · Delayed fractional-order systems · Razumikhin stability theorem · Linear matrix inequalities

1 Introduction

Fractional calculus is the subject of studying fractional integrals and fractional derivatives, which means that the orders of integration and differentiation are not integers. During the last few decades, fractional calculus has evolved into an interesting and useful area of research in view of the extensive application of its modeling tools in applied and technical sciences. Fractional-order differential systems are differential systems which involve fractional derivatives, and they have been successfully used to model many real-world phenomena such as electrical circuits [1], fractional-order chaotic Lu systems [2], and diffusion of heat through a semi-infinite solid where heat flow is equal to the half-derivative of the temperature [3] and quantum mechanics [4].

The problem of the state observer design for dynamical systems with or without time-delays has been an active research topic for the past few decades with many applications in realisation of state-feedback control, system supervision and monitoring, fault diagnosis of dynamic processes, fault detection and isolation (see, for example, [5–19]). In fact, most physical and biological systems are nonlinear (see, for example, [20–24]). Modeling such systems is generally difficult and often includes errors. In order to design state observers, it is also essential to consider the issue of unknown disturbances that is inherent in many control systems (see, for example, [5]). To deal with this issue, the problem of estimating simultaneously the state of a dynamical system and its unknown disturbances has been studied. The significance of this problem is that when the unknown disturbances measurement of the system is too expensive or physically not possible. Moreover, when the faults are regarded as the unknown disturbances, the state and unknown disturbances estimation can be significant for the purpose of fault detection and isolation, and fault-tolerant control (see, for example, [19]). State and unknown disturbances simultaneous estimation also reveals that it is useful in many industrial applications (see, for example, [25, 26]). As reported in the literature, some works have been carried out on the problem of estimating simultaneously the state of a dynamical system and its unknown disturbances. In particular, in [27], the authors

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proposed three types of reduced-order observers, namely observers with time delays, observers without internal delays and delay-free observers to estimate a partial set of the state vector. The paper [28] dealt with the problem of state unknown input estimation for a class of time-varying delay systems with additive uncertain, nonlinear disturbance. The problem of estimating simultaneously the state and input of a class of nonlinear systems was studied in [5]. In [29], the authors combined the highgain observer with sliding mode observer to develop a new observer, and the new observer estimates the state and fault for multiple-inputs multiple-outputs nonlinear systems simultaneously. In [30], a simultaneous state and unknown input signal estimation technique is developed for a class of nonlinear systems with time-delay in state and unknown input signal. As the nonlocal property and weakly singular kernels of fractional-order derivatives, nonlinear fractional-order systems own better description of dynamical behaviours to integer-order ones. It should be mentioned here that the most frequently used method to solve the problem of estimating simultaneously the state of a dynamic system and its unknown disturbances for integer-order systems, the so-called Lyapunov-Krasovskii functional method, can not be extended easily to fractional-order ones. In fact, it is difficult to construct a Lyapunov function, and calculate its fractional derivative since the Leibniz rule does not hold for fractional derivative. This is the main reason that there are very few practical algebraic criteria on estimating simultaneously the state of a dynamic system and its unknown disturbance of fractional-order systems in the current literature. To the best of our knowledge, no works on the simultaneous estimation of state vector and unknown disturbance vector of nonlinear fractional-order systems with time delays have been done. This motivates the present work.

In this paper, we consider the problem of estimating simultaneously the state and unknown disturbance of fractional-order one-sided Lipschitz systems for the first time. The main challenges and contributions of this paper are summarised as follows:

1. In order to overcome the difficulties of calculating the fractional-order derivative of a quadratic function, we construct a suitable Lyapunov functional and using Lyapunov-Razumikhin theory and linear matrix inequality approach to derive some sufficient conditions for existence of the fractional-order observer. The proposed criteria are quite general since many factors, such as the time delay in state and unknown disturbance, fractional-order systems, nonlinearity is assumed to be one-side Lipschitz are considered. Therefore, the results obtained in this paper generalize those given in the previous literature.

2. Compared with the paper [30], the fractional-order observer in this paper is established to reconstruct the state and unknown disturbance of the nonlinear system with the delay in state and input. In addition, the nonlinear part of the system is one-side Lipschitz that is the more general form in [30].

3. The stability condition is obtained in terms of linear matrix inequalities, which can be effectively solved by using existing convex algorithms.

This paper is organized as follows. In section 2, we present some preliminaries. The main results are given in Section 3. In section 4, we provide two numerical examples to demonstrate the effectiveness of the proposed design approach. A conclusion is drawn in Section 5.

Notation: I_n denotes the $n\times n$ identity matrix, $0_{m\times n}$ denotes the $m\times n$ zero matrix. For a real matrix M, M^T denotes the transpose. $\mathbb{R}^n$ denotes the $n-$dimensional linear vector space over the reals with the Euclidean norm $\|.\|$ given by $\|x\| = \sqrt{x_1^2+\ldots+x_n^2}, x=(x_1,\ldots,x_n)^T \in \mathbb{R}^n$; $\mathbb{R}^{n\times m}$ denotes the space of $n\times m$ matrices. For a real matrix $M, \lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the maximal and the minimal eigenvalue of M, respectively. A matrix P is positive definite ($P > 0$) if $x^TPx > 0, \forall x \neq 0$. The symmetric term in a matrix is denoted by $*$.

The Caputo derivative of function $f(t)$ with order $\alpha \in (0,1]$ is defined by

$$ {}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{f}(\tau)d\tau, $$

where ${}_0^CD_t^\alpha f$ is the first order derivative of function f and the function $\Gamma(\cdot)$ is defined as

$$ \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,\ z\in\mathbb{R}. $$

2 Preliminaries

This section presents some preliminary results which will be used throughout the paper.

Consider the following nonlinear fractional-order system

$$ \begin{aligned} {}_0^CD_t^\alpha x(t) &= Ax(t)+\sum_{i=1}^q A_i x(t-\tau_i)+Bu(t)+Dd(t) \\ &\quad +D_\delta d(t-\delta)+f\left(\begin{bmatrix} x(t)\\ d(t)\end{bmatrix}, y(t)\right), t\geq 0, \end{aligned} \quad (1) $$

$$ x(t) = \phi(t),\ \forall t\in[-\tau,0], \quad (2) $$

$$ y(t) = Cx(t)+Ed(t), \quad (3) $$

where $\alpha\in(0,1]$, $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control input vector, $y(t)\in\mathbb{R}^p$ is the output vector, $d(t)\in\mathbb{R}^{n_d}$ is the unknown disturbance and $\phi(t)\in\mathbb{R}^n$ is the initial function. By ${}_0^CD_t^\alpha x(t)$ we mean that ${}_0^CD_t^\alpha x(t) = \left[{}_0^CD_t^\alpha x_1(t)\ \ldots\ {}_0^CD_t^\alpha x_n(t)\right]^T$. $A\in\mathbb{R}^{n\times n}, A_i\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{n\times n_d}$, $D_\delta\in\mathbb{R}^{n\times n_d}$ and $E\in\mathbb{R}^{p\times n_d}$ are constant matrices, the time delays δ, τ_i, $i=1,2,\ldots,q$, are assumed to be known positive constants. In (2), τ is defined as

$\tau = \max\{\tau_i, \delta,\ j = 1,2,\ldots,q\}$. $f\left(\begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, y(t)\right)$ is a real nonlinear vector function on $\mathbb{R}^n$ satisfying

$$f\left(\begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, y(t)\right) = f_L\left(\begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, y(t)\right) + W f_U\left(\begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, y(t)\right), \tag{4}$$

where $f_L\left(\begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, y(t)\right) \in \mathbb{R}^n$ and $f_U\left(\begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, y(t)\right) \in \mathbb{R}^r$ are known and unknown nonlinear vectors, respectively. Matrix W is assumed to be real constant and have a full column rank of r. Furthermore, $f_L\left(\begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, y(t)\right)$ is assumed to be satisfied the following assumptions:

Assumption 1 *$f_L(\rho, y)$ is one-sided Lipschitz [31], that is,*

$$\langle f_L(\rho,y) - f_L(\hat{\rho},y), \rho - \hat{\rho}\rangle \leq \theta||\rho - \hat{\rho}||^2, \forall \rho, \hat{\rho} \in \mathbb{R}^{n+n_d}, \tag{5}$$

where θ is one-sided Lipschitz constant.

Assumption 2 *$f_L(\rho, y)$ is quadratically inter-bounded [31], that is,*

$$\begin{aligned} &\|f_L(\rho,y) - f_L(\hat{\rho},y)\|^2 \\ &\leq \beta\|\rho - \hat{\rho}\|^2 + \gamma\langle f_L(\rho,y) - f_L(\hat{\rho},y), \rho - \hat{\rho}\rangle, \end{aligned} \tag{6}$$

where β and γ are known constants.

Remark 1 From the decomposition (4), the nonlinear function is treated more generally as it can be divided into two parts in which one part is assumed to be under one-sided Lipschitz condition and the other is not necessary to be under one-sided Lipschitz condition. Moreover, from Assumptions 1 and 2, the classical Lipschitz constant is required to be positive; however, for the one-sided Lipschitz condition, the constants can be positive, zero, or negative and they are usually much smaller than that of the classical Lipschitz condition. Furthermore, if a function is Lipschitz, it is both one-sided Lipschitz and quadratically inner-bounded; nevertheless, the converse is not true. Thus, the considered nonlinear systems is more desirable than the traditional Lipschitz systems.

Let us define $\rho(t) = \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \in \mathbb{R}^{n+n_d}$ and the following matrices:

$$\begin{aligned} &F = \begin{bmatrix} I_n & 0_{n\times n_d} \end{bmatrix}, M = \begin{bmatrix} A & D \end{bmatrix}, \\ &M_i = \begin{bmatrix} A_i & 0_{n\times n_d} \end{bmatrix}, i = 1,\ldots,q, \end{aligned} \tag{7}$$

$$M_\delta = \begin{bmatrix} 0_{n\times n} & D_\delta \end{bmatrix}, \bar{C} = \begin{bmatrix} C & E \end{bmatrix}. \tag{8}$$

Then the system (1)-(3) can be rewritten as follow:

$$\begin{aligned} {}_0^C D_t^\alpha F\rho(t) = M\rho(t) + \sum_{i=1}^q M_i\rho(t-\tau_i) + M_\delta\rho(t-\delta) \\ + Bu(t) + f(\rho(t), y(t)),\ t \geq 0, \end{aligned} \tag{9}$$

$$x(\theta) = \phi(\theta),\ \forall\theta \in [-\tau, 0], \tag{10}$$

$$y(t) = \bar{C}\rho(t). \tag{11}$$

We now consider the following observer

$$\hat{\rho}(t) = w(t) + Qy(t), \tag{12}$$

$$\begin{aligned} {}_0^C D_t^\alpha w(t) = Nw(t) + \sum_{i=1}^q N_i w(t-\tau_i) + N_\delta w(t-\delta) + Jy(t) \\ + \sum_{i=1}^q J_i y(t-\tau_i) + J_\delta y(t-\delta) + Hu(t) \\ + Tf_L(\hat{\rho}(t), y(t)),\ t \geq 0, \end{aligned} \tag{13}$$

$$w(\zeta) = \psi(\zeta) \in \mathbb{R}^{n+n_d},\ \forall\zeta \in [-\tau, 0], \tag{14}$$

where $N \in \mathbb{R}^{(n+n_d)\times(n+n_d)}$, $J \in \mathbb{R}^{(n+n_d)\times p}$, $N_i \in \mathbb{R}^{(n+n_d)\times(n+n_d)}$, $N_\delta \in \mathbb{R}^{(n+n_d)\times(n+n_d)}$, $J_i \in \mathbb{R}^{(n+n_d)\times p}$ $(i = 1,2,\ldots,q)$, $J_\delta \in \mathbb{R}^{(n+n_d)\times p}$ are matrices to be determined and $\psi(\zeta) \in \mathbb{R}^{n+n_d}$ is the initial function.

3 Main results

Let us definite the error

$$e(t) = \hat{\rho}(t) - \rho(t) \in \mathbb{R}^{n+n_d}. \tag{15}$$

The following theorem provides conditions which guarantee the existence of the observers (12)-(14).

Theorem 1 *Assume that the following conditions hold:*

$$\begin{aligned} {}_0^C D_t^\alpha e(t) = Ne(t) + \sum_{i=1}^q N_i e(t-\tau_i) + N_\delta e(t-\delta) \\ + T[f_L(\hat{\rho}(t), y(t)) - f_L(\rho(t), y(t))] \end{aligned} \tag{16}$$

is asymptotically stable

$$TF + Q\bar{C} - I_{n+n_d} = 0, \tag{17}$$

$$N(Q\bar{C} - I_{n+n_d}) + TM - J\bar{C} = 0, \tag{18}$$

$$N_i(Q\bar{C} - I_{n+n_d}) + TM_i - J_i\bar{C} = 0,\ i = 1,2,\ldots,q, \tag{19}$$

$$N_\delta(Q\bar{C} - I_{n+n_d}) + TM_\delta - J_\delta\bar{C} = 0, \tag{20}$$

$$TB - H = 0, \tag{21}$$

$$TW = 0. \tag{22}$$

Then, the estimation $\hat{\rho}(t)$ will converge asymptotically to $\rho(t)$ as $t \to \infty$.

Proof Regarding (9), (13), (15) and (22), the fractional-order derivative of $e(t)$ is given by

$$\begin{aligned}
{}_0^C D_t^\alpha e(t) &= {}_0^C D_t^\alpha \hat{\rho}(t) - {}_0^C D_t^\alpha \rho(t) = {}_0^C D_t^\alpha w(t) - T\left[{}_0^C D_t^\alpha F\rho(t)\right] \\
&= Nw(t) + \sum_{i=1}^{q} N_i w(t-\tau_i) + N_\delta w(t-\delta) + Jy(t) \\
&\quad + \sum_{i=1}^{q} J_i y(t-\tau_i) + J_\delta y(t-\delta) + Hu(t) \\
&\quad + f_L(\hat{\rho}(t), y(t)) - TM\rho(t) \\
&\quad - T\sum_{i=1}^{q} M_i \rho(t-\tau_i) - TM_\delta \rho(t-\delta) - TBu(t) \\
&\quad - Tf_L(\rho(t), y(t)) \qquad (23) \\
&= Ne(t) + \sum_{i=1}^{q} N_i e(t-\tau_i) + N_\delta e(t-\delta) \\
&\quad + T[f_L(\hat{\rho}(t), y) - f_L\rho(t), y(t))] \\
&\quad + (J\bar{C} - N(Q\bar{C} - I_{n+n_d}) - TM)\rho(t) \\
&\quad + \sum_{i=1}^{q} (J_i\bar{C} - N_i(Q\bar{C} - I_{n+n_d}) - TM_i)\rho(t-\tau_i) \\
&\quad + (J_\delta\bar{C} - N_\delta(Q\bar{C} - I_{n+n_d}) - TM_\delta)\rho(t-\delta) \\
&\quad + (H - TB)u(t). \qquad (24)
\end{aligned}$$

It is clear from (24) that $e(t) \to 0$ as $t \to \infty$ if conditions (16)-(21) of Theorem 1 are satisfied. Hence, $\hat{\rho}(t)$ converges asymptotically to $\rho(t)$ as $t \to \infty$. This completes the proof of Theorem 1.

Our purpose in this paper is to design observer matrices N, N_i, N_δ, T, Q, J, J_i, J_δ and H such that $\hat{\rho}(t)$ converges asymptotically to $\rho(t)$ as $t \to \infty$. From equation (21) of Theorem 1, matrix H can be obtained as $H = TB$. Therefore, the remaining task is to solve equations (17)-(20) and (22) for unknown matrices N, N_i, N_δ, T, Q, J, J_i and J_δ. We first solve equations (17) and (22). For this, we represent these equations into the following form

$$\chi \mathscr{X} = \mathscr{Y}, \qquad (25)$$

where

$$\chi = \begin{bmatrix} T & Q \end{bmatrix}, \ \mathscr{X} = \begin{bmatrix} F & W \\ \bar{C} & 0 \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+n_d+r)},$$
$$\mathscr{Y} = \begin{bmatrix} I_{n+n_d} & 0 \end{bmatrix} \in \mathbb{R}^{(n+n_d)\times(n+n_d+r)}.$$

Since $\mathscr{X}$ and $\mathscr{Y}$ are two known constant matrices, a solution for χ always exists if and only if

$$\operatorname{rank}\begin{bmatrix} \mathscr{X} \\ \mathscr{Y} \end{bmatrix} = \operatorname{rank}(\mathscr{X}). \qquad (26)$$

Under condition (26), a general solution for χ is given by

$$\chi = \mathscr{Y}\mathscr{X}^+ + Z\left(I_{n+p} - \mathscr{X}\mathscr{X}^+\right), \qquad (27)$$

where $\mathscr{X}^+ \in \mathbb{R}^{(n+n_d+r)\times(n+p)}$ is the Moor-Penrose-inverse of $\mathscr{X}$ and $Z \in \mathbb{R}^{(n+n_d)\times(n+p)}$ is an arbitrary matrix to be determined. Moreover, matrices T and Q can now be extracted from (27) and are expressed as

$$T = \Phi e_T + Z\Psi e_T, \qquad (28)$$
$$Q = \Phi e_Q + Z\Psi e_Q, \qquad (29)$$

where

$$\Phi = \mathscr{Y}\mathscr{X}^+, \ \Psi = I_{n+p} - \mathscr{X}\mathscr{X}^+ \qquad (30)$$

and $e_T \in \mathbb{R}^{(n+p)\times n}$, $e_Q \in \mathbb{R}^{(n+p)\times p}$ are the following

$$e_T = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, e_Q = \begin{bmatrix} 0 \\ I_p \end{bmatrix}. \qquad (31)$$

Next, from (18)-(20) we obtain

$$N + (J - NQ)\bar{C} - TM = 0, \qquad (32)$$
$$N_i + (J_i - N_iQ)\bar{C} - TM_i = 0, \ i = 1, 2, \ldots, q, \qquad (33)$$
$$N_\delta + (J_\delta - N_\delta Q)\bar{C} - TM_\delta = 0. \qquad (34)$$

We now denote

$$R = J - NQ, \ R_i = J_i - N_iQ, \ i = 1, 2, \ldots, q,$$
$$R_\delta = J_\delta - N_\delta Q. \qquad (35)$$

Then, from (28)-(29) and (32)-(34) we obtain the following

$$N = \Phi e_T M - R\bar{C} + Z\Psi e_T M, \qquad (36)$$
$$N_i = \Phi e_T M_i - R_i\bar{C} + Z\Psi e_T M_i, \ i = 1, 2, \ldots, q, \qquad (37)$$
$$N_\delta = \Phi e_T M_\delta - R_\delta\bar{C} + Z\Psi e_T M_\delta. \qquad (38)$$

By substituting N, N_i, $(i = 1, 2, \ldots, q)$ and N_δ from (36)-(38) into (16), we obtain

$$\begin{aligned}
{}_0^C D_t^\alpha e(t) &= (\Phi e_T M - R\bar{C} + Z\Psi e_T M)e(t) \\
&\quad + \sum_{i=1}^{q} (\Phi e_T M_i - R_i\bar{C} + Z\Psi e_T M_i)e(t-\tau_i) \\
&\quad + (\Phi e_T M_\delta - R_\delta\bar{C} + Z\Psi e_T M_\delta)e(t-\delta) \\
&\quad + (\Phi e_T + Z\Psi e_T)[f_L(\hat{\rho}(t), y) - f_L(\rho(t), y)]. \qquad (39)
\end{aligned}$$

Now, the remaining task is to determine of matrices R, R_i $(i = 1, 2, \ldots, q)$, R_δ and Z such that the fractional-order time-delay system (39) is asymptotically stable. To solve this task, we first systematize the following auxiliary lemmas which are essential for determining the stability of (39).

Lemma 1 *([32]) Let $x(t) \in \mathbb{R}^n$ be a vector of diffrentiable function. Then, for any time instant $t \geq t_0$, the following relationship holds*

$${}_{t_0}^C D_t^\alpha (x^T(t)Px(t)) \leq 2x^T(t)P\,{}_{t_0}^C D_t^\alpha x(t), \quad \forall \alpha \in (0,1), \forall t \geq t_0 \geq 0,$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

Lemma 2 *(Razumikhin-type stability [33]) Assume that $u, v, w : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous nondecreasing functions, $u(s), v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$, and $q > 1$. If there exists a continuous function $V(t, x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that:*
(i) $u(\|x\|) \leq V(t,x) \leq v(\|x\|), \quad t \in \mathbb{R}^+, x \in \mathbb{R}^n$
(ii) $D^\alpha V(t, x(t)) \leq -w(\|x(t)\|)$ *if* $V(t+s, x(t+s)) < qV(t, x(t)), \quad \forall s \in [-\tau, 0], t \geq 0,$
then the zero solution of system (39) is asymptotically stable.

Theorem 2 *The zero solution of system* (39) *is asymptotically stable if there exist positive scalars $\varepsilon_1, \varepsilon_2, \zeta, \upsilon > 1$, a symmetric positive definite matrix P and matrices X, Y, Y_δ, $Y_i (i = 1, \ldots, q)$ with appropriate dimensions such that the following LMI holds:*

$$\begin{bmatrix} \mathscr{M}_{11} & \mathscr{M}_{12} \\ \mathscr{M}_{12}^T & -\mathscr{M}_{22} \end{bmatrix} < 0, \tag{40}$$

where

$$\begin{aligned}\mathscr{M}_{11} = {} & P\Phi e_T M + M^T e_T^T \Phi^T P - Y\overline{C} - \overline{C}^T Y^T + X\Psi e_T M \\ & + M^T e_T^T \Psi^T X^T + (\varepsilon_1+\varepsilon_2)(\beta+\gamma\zeta)I \\ & + (3q+3)(1+\upsilon)P,\end{aligned}$$

$$\begin{aligned}\mathscr{M}_{12} = \Big[& P\Phi e_T M_1 \ldots P\Phi e_T M_q \quad Y_1\overline{C} \ldots Y_q\overline{C} \\ & X\Psi e_T M_1 \ldots X\Psi e_T M_q \quad P\Phi e_T M_\delta \quad Y_\delta \overline{C} \\ & X\Psi e_T M_\delta \quad X\Psi e_T \quad P\Phi e_T \Big],\end{aligned}$$

$$\mathscr{M}_{22} = diag\{P, \underbrace{\ldots}_{3q+1}, P, \varepsilon_1 I, \varepsilon_2 I\}$$

Moreover, the matrices R, R_i ($i = 1, 2, \ldots, q$), R_δ and Z are obtained as follows:

$R = P^{-1}Y, Z = P^{-1}X, R_\delta = P^{-1}Y_\delta, R_i = P^{-1}Y_i (i = 1, 2, \ldots, q)$.

Proof We choose a Lyapunov functional candidate as follows:

$$V(t, e(t)) = e^T(t)Pe(t).$$

It is easy to verify that

$$\lambda_{\min}(P)\|e(t)\|^2 \le V(t, e(t)) \le \lambda_{\max}(P)\|e(t)\|^2. \tag{41}$$

Therefore, condition (i) in Lemma 2 is satisfied. It follows from Lemma 1 that we obtain the α-order Caputo derivative of $V(t, e(t))$ along the trajectories of the system (39) as follows:

$$\begin{aligned}& D^\alpha V(t, e(t)) \\ & \le 2e^T(t)PD^\alpha e(t) \\ & = e^T(t)\Big[P\Phi e_T M + M^T e_T^T \Phi^T P - PR\overline{C} - \overline{C}^T R^T P \\ & \quad + PZ\Psi e_T M + M^T e_T^T \Psi^T Z^T P\Big] e(t) \\ & \quad + \sum_{i=1}^{q} \Big[2e^T(t)P\Phi e_T M_i e(t-\tau_i) - 2e^T(t)PR_i\overline{C}e(t-\tau_i) \\ & \quad + 2e^T(t)PZ\Psi e_T M_i e(t-\tau_i)\Big] \\ & \quad + 2e^T(t)P\Phi e_T M_\delta e(t-\delta) - 2e^T(t)PR_\delta\overline{C}e(t-\delta) \\ & \quad + 2e^T(t)PZ\Psi e_T M_\delta e(t-\delta) \\ & \quad + 2e^T(t)PZ\Psi e_T \Big(f_L(\hat{\rho}(t), y) - f_L(\rho(t), y)\Big). \\ & \quad + 2e^T(t)P\Phi e_T \Big(f_L(\hat{\rho}(t), y) - f_L(\rho(t), y)\Big).\end{aligned} \tag{42}$$

Letting

$R = P^{-1}Y, Z = P^{-1}X, R_\delta = P^{-1}Y_\delta, R_i = P^{-1}Y_i (i = 1, \ldots, q)$

and using the Cauchy matrix inequality, we have the following estimate

$$\begin{aligned}& 2e^T(t)P\Phi e_T M_i e(t-\tau_i) \\ & \le e^T(t)P\Phi e_T M_i P^{-1} M_i^T e_T^T \Phi^T Pe(t) \\ & \quad + e^T(t-\tau_i)Pe(t-\tau_i),\end{aligned} \tag{43}$$

$$\begin{aligned}& -2e^T(t)PR_i\overline{C}e(t-\tau_i) \\ & \le e^T(t)PR_i\overline{C}P^{-1}\overline{C}^T R_i^T Pe(t) + e^T(t-\tau_i)Pe(t-\tau_i) \\ & \le e^T(t)Y_i\overline{C}P^{-1}\overline{C}^T Y_i^T e(t) + e^T(t-\tau_i)Pe(t-\tau_i),\end{aligned} \tag{44}$$

$$\begin{aligned}& 2e^T(t)PZ\Psi e_T M_i e(t-\tau_i) \\ & \le e^T(t)PZ\Psi e_T M_i P^{-1} M_i^T e_T^T \Psi^T Z^T Pe(t) \\ & \quad + e^T(t-\tau_i)Pe(t-\tau_i) \\ & \le e^T(t)X\Psi e_T M_i P^{-1} M_i^T e_T^T \Psi^T X^T e(t) \\ & \quad + e^T(t-\tau_i)Pe(t-\tau_i),\end{aligned} \tag{45}$$

$$\begin{aligned}& 2e^T(t)P\Phi e_T M_\delta e(t-\delta) \\ & \le e^T(t)P\Phi e_T M_\delta P^{-1} M_\delta^T e_T^T \Phi^T Pe(t) \\ & \quad + e^T(t-\delta)Pe(t-\delta),\end{aligned} \tag{46}$$

$$\begin{aligned}& -2e^T(t)PR_\delta\overline{C}e(t-\delta) \\ & \le e^T(t)PR_\delta\overline{C}P^{-1}\overline{C}^T R_\delta^T Pe(t) + e^T(t-\delta)Pe(t-\delta) \\ & \le e^T(t)Y_\delta\overline{C}P^{-1}\overline{C}^T Y_\delta^T e(t) + e^T(t-\delta)Pe(t-\delta),\end{aligned} \tag{47}$$

$$\begin{aligned}& 2e^T(t)PZ\Psi e_T M_\delta e(t-\delta) \\ & \le e^T(t)PZ\Psi e_T M_\delta P^{-1} M_\delta^T e_T^T \Psi^T Z^T Pe(t) \\ & \quad + e^T(t-\delta)Pe(t-\delta) \\ & \le e^T(t)X\Psi e_T M_\delta P^{-1} M_\delta^T e_T^T \Psi^T X^T e(t) \\ & \quad + e^T(t-\delta)Pe(t-\delta).\end{aligned} \tag{48}$$

Using the Cauchy matrix inequality, Assumptions 1 and Assumptions 2, we obtain

$$\begin{aligned}& 2e^T(t)PZ\Psi e_T \Big(f_L(\hat{\rho}(t), y) - f_L(\rho(t), y)\Big) \\ & \le \varepsilon_1^{-1} e^T(t)PZ\Psi e_T e_T^T \Psi^T Z^T Pe(t) \\ & \quad + \varepsilon_1 \|f_L(\hat{\rho}(t), y) - f_L(\rho(t), y)\|^2 \\ & \le \varepsilon_1^{-1} e^T(t)PZ\Psi e_T e_T^T \Psi^T Z^T Pe(t) + \varepsilon_1 \beta \|\hat{\rho}(t) - \rho(t)\|^2 \\ & \quad + \varepsilon_1 \gamma \langle f_L(\hat{\rho}(t), y) - f_L(\rho(t), y), \hat{\rho}(t) - \rho(t)\rangle \\ & \le \varepsilon_1^{-1} e^T(t)PZ\Psi e_T e_T^T \Psi^T Z^T Pe(t) + \varepsilon_1 \beta \|\hat{\rho}(t) - \rho(t)\|^2 \\ & \quad + \varepsilon_1 \gamma\zeta \|\hat{\rho}(t) - \rho(t)\|^2 \\ & = \varepsilon_1^{-1} e^T(t)PZ\Psi e_T e_T^T \Psi^T Z^T Pe(t) + \varepsilon_1 (\beta+\gamma\zeta) \|e(t)\|^2\end{aligned} \tag{49}$$

and

$$\begin{aligned}
&2e^T(t)P\Phi e_T\Big(f_L(\hat{\rho}(t),y)-f_L(\rho(t),y)\Big)\\
&\le \varepsilon_2^{-1}e^T(t)P\Phi e_T e_T^T\Phi^T Pe(t)\\
&\quad+\varepsilon_2\left\|f_L(\hat{\rho}(t),y)-f_L(\rho(t),y)\right\|^2\\
&\le \varepsilon_2^{-1}e^T(t)P\Phi e_T e_T^T\Phi^T Pe(t)+\varepsilon_2\beta\left\|\hat{\rho}(t)-\rho(t)\right\|^2\\
&\quad+\varepsilon_2\gamma\langle f_L(\hat{\rho}(t),y)-f_L(\rho(t),y),\hat{\rho}(t)-\rho(t)\rangle\\
&\le \varepsilon_2^{-1}e^T(t)P\Phi e_T e_T^T\Phi^T Pe(t)+\varepsilon_2\beta\left\|\hat{\rho}(t)-\rho(t)\right\|^2\\
&\quad+\varepsilon_2\gamma\zeta\left\|\hat{\rho}(t)-\rho(t)\right\|^2\\
&=\varepsilon_2^{-1}e^T(t)P\Phi e_T e_T^T\Phi^T Pe(t)+\varepsilon_2(\beta+\gamma\zeta)\left\|e(t)\right\|^2
\end{aligned} \tag{50}$$

Since $V(t,e(t))=e^T(t)Pe(t)$, in the light of the Razumikhin theorem, we assume that for some real number $\upsilon>0$ such that

$$V(t+s,e(t+s))<(1+\upsilon)V(t,e(t)),\quad \forall s\in[-\tau,0],$$
$$\tau=\max\{\tau_1,\dots,\tau_q,\delta\},$$

and using (42)–(50), it is easy to obtain

$$D^\alpha V(t,e(t))\le e^T(t)\mathscr{M}e(t), \tag{51}$$

where

$$\begin{aligned}
\mathscr{M}=\;&P\Phi e_T M+M^T e_T^T\Phi^T P-PR\overline{C}-\overline{C}^T R^T P+PZ\Psi e_T M\\
&+M^T e_T^T\Psi^T Z^T P+\varepsilon(\beta+\gamma\zeta)I+(3q+3)(1+\upsilon)P\\
&+P\Phi e_T M_i P^{-1}M_i^T e_T^T\Phi^T P+Y_i\overline{C}P^{-1}\overline{C}^T Y_i^T\\
&+X\Psi e_T M_i P^{-1}M_i^T e_T^T\Psi^T X^T+P\Phi e_T M_\delta P^{-1}M_\delta^T e_T^T\Phi^T P\\
&+Y_\delta\overline{C}P^{-1}\overline{C}^T Y_\delta^T+X\Psi e_T M_\delta P^{-1}M_\delta^T e_T^T\Psi^T X^T\\
&+\varepsilon_1^{-1}PZ\Psi e_T e_T^T\Psi^T Z^T P+\varepsilon_2^{-1}P\Phi e_T e_T^T\Phi^T P.
\end{aligned}$$

By using the Schur Complement Lemma, we see that the condition $\mathscr{M}<0$ is equivalent to the condition (40). Therefore, we have

$$D^\alpha V(t,e(t))\le\lambda_{\max}(\mathscr{M})\|e(t)\|^2. \tag{52}$$

From (40) we have $\mathscr{M}<0$. Hence $\lambda_{\max}(\mathscr{M})<0$. Thus, the condition (ii) in Lemma 2 is also satisfied. Therefore, the system (39) is asymptotically stable. This completes the proof of the theorem.

In the following, we propose an effective algorithm to obtain the observer parameters.

Algorithm 1

Step 1: Obtain matrices $\mathscr{X}$ and $\mathscr{Y}$ from (25). Check the existence condition (26).
Step 2: Solve LMI (39). Obtain matrix $R=P^{-1}Y, Z=P^{-1}X$, $R_\delta=P^{-1}Y_\delta, R_i=P^{-1}Y_i (i=1,2,\dots,q)$.
Step 3: From (28)-(29), compute matrices T and Q. Then, from (21), obtain matrix H. From (36)-(38), compute matrices N, N_i $(i=1,2,\dots,q)$ and N_δ. Finally, from (35), obtain J, J_i $(i=1,2,\dots,q)$ and J_δ. The observer design is complete.

4 Numerical examples

Example 1 Consider the nonlinear time-delay fractional-order systems described in (1)-(3), where $\alpha=0.87$,

$$x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}\in\mathbb{R}^3,\ A=\begin{bmatrix}-10&1&2\\-48&-2&1\\1&-1&-20\end{bmatrix}, \tag{53}$$

$$A_1=\begin{bmatrix}0.5&0&-1\\-0.5&1&0.5\\0.25&0&0.5\end{bmatrix},\ A_2=\begin{bmatrix}0.01&0.02&0.03\\0.03&0.02&0.01\\0.01&0.01&0.03\end{bmatrix}, \tag{54}$$

$$B=\begin{bmatrix}1\\2\\3\end{bmatrix},\ D=\begin{bmatrix}1\\4\\2\end{bmatrix},D_\delta=\begin{bmatrix}0.5\\0\\1\end{bmatrix},\ C=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}, \tag{55}$$

$$f\left(\begin{bmatrix}x(t)\\d(t)\end{bmatrix},y(t)\right)=\begin{bmatrix}0\\x_2(t)x_3(t)\\0.5\sin(x_2(t))\end{bmatrix},E=\begin{bmatrix}0.5\\1\end{bmatrix}. \tag{56}$$

From decomposition (4), the nonlinear function can be decomposed into two parts as

$$f_L\left(\begin{bmatrix}x(t)\\d(t)\end{bmatrix},y(t)\right)=\begin{bmatrix}0&0&0.5\sin(x_2(t))\end{bmatrix}^T, \tag{57}$$

$$f_U\left(\begin{bmatrix}x(t)\\d(t)\end{bmatrix},y(t)\right)=x_2(t)x_3(t), \tag{58}$$

$W=\begin{bmatrix}0&1&0\end{bmatrix}^T$, where the nonlinear function $f_L\left(\begin{bmatrix}x(t)\\d(t)\end{bmatrix},y(t)\right)$ is under one-sided Lipschitz condition in Assumption 1 with the constant $\theta=0.5$. By using the well-known mean value theorem, $f_L\left(\begin{bmatrix}x(t)\\d(t)\end{bmatrix},y(t)\right)$ also satisfies Assumption 2 with the constants $\beta=0.25$ and $\gamma=0$.

According to Step 1 of Algorithm 1, we obtain matrices $\mathscr{X}$ and $\mathscr{Y}$ from equation (25). Since rank $\begin{bmatrix}\mathscr{X}\\\mathscr{Y}\end{bmatrix}=5=$ rank $\begin{bmatrix}\mathscr{X}\end{bmatrix}$, condition (26) is satisfied. According to Step 2 of Algorithm 1, the LMI in (40) yield the following results

$$P=\begin{bmatrix}1.3737&-0.333&-0.0117&0.331\\-0.333&0.8739&-0.0207&0.6615\\-0.0117&-0.0207&0.5428&0.0217\\0.331&0.6615&0.0217&0.8732\end{bmatrix},$$

$$X=10^3\begin{bmatrix}-0.1519&0.0025&-0.2272&-0.3538&-1.1307\\-0.3039&0.0049&-0.4545&-0.7077&-2.2615\\-1.3759&0.5599&2.5183&0.4256&0.0862\\0.3039&-0.0049&0.4545&0.7077&2.2615\end{bmatrix},$$

$$Y=\begin{bmatrix}11.8987&-421.0298\\411.6777&9.1003\\3.9756&-0.7216\\424.4088&-203.0891\end{bmatrix},\ Y_1=Y_2=Y_\delta=0_{4\times2}.$$

Then, matrices Z, R, R_1, R_2, R_δ can be obtained as

$$Z=10^4\begin{bmatrix}-0.4067 & 0.0308 & -0.4026 & -0.784 & -2.5625\\ -0.8128 & 0.0613 & -0.8057 & -1.5678 & -5.124\\ -0.3257 & 0.1086 & 0.3923 & -0.061 & -0.4396\\ 0.8129 & -0.0614 & 0.8053 & 1.5675 & 5.1232\end{bmatrix},$$

$$R=\begin{bmatrix}-23.0438 & -266.0315\\ 205.8894 & 20.5263\\ 1.1665 & -0.3975\\ 338.7703 & -147.2844\end{bmatrix},\ R_1=R_2=R_\delta=0_{4\times 2}.$$

According to Step 3 of Algorithm 1, matrices T, Q, N, N_1, N_2, N_δ, J, J_1, J_2, J_δ can be obtained as

$$T=\begin{bmatrix}-1 & 1 & 0\\ -2 & 2 & 0\\ 0 & 0 & 1\\ 2 & -2 & 0\end{bmatrix},\ Q=\begin{bmatrix}2 & -1\\ 2 & -1\\ 0 & 0\\ -2 & 2\end{bmatrix},$$

$$N=\begin{bmatrix}-14.9562 & 263.0315 & -1 & 280.5534\\ -281.8894 & -26.5263 & -2 & -117.471\\ -0.1665 & -0.6025 & -20 & 1.8142\\ -262.7703 & 153.2844 & 2 & -28.1008\end{bmatrix},$$

$$N_1=\begin{bmatrix}-1 & 1 & 1.5 & 0\\ -2 & 2 & 3 & 0\\ 0.25 & 0 & 0.5 & 0\\ 2 & -2 & -3 & 0\end{bmatrix},\ N_2=\begin{bmatrix}0.02 & 0 & -0.02 & 0\\ 0.04 & 0 & -0.04 & 0\\ 0.01 & 0.01 & 0.03 & 0\\ -0.04 & 0 & 0.04 & 0\end{bmatrix},$$

$$N_\delta=\begin{bmatrix}0 & 0 & 0 & -0.5\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix},\ J=\begin{bmatrix}-88 & 47\\ -176 & 94\\ -4 & 4\\ 176 & -94\end{bmatrix},$$

$$J_1=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0.5 & -0.25\\ 0 & 0\end{bmatrix},\ J_2=\begin{bmatrix}0.04 & -0.02\\ 0.08 & -0.04\\ 0.04 & -0.02\\ -0.08 & 0.04\end{bmatrix},$$

$$J_\delta=\begin{bmatrix}1 & -1\\ 2 & -2\\ -2 & 2\\ -2 & 2\end{bmatrix},\ H=\begin{bmatrix}1\\ 2\\ 3\\ -2\end{bmatrix}.$$

Example 2 Consider the nonlinear time-delay fractional-order systems described in (1)-(3), where $\alpha=0.91$,

$$x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\in\mathbb{R}^2,\ A=\begin{bmatrix}-2 & 1\\ 4 & -1\end{bmatrix}, \tag{59}$$

$$A_1=\begin{bmatrix}1 & 0\\ -0.5 & 1\end{bmatrix},\ A_2=\begin{bmatrix}0 & 2\\ -1 & 0\end{bmatrix},\ B=\begin{bmatrix}1\\ 2\end{bmatrix}, \tag{60}$$

$$D=\begin{bmatrix}1\\ 4\end{bmatrix}, D_\delta=\begin{bmatrix}0.5\\ 0\end{bmatrix},\ C=\begin{bmatrix}0 & 1\end{bmatrix},\ E=0.5, \tag{61}$$

$$f\left(\begin{bmatrix}x(t)\\ d(t)\end{bmatrix},y(t)\right)=\begin{bmatrix}-x_1(t)(x_1^2(t)+x_2^2(t))\\ -x_2(t)(x_1^2(t)+x_2^2(t))\end{bmatrix}. \tag{62}$$

As reported in the work of [31], the nonlinear function $f_L\left(\begin{bmatrix}x(t)\\ d(t)\end{bmatrix},y(t)\right)=\begin{bmatrix}-x_1(t)(x_1^2(t)+x_2^2(t))\\ -x_2(t)(x_1^2(t)+x_2^2(t))\end{bmatrix}$ is globally one-sided Lipschitz with $\theta=0$. Consider region

$$S=\left\{x\in\mathbb{R}^2: ||x||\le 2\right\},$$

$f_L\left(\begin{bmatrix}x(t)\\ d(t)\end{bmatrix},y(t)\right)$ is quadratically inner-bounded in S with constants $\beta=-48$ and $\gamma=-16$. In region S, $f_L\left(\begin{bmatrix}x(t)\\ d(t)\end{bmatrix},y(t)\right)$ is locally Lipschitz with the Lipschitz constant 12.

According to Step 1 of Algorithm 1, we obtain matrices $\mathscr{X}$ and $\mathscr{Y}$ from equation (25). Since rank $\begin{bmatrix}\mathscr{X}\\ \mathscr{Y}\end{bmatrix}=3=$ rank $\begin{bmatrix}\mathscr{X}\end{bmatrix}$, condition (26) is satisfied. According to Step 2 of Algorithm 1, the LMI in (40) yield the following results

$$P=10^4\begin{bmatrix}0.0007 & 0 & 0\\ 0 & 8.4588 & 4.2292\\ 0 & 4.2292 & 2.1151\end{bmatrix},$$

$$X=\begin{bmatrix}0 & -19.4767 & -58.4301\\ 0 & 7.0738 & 21.2214\\ 0 & -14.1476 & -42.4427\end{bmatrix},$$

$$Y=10^5\begin{bmatrix}0\\ 8.0738\\ 4.0368\end{bmatrix},\ Y_1=Y_2=Y_\delta=0_{3\times 1}.$$

Then, matrices Z, R, R_1, R_2, R_δ can be obtained as

$$Z=\begin{bmatrix}0 & -2.5531 & -7.6592\\ 0 & 1.3557 & 4.067\\ 0 & -2.7113 & -8.134\end{bmatrix},$$

$$R=\begin{bmatrix}0.868\\ 8.6242\\ 1.8415\end{bmatrix},\ R_1=R_2=R_\delta=0_{3\times 1}.$$

According to Step 3 of Algorithm 1, matrices T, Q, N, N_1, N_2, N_δ, J, J_1, J_2, J_δ can be obtained as

$$T=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & -2\end{bmatrix},\ Q=\begin{bmatrix}0\\ 0\\ 2\end{bmatrix},\ N=\begin{bmatrix}-2 & 0.132 & 0.566\\ 4 & -9.6242 & -0.3121\\ -8 & 0.1585 & -8.9208\end{bmatrix},$$

$$N_1=\begin{bmatrix}1 & 0 & 0\\ -0.5 & 1 & 0\\ 1 & -2 & 0\end{bmatrix},\ N_2=\begin{bmatrix}0 & 2 & 0\\ -1 & 0 & 0\\ 2 & 0 & 0\end{bmatrix},\ N_\delta=\begin{bmatrix}0 & 0 & 0.5\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$J=\begin{bmatrix}2\\ 8\\ -16\end{bmatrix},\ J_1=J_2=0_{3\times 1},\ J_\delta=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},\ H=\begin{bmatrix}1\\ 2\\ -4\end{bmatrix}.$$

5 Conclusion

This paper has considered the problem of estimating simultaneously the state and unknown disturbance of one-sided Lipschitz fractional-order systems with time-delay. The nominal models of nonlinearities are assumed to satisfy both the one-sided Lipschitz condition and the quadratically inner-bounded condition. Based on the Razumikhin stability theorem and a recent result on the Caputo fractional derivative of a quadratic function, a sufficient condition for the asymptotic stability of the observer error dynamic system has been derived. The stability condition is obtained in terms of linear matrix inequalities, which can be effectively solved by using existing convex algorithms. Two numerical examples have been provided to demonstrate effectiveness of the obtained results.

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