ON THE GAUSS MAP OF COMPLETE MINIMAL SURFACES WITH FINITE TOTAL CURVATURE INTO PROJECTIVE VARIETIES RAMIFIED OVER HYPERSURFACES IN SUBGENERAL POSITION

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Abstract. This paper is a continuation of the recent studies of L. Jin - M. Ru [13] and D. D. Thai - P. D. Thoan [5], [6]. The first aim of this paper is to show the second main theorem for linearly non-degenerate holomorphic maps from a compact Riemann surface into a projective algebraic variety which are ramified over hypersurfaces located in subgeneral position. We then use it to study the ramification over hypersurfaces located in subgeneral position of the linearly non-degenerate generalized Gauss maps of complete regular minimal surfaces in $\mathbb{R}^m$ with finite total curvature into projective algebraic varieties in $\mathbb{P}^{m-1}$. Finally, we study the unicity problem of the generalized Gauss maps of complete regular minimal surfaces in $\mathbb{R}^m$ with finite total curvature sharing hypersurfaces located in subgeneral position without the linear non-degeneracy (or algebraic non-degeneracy) assumption of these maps. Our results complete the previous results in [13], [5], [6].

1. Introduction

The second main theorem for holomorphic curves from a compact Riemann surface into the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$ is studied intensively in recent years. For instance, in 2007, L. Jin-M. Ru [13] established the second main theorem for linearly non-degenerate holomorphic curves from a compact Riemann surface into $\mathbb{P}^n(\mathbb{C})$ sharing hyperplanes in general position. Namely, they showed the following.

Theorem A [13, Theorem 2.4] *Let S be a compact Riemann surface of genus g. Let $f : S \to \mathbb{P}^n(\mathbb{C})$ be non-constant algebraic curve. Assume that $f(S)$ is contained in some k-dimensional projective subspace of $\mathbb{P}^n(\mathbb{C})$, but not in any subspace of dimension lower than k, where $1 \le k \le n$. Let $H_1, \cdots, H_q$ be the hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position and let $L_1, \cdots, L_q$ be the corresponding linear forms. Let E be a finite subset of*

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S. Then

$$(q-2n+k-1)\deg(f) \leq \sum_{j=1}^{q}\sum_{P\notin E}\min\{k,\nu_P(L_j(f))\}+\frac{1}{2}k(2n-k+1)\{2(g-1)+|E|\},$$

where $|E|$ is the number of elements of E and $\nu_P(L_j(f))$ is the vanishing order of $L_j(f)$ at the point P.

Recently, D. D. Thai - P. D. Thoan [6] showed a second main theorem for algebraically non-degenerate holomorphic curves from a compact Riemann surface into $\mathbb{P}^n(\mathbb{C})$ which are ramified over hypersurfaces located in subgeneral position.

Theorem B [6, Theorem 1] *Let S be a compact complex Riemann surface of genus g. Let $f : S \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve such that $f(S) \subset \mathbb{P}^k(\mathbb{C})$ and $f(S)$ is not contained in any hypersurface in $\mathbb{P}^k(\mathbb{C})$ for some $1 \leq k \leq n$. Let $Q_1, \cdots, Q_q$ be the hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, located in $N-$subgeneral position with $d_i := \deg Q_i$ $(1 \leq i \leq q)$. Put $d =$ lcm $(d_1, \cdots, d_q)$ and $M = \binom{k+d}{k} - 1$. Let E be a finite subset of S. Then*

$$\left(q-\frac{(2N-k+1)(M+1)}{k+1}\right)\deg(f) \leq \sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d_j}\min\{M,\nu_{Q_j(f)}(P)\} + \frac{(2N-k+1)M(M+1)}{2(k+1)}\cdot\frac{2(g-1)+|E|}{d},$$

where $\nu_{Q_j(f)} = f^*Q_j$ $(1 \leq j \leq q)$ is the vanishing order of $Q(f)$.

The first question is arised naturally at this moment.

Let S be a compact complex Riemann surface of genus g. Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k $(1 \leq k \leq n)$. Let f be a holomorphic curve of S into V such that f is linearly non-degenerate, i.e. $f(S)$ is not contained in any complex projective subspace of $\mathbb{P}^n(\mathbb{C})$ of dimension lower than k. How to state the second main theorem for f sharing hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, located in subgeneral position with respect to V?

Using the second main theorem, L. Jin-M. Ru [13] also showed the following theorem on the ramification over hyperplanes located in general position of the generalized Gauss map of complete regular minimal surfaces immersed in $\mathbb{R}^m$ with finite total curvature.

Theorem C [13, Theorem 3.1] *Let $x : S \to \mathbb{R}^m$ be a non-flat complete regular minimal surface with finite total curvature. Let $G : S \to \mathbb{P}^{m-1}(\mathbb{C})$ be its generalized Gauss map. Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$, located in general position in $\mathbb{P}^{m-1}(\mathbb{C})$, $(1 \leq i \leq q)$. If G is ramified over H_j with multiplicity at least m_j for each j (note that if $G(S)$*

omits H_j, then we take $m_j = \infty$), we obtain that

$$\sum_{j=1}^{q}\left(1-\frac{m-1}{m_j}\right) < \frac{1}{2}m(m+1).$$

In particular, $G(S)$ can fail to intersect at most $m(m+1)/2$ hyperplanes in general position in $\mathbb{P}^{m-1}(\mathbb{C})$.

Also in [6], they showed the following theorem on the ramification over hypersurfaces located in subgeneral position of the generalized Gauss map of complete regular minimal surfaces immersed in $\mathbb{R}^m$ with finite total curvature with the additional assumption on the algebraic non-degeneracy of the generalized Gauss map.

Theorem D [6, Theorem 2] *Let $x : S \to \mathbb{R}^m$ be a non-flat complete regular minimal surface with finite total curvature. Let $G : S \to \mathbb{P}^{m-1}(\mathbb{C})$ be its generalized Gauss map. Assume that $G(S) \subset \mathbb{P}^k(\mathbb{C})$ and $G(S)$ is not contained in any hypersurface in $\mathbb{P}^k(\mathbb{C})$ for some $1 \le k \le m-1$. Let $Q_1, \cdots, Q_q$ be hypersurfaces in $\mathbb{P}^{m-1}(\mathbb{C})$, located in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ with $\deg Q_i = d_i$ $(1 \le i \le q)$. Let $d =$ lcm $(d_1, \cdots, d_q)$. Assume that G is ramified over hypersurfaces Q_j with multiplicity at least m_j for each j and $M_k = \begin{pmatrix} k+d \\ k \end{pmatrix} - 1$. Then*

$$\sum_{j=1}^{q}\left(1-\frac{M_k}{m_j}\right) < \frac{(2N-k+1)(M_k+1)(M_k+2d)}{2(k+1)d}.$$

In particular, for each $1 \le k \le m-1$, then $\sum_{j=1}^{q}\left(1-\frac{m-1}{m_j}\right) < \frac{(2N-m+2)(m+1)}{2}$ if $d = 1$ and $\sum_{j=1}^{q}\left(1-\frac{M}{m_j}\right) < \frac{(2N-m+2)(M+1)(M+2d)}{2dm}$ if $d > 1$, where $M = \begin{pmatrix} m-1+d \\ m-1 \end{pmatrix} - 1$.

The second question is arised naturally at this moment.

Let V be a complex projective subvariety of $\mathbb{P}^{m-1}(\mathbb{C})$ of dimension k $(1 \le k \le m-1)$. Let $x : S \to \mathbb{R}^m$ be a non-flat complete regular minimal surface with finite total curvature. Assume that $G : S \to \mathbb{P}^{m-1}(\mathbb{C})$ is its generalized Gauss map such that $G(S) \subset V$. How to state the theorem on the ramification over hypersurfaces in $\mathbb{P}^{m-1}(\mathbb{C})$, located in subgeneral position with respect to V for the map G without the algebraic non-degeneracy assumption of this map?

Using Theorem A and Theorem C, L. Jin-M. Ru [13] also obtained the following unicity theorem for the generalized Gauss maps of complete regular minimal surfaces immersed in $\mathbb{R}^m$ with finite total curvature sharing hyperplanes located in general position with the additional assumption on the linear non-degeneracy of these maps.

Theorem E [13, Theorem 4.1] *Consider two algebraic minimal surfaces M_1, M_2 immersed in $\mathbb{R}^m$ with the same basic domain $M = \overline{M} \setminus \{P1, \cdots, Pr\}$. Let G_1, G_2 be the generalized Gauss map of M_1, M_2 respectively. Assume that G_1, G_2 are linearly non-degenerate and assume that $G1 \not\equiv G2$. Let $H_1, \cdots, H_q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position, and let $L_1, \cdots, L_q$ be the corresponding linear forms. Assume that*

(i) $\min\{\nu_P(L_j(G_1)), 1\} = \min\{\nu_P(L_j(G_2)), 1\}$, *for $P \in M$ and $j = 1, \cdots, q$;*

(ii) For every $i \neq j$, $G_1^{-1}(H_i) \bigcap G_1^{-1}(H_j) = \emptyset$;

(iii) $G_1 = G_2$ on $\bigcup_{j=1}^q G_1^{-1}(H_j)$.

Then $q < \dfrac{1}{2}(m^2 + 5m - 4)$.

Using Theorem B and Theorem D, in [5] they also obtained the following unicity theorem for the generalized Gauss maps of complete regular minimal surfaces immersed in $\mathbb{R}^m$ with finite total curvature sharing hypersurfaces located in subgeneral position with the additional assumption on the algebraic non-degeneracy of these maps.

Theorem F [5, Theorem 3] *Consider two algebraic minimal surfaces S_1, S_2 immersed in $\mathbb{R}^m$ with the same basic domain $S = \bar{S} \setminus \{P_1, \cdots, P_r\}$. Let G_1, G_2 be the generalized Gauss map of S_1, S_2 respectively. Assume that $G_1(S_1), G_2(S_2)$ are not contained in any hypersurface in $\mathbb{P}^{m-1}(\mathbb{C})$. Let $\{Q_i\}_{i=1}^q$ be the hypersurfaces in $\mathbb{P}^{m-1}(\mathbb{C})$, located in N-subgeneral position with $\deg Q_i = d_i$ $(1 \leq i \leq q)$. Assume that*

(i) $\min\{\nu_{(Q_j(G_1))}, 1\} = \min\{\nu_{(Q_j(G_2))}, 1\}$, *for all $P \in S$ and $1 \leq j \leq q$*

(ii) $G_1 \equiv G_2$ on $\bigcup_{i=1}^q G_1^{-1}(Q_j)$.

Then $G_1 \equiv G_2$ if

$$q \geq \frac{(2N - m + 2)(M + 1)[(2d + 1)M + 2d]}{2dm},$$

where $d =$ lcm $(d_1, \cdots, d_q)$ and $M = \begin{pmatrix} m - 1 + d \\ m - 1 \end{pmatrix} - 1$.

The third question is arised naturally at this moment.

How to state the unicity theorem for the generalized Gauss maps of complete regular minimal surfaces immersed in $\mathbb{R}^m$ with finite total curvature sharing hypersurfaces located in subgeneral position without the additional assumption on the linear non-degeneracy or the algebraic non-degeneracy of these maps?

The main aim of this paper is to give compete answers for the above-mentioned problems. To state our results, we now recall some notations.

Let M be a complete immersed minimal surface in $\mathbb{R}^m$. Take an immersion $x = (x_0, ..., x_{m-1}) : M \to \mathbb{R}^m$. Then M has the structure of a Riemann surface and any local isothermal coordinate (x, y) of M gives a local holomorphic coordinate $z = x + \sqrt{-1}y$. The generalized Gauss map of x is defined to be

$$G : M \to \mathbb{P}^{m-1}(\mathbb{C}), G = \mathbb{P}\Big(\frac{\partial x}{\partial z}\Big) = \Big(\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}\Big).$$

Since $x : M \to \mathbb{R}^m$ is immersed, it implies that

$$g = g_z := (g_0, \ldots, g_{m-1}) = ((g_0)_z, \ldots, (g_{m-1})_z) = \Big(\frac{\partial x_0}{\partial z}, \ldots, \frac{\partial x_{m-1}}{\partial z}\Big)$$

is a (local) reduced representation of G. Moreover, for another local holomorphic coordinate ξ on M, we have $g_\xi = g_z \cdot \Big(\dfrac{dz}{d\xi}\Big)$ and hence, g is well defined (independently of the local holomorphic coordinate). Since M is minimal, G is a holomorphic map.

We now consider the hypersurface Q given by

$$\sum_{I \in \mathcal{I}_d} a_I z^I = 0,$$

where $\mathcal{I}_d = \{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1} : i_0 + \cdots + i_n = d\}, I = (i_0, \ldots, i_n) \in \mathcal{I}_d, z^I = z_0^{i_0} \cdots z_n^{i_n}$ and $a_I \in \mathbb{C}$ $(I \in \mathcal{I}_d)$. Put $M = \begin{pmatrix} n+d \\ n \end{pmatrix} - 1$ and denote by

$$H = \{(z_0, \ldots, z_M) \in \mathbb{C}^{M+1} : \sum_{I_j \in \mathcal{I}_d} a_{I_j} z_{I_j} = 0\}$$

the hyperplane in $\mathbb{C}^{M+1}$ associated with Q_i.

Let $f : S \to \mathbb{P}^n(\mathbb{C})$ be an holomorphic map with a reduced (local) representation $f(z) = (f_0(z), \ldots, f_n(z))$. For each d, define $F : S \to \mathbb{P}^M(\mathbb{C})$ by

$$F(z) = (f^{I_0}(z), \ldots, f^{I_M}(z)),$$

where $\{I_0, \ldots, I_M\} = \mathcal{I}_d$ and $f^I(z) = f_0^{i_0}(z) \cdots f_n^{i_n}(z)$ for $I = (i_0, \ldots, i_n) \in \mathcal{I}_d$. Such definition is independent of the choice of the representation of f and of the parameter z. We call F the associated map with f of degree d. Put $Q(f) = H(F) = \sum_{I \in \mathcal{I}_d} a_I f^I$. We will consider $f^*Q = \nu_{Q(f)}$ as a divisor.

Definition 1. *The map f is said to be ramified over a hypersurface Q in $\mathbb{P}^{m-1}(\mathbb{C})$ with multiplicity at least e if all the zeros of the function $Q(f)$ have orders at least e.*

If the image of f omits Q, one will say that f is ramified over Q with multiplicity ∞.

Now, let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k $(k \leq n)$. Let d be a positive integer. We denote by $I(V)$ the ideal of homogeneous polynomials in

$\mathbb{C}[x_0,...,x_n]$ defining V and by $\mathbb{C}[x_0,...,x_n]_d$ the vector space of all homogeneous polynomials in $\mathbb{C}[x_0,...,x_n]$ of degree d including the zero polynomial. Define

$$\mathbb{C}_d(V) := \frac{\mathbb{C}[x_0,...,x_n]_d}{I(V)\cap \mathbb{C}[x_0,...,x_n]_d} \text{ and } H_V(d) := \dim \mathbb{C}_d(V).$$

Then $H_V(d)$ is called the Hilbert function of V. Each element of $\mathbb{C}_d(V)$ which is an equivalent class of an element $Q \in \mathbb{C}[x_0,...,x_n]_d$, will be denoted by $[Q]$.

Definition 2. *Let $Q_1,\dots,Q_q$ $(q \geq k+1)$ be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. The family of hypersurfaces $\{Q_i\}_{i=1}^q$ is said to be in $N-$subgeneral position with respect to V if for any $1 \leq i_1 < \ldots < i_{N+1}$,*

$$(\bigcap_{j=1}^{N+1} Q_{i_j}) \cap V = \emptyset.$$

If $\{Q_i\}_{i=1}^q$ is in $n-$subgeneral position with respect to V, then we say that it is in general position with respect to V.

We now state the first result.

Theorem 1. *Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k $(k \leq n)$. Let $Q_1,\cdots,Q_q$ be the hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, located in N-subgeneral position with respect to V and $d_i := \deg Q_i$ $(1 \leq i \leq q)$. Put $d =$ lcm $(d_1,\cdots,d_q)$. Let S be a compact Riemann surface of genus g and let E be a finite subset of S. Let f be a holomorphic curve of S into $\mathbb{P}^n(\mathbb{C})$ such that $f(S)$ is contained in V. Assum that the map f is linearly non-degenerate in V, i.e. its image $f(S)$ is not contained in any complex projective subspace of dimension lower than k of $\mathbb{P}^n(\mathbb{C})$. Then*

$$\left(q - \frac{(2N-k+1)H_V(d)}{k+1}\right)\deg(f) \leq \sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d_j}\min\{\nu_{Q_j(f)}(P), H_V(d)-1\} + \frac{(2N-k+1)(H_V(d)-1)H_V(d)}{2(k+1)}\cdot\frac{2(g-1)+|E|}{d},$$

where $\nu_{Q_j(f)} = f^*Q_j$ $(1 \leq j \leq q)$ is the vanishing order of $Q(f)$ and $H_V(d)$ is the Hilbert function of V.

It is easy to see that Theorem A is deduced immediately from Theorem 1 by considering $V = \mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ and Q_j are hyperplanes, because $d = 1$ and $H_V(d) = k+1$ in this case. Moreover, Theorem B is deduced immediately from Theorem 1 by considering $V = \mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$.

We now state the second result.

Theorem 2. *Let V be a complex projective subvariety of $\mathbb{P}^{m-1}(\mathbb{C})$ of dimension k ($1 \leq k \leq m-1$). Let $Q_1, \cdots, Q_q$ be hypersurfaces in $\mathbb{P}^{m-1}(\mathbb{C})$, located in N-subgeneral position with respect to V and $d_i := \deg Q_i$ ($1 \leq i \leq q$). Put $d =$ lcm $(d_1, \cdots, d_q)$. Let $x : S \to \mathbb{R}^m$ be a non-flat complete regular minimal surface with finite total curvature. Let $G : S \to \mathbb{P}^{m-1}(\mathbb{C})$ be its generalized Gauss map. Assume that $G(S)$ is contained in V and the map G is linearly non-degenerate in V, i.e. its image $f(S)$ is not contained in any complex projective subspace of dimension lower than k of $\mathbb{P}^{m-1}(\mathbb{C})$. Assume that G is ramified over hypersurfaces Q_j with multiplicity at least m_j for each j. Then*

$$\sum_{j=1}^{q}\left(1-\frac{H_V(d)-1}{m_j}\right) < \frac{(2N-k+1)H_V(d)(H_V(d)-1+2d)}{2(k+1)d}.$$

We now consider $V = \mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$. It is easy to see that $H_V(d) = \binom{k+d}{k}$.

For each $1 \leq k \leq m-1$, put

$$\begin{aligned}
a_k &= \binom{k+d}{k},\\
M_k &= a_k - 1,\\
M &= M_{m-1} = \binom{m-1+d}{m-1} - 1,\\
A_k &= (2N-k+1)\frac{H_V(d)}{k+1},\\
B_k &= \frac{A_k(M_k+2d)}{2d} = \frac{(2N-k+1)H_V(d)(H_V(d)-1+2d)}{2(k+1)d}.
\end{aligned}$$

Then $B_k \leq B_{m-1} = \dfrac{(2N-m+2)(M+1)(M+2d)}{2md}$ for all $1 \leq k \leq m-1$.
Indeed, we consider two cases.
Case 1: Assume that $d > 1$.
Then

$$\begin{aligned}
A_k &= (2N-k+1)\frac{a_k}{k+1}\\
&= \frac{2N-k+1}{k+1} \cdot \frac{k+d}{k} a_{k-1}\\
&= (2N-k+1)\Big(\frac{a_{k-1}}{k} + \frac{d-1}{k+1} \cdot \frac{a_{k-1}}{k}\Big)\\
&= (2N-k+2)\frac{a_{k-1}}{k} + \frac{a_{k-1}}{k}\Big[\frac{d-1}{k+1}(2N-k+1) - 1\Big]\\
&\geq A_{k-1}.
\end{aligned}$$

Hence, for each $1 \le k \le m-1$, we have

$$B_k = \frac{A_k(M_k + 2d)}{2d} \ge \frac{A_{k-1}(M_{k-1} + 2d)}{2d} = B_{k-1}.$$

It yields that $B_k \le B_{m-1}$ for all $1 \le k \le m-1$.

Case 2: Assume that $d = 1$.

Then $d_i = 1$ $(1 \le i \le q)$ and hence, $M_k = k$. Since

$$(2N - k + 1)(k + 2) \le (2N - m + 1)(m + 1)$$

for $1 \le k \le m-1$, we also have $B_k \le B_{m-1}$ for all $1 \le k \le m-1$.

From Theorem 2, we now have the following theorem on the ramification over hypersurfaces located in subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ for the map G without any additional assumption of this map.

Corollary 3. *Let $x : S \to \mathbb{R}^m$ be a non-flat complete regular minimal surface with finite total curvature. Let $G : S \to \mathbb{P}^{m-1}(\mathbb{C})$ be its generalized Gauss map. Let $Q_1, \cdots, Q_q$ be hypersurfaces located in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ and $d_i := \deg Q_i$ $(1 \le i \le q)$. Let $d =$ lcm $(d_1, \ldots, d_q)$ and $M = \binom{m-1+d}{m-1} - 1$. Assume that G is ramified over hypersurfaces Q_j with multiplicity at least m_j for each j. Then*

$$\sum_{j=1}^{q}\left(1 - \frac{M}{m_j}\right) < \frac{(2N - m + 2)(M+1)(M+2d)}{2md}.$$

It is easy to see that Theorem C is deduced immediately from Corollary 3 by considering $V = \mathbb{P}^{m-1}(\mathbb{C})$ and Q_j are hyperplanes located in general position in $\mathbb{P}^{m-1}(\mathbb{C})$, because $d = 1$ and $M = m - 1$ in this case. Moreover, Theorem D is deduced immediately from Theorem 2 by considering $V = \mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^{m-1}(\mathbb{C})$ and remarking that if $Q_1, \cdots, Q_q$ are located in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ then they are also located in N-subgeneral position in $\mathbb{P}^k(\mathbb{C})$.

Let $x : S \to \mathbb{R}^m$ be a complete regular minimal surface with finite total curvature. Let $G : S \to \mathbb{P}^{m-1}(\mathbb{C})$ be its generalized Gauss map. By the result of S.S. Chern and R. Osserman (see [3]), S is conformally equivalent to a compact surfaces $\bar{S}$ punctured at a finite number of points $P_1, \cdots, P_r$. Hence, $G : S = \bar{S} \setminus \{P_1, \ldots, P_r\} \to \mathbb{P}^{m-1}(\mathbb{C})$ is algebraic. We call S the basic domain of the minimal surface.

By using the arguments in [4, 5, 8, 12, 13], we have the following.

Theorem 4. *Consider two complete regular minimal surfaces with finite total curvature S_1 and S_2 immersed in $\mathbb{R}^m$ with the same basic domain $S = \bar{S} \setminus \{P_1, \cdots, P_r\}$. Let G_1 and G_2 be the generalized Gauss maps of S_1 and S_2 respectively. Let $\{Q_i\}_{i=1}^q$ be the hypersurfaces in $\mathbb{P}^{m-1}(\mathbb{C})$ located in N-subgeneral position with common degree of d. Assume that* *(i)* $\min\{\nu_{Q_j(G_1)}(P), 1\} = \min\{\nu_{Q_j(G_2)}(P), 1\}$ *for all* $P \in S$ *and* $1 \le j \le q$,

(ii) there exist a positive integer mumber k *such that* $\bigcap_{j=1}^{k+1} G_1^{-1}(Q_{i_j}) = \emptyset$ *for any* $\{i_1, \ldots, i_{k+1}\} \subset \{1, \ldots, q\}$,

(iii) $G_1 \equiv G_2$ *on* $\bigcup_{i=1}^{q} G_1^{-1}(Q_j)$.

Then $G_1 \equiv G_2$ *if*

$$q \ge \frac{(2N-m+2)(M+1)(M+2d)}{2md} + \frac{2kMq}{q-2k+2Mk},$$

where $M = \binom{m-1+d}{m-1} - 1$.

In the case $k = 1$, since $\frac{2kMq}{q-2k+2Mk} < 2M$, we obtain the following corollary.

Corollary 5. *Consider two complete regular minimal surfaces with finite total curvature* S_1 *and* S_2 *immersed in* $\mathbb{R}^m$ *with the same basic domain* $S = \bar{S} \setminus \{P_1, \cdots, P_r\}$. *Let* G_1 *and* G_2 *be the generalized Gauss maps of* S_1 *and* S_2 *respectively. Let* $\{Q_i\}_{i=1}^q$ *be the hypersurfaces in* $\mathbb{P}^{m-1}(\mathbb{C})$ *located in* N*-subgeneral position with common degree of* d. *Assume that*

(i) $\min\{\nu_{(Q_j(G_1))}, 1\} = \min\{\nu_{(Q_j(G_2))}, 1\}$ *for all* $P \in S$ *and* $1 \le j \le q$,

(ii) for every $i \ne j$, $G_1^{-1}(Q_j) \bigcap G_1^{-1}(Q_i) = \emptyset$,

(iii) $G_1 \equiv G_2$ *on* $\bigcup_{i=1}^{q} G_1^{-1}(Q_j)$.

Then $G_1 \equiv G_2$ *if*

$$q \ge \frac{(2N-m+2)(M+1)(M+2d)}{2md} + 2M,$$

where $M = \binom{m-1+d}{m-1} - 1$.

In Corollary 5, if $\{Q_i\}_{i=1}^q$ are the hyperplanes in general position in $\mathbb{P}^{m-1}(\mathbb{C})$, then

$$d = 1, M = N = m-1, \frac{(2N-m+2)(M+1)(M+2d)}{2md} + 2M = \frac{1}{2}(m^2 + 5m - 4)$$

and hence, Corollary 5 gave a nice improvement of Theorem E by omitting the linear non-degeneracy assumption of the maps G_1 and G_2 in this theorem.

In Theorem 4, if we choose $k = N$ then condition (ii) automatically holds when the hypersurfaces are in N-subgeneral position. Since $\frac{2kMq}{q-2k+2Mk} < 2MN$, it implies that the following corollary holds.

Corollary 6. *Consider two complete regular minimal surfaces with finite total curvature* S_1 *and* S_2 *immersed in* $\mathbb{R}^m$ *with the same basic domain* $S = \bar{S} \setminus \{P_1, \cdots, P_r\}$. *Let* G_1 *and* G_2 *be the generalized Gauss maps of* S_1 *and* S_2 *respectively. Let* $\{Q_i\}_{i=1}^q$ *be* *the hypersurfaces located in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ with common degree of d. Assume that*

(i) $\min\{\nu_{Q_j(G_1)}(P), 1\} = \min\{\nu_{Q_j(G_2)}(P), 1\}$ *for all* $P \in S$ *and* $1 \leq j \leq q$,

(ii) $G_1 \equiv G_2$ *on* $\bigcup_{i=1}^{q} G_1^{-1}(Q_j)$.

Then $G_1 \equiv G_2$ *if*

$$q \geq \frac{(2N - m + 2)(M + 1)(M + 2d)}{2md} + 2MN,$$

where $M = \binom{m-1+d}{m-1} - 1$.

Finally, we would like to emphasize that, by the another approach, D. D. Thai and V. D. Viet in [7] showed the second main theorem and a unicity theorem for holomorphic curves of a compact Riemann surface into a compact complex manifold sharing divisors in subgeneral position in this manifold.

2. Auxiliary lemmas

Assume that $f : S \to \mathbb{P}^n(\mathbb{C})$ is a linearly non-degenerate holomorphic curve (that is, $f(S)$ is not contained in any hyperplane in $\mathbb{P}^n(\mathbb{C})$). For every point $P \in S$, in a neighborhood of P, let $f(z) = (f_0(z), \cdots, f_n(z))$ be a reduced representation of f at P with $z(P) = 0$, where z is a local parameter for S at P and $f_0, \cdots, f_n$ are holomorphic functions without common zeros. Take a hyperplane $H : a_0z_0 + \cdots + a_nz_n = 0$ in $\mathbb{P}^n(\mathbb{C})$ and put

$$H(f) = a_0f_0 + \cdots + a_nf_n.$$

Then $\sum_{z\in S} \nu_{H(f)}(z)$ does not depend on the choice of H, where $\nu_{H(f)}(z)$ is the intersection multiplicity of the images of f and H at $f(z)$. We define the degree of f by

$$\deg(f) = \sum_{P\in S} \nu_{H(f)}(P).$$

It is easy to see that if $f^{-1}(H) = \{P_1, \cdots, P_r\}$, then

$$\deg(f) = \sum_{j=1}^{r} \nu_{H(f)}(P_j) \geq r. \tag{2.1}$$

Now we may assume that $f(0) = (1, 0, \cdots, 0)$ by making a linear change of coordinates in $\mathbb{C}^{n+1}$. We have $f_1(0) = \cdots = f_n(0) = 0$. Write $(f_1(z), \cdots, f_n(z)) = z^{\delta_1}(f_1^1(z), \cdots, f_n^1(z))$ with $(f_1^1(0), \cdots, f_n^1(0)) \neq 0$. Make a linear change of the last n coordinate $\mathbb{C}^{n+1}$ so that $(f_1^1(0), \cdots, f_n^1(0)) = (1, 0, \cdots, 0)$. Write $(f_2^1(z), \cdots, f_n^1(z)) = z^{\delta_2-\delta_1}(f_2^2(z), \cdots, f_n^2(z))$

with $(f_2^2(0), \cdots, f_n^2(0)) \neq 0$. Continuing in this way we end up with a system of coordinate for $\mathbb{C}^{n+1}$ in terms of which

$$f(z) = (z^{\delta_0} + \cdots, z^{\delta_1} + \cdots, \cdots, z^{\delta_n} + \cdots), \tag{2.2}$$

where $0 = \delta_0 < \delta_1 < \cdots < \delta_n$. Put $\nu_i = \delta_{i+1} - \delta_i - 1,\ 0 \le i \le n-1$ and note that, for $P \in S$, we have

$$\sum_{i=0}^{n}(n-i)\nu_i(P) + \frac{1}{2}n(n+1) = \delta_0(P) + \delta_1(P) + \cdots + \delta_n(P). \tag{2.3}$$

Let

$$\sigma_i = \sum_{P \in S} \nu_i(P). \tag{2.4}$$

By Plücker formula which is a generalization of the Riemann-Hurwitz's theorem (see [11]), we have

$$\sum_{i=0}^{n}(n-i)\sigma_i = (n+1)\deg(f) + n(n+1)(g-1). \tag{2.5}$$

Here g stands for the genus of S.

Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k $(k \le n)$. Let $\{Q_i\}_{i=1}^q$ be a family hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d. Each Q_i is defined by some homogeneous polynomial $Q_i^* \in \mathbb{C}[x_0, x_1, \ldots, x_n]$. Consider the set $\mathbb{C}_d(V) := \dfrac{\mathbb{C}[x_0, ..., x_n]_d}{I(V) \cap \mathbb{C}[x_0, ..., x_n]_d}$ as a vector space and define

$$\operatorname{rank}\{Q_i\}_{i \in R} = \operatorname{rank}\{[Q_i^*]\}_{i \in R}$$

for every subset $R \subset \{1, \ldots, q\}$. It is easy to see that

$$\operatorname{rank}\{Q_i\}_{i \in R} = \operatorname{rank}\{[Q_i^*]\}_{i \in R} \ge \dim V - \dim(\bigcap_{i \in R} Q_i \cap V),$$

with $\dim \emptyset := -1$. Hence, if $\{Q_i\}_{i=1}^q$ is N-subgeneral position, then

$$\operatorname{rank}\{Q_i\}_{i \in R} = \operatorname{rank}\{[Q_i^*]\}_{i \in R} \ge \dim V - \dim(\bigcap_{i \in R} Q_i \cap V) = k+1$$

for any subset $R \subset \{1, \ldots, q\}$ with $|R| = N+1$.

Similar to [2, Lemma 4.2], we have the following.

Lemma 7. *Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of the common degree d in $\mathbb{P}^n(\mathbb{C})$. Then, there exist $(H_V(d) - k - 1)$ hypersurfaces $\{T_i\}_{i=1}^{H_V(d)-k-1}$ such that for any subset $R \subset \{1, \cdots, q\}$ with $|R| = \operatorname{rank}\{H_i\}_{i \in R} = k+1$, we get $\operatorname{rank}\{\{Q_i\}_{i \in R} \cup \{T_i\}_{i=1}^{H_V(d)-k-1}\} = H_V(d)$.*

By [2, Lemma 3.3], we have the following.

Lemma 8. *([2, Lemma 3.3]) Let V be a complex projective subvariety of dimension k of $\mathbb{P}^n(\mathbb{C})$ $(k \le n)$. Let $Q_1, \cdots, Q_q$ $(q > 2N-n+1)$ be hypersurfaces of the common degree d in $\mathbb{P}^n(\mathbb{C})$, located in N-subgeneral position with respect to V. Then there exists a function $\omega : \{1, \cdots, q\} \to (0,1]$ called a Nochka weight and a real number $\theta \ge 1$ called a Nochka constant satisfying the following conditions:*

(i) If $j \in \{1, \cdots, q\}$, then $0 < \omega(j)\theta \le 1$.

(ii) $q - 2N + n - 1 = \theta(\sum_{j=1}^{q} \omega(j) - n - 1)$.

(iii) For $R \subset \{1, \cdots, q\}$ with $|R| = N+1$, then $\sum_{i \in R} \omega(i) \le n+1$.

(iv) $\frac{N+1}{n+1} \le \theta \le \frac{2N-n+1}{n+1}$.

(v) Given real numbers $\lambda_1, \cdots, \lambda_q$ with $\lambda_j \ge 1$ for $1 \le j \le q$ and given any $R \subset \{1, \cdots, q\}$ and $|R| = N+1$, there exists a subset $R^0 \subset R$ such that $|R^0| = \mathrm{rank}\{Q_i\}_{i \in R^0} = n+1$ and

$$\prod_{i \in R} \lambda_i^{\omega(i)} \le \prod_{i \in R^0} \lambda_i.$$

Taking a $\mathbb{C}$-basis $\{[\Phi_i]\}_{i=0}^{H_V(d)-1}$ of $\mathbb{C}_d(V)$ with $\Phi_i \in H_d$, we may consider $\mathbb{C}_d(V)$ as a $\mathbb{C}$-vector space $\mathbb{C}^{H_V(d)}$.

We consider $[Q] \in \mathbb{C}_d(V)$, where $Q \in \mathbb{C}[x_0, ..., x_n]_d$ is a hypersurface of degree d. Then

$$[Q] = \sum_{i=0}^{H_V(d)-1} a_i[\Phi_i] = \sum_{i=0}^{H_V(d)-1} [a_i \Phi_i]$$

with $a_i \in \mathbb{C}$ $(1 \le i \le H_V(d))$. Denote by

$$H = (a_0 : \cdots : a_{H_V(d)-1}) \in \mathbb{P}^{H_V(d)-1}(\mathbb{C})$$

the hyperplane in $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$ which is called the associated hyperplane of Q with respect to the basis $\{[\Phi_i]\}_{i=0}^{H_V(d)-1}$.

We now consider a holomorphic curve $f : S \to V$. Also consider the holomorphic map $F = (\Phi_0(f) : \cdots : \Phi_{H_V(d)-1}(f))$ of S to $\mathbb{P}^{H_V(d)-1}$. Take a reduced representation of $\tilde{f} = (f_0 : \cdots : f_n)$ of f on a neighborhood of $P \in S$, then $\tilde{F} = (\Phi_0(\tilde{f}) : \cdots : \Phi_{H_V(d)-1}(\tilde{f}))$ is a reduced representation of F. The map F said to be the associated map of f with respect to the basis $\{[\Phi_i]\}_{i=0}^{H_V(d)-1}$.

It is easy to see that $Q(f) = F(H) = a_0\Phi_0(f) + \cdots + a_{H_V(d)-1}\Phi_{H_V(d)-1}(f)$. We need the following.

Lemma 9. *Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $f : S \to V$ be a holomorphic curve and $F : S \to \mathbb{P}^{H_V(d)-1}(\mathbb{C})$ be the associated map of f with respect to a some basis of $\mathbb{C}_d(V)$. Let Q be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. If f is linearly* *non-degenerate with respect to V, then*

$$\deg(F) = \sum_{P \in S} \nu_{Q(f)}(P) = d \deg(f). \tag{2.6}$$

Proof. Taking a basis $\{[\Phi_i]\}_{i=0}^{H_V(d)-1}$ of $\mathbb{C}_d(V)$. Let $H^0 = \{\omega_0 = 0\}$ be a hyperplane in $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$. Then $F(H^0) = \Phi_0(f)$, where $\Phi_0 \in \mathbb{C}[x_0, ..., x_n]_d$. Hence

$$\deg(F) = \sum_{P \in S} \nu_{\Phi_0(f)}(P). \tag{2.7}$$

Now, assume that each Q is given by

$$\sum_{I \in \mathcal{I}_d} a_I z^I = 0,$$

where $\mathcal{I}_d = \{(i_0, \cdots, i_n) \in \mathbb{N}^{n+1} : i_0 + \cdots + i_n = d\}, I = (i_0, \cdots, i_n) \in \mathcal{I}_d, z^I = z_0^{i_0} \cdots z_n^{i_n}$, $a_I \in \mathbb{C}$ $(1 \leq I \leq M+1, I \in \mathcal{I}_d)$ and $M = \begin{pmatrix} n+d \\ n \end{pmatrix} - 1$.

Denote by $H = \left\{(z_0, \cdots, z_M) \in \mathbb{C}^{M+1} : \sum_{I_j \in \mathcal{I}_d} a_{I_j} z_{I_j} = 0\right\}$ the hyperplane in $\mathbb{C}^{M+1}$ associated to Q.

Put $G : S \to \mathbb{P}^M(\mathbb{C})$ by

$$G(z) = (f^{I_0}(z), \cdots, f^{I_M}(z)),$$

where $\{I_0, \cdots, I_M\} = \mathcal{I}_d$ and $f^I(z) = f_0^{i_0}(z) \cdots f_n^{i_n}(z)$ for $I = (i_0, \cdots, i_n) \in \mathcal{I}_d$. Such definition is independent of the choice of the representation $\tilde{f}(z) = (f_0(z), \cdots, f_n(z))$ of f and of the parameter z. Put $Q(f) = H(G) = \sum_{I \in \mathcal{I}_d} a_I f^I$. We will consider $f^*Q = \nu_{Q(f)}$ as a divisor.

Consider the hyperplane $\hat{H} = \{\omega_0 = 0\}$ in $\mathbb{P}^M(\mathbb{C})$. Assume that $G^{-1}(\hat{H}) = \{P_1, \cdots, P_r\}$. For each $1 \leq j \leq r$, take a holomorphic local parameter z_j with $z_j(P_j) = 0$ in a neighborhood of P_j in S. Consider a sufficiently small positive number ϵ such that $\bar{U}_j(\epsilon) := \{z_j : |z_j| \leq \epsilon\}$ are mutually disjoint. Now take a reduced representation $\tilde{f}(z) = (f_0(z), \cdots, f_n(z))$ of f on $\bigcup_j U_j(\epsilon)$. We obtain $\hat{H}(G)(z) = f^{I_0}(z) = (f_0(z))^d$, where $I_0 = (1, 0, \cdots, 0) \in \mathcal{I}_d$. This implies that

$$\begin{aligned} \deg(G) &= \sum_{j=1}^{r} \nu_{\hat{H}(G)}(P_j) = \sum_{j=1}^{r} \nu_{f_0^d}(P_j) = \sum_{j=1}^{r} d \cdot \nu_{f_0}(P_j) \\ &= \sum_{j=1}^{r} d \cdot \nu_{\tilde{H}(f)}(P_j) = d \deg(f), \end{aligned} \tag{2.8}$$

where $\tilde{H} : \omega_0 = 0$ is a hyperplane in $\mathbb{P}^n(\mathbb{C})$. By taking the associated hyperplane H of Q and K of Φ_0 in $\mathbb{P}^M(\mathbb{C})$, we have

$$\deg(G) = \sum_{P\in S} \nu_{H(G)}(P) = \sum_{P\in S} \nu_{K(G)}(P) = \sum_{P\in S} \nu_{Q(f)}(P) = \sum_{P\in S} \nu_{\Phi_0(f)}(P). \tag{2.9}$$

Combining (2.7) with (2.8) and (2.9), we complete the proof of Lemma 9. $\square$

3. The proof of Theorem 1

Step 1. First of all, we prove the theorem in the case where all hypersurfaces Q_i $(1 \le i \le q)$ have the same degree d.

Fix a $\mathbb{C}-$basis $\{[\Phi_i]\}_{i=0}^{H_V(d)-1}$ of $\mathbb{C}_d(V)$, where $\Phi_i \in \mathbb{C}[x_0, ..., x_n]_d$. Assume that the image $F(S)$ is contained in the l-dimensional projective subspace $\mathbb{P}^l(\mathbb{C})$ of $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$, but not in any subspace of dimension lower than l, where $1 \le l \le H_V(d) - 1$. Consider a linear equation system determining $\mathbb{P}^l(\mathbb{C})$:

$$\begin{cases} a_{10}\omega_0 + \cdots + a_{1,H_d(V)-1}\omega_{H_d(V)-1} = 0 \\ \ldots \\ a_{H_d(V)-1-l,0}\omega_0 + \cdots + a_{H_d(V)-1-l,H_d(V)-1}\omega_{H_d(V)-1} = 0 \end{cases} \tag{3.10}$$

Without loss of generality, assume that

$$\text{rank } (a_{ij})_{1\le i\le H_d(V)-1, l+1\le j\le H_d(V)-1} = H_d(V) - 1 - l.$$

By solving the above linear equation system (3.10), it implies that $\mathbb{P}^l(\mathbb{C})$ is determined by

$$\begin{cases} \omega_{l+1} = b_{l+1,0}\omega_0 + \cdots + b_{l+1,l}\omega_l \\ \ldots \\ \omega_{H_d(V)-1} = b_{H_d(V)-1,0}\omega_0 + \cdots + b_{H_d(V)-1,l}\omega_l \end{cases}$$

Since $F(S) \subset \mathbb{P}^l(\mathbb{C})$, it follows that

$$\begin{cases} \Phi_{l+1}(f) = b_{l+1,0}\Phi_0(f) + \cdots + b_{l+1,l}\Phi_l(f) \\ \ldots \\ \Phi_{H_d(V)-1}(f) = b_{H_d(V)-1,0}\Phi_0(f) + \cdots + b_{H_d(V)-1,l}\Phi_l(f) \end{cases}.$$

Put $B = (b_{ij})_{0\le i\le l, l+1\le j\le H_d(V)-1}$. Then, the above linear equation system can be re-written as follows

$$\begin{pmatrix} \Phi_{l+1}(f) \\ \ldots \\ \Phi_{H_d(V)-1}(f) \end{pmatrix} = B \begin{pmatrix} \Phi_0(f) \\ \ldots \\ \Phi_l(f) \end{pmatrix}.$$

Consider the meromorphic map $F^* = (\Phi_0(f) : \cdots : \Phi_l(f)) : S \to \mathbb{P}^l(\mathbb{C})$. Then, the map F^* is linearly non-degenerate.

For each hypersurface Q of degree d in $\mathbb{C}[x_0, ..., x_n]_d$, take the associated hyperplane $H : a_0\omega_0 + \cdots + a_{H_V(d)-1}\omega_{H_d(V)-1} = 0$ in $\mathbb{P}^{H_V(d)-1}(\mathbb{C})$ of Q with respect to the basis $\{[\Phi_i]\}_{i=0}^{H_V(d)-1}$. We have

$$\begin{aligned} Q(f) = F(H) &= a_0\Phi_0(f) + \cdots + a_{H_V(d)-1}\Phi_{H_V(d)-1}(f) \\ &= (a_0 \cdots a_l)\begin{pmatrix}\Phi_0(f)\\ ...\\ \Phi_l(f)\end{pmatrix} + (a_{l+1} \cdots a_{H_d(V)-1})\begin{pmatrix}\Phi_{l+1}(f)\\ ...\\ \Phi_{H_d(V)-1}(f)\end{pmatrix} \\ &= ((a_0 \cdots a_l) + (a_{l+1} \cdots a_{H_d(V)-1})B)\begin{pmatrix}\Phi_0(f)\\ ...\\ \Phi_l(f)\end{pmatrix}. \end{aligned} \tag{3.11}$$

Put $Q^* = H \cap \mathbb{P}^l(\mathbb{C})$. By a simple calculation, we can see that the equation of Q^* in $\mathbb{P}^l(\mathbb{C})$ is

$$((a_0 \cdots a_l) + (a_{l+1} \cdots a_{H_d(V)-1})B)\begin{pmatrix}\omega_0\\ ...\\ \omega_l\end{pmatrix} = 0.$$

It follows that $Q^*(F^*) = H(F) = Q(f)$ and $Q \cap V = H \cap \mathbb{P}^l(\mathbb{C}) = Q^*$.

Repeating the above way for each hypersurface Q_j, we get the family hyperplanes $\{Q_j^*\}_{j=1}^q$. By the assumption, it is easy to see that

$$\emptyset = \left(\bigcap_{j\in R} Q_j\right) \cap V = \left(\bigcap_{j\in R} Q_j \cap V\right) = \left(\bigcap_{j\in R} Q_j^*\right)$$

for any subset $R \in \{1, \ldots, q\}$ with $|R| = N+1$. Note that $\operatorname{rank}\{Q_j\}_{j\in R} = \operatorname{rank}\{[Q_j]\}_{j\in R} \geq \dim V + 1 = k + 1$. We consider two cases as follows.

Case 1: $l \leq k$.

Then $\operatorname{rank}\{Q_j^*\}_{j\in R} = l+1$. This yields that the hyperplanes $\{Q_j^*\}_{j=1}^q$ are in N-subgeneral position in $\mathbb{P}^l(\mathbb{C})$. Applying Theorem A of L. Jin-M. Ru [13], we have

$$(q - 2N + l - 1)\deg(F^*) \leq \sum_{j=1}^{q}\sum_{P\notin E} \min\{l, \nu_{Q_j^*(F^*)}(P)\} + \frac{l(2N - l + 1)}{2}(2(g - 1) + |E|). \tag{3.12}$$

By Lemma 9, $\deg(F^*) = d\deg(f)$.

We now consider $d > 1$. Since $H_d(V) \geq \binom{k+d}{d} \geq 2k+1$ and $2N \leq \frac{2N-k+1}{k+1} \cdot 2k$, we get $2N - l + 1 \leq \frac{2N-k+1}{k+1} \cdot H_d(V)$ for $l \leq k$. Combining these to (3.12), we obtain

$$\left(q - \frac{(2N-k+1)H_d(V)}{k+1}\right)\deg(f) \leq \frac{1}{d}\sum_{j=1}^{q}\sum_{P\notin E}\min\{H_d(V)-1, \nu_{Q_j(f)}(P)\} + \frac{(2N-k+1)(H_d(V)-1)H_d(V)}{2(k+1)} \cdot \frac{2(g-1)+|E|}{d}. \tag{3.13}$$

We now consider $d = 1$. Since $f(S)$ is not contained in any complex projective subspace of dimension lower than k, it implies that $H_d(V) \geq l+1 \geq k+1$. And hence, $H_d(V) \geq l+1 = k+1$. From (3.12), we also obtain (3.13).

Case 2: $l > k$.

We have $\operatorname{rank}\{Q_j^*\}_{j\in R} = k+1$. By Lemma 7, we can choose a family of hypersurfaces $\{U_i\}_{i=1}^{l-k}$ in $\mathbb{P}^n(\mathbb{C})$ such that for any subset $R \subset \{1,\ldots,q\}$ with $|R| = \operatorname{rank}\{Q_i\}_{i\in R} = k+1$, we get $\operatorname{rank}\{\{Q_i\}_{i\in R} \cup \{U_i\}_{i=1}^{l-k}\} = l+1$. By the assumption, it is easy to see that $\operatorname{rank}\{\{Q_j^*\}_{j\in R}\cup\{U_i^*\}_{i=1}^{l-k}\} = l+1$ for any subset $R \subset \{1,\cdots,q\}$ with $|R| = \operatorname{rank}\{H_j\}_{j\in R} = k+1$.

Consider a point $P \in E$. Since $\{Q_j\}_{j=1}^q$ are in N-subgeneral position, there exist at most N hypersurfaces which can intersect $F^*(S)$ at P. Without loss of generality, we may assume that $f(S)$ intersects Q_j $(1 \leq j \leq N)$ and $f(S)$ does not intersect Q_j with $j > N$. Put $R = \{1,\cdots,N+1\}$ and choose $R^0 \subset R$ with $|R^0| = \operatorname{rank}\{Q_j\}_{j\in R^0} = k+1$ such that R^0 satisfies Lemma 8 (v) with respect to the numbers $\lambda_j = e^{\nu_{Q_j(f)}(P)}$. Then, we have

$$\prod_{j\in R} e^{\omega(j)\nu_{Q_j(f)}(P)} \leq \prod_{j\in R^0} e^{\nu_{Q_j(f)}(P)},$$

where $\omega(j)$ are the Nochka weights associated to the hypersurfaces Q_j $(1 \leq j \leq q)$. This deduces that

$$\sum_{j=1}^{q}\omega(j)\nu_{Q_j(f)}(P) = \sum_{j\in R}\omega(j)\nu_{Q_j(f)}(P) \leq \sum_{j\in R_0}\nu_{Q_j(f)}(P). \tag{3.14}$$

For the linearly independent family of hyperplanes $\{\{Q_j^*\}_{j\in R^0}, \{U_i^*\}_{i=1}^{l-k}\}$ in $\mathbb{P}^l(\mathbb{C})$, take a local parameter z for S at P such that $z(P) = 0$ and write F^* in the form in (2.2). At P the maximum possible value of $\nu_{Q_j(f)}(P) = \nu_{Q_j^*(F^*)}(P)$ $(j \in R^0)$ or $\nu_{U_i^*(F^*)}(P)$ $(1 \leq i \leq l-k)$ is $\delta_l(P)$, and for the unique hyperplane $z_l = 0$. A second hyperplane can intersect $f(S)$

at P with multiplicities at most $\delta_{l-1}(P),\dots$. It follows that

$$\sum_{i=1}^{l-k}\nu_{U_i^*(F^*)}(P)+\sum_{j\in R^0}\nu_{Q_j^*(F^*)}(P)\le\delta_0(P)+\delta_1(P)+\cdots+\delta_l(P).$$

By (2.3), we have

$$\sum_{i=1}^{l-k}\nu_{U_i(f))}(P)+\sum_{j\in R^0}\nu_{Q_j(f)}(P)\le\sum_{i=0}^{l}(l-i)\nu_i(P)+\frac{1}{2}l(l+1).\tag{3.15}$$

Combining (3.15) with (3.14), we get

$$\sum_{j=1}^{q}\omega(j)\nu_{Q_j(f)}(P)+\sum_{i=1}^{l-k}\nu_{U_i(f)}(P)\le\sum_{i=0}^{l}(l-i)\nu_i(P)+\frac{1}{2}l(l+1).$$

Hence, we have

$$\sum_{i=0}^{l}\sum_{P\in E}(l-i)\nu_i(P)\ge\sum_{j=1}^{q}\sum_{P\in E}\omega(j)\nu_{Q_j(f)}(P)-\frac{1}{2}l(l+1)|E|.\tag{3.16}$$

Consider a point $P\notin E$. Then, there exist at most N hypersurfaces which can intersect $F(S)$ at P. We may assume that $F(S)$ intersects Q_j, $j\in A\subset\{1,\cdots,q\}$ with $|A|=N$ and $F(S)$ does not intersect Q_j with $j\notin A$. Take $R_1\subset\{1,\cdots,q\}$ such that $R_1\supset A$ and $|R_1|=N+1$. We choose $R_1^0\subset R_1$ with $|R_1^0|=\operatorname{rank}\{Q_j\}_{j\in R_1^0}=k+1$ such that R_1^0 satisfies Lemma 8 (v) with respect to the numbers $\lambda_j=e^{\max\{\nu_{Q_j(f)}(P)-l,0\}}$ $(1\le j\le q)$. Then, we have

$$\prod_{j\in R_1}e^{\omega(j)\max\{\nu_{Q_j(f)}(P)-l,0\}}\le\prod_{j\in R_1^0}e^{\max\{\nu_{Q_j(f)}(P)-l,0\}}.$$

This yields that

$$\begin{aligned}\sum_{j=1}^{q}\omega(j)\max\{\nu_{Q_j(f)}(P)-l,0\}&=\sum_{j\in R_1}\omega(j)\max\{\nu_{Q_j(f)}(P)-l,0\}\\&\le\sum_{j\in R_1^0}\max\{\nu_{Q_j(f)}(P)-l,0\}.\end{aligned}\tag{3.17}$$

Denoting $k+1$ hypersurfaces Q_j $(j\in R_1^0)$ by $Q_{P,l+1-k},\cdots,Q_{P,l+1}$, we have the linearly independent family of hyperplanes $\{\{Q_{P,j}^*\}_{j=l+1-k}^{l+1},\{U_i^*\}_{i=1}^{l-k}\}$. Without loss of generality, we may assume that

$$\nu_{U_1(f)}(P)\le\cdots\le\nu_{U_{l-k}(f)}(P)\le\nu_{Q_{P,l+1-k}(f)}(P)\le\cdots\le\nu_{Q_{P,l+1}(f)}(P).$$

Then for each $1\le i\le l-k$, we have $\nu_{U_i(f)}(P)\le\delta_{i-1}(P)$ and for each $0\le j\le k$, we have $\nu_{Q_{P,l+1-k+j}(f)}(P)\le\delta_{l-k+j}(P)$. Since $\delta_i\ge i$ for $0\le i\le l$ and $\nu_{Q_{P,l+1-k+j}(f)}(P)\le\delta_{l-k+j}(P)$

for $0 \leq j \leq k$, it is easy to see that

$$\begin{aligned}\sum_{j=0}^{k}[\delta_{l-k+j}(P)-(l-k+j)] &\geq \sum_{j=0}^{k}\max\{\delta_{l-k+j}(P)-l,0\}\\ &\geq \sum_{j=0}^{k}\max\{\nu_{Q_{l-1-k+j}}(P)-l,0\}.\end{aligned} \tag{3.18}$$

Combining (2.3) with (3.17) and (3.18), we get

$$\begin{aligned}\sum_{i=0}^{l}(l-i)\nu_i(P) &= \sum_{i=0}^{l}(\delta_i(P)-i)\\ &\geq \sum_{j=0}^{k}[\delta_{l-k+j}(P)-(l-k+j)]\\ &\geq \sum_{j=1}^{q}\omega(j)\max\{\nu_{Q_j(f)}(P)-l,0\}]\\ &= \sum_{j=1}^{q}\omega(j)[\nu_{Q_j(f)}(P)-\min\{\nu_{Q_j(f)}(P),l\}].\end{aligned}$$

Therefore, we get

$$\sum_{i=0}^{l}\sum_{P\notin E}(l-i)\nu_i(P) \geq \sum_{j=1}^{q}\sum_{P\notin E}\omega(j)\nu_{Q_j(f)}(P) \quad -\sum_{j=1}^{q}\sum_{P\notin E}\omega(j)\min\{\nu_{Q_j(f)}(P),l\}.$$

From (3.16) and by above inequality, we get

$$\begin{aligned}\sum_{i=0}^{l}\sum_{P\in S}(l-i)\nu_i(P) \geq \sum_{j=1}^{q}\sum_{P\in S}\omega(j)\nu_{Q_j(f)}(P) \quad &-\sum_{j=1}^{q}\sum_{P\notin E}\omega(j)\min\{\nu_{Q_j(f)}(P),l\}\\ -\frac{1}{2}l(l+1)|E|.\end{aligned} \tag{3.19}$$

Combining this inequality with (2.4) and (2.5), we get

$$\begin{aligned}(l+1)\deg(F^*)+l(l+1)(g-1) \geq \sum_{j=1}^{q}\sum_{P\in S}\omega(j)\nu_{Q_j(f)}(P) \quad &-\sum_{j=1}^{q}\sum_{P\notin E}\omega(j)\min\{\nu_{Q_j(f)}(P),l\}\\ -\frac{1}{2}l(l+1)|E|.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{j=1}^{q}\sum_{P\in S}\omega(j)\nu_{Q_j(f)}(P)-(l+1)\deg(F^*) \leq &\sum_{j=1}^{q}\sum_{P\notin E}\omega(j)\min\{\nu_{Q_j(f)}(P),l\}\\ &+\frac{1}{2}l(l+1)\cdot\{2(g-1)+|E|\}.\end{aligned}$$

By Lemma 9, this inequality implies that

$$\begin{aligned}\sum_{j=1}^{q}(\omega(j)-(l+1))d\deg(f) &\le \sum_{j=1}^{q}\sum_{P\notin E}\omega(j)\min\{\nu_{Q_j(f)}(P),l\}\\ &\quad+\frac{1}{2}l(l+1)\cdot\{2(g-1)+|E|\}.\end{aligned}\tag{3.20}$$

Using (ii) and (iv) in Lemma 8, we get

$$\begin{aligned}\theta\Big(\sum_{j=1}^{q}\omega(j)-(l+1)\Big) &= \theta\Big(\sum_{j=1}^{q}\omega(j)-k-1\Big)-\theta(l-k)\\ &=(q-2N+k-1)-\theta(l-k)\\ &\ge q-\frac{(2N-k+1)(l+1)}{k+1}.\end{aligned}$$

Combining this inequality with (3.20), we have

$$\begin{aligned}\Big(q-\frac{(2N-k+1)(l+1)}{k+1}\Big)d\deg(f) &\le \sum_{j=1}^{q}\sum_{P\notin E}\theta\omega(j)\min\{\nu_{Q_j(f)}(P),l\}\\ &\quad+\frac{1}{2}\theta l(l+1)\cdot\{2(g-1)+|E|\}.\end{aligned}$$

It follows from (i) and (iv) in Lemma 8 that

$$\begin{aligned}\left(q-\frac{(2N-k+1)(l+1)}{k+1}\right)\deg(f) &\le \frac{1}{d}\sum_{j=1}^{q}\sum_{P\notin E}\min\{\nu_{Q_j(f)}(P),l\}\\ &\quad+\frac{(2N-k+1)l(l+1)}{2(k+1)}\cdot\frac{2(g-1)+|E|}{d}.\end{aligned}$$

Since $l\le H_d(V)-1$, we obtain again the inequality (3.13) from the above inequality. Hence, the theorem is proved in the case where all Q_i have the same degree.

Step 2. We now prove the theorem in the general case where $\deg Q_i=d_i$ $(1\le i\le q)$. We put $T_i=Q_i^{\frac{d}{d_i}}$ $(1\le i\le q)$. It is easy to see that the hypersurfaces $T_1,\cdots,T_q$ have the same degree d and they are still in N-subgeneral position with respect to V. By (3.13) in

Step 1, we have

$$\begin{aligned}\left(q-\frac{(2N-k+1)H_V(d)}{k+1}\right)\deg(f) &\le \sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d}\min\{\nu_{Q_j^{\frac{d}{d_j}}(f)}(P),H_V(d)-1\}\\ &\quad+\frac{(2N-k+1)(H_V(d)-1)H_V(d)}{2(k+1)}\cdot\frac{2(g-1)+|E|}{d}\\ &\le \sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d_j}\min\{\nu_{Q_j(f)}(P),H_V(d)-1\}\\ &\quad+\frac{(2N-k+1)(H_V(d)-1)H_V(d)}{2(k+1)}\cdot\frac{2(g-1)+|E|}{d}.\end{aligned}$$

The proof of the Theorem 1 is completed. □

4. The proof of Theorem 2

Since S is a complete regular minimal surfaces with finite total curvature, S is conformally equivalent to a compact surface $\bar{S}$ punctured at a finite mumber of points $P_1,\dots,P_r$ and the generalized Gauss map G extends holomorphically to $\bar{G}:\bar{S}\to\mathbb{P}^{m-1}(\mathbb{C})$ (see [3]). Let $\{Q_1,\dots,Q_{r_0},Q_{r_0+1},\dots,Q_q\}$ be the set of totally ramified hypersurfaces of $\bar{G}$, located in N-subgeneral position, where $Q_{r_0+1},\dots,Q_q$ are exceptional hypersurfaces. Put

$$E=\{P_1,\dots,P_r\}.$$

By the results of S.S. Chern and R. Osserman (see [3]), we have

$$C(S)=-2\pi\deg(\bar{G})\le 2\pi(\mathcal{X}-r)=2\pi(2-2g-r-r),$$

where $\mathcal{X}$ is the Euler characteristic of $\bar{S}$ and g is genus of $\bar{S}$. Hence,

$$2(g-1)\le\deg(\bar{G})-2r.$$

This implies that

$$2(g-1)+|E|\le\deg(\bar{G})-r<\deg(\bar{G}). \tag{4.21}$$

Applying the second main theorem for the holomorphic curve $\overline{G}$ with $E=\{P_1,\dots,P_r\}$ and by (4.21), we have

$$\begin{aligned}\left(q-\frac{(2N-k+1)H_V(d)}{k+1}\right)\deg(\bar{G}) &< \sum_{j=1}^{r_0}\sum_{P\notin E}\frac{1}{d_j}\min\{\nu_{Q_j(\bar{G})}(P),H_V(d)-1\}\\ &\quad+\sum_{j=r_0+1}^{q}\sum_{P\notin E}\frac{1}{d_j}\min\{\nu_{Q_j(\bar{G})}(P),H_V(d)-1\}\\ &\quad+\frac{(2N-k+1)(H_V(d)-1)H_V(d)}{2(k+1)}\cdot\frac{\deg\bar{G}}{d}\end{aligned} \tag{4.22}$$

Since $Q_{r_0+1},\dots,Q_q$ are exceptional hypersurfaces, for $P\notin E$, $\nu_{Q_j(\bar G)}(P)=0$ for $r_0+1\le j\le q$. On the other hand, for every $P\in S$ and $1\le j\le r_0$, we have

$$\min\{\nu_{Q_j(\bar G)}(P),H_V(d)-1\}\le (H_V(d)-1)\cdot\min\{\nu_{Q_j(\bar G)}(P),1\}\le \frac{H_V(d)-1}{m_j}\nu_{Q_j(\bar G)}(P). \tag{4.23}$$

By Lemma 9, we get

$$\begin{aligned}\sum_{j=1}^{r_0}\sum_{P\notin E}\frac{H_V(d)-1}{m_jd_j}\nu_{Q_j(\bar G)}(P)&\le\sum_{j=1}^{r_0}\sum_{P\in\bar S}\frac{H_V(d)-1}{m_jd_j}\nu_{Q_j(\bar G)}(P)\\ &=\sum_{j=1}^{r_0}\frac{(H_V(d)-1)d_j\deg(\bar G)}{m_jd_j}\\ &=\sum_{j=1}^{r_0}\frac{(H_V(d)-1)\deg(\bar G)}{m_j}.\end{aligned}\tag{4.24}$$

Combining this with (4.22) and (4.23), we have

$$\begin{aligned}\left(q-\frac{(2N-k+1)H_V(d)}{k+1}\right)\deg(\bar G)<&\sum_{j=1}^{r_0}\frac{(H_V(d)-1)\deg(\bar G)}{m_j}\\ &+\frac{(2N-k+1)(H_V(d)-1)H_V(d)}{2(k+1)}\cdot\frac{\deg(\bar G)}{d}.\end{aligned}$$

For all $1\le k\le m-1$, the above inequality implies that

$$\sum_{j=1}^{q}\left(1-\frac{H_V(d)-1}{m_j}\right)\le q-\sum_{j=1}^{r_0}\frac{H_V(d)-1}{m_j}<\frac{(2N-k+1)H_V(d)(H_V(d)+2d-1)}{2(k+1)d}.$$

The proof of Theorem 2 is completed. $\square$

5. The proof of Theorem 4

Replacing Q_j by Q_j^{d/d_j} if necessary, without loss of generallity, we may assume that $d_j=d$ for $1\le j\le q$.

Assume that $G_1\not\equiv G_2$ on S. Consider the equivalence relation on $Q=\{1,\dots,q\}$ given by

$$i\sim j\ \text{ if and only if }\ \frac{Q_i(G_1)}{Q_i(G_2)}-\frac{Q_j(G_1)}{Q_j(G_2)}\equiv 0.$$

Therefore, the set of indexes Q may be split up disjoint equivalence classes $S_1,\dots,S_t$. Since $Q_1,Q_2\dots,Q_q$ are in N-subgeneral position, we have $|S_k|\le N$ for all $1\le k\le t$. Without loss of generality, we can assume that $S_k=\{i_{k-1}+1,i_{k-1}+2,\cdots,i_k,\}$ for

$1 \le k \le t$, where $1 = i_1 < i_2 < \cdots < i_t = q$. It mean that

$$\underbrace{\frac{Q_1(G_1)}{Q_1(G_2)} \equiv \frac{Q_2(G_1)}{Q_2(G_2)} \equiv \cdots \equiv \frac{Q_{i_1}(G_1)}{Q_{i_1}(G_2)}}_{S_1 \text{ group}} \not\equiv \underbrace{\frac{Q_{i_1+1}(G_1)}{Q_{i_1+1}(G_2)} \equiv \frac{Q_{i_1+2}(G_1)}{Q_{i_1+2}(G_2)} \equiv \cdots \equiv \frac{Q_{i_2}(G_1)}{Q_{i_2}(G_2)}}_{S_2 \text{ group}}$$

$$\underbrace{\frac{Q_{i_2+1}(G_1)}{Q_{i_2+1}(G_2)} \equiv \frac{Q_{i_2+2}(G_1)}{Q_{i_2+2}(G_2)} \equiv \cdots \equiv \frac{Q_{i_3}(G_1)}{Q_{i_3}(G_2)}}_{S_3 \text{ group}} \not\equiv \cdots \not\equiv \underbrace{\frac{Q_{i_{t-1}+1}(G_1)}{Q_{i_{t-1}+1}+1}(G_2) \equiv \cdots \equiv \frac{Q_{i_t}(G_1)}{Q_{i_t}(G_2)}}_{S_t \text{ group}}.$$

Define the map $\sigma : \{1, \ldots, q\} \to \{1, \ldots, q\}$ by

$$\sigma(i) = \begin{cases} i + M \text{ if } i + M \le q, \\ i + M - q \text{ if } i + M > q. \end{cases}$$

Then obviously σ is bijective and $|\sigma(i) - i| \ge M$. This implies that i and $\sigma(i)$ belong two distinct elements of $\{S_1, \ldots, S_k\}$. So we have

$$\frac{Q_i(G_1)}{Q_i(G_2)} - \frac{Q_{\sigma(i)}(G_1)}{Q_{\sigma(i)}(G_2)} \not\equiv 0.$$

Put $\chi_i := Q_i(G_1)Q_{\sigma(i)}(G_2) - Q_{\sigma(i)}(G_1)Q_i(G_2)$. Then $\chi \not\equiv 0$. Define

$$\chi := \prod_{j=1}^{q} \chi_i = \prod_{j=1}^{q} (Q_i(G_1)Q_{\sigma(i)}(G_2) - Q_{\sigma(i)}(G_1)Q_i(G_2)) \not\equiv 0.$$

It is easy to see that

$$\sum_{P \in S} \nu_\chi(P) \le dq(\deg(G_1) + \deg(G_2)). \tag{5.25}$$

By the same arguments as in Lemma [4, 8] or [12], we have the following lemma.

Lemma 10. *Under the conditions of Theorem 4, we get*

$$\nu_\chi(P) \ge \left(\frac{q - 2k + 2kM}{2kM}\right) \sum_{j=1}^{q} \left(\min\{\nu_{\overline{G}_1}(P), M\} + \min\{\nu_{\overline{G}_2}(P), M\}\right)$$

for all $P \notin E$.

Then from this Lemma and (5.25), we get

$$\sum_{j=1}^{q} \sum_{P \notin E} \frac{1}{d} \left(\min\{\nu_{\overline{G}_1}(P), M\} + \min\{\nu_{\overline{G}_2}(P), M\}\right) \le \frac{2kMq}{q - 2k + 2kM} (\deg(\overline{G}_1) + \deg(\overline{G}_2)). \tag{5.26}$$

Let $E_{G_1} = \bigcup_{i=1}^{q} G_1^{-1}(Q_j)$. By assumption (i), $E_{G_2} = \bigcup_{i=1}^{q} G_2^{-1}(Q_j)$.

We can assume that $G_1(S)$ is contained in a complex projective subspace V of dimension k, but not in any complex projective subspace of lower dimension k.

Case $d > 1$.

By Theorem 1, we immediately obtain

$$\left(q-\frac{(2N-n+1)(M+1)}{n+1}\right)\deg(G_1)\leq\sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d}\min\{\nu_{Q_j(f)}(P),M\}$$
$$+\frac{(2N-n+1)M(M+1)}{2(n+1)}\cdot\frac{2(g-1)+|E|}{d},$$

where $M=\binom{n+d}{n}-1$ and $E=\{P_1,\ldots,P_r\}$. By using (4.21), we have

$$\begin{aligned}\left(q-\frac{(2N-m+2)(M+1)}{m}\right)\deg(\overline{G}_1)&\leq\sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d}\min\{\nu_{Q_j(\overline{G}_1)}(P),M\}\\&\quad+\frac{(2N-m+2)M(M+1)}{2m}\cdot\frac{2(g-1)+|E|}{d}\\&<\sum_{j=1}^{q}\sum_{P\in S}\frac{1}{d}\min\{M,\nu_{Q_j(\overline{G}_1)}(P)\}\\&\quad+\frac{(2N-m+2)M(M+1)}{2m}\cdot\frac{\deg(\overline{G}_1)}{d}.\end{aligned}$$

So

$$\left(q-\frac{(2N-m+2)(M+1)(M+2d)}{2dm}\right)\deg(\overline{G}_1)\leq\sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d}\min\{\nu_{Q_j(\overline{G}_1)}(P),M\}.$$

Similarly, we have

$$\left(q-\frac{(2N-m+2)(M+1)(M+2d)}{2dm}\right)\deg(\overline{G}_2)\leq\sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d}\min\{\nu_{Q_j(\overline{G}_2)}(P),M\}.$$

Combining two above equalities with Lemma 10 and (5.26), we get

$$q-\frac{(2N-m+2)(M+1)(M+2d)}{2dm}<\frac{2kMq}{q-2k+2kM}.\tag{5.27}$$

This is a contradiction.

Case $d=1$.

We have $H_V(d)=k+1$. Applying Theorem 1 for the holomorphic curve $\overline{G}_1$ and using the above argument, we get

$$\left(q-\frac{(2N-k+1)(k+2)}{2}\right)\deg(\overline{G}_1)\leq\sum_{j=1}^{q}\sum_{P\notin E}\frac{1}{d}\min\{\nu_{Q_j(\overline{G}_1)}(P),k\}.$$

It follows from $(2N-k+1)(k+2)\leq(2N-m+2)(m+1)$ for all $0<k\leq m-1$ that

$$\left(q-\frac{(2N-m+2)(m+1)}{2}\right)\deg(\overline{G}_1)\leq\sum_{j=1}^{q}\sum_{P\notin E}\min\{\nu_{Q_j(\overline{G}_1)}(P),m-1\}$$

and also

$$\left(q - \frac{(2N-m+2)(m+1)}{2}\right)\deg(\overline{G}_2) \leq \sum_{j=1}^{q}\sum_{P\notin E}\min\{\nu_{Q_j(\overline{G}_2)}(P), m-1\}$$

These inequalities will lead us to the inequality (5.27) for the case $d = 1$. So the proof of Theorem 4 is completed. □

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