SINGULAR DIRECTIONS OF BRODY CURVES

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ABSTRACT. In this paper, we establish the existence of singular directions of Brody curves into algebraic varieties. Moreover, we also give a version of "angular domain" type for the results of B. F. P. Da Costa and J. Duval [2] for Brody curves into a complex projective variety in $P^N(\mathbb{C})$ intersecting hypersurfaces.

1. INTRODUCTION

The problem of singular directions of meromorphic functions on $\mathbb{C}$ has a long history, dating back to G. Julia [6], H. Milloux [7], G. Valiron [10].

In 1919, G. Julia [6] proved the following famous theorem.

Theorem A. *Let $f(z)$ be a transcendental entire function on $\mathbb{C}$. Then there exists a ray $J(\theta) = \{z : \arg z = \theta\}$ such that for any ε with $0 < \varepsilon < \pi$ and for all a with at most one exception on $\mathbb{C}$,*

$$\lim_{r\to\infty} n(r, \Omega(\theta, \varepsilon), f = a) = \infty,$$

where $\Omega(\theta, \varepsilon) = \{z : \ \theta - \varepsilon < \arg z < \theta + \varepsilon\}$ and $n(r, \Omega(\theta, \varepsilon), f = a)$ is the number of solutions of $f(z) = a$ in $\Omega(\theta, \varepsilon) \cap \{|z| < r\}$ counting multiplicities.

H. Milloux [7] generalized Theorem A to meromorphic functions on $\mathbb{C}$.

Theorem B. *Let $f(z)$ be a transcendental meromorphic function on $\mathbb{C}$ with an asymptotic value in $\mathbb{P}^1(\mathbb{C})$. Then there exists a ray $J(\theta) = \{z : \arg z = \theta\}$ such that for any ε with $0 < \varepsilon < \pi$ and for all a with at most two exceptions on $\mathbb{P}^1(\mathbb{C})$,*

$$\lim_{r\to\infty} n(r, \Omega(\theta, \varepsilon), f = a) = \infty,$$

where $\Omega(\theta, \varepsilon) = \{z : \ \theta - \varepsilon < \arg z < \theta + \varepsilon\}$ and $n(r, \Omega(\theta, \varepsilon), f = a)$ is the number of solutions of $f(z) = a$ in $\Omega(\theta, \varepsilon) \cap \{|z| < r\}$ counting multiplicities.

Here $\alpha \in \mathbb{P}^1(\mathbb{C})$ is called an asymptotic value for a meromorphic function $f(z)$ on $\mathbb{C}$ at a point a if there exists a continuous path $L : z = z(t), 0 \le t < 1$ such that $\lim_{t\to 1-0} z(t) = a$ and $\lim_{t\to 1-0} f(z(t)) = \alpha$. Since a transcendental entire function always has the asymptotic value ∞ in $\mathbb{P}^1(\mathbb{C})$, it implies that Theorem B is a generalization of

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Theorem A. Without the assumption on the existence of an asymptotic value in $\mathbb{P}^1(\mathbb{C})$, A. Ostrowski [8] gave a counterexample to Theorem B.

The ray $J(\theta)$ in Theorem A or Theorem B is called to be *a Julia direction* of f.

Theorem A is a refinement of the Picard theorem for transcendental entire functions. In order to get a similar refinement for the Borel theorem, a more refined notion of Borel directions was introduced by G. Valiron in 1928. Namely, a ray $J(\theta) = \{z : \arg z = \theta\}$ is called *a Borel direction* of order ρ for f if for every ε with $0 < \varepsilon < \pi$,

$$\limsup_{r\to\infty} \frac{n(r, \Omega(\theta, \varepsilon), f = a)}{\log r} \geq \rho,$$

for all a on $\mathbb{P}^1(\mathbb{C})$ with at most two exceptions. It is well known that f has at least one Borel direction in the case where the growth ρ of Nevanlinna characteristic $T(r, f)$ satisfying $0 < \rho < \infty$ (see G. Valiron [10]).

Much attention has been given to the study of singular directions in general context for non-constant holomorphic curves on $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, and several remarkable results on this topic have obtained (see A. Eremenko [3], Zh-H. Tu [9], J. Zheng [11],...).

For instance, in 1996, Zh-H. Tu [9] defined that a ray $J(\theta) = \{z : \arg z = \theta\}$ is called a Julia direction for a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ if in any open sector with vertex $z = 0$ containing $J(\theta)$, f misses at most $2n$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. He showed that if $f(z)$ is a transcendental entire holomorphic curve with an asymptotic value in $\mathbb{P}^n(\mathbb{C})$, then there exists a Julia direction for $f(z)$. Here, we say that a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ has an asymptotic value in $\mathbb{P}^n(\mathbb{C})$ if there exist a continuous path $z = z(t)$ $(0 \leq t < 1)$ satisfying $\lim_{t\to 1} z(t) = \infty$ and a reduced representation $\widetilde{f}(z) = (f_0(z), f_1(z), \cdots, f_n(z))$ such that $\lim_{t\to 1} f_i(z(t)) = a_i$ $(0 \leq i \leq n)$ with the property that $(a_0, a_1, \cdots, a_n)$ induces a point in $\mathbb{P}^n(\mathbb{C})$.

We now formulate the recent result of J. Zheng [11] which is the best result available at present. First of all, recall the following.

Let $f : \mathbb{C} \to P^N(\mathbb{C})$ be a holomorphic curve. Let $\widetilde{f} = (f_0, ..., f_N)$ be a reduced representation of f, where $f_0, ..., f_N$ are entire functions on $\mathbb{C}$ and have no common zeros. Put

$$\nu(z) = \max\{\log|f_0(z)|, \cdots, \log|f_N(z)|\}, \ z \in \mathbb{C}.$$

The Nevanlinna-Cartan characteristic function $T(r, f)$ is defined by

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \nu(re^{i\theta})d\theta - \nu(0).$$

For $0 \leq \theta \leq 2\pi$, by $\Omega(\theta, \epsilon)$ we denote by the angular domain

$$\Omega(\theta, \epsilon) = \{z : \theta - \epsilon < \arg z < \theta + \epsilon\}$$

and by $\overline{\Omega}(\theta, \epsilon)$ its closure. Sometimes, without occurrence of any confusion in the context, we write simply Ω instead of $\Omega(\theta, \epsilon)$.

Let H be a hyperplane in $P^N(\mathbb{C})$ given by

$$H := \{[z_0 : z_1 : \cdots : z_N] \in P^N(\mathbb{C}) : a_0 z_0 + a_1 z_1 + \cdots + a_N z_N = 0\}.$$

Put

$$H(f)(z) = a_0 f_0(z) + a_1 f_1(z) + \cdots + a_N f_N(z).$$

Denote by $n_{\Omega(\theta,\epsilon)}(r, H, f)$ the number of zeros of $H(f)$ in the domain $\{|z| < r\} \cap \Omega(\theta, \epsilon)$, counting multiplicity. We also define the counting function

$$N_{\Omega(\theta,\epsilon)}(r, H, f) = \int_0^r \frac{n_{\Omega(\theta,\epsilon)}(t, H, f) - n_{\Omega(\theta,\epsilon)}(0, H, f)}{t} dt + n_{\Omega(\theta,\epsilon)}(0, H, f) \log r.$$

Definition 1.1. (see [11]) A ray $J(\theta) = \{z : \arg z = \theta\}$ is a T-direction for a holomorphic curve $f : \mathbb{C} \to P^N(\mathbb{C})$ if for any ϵ $(0 < \epsilon < \pi)$, we have

$$\limsup_{r\to\infty} \frac{N_{\Omega(\theta,\epsilon)}(r, H, f)}{T(r, f)} = 0,$$

for at most $2N$ hyperplanes H in general position in $P^N(\mathbb{C})$.

Theorem C. (see [11]) *Let $f : \mathbb{C} \to P^N(\mathbb{C})$ be a holomorphic curve such that*

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^2} = +\infty.$$

Then the holomorphic curve f has at least one T-direction.

Our main aim in this paper is to study singular directions for Brody holomorphic curves into a complex projective variety V in $\mathbb{P}^N(\mathbb{C})$ sharing hypersurfaces in general position in V. To state our results, we recall the following.

Let $f : \mathbb{C} \to P^N(\mathbb{C})$ be a holomorphic curve. Let $\widetilde{f} = (f_0, ..., f_N)$ be a reduced representation of f, where $f_0, ..., f_N$ are entire functions on $\mathbb{C}$ and have no common zeros. Put

$$\|f\|^2 = \sum_{j=0}^{N} |f_j|^2.$$

The Fubini-Study derivative $\|f'\|$ measures the length distortion from the Euclidean metric in $\mathbb{C}$ to the Fubini-Study metric in $P^N(\mathbb{C})$. The explicit expression is

$$\|f'\|^2 = \|f\|^{-4} \sum_{i<j} |f_i' f_j - f_j' f_i|^2.$$

A holomorphic curve is called a Brody curve if its Fubini-Study derivative is bounded.

It is well-known that the Nevanlinna-Cartan characteristic function $T(r, f)$ is also given by

$$T(r,f) = \int_0^r \frac{dt}{t}\left(\frac{1}{\pi}\int_{|z|\le t} \|f'\|^2(z)dm(z)\right),$$

where dm is the area element in $\mathbb{C}$.

Let D be a hypersurface in $P^N(\mathbb{C})$ of degree d. Let Q be the homogeneous polynomial (form) of degree d defining D. Denote by $n_{\Omega(\theta,\epsilon)}(r, D, f)$ the number of zeros of $Q \circ \widetilde{f}$ in the domain $\{|z| < r\} \cap \Omega(\theta, \epsilon)$, counting multiplicity, where $0 \le \theta \le 2\pi$ and $\Omega(\theta, \epsilon) = \{z : \theta - \epsilon < \arg z < \theta + \epsilon\}$. We also define the counting function

$$N_{\Omega(\theta,\epsilon)}(r,D,f) = \int_0^r \frac{n_{\Omega(\theta,\epsilon)}(t,D,f) - n_{\Omega(\theta,\epsilon)}(0,D,f)}{t}dt + n_{\Omega(\theta,\epsilon)}(0,D,f)\log r.$$

Now we give the following definition of T_m-direction for a holomorphic curve.

Definition 1.2. Let m be a natural number. A ray $J(\theta)$ is said to be a T_m-direction for a holomorphic curve $f : \mathbb{C} \to P^N(\mathbb{C})$ if for any $\epsilon(0 < \epsilon < \pi)$, we have

$$\limsup_{r\to\infty} \frac{N_{\Omega(\theta,\epsilon)}(r,D,f)}{T(r,f)} = 0,$$

for at most m hypersurfaces D in general position in $P^N(\mathbb{C})$.

It is clear that $J(\theta)$ is a T-direction if $J(\theta)$ is a T_{2N}-direction; and if $J(\theta)$ is a T_m-direction, then $J(\theta)$ is also a T_k-direction for all $k \ge m$. Moreover, a T-direction must be a Julia direction.

Definition 1.3. (see [2]) Let $f : \mathbb{C} \to P^N(\mathbb{C})$ be a Brody curve. We say that f has a positive energy if

$$\limsup_{r\to\infty} \frac{T(r,f)}{r^2} > 0.$$

Definition 1.4. Let V be a complex projective variety in $P^N(\mathbb{C})$ of dimension $n \ge 1$.
(i) Let $D_1, ..., D_q$ be hypersurfaces in $P^N(\mathbb{C})$, where $q > n$. The hypersurfaces $D_1, ..., D_q$ are said to be in general position in V if for every subset $\{i_0, ..., i_n\} \subset \{1, ..., q\}$, we have

$$V \cap \mathrm{supp} D_{i_0} \cap ... \cap \mathrm{supp} D_{i_n} = \varnothing,$$

where supp$D$ means the support of the hypersurface D.
(ii) Let $D_1, ..., D_k (k \le n)$ be hypersurfaces in $P^N(\mathbb{C})$. The hypersurfaces $D_1, ..., D_k$ are said to be in general position in V if $\dim\{V \cap \mathrm{supp} D_1 \cap ... \cap \mathrm{supp} D_k\} = n - k$.

Definition 1.5. Let V be a complex projective variety in $P^N(\mathbb{C})$ of dimension $n \geq 1$. Let $f : \mathbb{C} \to V$ be a Brody curve. A ray $J(\theta)$ is called a $\overline{T}$-direction for f if for any ϵ $(0 < \epsilon < \pi)$, we have

$$\limsup_{r\to\infty} \frac{N_{\Omega(\theta,\epsilon)}(r, D, f)}{T(r, f)} = 0,$$

for at most $n-1$ hypersurfaces D located in general position in V.

It is clear that if $J(\theta)$ is a $\overline{T}$-direction, then $J(\theta)$ is also a T_{N-1}-direction.

We now prove the main result of this paper.

Theorem 1.6. *Let V be a complex projective variety in $P^N(\mathbb{C})$ of dimension $n \geq 1$. Let $f : \mathbb{C} \to V$ be a Brody curve. If f has a positive energy, then f has at least one $\overline{T}$-direction. In particular, if $f : \mathbb{C} \to P^N(\mathbb{C})$ is a Brody curve having a positive energy, then f has at least one T_{N-1}-direction.*

In the second part of this paper, we give a version of "angular domain" type for the results of B. F. P. Da Costa and J. Duval [2] for Brody curves into a complex projective variety in $P^N(\mathbb{C})$ intersecting hypersurfaces. Recall now the following.

Let $f : \mathbb{C} \to P^N(\mathbb{C})$ be a holomorphic curve. Let $\widetilde{f} = (f_0, ..., f_N)$ be a reduced representation of f. Consider the subharmonic function

$$\nu(z) = \max\{\log|f_0(z)|, \cdots, \log|f_N(z)|\}, \; z \in \mathbb{C}.$$

Denote by $\triangle\nu$ the Riesz measure of ν, i.e. $\triangle\nu := \dfrac{1}{2\pi}\mathcal{D}\nu$, where $\mathcal{D}$ is the Laplacian.

Denote $\mathbb{D} = \{z : |z| \leq 1\}$ and $r\mathbb{D} = \{z : |z| \leq r\}$. It is well-known that

$$T(r, f) = \int_1^r \triangle\nu(r\mathbb{D})dr + O(1).$$

We also define

$$T(r, \Omega(\theta, \epsilon), f) = \int_1^r \triangle\nu(\Omega(\theta, \epsilon) \cap r\mathbb{D})dr + O(1)$$

for an angular domain $\Omega(\theta, \epsilon) \subset \mathbb{C}$. It is well-known that

$$T(r, \Omega(\theta, \epsilon), f) = \int_0^r \frac{dt}{t}\left(\frac{1}{\pi}\int_{\{|z|\leq t\}\cap\Omega(\theta,\epsilon)} \|f'\|^2(z)dm(z)\right).$$

We will prove the following.

Theorem 1.7. *Let V be a complex projective variety in $P^N(\mathbb{C})$ of dimension $n \geq 1$. Let $f : \mathbb{C} \to V$ be a Brody curve. Let $D_1, ..., D_q$ be q hypersurfaces of degree $d_1, ..., d_q$ in*

$P^N(\mathbb{C})$ located in general position in V. Let $\Omega(\theta,\epsilon)$ be any angular domain in $\mathbb{C}$. Then we have

$$(q-n+1)T(r,\Omega(\theta,\epsilon),f) \le \sum_{1\le i\le q} \frac{1}{d_i} N_{\Omega(\theta,\epsilon)}(r,D_i,f) + o(r^2).$$

2. Some lemmas

First of all, we recall some definitions concerning subharmonic functions.

Let u be a subharmonic function in a domain $\Omega \subset \mathbb{C}$. Let $\triangle u$ be the Riesz measure of u. The term "quasi-everywhere" means "everywhere except for a set of capacity 0". A function ν defined quasi-everywhere in Ω is called δ-subharmonic if ν can be represented as the difference of two functions that are subharmonic in Ω. The Riesz charge of ν is the difference of the Riesz measures.

We need some following lemmas

Lemma 2.1. *(Grishin [4]). If $\nu \ge 0$ is δ-subharmonic in Ω and $\nu(z) = 0$ on some Borel set X, then the restriction to X of the Riesz charge of ν is a nonnegative measure.*

Lemma 2.2. *(Da Costa-Duval [2]). If $\nu \ge 0$ is δ-subharmonic in Ω such that $\triangle\nu$ is L^∞ on $\nu = 0$, then $\triangle\nu = 0$ on $\{\nu = 0\}$.*

Lemma 2.3. *Let q, n be positive integers. Let ν_i and ν be q subharmonic functions in Ω. Assume that $\triangle\nu$ is L^∞ on Ω and $\nu = \max_I \nu_i$ for any subset $I \subset \{1, ..., q\}, |I| = n$. Then $\sum \nu_i - (q-n+1)\nu$ is subharmonic in Ω.*

Proof. Without loss of generality we can assume that $q \ge n$. Put $\omega_i = \nu - \nu_i$. Then ω_i is nonnegative δ-subharmonic in Ω. By Lemma 2.1, we have $\triangle\omega_i = \triangle v - \triangle v_i \ge 0$ on $\{\omega_i = 0\}$. Since $\triangle\nu$ is L^∞ on Ω, it follows that $\triangle\nu_i$ is L^∞ on $\{\nu = \nu_i\}$.

Since $\nu = \max_I \nu_i$ for any subset $I \subset \{1, ..., q\}, |I| = n$, it implies that, for each $z \in \Omega$, there exist at most $n-1$ indeces i such that $\nu_i(z) < \nu(z)$. In the other words, there are at least $q-n+1$ indeces j such that $\nu_j(z) = \nu(z)$. For each subset $J \subset \{1, ..., q\}, |J| = q-n+1$, put $\Omega_J = \{z \in \Omega : \nu_j(z) = \nu(z), \forall j \in J\}$. Then Ω_J is a Borel subset of Ω. Repeating the above argument, we have $\Omega = \cup\Omega_J$.

By using Lemma 2.2, we obtain

$$\triangle\left[\sum \nu_i - (q-n+1)\nu\right] = \sum_{i\in J} \triangle(\nu_i - \nu) + \sum_{i\notin J} \triangle\nu_i$$

is a nonnegative measure in Ω_J. Therefore $\sum \nu_i - (q-n+1)\nu$ is subharmonic in Ω. This completes the proof of Lemma 2.3. □

Let $V \subset P^N(\mathbb{C})$ be a complex projective variety of dimension $n \geq 1$. Let $f : \mathbb{C} \to V$ be a holomorphic curve. Let $\widetilde{f} = (f_0, ..., f_N)$ be a reduced representation of f, where $f_0, ..., f_N$ are entire functions on $\mathbb{C}$ and have no common zeros. Put $u = \log \|\widetilde{f}\|$.

Let $D_1, ..., D_q$ be q hypersurfaces in $P^N(\mathbb{C})$ of degree d_j, located in general position in V. Let Q_j $(1 \leq j \leq q)$ be the homogeneous polynomials in $[X_0, ..., X_N]$ of degree d_j defining D_j. Replacing Q_j by Q^{d/d_j} if necessary, where d is the l.c.m of $d_j's$, we can assume that $Q_1, ..., Q_q$ have the same degree d. For every $1 \leq i \leq q$, define

$$u_i = \frac{1}{d} \log |Q_i \circ \widetilde{f}| = \frac{1}{d} \log |Q_i(f_0, ..., f_N)|.$$

Lemma 2.4. *Let u_i, u be $q+1$ subharmonic functions defined as above. Then* $\max_{i\in I} u_i = u + O(1)$ *for all subset $I \subset \{1, ..., q\}, |I| = n + 1$.*

Proof. Let $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to P^N(\mathbb{C})$ be the standard projection. Let $I \subset \{1, ..., q\}, |I| = n+1$. The set $K = \{z \in \pi^{-1}(V) : \|z\| = 1\}$ is a compact subset of $\mathbb{C}^{N+1}$. Since $D_1, ..., D_q$ are hypersurfaces located in general position in V and K is compact, it follows that there exist two positive constants C_1 and C_2 such that

$$C_1 \leq \max_{i\in I} |Q_i(z)|^{1/d} \leq C_2.$$

Moreover, since Q_j are homogeneous polynomials with the same degree d, we obtain

$$C_1 \|\widetilde{f}(z)\| \leq \max_{i\in I} |Q_i \circ \widetilde{f}(z)|^{1/d} \leq C_2 \|\widetilde{f}(z)\|, \forall z \in \mathbb{C}.$$

It implies that $\max_{i\in I} u_i = u + O(1)$. This completes the proof of Lemma 2.4. □

3. Proof of Theorem 1.6

Since the Brody curve f has a positive energy, there exists a sequence $r_k \to \infty$ such that

$$\limsup_{k\to\infty} \frac{T(r_k, f)}{r_k^2} > 0. \ (1)$$

We now prove that there exists a ray $J(\theta)$ such that

$$\limsup_{k\to\infty} \frac{T(r_k, \Omega(\theta, \epsilon), f)}{T(r_k, f)} > 0, \forall \epsilon > 0, \ (2)$$

where $\Omega(\theta, \epsilon) = \{z : \theta - \epsilon < \arg z < \theta + \epsilon\}$.

Indeed, suppose that this fact does not hold. This means that for each ray $J(\theta)$, there exists $\epsilon(\theta)$ such that

$$\limsup_{k\to\infty} \frac{T(r_k, \Omega(\theta, \epsilon(\theta)), f)}{T(r_k, f)} = 0.$$

By the compactness of $[0, 2\pi]$, it follows that there exists a finite family of angles $\theta_1, ..., \theta_m$ such that

$$\mathbb{C} \subset \cup_{1\leq i\leq m} \Omega(\theta_i, \epsilon(\theta_i)).$$

Therefore

$$1 \leq \limsup_{k\to\infty} \frac{\sum T(r_k, \Omega(\theta_i, \epsilon(\theta_i)), f)}{T(r_k, f)} \leq \sum \limsup_{k\to\infty} \frac{\sum T(r_k, \Omega(\theta_i, \epsilon(\theta_i)), f)}{T(r_k, f)} = 0.$$

This is a contradiction. Hence (2) is proved. Consequently, by combining (1) and (2), there exists a ray $J(\theta)$ such that

$$\limsup_{k\to\infty} \frac{T(r_k, \Omega(\theta, \epsilon), f)}{r_k^2} > 0. \quad (3)$$

Since θ is constant, we set $\Omega_\epsilon = \Omega(\theta, \epsilon)$.

Let $D_0, ..., D_{n-1}$ be any n hypersurfaces in $P^N(\mathbb{C})$ of degree d_j, located in general position in V. Let D_n be a hypersurface in $P^N(\mathbb{C})$ of degree d_n such that $D_0, ..., D_n$ located in general position in V. Let Q_j $(0 \leq j \leq n)$ be the homogeneous polynomials in $[X_0, .., X_N]$ of degree d_j defining D_j.

The following lemma is an immediate corollary of Theorem 1.7

Lemma 3.1. *Let V be a complex projective variety in $P^N(\mathbb{C})$ of dimension $n \geq 1$. Let $f : \mathbb{C} \to V$ be a Brody curve. Let $D_0, ..., D_{n-1}$ be any n hypersurfaces in $P^N(\mathbb{C})$ of degree d_j, located in general position in V. Then, for each $\epsilon > 0$, we have*

$$T(r, \Omega_\epsilon, f) \leq \sum_{0\leq i\leq n-1} \frac{1}{d_i} N_{\Omega_\epsilon}(r, D_i, f) + o(r^2).$$

From (3) and Lemma 3.1, it implies that

$$0 < \limsup_{k\to\infty} \sum_{0\leq i\leq n-1} \frac{N_{\Omega_\epsilon}(r_k, D_i, f)}{d_i T(r_k, \Omega_\epsilon, f)}.$$

So $J(\theta)$ is a $\overline{T}$-direction.

This completes the proof of Theorem 1.6.

4. Proof of Theorem 1.7

Let $\widetilde{f} = (f_0, ..., f_N)$ be a reduced representation of the Brody curve f. Put $u = \log \|\widetilde{f}\|$. Let $\pi_1 : \mathbb{C}^{N+1} \setminus \{0\} \to P^N(\mathbb{C})$ and $\pi_2 : \mathbb{C}^{n+1} \setminus \{0\} \to P^n(\mathbb{C})$ be the standard projections. Without loss of generality, we also assume that $d_j = d,\ 1 \leq j \leq q$.

Lemma 4.1. *Let $\overline{D}_0, ..., \overline{D}_{n-1}$ be any n hypersurfaces in $P^N(\mathbb{C})$ of degree d, located in general position in V. Let $\overline{D}_n$ be a hypersurface in $P^N(\mathbb{C})$ of degree d such that $\overline{D}_0, ..., \overline{D}_n$ located in general position in V. Let $\overline{Q}_j$ $(0 \leq j \leq n)$ be homogeneous polynomials in $\mathbb{C}[X_0, .., X_N]$ of degree d defining $\overline{D}_j$. For every $0 \leq i \leq n$, define*

$$\overline{u}_i = \frac{1}{d} \log |\overline{Q}_i \circ \widetilde{f}| = \frac{1}{d} \log |\overline{Q}_i(f_0, ..., f_N)|.$$

Let r_k be a sequence of positive real numbers which converges to $+\infty$. Let h_k be the smallest harmonic majorant of u in the disk $2\mathbb{D}_{r_k}$ and $\lambda_k : \mathbb{C} \to \mathbb{C}$ be the map given by $\lambda_k(z) = 2r_k z, \forall z \in \mathbb{C}$. Then, there exist subsequences of the sequences $\frac{1}{r_k^2}(\overline{u}_i - h_k) \circ \lambda_k$ and $\frac{1}{r_k^2}(u - h_k) \circ \lambda_k$ which converge to subharmonic functions $\overline{\nu}_i$ and ν respectively, and $\max_{0 \le i \le n-1} \overline{\nu}_i = \nu$.

Note that the functions u and ν depend only on f.

Proof. Denote $\widetilde{Q} = (\overline{Q}_0, ..., \overline{Q}_n) : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}$. Let $Q : P^N(\mathbb{C}) \to P^n(\mathbb{C})$ be a morphism defined by $\pi_2 \circ \widetilde{Q} = Q \circ \pi_1$. Note that Q is a meromorphic mapping, but Q is a holomorphic mapping on V. Denote $g = Q \circ f : \mathbb{C} \to P^n(\mathbb{C})$ and $\widetilde{g} = \widetilde{Q} \circ \widetilde{f} : \mathbb{C} \to \mathbb{C}^{n+1}$. It is clear that $g = Q \circ f = \pi_2 \circ \widetilde{g}$ is a holomorphic curve and $\widetilde{g} = (g_0, g_1, ..., g_n) : \mathbb{C} \to \mathbb{C}^{n+1}$ is a reduced representation of the holomorphic curve g. Since f is a Brody curve and Q is a holomorphic mapping on the compact complex projective variety V and $g = Q \circ f$, it follows that $g : \mathbb{C} \to P^n(\mathbb{C})$ is also a Brody curve.

By Lemma 2.4, we have $u = \frac{1}{d} \log \|\widetilde{g}\| + O(1)$ and $u_i = \frac{1}{d} \log |g_i|$.

Consider the sequence of subharmonic functions ≤ 0 on $2\mathbb{D}$ defined by $\frac{1}{r_k^2}(u - h_k) \circ \lambda_k$. Note that $h_k(0) = \oint_{2r_k} u = O(r_k^2)$ because f is a Brody curve. Hence $\frac{1}{r_k^2}(u(0) - h_k(0)) > -\infty$. So the sequence $\frac{1}{r_k^2}(u - h_k) \circ \lambda_k$ does not converge locally uniformly to $-\infty$. By using a result of Hormander [5, Theorem 4.1.9], it follows that there exists a subsequence of the sequence $\frac{1}{r_k^2}(u - h_k) \circ \lambda_k$ which converges to a subharmonic function ν. By the same argument, there exists a subsequences of the sequence $\frac{1}{r_k^2}(\overline{u}_i - h_k) \circ \lambda_k$ which converges to a subharmonic functions $\overline{\nu}_i$. By applying a result of Da Costa [1, Lemma 22] to the Brody curve g, we have $\max_{0 \le i \le n-1} \overline{\nu}_i = \nu$. □

Lemma 4.2. *The Riesz measure $\triangle\nu$ is L^∞ on $2\mathbb{D}$.*

Proof. Take any $D(z, \delta) \subset 2\mathbb{D}$. Then we have

$$\triangle\nu(D(z,\delta)) \le \liminf_{r\to\infty} \frac{\triangle u(D(rz, r\delta))}{\rho^2 r^2} \le \liminf_{r\to\infty} \frac{(r\delta)^2 O(1)}{r^2} = O(1)\delta^2,$$

where $O(1)$ depends only on the absolute value of the Fubini-Study derivative $\|f'\|$. Hence $\triangle\nu$ is L^∞ on $2\mathbb{D}$. This completes the proof of Lemma 4.2. □

We now come back to the proof of Theorem 1.7.

Without loss of generality, we also assume that $d_j = d,\ 1 \le j \le q$.

For every $1 \le i \le q$, define

$$u_i = \frac{1}{d} \log |Q_i \circ \widetilde{f}| = \frac{1}{d} \log |Q_i(f_0, ..., f_N)|.$$

It is easy to see that the conclusion of Theorem 1.7 can be written in the following

$$(q-n+1)T(r,\Omega(\theta,\epsilon),f) \le \sum_{1\le i\le q} \frac{1}{d} N_{\Omega(\theta,\epsilon)}(r,D_i,f) + o(r^2). \quad (4)$$

By the argument as in Lemma 4.1, for each $1 \le i \le q$, we also define ν_i corresponding to u_i and D_i, i.e. there exists a subsequence of the sequence $\frac{1}{r_k^2}(u_i - h_k)\circ\lambda_k$ which converges to a subharmonic function ν_i. Also by Lemma 4.1, we obtain $\max_{i\in I}\nu_i = \nu$ for all subset $I \subset \{1,...,q\}, |I| = n$.

We now prove that

$$\limsup \frac{1}{r^2}\left[(q-n+1)\triangle u(\mathbb{D}_r^\epsilon) - \sum \triangle u_i(\mathbb{D}_r^\epsilon)\right] \le 0, \text{ where } \mathbb{D}_r^\epsilon = \Omega_\epsilon \cap \{|z|<r\}. \quad (5)$$

Indeed, assume that $r_k \to +\infty$ such that $\frac{1}{r_k^2}\left[(q-n+1)\triangle u(\mathbb{D}_{r_k}^\epsilon) - \sum \triangle u_i(\mathbb{D}_{r_k}^\epsilon)\right]$ converges. Combining Lemmas 2.2, 2.3, 4.1 and 4.2, we obtain that $\sum\limits_{1\le i\le q} \nu_i - (q-n+1)\nu$ is subharmonic in $2\mathbb{D}$. So $\sum\limits_{1\le i\le q} \nu_i - (q-n+1)\nu$ is also subharmonic in $2\mathbb{D}^\epsilon(= 2\mathbb{D}\cap\Omega_\epsilon)$. Fix $\delta, \epsilon > 0$ with $0 < \delta < \epsilon^2$ and a nonnegative smooth function χ in $\mathbb{D}^\epsilon$ such that $\chi = 1$ in $\mathbb{D}_\delta^\epsilon = (1-\delta)\mathbb{D}\cap\mathbb{D}^{\epsilon-\delta}\cap\{|z|>\delta\}$ and $\mathrm{supp}\chi$ is a compact subset in $\mathbb{D}^\epsilon$. Since $\sum\limits_{1\le i\le q} \nu_i - (q-n+1)\nu$ is subharmonic, it implies that for k large enough, we get

$$(q-n+1)\int \chi\triangle(u\circ\lambda_k) \le \sum\int \chi\triangle(u_i\circ\lambda_k) + \epsilon r_k^2, \text{ where integration is taken on } \mathbb{D}^\epsilon.$$

Hence $(q-n+1)\triangle\nu(r_k\mathbb{D}_\delta^\epsilon) \le \frac{1}{d}\sum n_{r_k\mathbb{D}^\epsilon}(r_k, D_i, f) + \epsilon r_k^2$, where $r_k\mathbb{D}^\epsilon = r_k\mathbb{D}\cap\Omega_\epsilon$ and $n_{r_k\mathbb{D}^\epsilon}(r_k,D_i,f)$ is the number of zeros of $Q_i(\widetilde{f})$ in the domain $r_k\mathbb{D}^\epsilon$, counting multiplicity. Moreover, since the holomorphic curve f is Brody, we deduce that $\triangle\nu(r_k\mathbb{D}^\epsilon) \le \triangle\nu(r_k\mathbb{D}_\delta^\epsilon) + \delta O(r_k^2)$. Thus, we have

$$\lim_{k\to+\infty}\frac{1}{r_k^2}\left[(q-n+1)\triangle u(\mathbb{D}_{r_k}^\epsilon) - \sum\triangle u_i(\mathbb{D}_{r_k}^\epsilon)\right] \le 0.$$

This yields that the assertion (5) holds. Therefore, we get

$$(q-n+1)\triangle\nu(r_k\mathbb{D}^\epsilon) \le \frac{1}{d}\sum n_{r_k\mathbb{D}^\epsilon}(r_k,D_i,f) + o(r_k^2).$$

Taking integration of two hand sides, we deduce that (4) holds. This completes the proof of Theorem 1.7.

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