Noname manuscript No.
(will be inserted by the editor)

A new concept of slope for set-valued maps and applications in set optimization studied with Kuroiwa's set approach

Truong Xuan Duc Ha

Received: date / Accepted: date

Abstract In this paper, scalarizing functions defined with the help of the Hiriart-Urruty signed distance are used to characterize set order relations and weak optimal solutions in set optimization studied with Kuroiwa's set approach and to introduce a new concept of slope for a set-valued map. It turns out that this slope possesses most properties of the strong slope of a scalar-valued function. As applications, we obtain criteria for error bounds of a lower level set and the existence of weak optimal solutions under a Palais-Smale type condition.

Keywords Set-valued map · slope · set optimization · error bound · optimal solution · Palais-Smale condition

1 Introduction

Optimization problems with set-valued data have been recently attracted more attention due to their important real-world applications in socio-economics, see [5], [20].

Consider an optimization problem (SP) with the objective map being a set-valued map F from a set X into a vector space Y. There are several approaches to defining an optimal solution, say $\bar{x}$, for such a problem, and we restrict ourselves to the two ones of them. In the first approach, one take $\bar{y} \in F(\bar{x})$ and compares $\bar{y}$ to elements y of the image set $F(X)$ with respect to an order in Y, where $F(X) := \{y \mid y \in F(x), x \in X\}$. Meanwhile, in the second approach proposed by Kuroiwa [21], the set $F(\bar{x})$ is compared to other sets $F(x)$ $(x \in X)$ with respect to set order relations in 2^Y.

To address the needs which arose in set optimization, some useful tools from Variational Analysis such as derivative, coderivative, slope for set-valued

Truong Xuan Duc Ha
Institute of Mathematics, Vietnam Academy of Science and Technology
E-mail: txdha@math.ac.vn

maps have been extensively developed (we refer the interested reader to the monographs [20] and [23] for literature in this field).

Recall that the concept of so called strong slope for a real-valued function has been introduced by De Giorgi, Marino, and Tosques in [8]. Later, several kinds of local and nonlocal slopes are defined in [10], and extended to vector-valued maps in [6]-[7] and to set-valued maps in [25]. Numerous applications of slopes can be found in the study of error bounds, sharp and weak sharp minima, metric regularity..., see for instance, [2], [3], [4], [25].

It should be noted that the slopes introduced in [25] for a set-valued map are useful in the study of its local properties (various kinds of metric regularity) around some point of its graph but they may not be a right tool for dealing with set optimization problems studied with Kuroiwa's set approach. Motivated by this fact, we use in this paper the scalarizing function, which has been defined in [16] with the help of the Hiriart-Urruty signed distance, to introduce a new concept of slope for a set-valued map. It turns out that this slope possesses most properties of the strong slope of a scalar-valued function. This slope together with obtained here scalar characterizations of set order relations and weak optimal solutions allow us to establish criteria for error bounds for a lower level set and the existence of these solutions under a Palais-Smale type condition.

The paper is organized as follows. In Section 2, we recall some notions from vector optimization and set optimization. Section 3 is devoted to scalarization for set order relations and weak optimal solutions of set-valued optimization problem. Section 4 concerns with the new concept of slope of a set-valued map. Criteria for error bounds and the existence of weak optimal solutions are established in the last section.

2 Preliminaries

In this section, we recall some concepts from vector optimization and set optimization [20], [24] such as efficiency, set order relations, cone continuity, cone convexity and coderivative for a set-valued map.

Throughout the paper, $\mathbb{R}^n$ is the n-dimensional euclidean space, $\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$, $\mathbb{R} := \mathbb{R}^1$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. For any $t \in \mathbb{R}$, $[t]^+ := \max\{t, 0\}$. Unless otherwise specified, X is a complete metric space, Y is a Banach space with the dual Y^* and the dual pairing $\langle ., . \rangle$. By $\mathbb{B}$ and $\mathring{\mathbb{B}}$ we mean the closed unit ball and the open unit ball in a normed space, respectively. For a nonempty subset U of X or Y, int$U$ and bd$U$ stand for its interior and boundary, respectively, and both the notations $d_U(u)$ and $d(u; U)$ stand for the distance from u to U.

Let $K \subset Y$ be a nontrivial closed pointed convex cone (pointedness means $K \cap (-K) = \{0\}$). The cone K induces a partial order in Y as follows: for $y_1, y_2 \in Y$,

$$y_1 \leq y_2 \quad \text{iff} \quad y_2 - y_1 \in K.$$

Let $K^* := \{k^* \in Y^* \ : \ \langle k^*, k \rangle \geq 0, \ \forall\, k \in K\}$.

Let $A \subset Y$ be a nonempty set. We say that $\bar{a} \in A$ is a Pareto minimal point of A if $a \not\leq \bar{a}$ for all $a \in A$, $a \neq \bar{a}$. By Min$A$ we mean the set of all Pareto minimal points of A. The *lower less oder* $\preceq_l$ [21] is a set order relation induced by K in the family of nonempty subsets of Y: for any nonempty subsets A_1, A_2 of Y,

$$A_1 \preceq_l A_2 \quad \text{iff} \quad A_2 \subseteq A_1 + K.$$

Assuming that K has nonempty interior int$K$, we will also consider the *strict lower less oder* $\prec_l$ given by

$$A_1 \prec_l A_2 \quad \text{iff} \quad A_2 \subseteq A_1 + \text{int}K.$$

From now on, it is assumed that int$K \neq \emptyset$ whenever the relation $\prec_l$ is under consideration. It is easy to see that

$$A_1 \preceq_l A_2 \text{ and } A_2 \preceq_l A_1 \quad \text{iff} \quad A_1 + K = A_2 + K.$$

We use the symbol $\equiv_l$ in the following sense

$$A_1 \equiv_l A_2 \text{ iff} \quad A_1 + K = A_2 + K.$$

Further, recall [24] that A is K-*bounded* if there exists a nonempty bounded subset M of Y such that $A \subset M + K$ (or, in terms of the lower set order relation, if $M \preceq_l A$) and K-*compact* if any cover of A of the form

$$\{U_\alpha \ : \ \alpha \in I, U_\alpha \text{ is open}\}$$

admits a finite subcover. It is known that if A is K-compact, then it is K-bounded [24].

Let $f : X \to \overline{\mathbb{R}}$ be a function and $F : X \rightrightarrows Y$ a set-valued map. The domains of f and F are the sets given by dom$f := \{x \in X \ : \ f(x) < +\infty\}$ and dom$F := \{x \in X \ : \ F(x) \neq \emptyset\}$, respectively. The graph of F is the set

$$\text{gr}F := \{(x, y) \in X \times Y \mid x \in \text{dom}F,\ y \in F(x)\}.$$

Let t be a scalar and A be a nonempty subset A of Y. The *lower level* set of f at t and the *lower level* set of F at A are defined by

$$[f \leq t] := \{x \in X \mid f(x) \leq t\}$$

and

$$[F \preceq_l A] := \{x \in X \mid F(x) \preceq_l A\},$$

respectively.

For an extended-valued function $f : X \to \overline{\mathbb{R}}$, we say that $\bar{x} \in \text{dom}f$ is a *local minimizer* of f if there exists a neighborhood U of $\bar{x}$ such that

$$f(x) \geq f(\bar{x}), \quad \forall x \in \text{dom}f \cap U.$$

For a map $f : X \to Y$, we say that $\bar{x} \in X$ is a *local Pareto minimizer* of f if there exists a neighborhood U of $\bar{x}$ such that

$$f(x) \not\leq f(\bar{x}), \quad \forall x \in U, f(x) \neq f(\bar{x}).$$

Let be given a set-valued map $F : X \rightrightarrows Y$. Following Kuroiwa in his set approach to a set-valued optimization problem [21], we say that $\bar{x} \in \mathrm{dom}F$ is a *local $\preceq_l$-minimizer* of F (or a *local optimal solution* of (SP)) if there exists a neighborhood U of $\bar{x}$ such that

$$F(x) \preceq_l F(\bar{x}) \text{ for some } x \in \mathrm{dom}F \cap U \text{ implies } F(\bar{x}) \preceq_l F(x).$$

Assume that K has nonempty interior. We say that $\bar{x}$ is a *local weak $\preceq_l$-minimizer* of F (or a *local weak optimal solution* of (SP)) if there exists a neighborhood U of $\bar{x}$ such that for all $x \in \mathrm{dom}F \cap U$

$$\text{either } F(x) \equiv_l F(\bar{x}) \text{ or } F(x) \not\prec_l F(\bar{x}).$$

When $U = X$, we have corresponding global concepts of minimizer/solutions. From now on, it is assumed that $\mathrm{int}K \neq \emptyset$ whenever a local/global weak $\preceq_l$-minimizer is under consideration.

Observe that $\bar{x}$ is a local $\preceq_l$-minimizer iff there exists a neighborhood U of $\bar{x}$ such that for all $x \in \mathrm{dom}F \cap U$

$$\text{either } F(x) \equiv_l F(\bar{x}) \text{ or } F(x) \not\preceq_l F(\bar{x}).$$

It is clear that if $\bar{x}$ is a local $\preceq_l$-minimizer of F, then it is a local weak $\preceq_l$-minimizer of F.

Next, we recall some concepts for a set-valued map such as continuity, convexity, boundedness.

Definition 1 (see [24]) We say that

(i) F is *K-upper semicontinuous* (K-u.s.c.) at $x \in \mathrm{dom}F$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that for all $u \in \mathrm{dom}F$ satisfying $d(u; x) \leq \delta$, we have

$$F(u) \subset F(x) + \epsilon \mathring{\mathbb{B}} + K$$

or $F(x) + \epsilon \mathring{\mathbb{B}} \preceq_l F(u)$, in terms of the set order relation $\preceq_l$.

(ii) F is *K-bounded from below* (in short, K-bounded) if the set $F(X)$ is K-bounded , i.e., if there exists a nonempty bounded set $M \subset Y$ such that

$$M \preceq_l F(x), \quad \forall x \in \mathrm{dom}F.$$

Definition 2 Assume that X is a linear space and $\mathrm{dom}F$ is convex. We say that

(i) F is *convex* [1] if for any $x_1, x_2 \in \mathrm{dom}F$ and $\lambda \in [0,1]$, one has

$$\lambda F(x_1) + (1-\lambda) F(x_2) \subseteq F(\lambda x_1 + (1-\lambda) x_2)$$

(ii) F is *K-convex* [22] if for any $x_1, x_2 \in \mathrm{dom}F$ and $\lambda \in [0,1]$, one has

$$\lambda F(x_1) + (1-\lambda) F(x_2) \subseteq F(\lambda x_1 + (1-\lambda) x_2) + K$$

or, in terms of the order relation $\preceq_l$,

$$F(\lambda x_1 + (1-\lambda) x_2) \preceq_l \lambda F(x_1) + (1-\lambda) F(x_2).$$

In what follows we always assume that X is a linear space and dom$F$ is convex whenever the above concepts of convexity are involved. One can check that if F is convex, then it is K-convex and F is convex iff its graph is convex.

We conclude this section by recalling the concepts of subdifferential and coderivative. Assume that X is a Banach space and X^* is its dual. Assume that f is convex. The subdifferential of f at $x \in \mathrm{dom} f$ denoted by $\partial f(x)$ is given by

$$\partial f(x) := \{x^* \in X^* \mid \langle x^*, u - x\rangle \leq f(u) - f(x),\ \forall u \in \mathrm{dom} f\}.$$

Let V be a nonempty closed convex subset of Y and $\bar{v} \in V$. Recall that the *normal cone* $N(\bar{v}; V)$ to V at $\bar{v}$ is defined by

$$N(\bar{v}; V) = \{v^* \in V^* \mid \langle v^*, v - \bar{v}\rangle \leq 0 \text{ for all } v \in V\}.$$

Assume that the set-valued map F is convex, i.e. its graph is convex. For $(x, y) \in \mathrm{gr}F$, the *coderivative* of convex analysis $D^*F(x, y)$ at (x, y) is a set-valued map between the spaces Y^* and X^* defined as follows (see [1]): for any $y^* \in Y^*$,

$$D^*F(x, y)(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N((x, y); \mathrm{gr}F)\}.$$

3 Scalar characterizations of the lower less order and of weak optimal solutions of (SP)

In this section, we recall the Hiriart-Urruty signed distance [18], some scalarizing functions considered in [16] and establish scalar characterizations of the lower less order and of weak optimal solutions of (SP).

The Hiriart-Urruty *signed distance* function Δ_U associated to a nonempty set $U \subset Y$ in the special case $U = -K$ is given by

$$\Delta_{-K}(y) := d_{-K}(y) - d_{Y\setminus(-K)}(y) = \begin{cases} -d_{Y\setminus(-K)}(y) & \text{if } y \in -K \\ d_{-K}(y) & \text{otherwise.} \end{cases}$$

Some useful properties of Δ_{-K} are collected in the following proposition.

Proposition 1 *The function Δ_{-K} has the properties:*

(i) It is Lipschitz of rank 1 on Y, convex and positively homogeneous.

(ii) (Triangle inequality) For any $y_1, y_2 \in Y$, we have

$$\Delta_{-K}(y_1 + y_2) \leq \Delta_{-K}(y_1) + \Delta_{-K}(y_2).$$

(iii) (K-monotonicity) For any $y_1, y_2 \in Y$, we have

$$y_1 \leq y_2 \ \text{ implies } \ \Delta_{-K}(y_1) \leq \Delta_{-K}(y_2).$$

(iv) (Boundedness and positivity of subdifferential) For any $y \in Y$,we have

$$\partial\Delta_{-K}(y) \subset K^* \cap \mathbb{B}.$$

Proof The properties (i)-(iii) are known, see e.g. [26], and the last one can be derived from [18, Prop. 2 and 5]. □

Below are some illustrating examples , see [16, Example 3.1] and [15, Proposition 2.3].

Example 1 (i) If $K = \{0\}$, then $\Delta_{-K}(y) = \|y\|$ for all $y \in Y$.
(ii) If $Y = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, then for all $y = (y_i) \in \mathbb{R}^n$

$$\Delta_{-\mathbb{R}^n_+}(y) = \begin{cases} -\min_i |y_i| & \text{if } y \in -\mathbb{R}^n_+ \\ \sqrt{\sum_{i=1}^n ([y_i]^+)^2} & \text{otherwise.} \end{cases}$$

Moreover, $\partial \Delta_{-\mathbb{R}^+_n}(0) = \mathrm{conv}\{v \in \mathbb{R}^n_+ : \|v\| = 1\}$.
(iii) If $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, then $\Delta_{-K}(y) = y$ and $\partial \Delta_{-K}(y) = 1$ for all $y \in \mathbb{R}$.

Let be given two nonempty K-bounded subsets A and B of Y. Define

$$h_K(A, B) := \sup_{b\in B} \inf_{a\in A} \Delta_{-K}(a - b).$$

Remark 1 The bifunction $h_K(.,.)$ has been introduced in [16] and used there to define a Hausdorff-type distance as follows. Let A, B be nonempty K-bounded subsets of Y. A *Hausdorff-type distance* relative to the ordering cone K between A and B, denoted by $d_K(A, B)$, is defined as follows:

$$d_K(A, B) := \max\{h_K(A, B), h_K(B, A)\}.$$

The name is originated from the fact that when $K = \{0\}$, this distance reduces to the classical Hausdorff distance given by

$$d(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

In the mentioned paper, a concept of directional derivative for a set-valued map has been introduced with the help of the Hausdorff-type distance.

In what follows, we **abbreviate** $h_K(A, B)$ to $h(A, B)$. We recall nice properties of the bifunction h.

Proposition 2 *Let A, B and C be nonempty K-bounded subsets of Y. Then*

(i) $h(A, B)$ *is finite.*
(ii) The triangle inequality holds:

$$h(A, B) \le h(A, C) + h(C, B).$$

(iii) $h(A, B) = h(A + K, B + K)$.
(iv) $h(A, A) = 0$ *if* $\mathrm{int}K = \emptyset$ *or* $\mathrm{Min}A \neq \emptyset$ *(for instance, if A is K-compact).*
(v) $h(A, B) = 0$ *if A, B are K-compact and* $A \equiv_l B$.
(vi) $h(A, B) = h(A_1, B_1)$ *if A, B, A_1, B_1 are K-compact and* $A \equiv_l A_1$, $B \equiv_l B_1$.

Proof The assertions (i)-(iv) have been established in [16, Lemmas 3.1, 3.4 and 3.5]. The assertions (v)-(vi) follow the assertions (iii)-(iv): $h(A,B) = h(A+K,B+K) = h(A+K,A+K) = 0$ and $h(A,B) = h(A+K,B+K) = h(A_1+K,B_1+K) = h(A_1,B_1)$. □

Recall that a nonempty subset A of Y is said to have the *domination property* [24] if $\mathrm{Min}A \neq \emptyset$ and

$$A \subseteq \mathrm{Min}A + K.$$

The above inclusion is equivalent to $A + K = \mathrm{Min}A + K$ or $A \equiv_l \mathrm{Min}A$.

Proposition 3 [16, Lemma 3.2] *Let A and B be nonempty K-compact subsets of Y. Then*

$$h(A,B) = h(\mathrm{Min}A, \mathrm{Min}B).$$

Proof Lemma 3.1 in [11] and Theorem 4.3 in [24] imply that the sets A and B being K-compact have the domination property. Hence, $A \equiv_l \mathrm{Min}A$ and $B \equiv_l \mathrm{Min}B$ and the assertion follows from Proposition 2 (vi). □

Proposition 4 [16, Lemma 3.2] *Let A and B be nonempty subsets of Y.*

(i) If A is K-compact, then for any $b \in Y$ the function

$$\Delta_{-K}(. - b) : \; A \to \mathbb{R}$$

attains its finite infimum on A.

(ii) If A is K-bounded and B is K-compact, then the function

$$\inf_{a\in A} \Delta_{-K}(a - .) : \; B \to \mathbb{R}$$

attains its finite maximum on B.

(iii) If A and B are K-compact, then

$$h(A,B) = \max_{b\in B} \min_{a\in A} \Delta_{-K}(a-b).$$

We will characterize the set order relation $\preceq_l$ in term of the bifunction h. For a nonempty subsets M of Y, let

$$\mu_1(M) := \sup_{m\in M} \Delta_{-K}(-m)$$

and

$$\mu_2(M) := \sup_{m\in M} \Delta_{-K}(m).$$

Lemma 1 *Let A, B and M be nonempty subsets of Y and $\rho \geq 0$ be a scalar.*

(i) Assume that $\mu_1(M)$ is finite and $\mu_1(M) \geq 0$. Then

$$A + \rho M \preceq_l B \Rightarrow h(A,B) \leq \rho\mu_1(M).$$

If B is K-compact, then

$$A + \rho M \prec_l B \Rightarrow h(A,B) < \rho\mu_1(M).$$

(ii) Assume that $\mu_2(M)$ is finite and $\mu_2(M) \geq 0$. Then

$$A + \rho M \not\prec_l B \Rightarrow h(A,B) \geq -\rho\mu_2(M).$$

If A is K-compact, then

$$A + \rho M \not\preceq_l B \Rightarrow h(A,B) > -\rho\mu_2(M).$$

Proof (i) It follows from $A + \rho M \preceq_l B$ that $B \subset A + \rho M + K$. For any $b \in B$ there is $a \in A$ and $m \in M$ such that $a - b + \rho m \in -K$. Hence, $\Delta_{-K}(a - b + \rho m) \leq 0$ and

$$\Delta_{-K}(a-b) \leq \Delta_{-K}(a-b+\rho m) + \Delta_{-K}(-\rho m) \leq \rho\Delta_{-K}(-m) \leq \rho\mu_1(M).$$

Therefore, $\inf_{a\in A} \Delta_{-K}(a-b) \leq \Delta_{-K}(a-b) \leq \rho\mu_1$. As $b \in B$ is arbitrarily chosen, we get

$$h(A,B) = \sup_{b\in B} \inf_{a\in A} \Delta_{-K}(a-b) \leq \rho\mu_1(M).$$

Next, assume that $A + \rho M \prec_l B$. Then for any $b \in B$ there is $a \in A$ and $m \in M$ such that $a - b + \rho m \in -\mathrm{int}K$. Hence, $\Delta_{-K}(a - b + \rho m) < 0$ and

$$\Delta_{-K}(a-b) \leq \Delta_{-K}(a-b+\rho m) + \Delta_{-K}(-\rho m) < \rho\Delta_{-K}(-m) \leq \rho\mu_1(M).$$

Therefore, $\inf_{a\in A} \Delta_{-K}(a-b) \leq \Delta_{-K}(a-b) < \rho\mu_1$. Thus, for any $b \in B$ we have $\inf_{a\in A} \Delta_{-K}(a-b) < \rho\mu_1$. By Lemma 4, the function $\inf_{a\in A} \Delta_{-K}(.-b)$ attains its maximum on B. Hence, we get

$$h(A,B) = \sup_{b\in B} \inf_{a\in A} \Delta_{-K}(a-b) < \rho\mu_1(M).$$

(ii) It follows from $A+\rho M \not\prec_l B$ that $B \not\subset A+\rho M+\mathrm{int}K$. Then there exists $b \in B$ such that for any $a \in A$ and any $m \in M$ one has $a - b + \rho m \notin -\mathrm{int}K$. Then we have $\Delta_{-K}(a - b + \rho m) \geq 0$ and therefore,

$$\Delta_{-K}(a-b) \geq \Delta_{-K}(a-b+\rho m) - \Delta_{-K}(\rho m) \geq -\rho\Delta_{-K}(m) \geq -\rho\mu_2(M).$$

It is easy to see now that

$$h(A,B) = \sup_{b\in B} \inf_{a\in A} \Delta_{-K}(a-b) \geq -\rho\mu_2.$$

Next, assume that $A + \rho M \not\preceq_l B$. Then there exists $b \in B$ such that for any $a \in A$ and any $m \in M$ one has $a - b + \rho m \notin -K$. Since K is closed, we have $\Delta_{-K}(a - b + \rho m) > 0$ and therefore,

$$\Delta_{-K}(a-b) \geq \Delta_{-K}(a-b+\rho m) - \Delta_{-K}(\rho m) > -\rho\Delta_{-K}(m) \geq -\rho\mu_2(M).$$

Since A is K-compact, Proposition 4 implies that $\Delta_{-K}(.-b)$ attains its minimum on A, and we get $\inf_{a\in A} \Delta_{-K}(a-b) = \min_{a\in A} \Delta_{-K}(a-b) > -\rho\mu_2(M)$. Hence,

$$h(A,B) = \sup_{b\in B} \inf_{a\in A} \Delta_{-K}(a-b) > -\rho\mu_2.$$

□

As a consequence of Lemma 1, we obtain the following important characterization of the relations $\preceq_l$ and $\prec_l$ in terms of the bifunction h.

Proposition 5 *Assume that A and B are nonempty K-compact subsets of Y. Then the following implications hold:*

$$A \preceq_l B \Leftrightarrow h(A,B) \leq 0$$

and

$$A \prec_l B \Leftrightarrow h(A,B) < 0.$$

Next, we will study some scalarizing functions. Let a be a vector of Y and A be a nonempty K-bounded subset of Y. Assume that F has K-bounded values. To the map F, the vector a and the set A, we associate two scalar-valued functions $g_{F,a},\ g_{F,A} : \mathrm{dom}F \to \mathbb{R}$ defined as follows: For $x \in \mathrm{dom}F$

$$g_{F,a}(x) = \inf_{y \in F(x)} \Delta_{-K}(y-a)$$

and

$$g_{F,A}(x) = \sup_{a \in A} \inf_{y \in F(x)} \Delta_{-K}(y-a).$$

One can see that

$$g_{F,A}(x) = \sup_{a \in A} g_{F,a}(x) = h(F(x), A).$$

The following result states that the function $g_{F,A}$ inherits continuity and convexity of the map F.

Proposition 6 *Assume that K has nonempty interior, F has K-bounded values and $x \in \mathrm{dom}F$. Assume further that A is a nonempty K-bounded subset of Y. If F is K-u.s.c. at x, then $g_{F,A}$ is lower semicontinuous (in brief, l.s.c.) at this point and if F is K-convex, then $g_{F,A}$ is convex.*

Proof Observe that $\mu_1(\mathring{\mathbb{B}}) := \sup_{b \in (\mathring{\mathbb{B}}} \Delta_{-K}(-b)$ is finite because the function Δ_{-K} is 1-Lipschitz. Moreover, $\mu_1(\mathring{\mathbb{B}}) > 0$ because $\Delta_{-K}(\bar{k}) > 0$ for any $\bar{k} \in \mathrm{int}K \cap \mathring{\mathbb{B}}$.

Now, suppose that F is K-u.s.c. at x. Then $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$F(x) + \epsilon\mathring{\mathbb{B}} \preceq_l F(u), \quad \forall u \in \mathbb{B}(x,\delta) \cap \mathrm{dom}F.$$

By Lemma 1, we have $h(F(x), F(u)) \leq \epsilon\mu_1(\mathring{\mathbb{B}})$ for $u \in \mathbb{B}(x,\delta) \cap \mathrm{dom}F$. The triangle inequality, see Proposition 2, implies $h(F(x), F(u)) \geq h(F(x), A) - h(F(u), A)$. Therefore, for $u \in \mathbb{B}(x,\delta) \cap \mathrm{dom}F$ we have

$$-\epsilon\mu_1(\mathring{\mathbb{B}}) \leq -h(F(x), F(u)) \leq h(F(u), A) - h(F(x), A) = g_{F,A}(u) - g_{F,A}(x),$$

which means that $g_{F,A}$ is l.s.c. at x.

Next, assume that F is K-convex. It follows from [14, Proposition 2.2(c)] that for any $a \in A$, the function $g_{F,a}$ is convex. Therefore, $g_{F,A}$ is convex because it is the supremum of a family of convex functions $g_{F,A}(x) = \sup_{a \in A} g_{F,a}(x)$. $\square$

We deduce from Proposition 5 the following characterizations of a local/glocal weak $\preceq_l$-minimizer of F.

Proposition 7 *Assume that F has K-compact values and $\bar{x} \in \mathrm{dom}F$. Denote $\bar{A} := F(\bar{x})$. Then $\bar{x}$ is a local weak $\preceq_l$-minimizer of F iff it is a local minimizer of $g_{F,\bar{A}}$: for some neighborhood U of $\bar{x}$, one has*

$$g_{F,\bar{A}}(x) \geq 0 = g_{F,\bar{A}}(\bar{x}), \quad \forall x \in \mathrm{dom}F \cap U.$$

The assertion holds true if "local" is replaced by "global".

Proof Recall that $g_{F,\bar{A}}(x) = h((F(x), \bar{A}) = h(F(x), F(\bar{x}))$. Observe that by Proposition 2 (iv) we have $g_{F,\bar{A}}(\bar{x}) = 0$. The assertions follow from the definitions of a local/globall weak $\preceq_l$-minimizer and Proposition 5. □

Proposition 8 *Assume that F is K-convex and K-compact-valued. Then $\bar{x} \in \mathrm{dom}F$ is a local weak $\preceq_l$-minimizer of F iff it is its global weak $\preceq_l$-minimizer.*

Proof Assume that $\bar{x}$ is a local weak $\preceq_l$-minimizer of F. Denote $\bar{A} := F(\bar{x})$. Proposition 7 implies that $\bar{x}$ is a local minimizer of the function $g_{F,\bar{A}}$. Since this function is convex by Proposition 6, $\bar{x}$ is then its global minimizer, i.e., $g_{F,\bar{A}}(x) \geq g_{F,\bar{A}}(\bar{x}) = 0$ for all $x \in \mathrm{dom}F$. Proposition 5 implies that $\bar{x}$ is a global weak $\preceq_l$-minimizer of F. □

4 A new concept of slope for a set-valued map

We first recall the notion of strong slope introduced by De Giorgi, Marino, and Tosques in [8].

Definition 3 Let $f : X \to \overline{\mathbb{R}}$ be a function, and let $x \in \mathrm{dom}f$. The strong slope $|\nabla f|(x)$ of f at x is defined by

$$|\nabla f|(x) := \begin{cases} 0 & x \text{ is a local minimizer of } f \\ \limsup_{u \to x} \frac{f(x)-f(u)}{d(x;u)} & \text{otherwise} \end{cases}$$

Extending the concept of strong slope to a single- vector-valued map $f : X \to Y$, Bednarczuk and Kruger [6]-[7] introduced the concepts of *lower* and *upper* slopes for f. We recall here the lower slope, which is closely related to our concept of slope in the set-valued case. Assume that K has a nonempty interior. The lower slope $^{-}|\nabla f|(x)$ of f at $x \in X$ is defined by

$$^{-}|\nabla f|(x) := \limsup_{x' \to x} \sup\{r > 0 : \frac{f(x) - f(x')}{d(x;x')} \in K + r\mathring{K}\},$$

where $\mathring{K} := \{k \in \mathrm{int}K : d(k; \mathrm{bd}K) = 1\}$. Theorem 3.1 in [15] shows that the lower slope can also be represented in the form

$$^{-}|\nabla f|(x) = \limsup_{u \to x} \frac{-\Delta_{-K}(f(u) - f(x))}{d(x;u)}.$$

Now, let us introduce a new concept of slope for the set-valued case. From now on, we will assume that **the cone K has nonempty interior**.

Definition 4 Assume that F has K-bounded values, and let $x \in \mathrm{dom}F$. The slope $|\nabla F|(x)$ of F at x is defined by

$$|\nabla F|(x) := \begin{cases} 0 & x \text{ is a local weak } \preceq_l - \text{minimizer of } F \\ \limsup_{u \to x} \frac{-h(F(u),F(x))}{d(x;u)} & \text{otherwise} \end{cases}$$

Remark 2 (a) When F is single-valued, the above slope reduces to the lower slope and when $Y = \mathbb{R}$, $K = \mathbb{R}_+$ and F is scalar- single-valued, Definition 4 reduces to Definition 3.

(b) Consider the case when F is "constant" around a point $x \in \mathrm{dom}F$ in the sense that $F(u) \equiv_l A$ for all $u \in \mathrm{dom}F \cap U$, where U is a neighborhood of x and A is K-bounded (note that it may happen that $F(u) \neq F(u')$ for some $u, u' \in \mathrm{dom}F \cap U$). Then x is a local $\preceq_l$-minimizer of F and we have $|\nabla F|(x) = 0$.

Proposition 9 *Suppose that F has K-compact-values and $x \in \mathrm{dom}F$ is not a local weak $\preceq_l$-minimizer of F. Then*

$$|\nabla F|(x) = \limsup_{u \to x, F(u) \prec_l F(x)} \frac{-h(F(u),F(x))}{d(x;u)}$$

and if $|\nabla F|(x)$ is finite, then it is nonnegative. We also have

$$|\nabla F|(x) = \limsup_{u \to x, F(u) \prec_l F(x)} \frac{-h(\mathrm{Min}F(u),\mathrm{Min}F(x))}{d(x;u)}.$$

Proof Since x is not a local weak $\preceq_l$-minimizer of F, it is not constant around x in the sense explained in Remark 2(b), and there exists a sequence $\{x_i\}$ converging to x such that $F(x_i) \prec_l F(x)$ for all i. By Lemma 1, we have $h(F(x_i),F(x)) < 0$ and therefore, $|\nabla F|(x) \geq 0$. For any u satisfying $F(u) \not\prec_l F(x)$, one has $-h(F(u),F(x)) \leq 0$ due to Lemma 1. The desired equality follows. It is clear now that if $|\nabla F|(x)$ is finite, then it is nonnegative. The last equality follows from Proposition 3. $\square$

We illustrate the notion of slope for set-valued maps by some examples.

Example 2 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$.

(a) Let F be a set-valued map defined by

$$F(x) = \begin{cases} \{x\} \times [x^2, 2x^2] & \text{if } x > 0 \\ \{0\} \times [-1, 1] & \text{if } x = 0 \\ \{-x\} \times [-0.5x, -2x] & \text{if } x < 0 \end{cases}$$

This map is K-u.s.c. on $\mathbb{R}$ and $\mathbb{R}^2_+$-bounded. One can check that

$$|\nabla F|(x) = \begin{cases} 1 & \text{if } x \geq 1/2 \\ 2x & \text{if } 0 < x < 1/2 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x < 0 \end{cases}$$

Note that $x = 0$ is the only local weak $\preceq_l$-minimizer of F and hence, $|\nabla F|(0) = 0$. For the reader's convenience, we provide detailed calculation of $|\nabla F|(x)$ at $x > 0$. Recall that by Proposition 9, we have

$$|\nabla F|(x) = \limsup_{u \to x, F(u) \prec_l F(x)} \frac{-h(\mathrm{Min}F(u), \mathrm{Min}F(x))}{d(x;u)}.$$

Since $F(x+t) \not\prec_l F(x)$ for $t > 0$ and $F(x+t) \prec_l F(x)$ for $t < 0$, we will consider $u = x + t$ with $t < 0$. Observe that $\mathrm{Min}F(u) = \{(u, u^2)\}$ for $u > 0$. It follows from Example 1 (ii) that

$$h(\mathrm{Min}F(x+t), \mathrm{Min}F(x)) = \varDelta_{-\mathbb{R}^2_+}((t, 2xt + t^2)) = -\min\{|t|, |2xt + t^2|\}.$$

As $x > 0$, for $|t|$ sufficiently small, one has

$$h(\mathrm{Min}F(x+t), \mathrm{Min}F(x)) = t\min\{1, 2x + t\} = \begin{cases} t(2x+t) & \text{if } 0 < x < 1/2 \\ t & \text{if } 1/2 \le x \end{cases}$$

Therefore,

$$|\nabla F|(x) = \begin{cases} 2x & \text{if } 0 < x < 1/2 \\ 1 & \text{if } 1/2 \le x \end{cases}$$

(b) Let F be a set-valued map defined by

$$F(x) = \begin{cases} \{(x, x^2), (x, 2x^2)\} & \text{if } x > 0 \\ \{(x, -x^2)\} & \text{if } x \le 0 \end{cases}$$

Then

$$|\nabla F|(x) = \begin{cases} 1 & \text{if } x \ge 1/2 \\ 2x & \text{if } 0 < x < 1/2 \\ -2x & \text{if } -1/2 \le x \le 0 \\ 1 & \text{if } x < -1/2 \end{cases}$$

Note that $|\nabla F|(0) = 0$, but $x = 0$ is not a local weak $\preceq_l$-minimizer of F.

Remark 3 Example 2 (b) shows that the slope of F may be equal to zero at a point which is not a local weak $\preceq_l$-minimizer of F. As the reader will see, when F is K-convex, its slope equals to zero at a point x iff x is a local weak $\preceq_l$-minimizer.

We establish some properties of the slope.

Proposition 10 *Suppose that F has K-compact values and $\bar{x} \in \mathrm{dom}F$. Denote $\bar{A} := F(\bar{x})$. Then*

$$|\nabla F|(\bar{x}) = |\nabla g_{F,\bar{A}}|(\bar{x}).$$

Proof If $\bar{x}$ is a local weak $\preceq_l$-minimizer of F, Proposition 7 implies that it is a local minimizer of the function $g_{F,\bar{A}}$. Definitions 3 and 4 imply $|\nabla F|(\bar{x}) = |\nabla g_{F,\bar{A}}|(\bar{x}) = 0$. Next, if $\bar{x}$ is not a local weak $\preceq_l$-minimizer of F, Proposition 7 implies that it is not a local minimizer of the function $g_{F,\bar{A}}$. From Definitions 3 and 4, we get

$$\begin{aligned} |\nabla F|(\bar{x}) &= \limsup_{x\to\bar{x}} \tfrac{-h(F(x),F(\bar{x}))}{d(x;\bar{x})} = \limsup_{x\to\bar{x}} \tfrac{g_{F,\bar{A}}(\bar{x})-g_{F,\bar{A}}(x)}{d(x;\bar{x})} \\ &= |\nabla g_{F,\bar{A}}|(\bar{x}). \end{aligned}$$

□

Example 3 Let F be the map considered in Example 2 (a). We illustrate the equality stated in Proposition 10 for $x > 0$. By the definition and Proposition 3, we have

$$g_{F,x}(u) = h(\mathrm{Min}F(u), \mathrm{Min}F(x)) = \begin{cases} u - x & \text{if } x \geq 1/2 \\ u^2 - x^2 & \text{if } 0 < x < 1/2 \end{cases}$$

Therefore, for $x > 0$, we have

$$|\nabla F|(x) = \nabla g_{F,x}(x) = \begin{cases} 1 & \text{if } x \geq 1/2 \\ 2x & \text{if } 0 < x < 1/2 \end{cases}$$

It is well known that the strong slope of a convex l.s.c. function possesses some nice properties. We will show that the slope of set-valued maps enjoys similar properties. Recall that for a convex l.s.c. function, any local minimizer is the global one. Moreover, we have the following result.

Proposition 11 *Assume that X is a Banach space. Let $f : X \to \overline{\mathbb{R}}$ be a proper convex l.s.c. function and $\bar{x} \in \mathrm{dom}f$. Then $\bar{x}$ is a global minimizer of f iff $|\nabla f|(\bar{x}) = 0$.*

Proof It suffices to consider the "if" part. Without loss of generality, we may assume that $f(\bar{x}) = 0$. Suppose to the contrary that there exists $u \in \mathrm{dom}f$ such that $f(u) < f(\bar{x}) = 0$. Let $t_i = 1/(i+1)$ for $i = 1, 2, ...$ and $x_i := \bar{x} + t_i(u - \bar{x}) = (1 - t_i)\bar{x} + t_i u$. Note that $i \in]0, 1[$ and $x_i \to \bar{x}$. Since f is convex, we have $f(x_i) \leq (1-t_i)f(\bar{x}) + t_i f(u) = t_i f(u) < 0$. It is easy to check that

$$\frac{f(\bar{x}) - f(x_i)}{d(x_i;\bar{x})} \geq \frac{f(\bar{x}) - f(u)}{d(u;\bar{x})} = \frac{-f(u)}{d(u;\bar{x})}.$$

Letting $i \to +\infty$, we deduce from the above inequality that

$$|\nabla f|(\bar{x}) = \limsup_{x'\to\bar{x}} \frac{f(\bar{x}) - f(x')}{d(x';\bar{x})} \geq \limsup_{i\to\infty} \frac{f(\bar{x}) - f(x_i)}{d(x_i;\bar{x})} \geq \frac{-f(u)}{d(u;\bar{x})} > 0,$$

which is a contradiction to the assumption that $|\nabla f|(\bar{x}) = 0$. □

First we provide a sufficient condition of a global weak $\preceq_l$-minimizer of F.

Theorem 1 *Assume that X is a Banach space and F is K-convex K-u.s.c. K-compact-valued. Then $\bar{x} \in \mathrm{dom} F$ is a global weak $\preceq_l$-minimizer of F iff $|\nabla F|(\bar{x}) = 0$.*

Proof We prove the "if" part. Let $\bar{A} := F(\bar{x})$. Proposition 6 yields that the function $g_{F,\bar{A}}$ is convex, l.s.c. It follows from Proposition 10 that

$$|\nabla g_{F,\bar{A}}|(\bar{x}) = |\nabla F|(\bar{x}) = 0,$$

which together with Proposition 11 yield that $\bar{x}$ is a global minimizer of the function $g_{F,\bar{A}}$. Then $\bar{x}$ also is a local weak $\preceq_l$-minimizer of F by Proposition 7. Proposition 8 implies that $\bar{x}$ is a global weak $\preceq_l$-minimizer of F. □

In the convex case, the strong slope of a convex function can be expressed in a more simple form and can also be calculated be means of the subdifferential of convex analysis. Namely, we have the following.

Proposition 12 [4, Proposition 3.1] *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper convex l.s.c. function. Suppose that x is not a (global) minimizer of f. Then*

$$|\nabla f|(x) = \sup_{f(u))<f(x)} \frac{f(x) - f(u)}{\|x-u\|} = d(0, \partial f(x)).$$

It turns out that a set-valued version of Proposition 12 holds. First we show that "limsup" in the definition of the slope can be replaced by "sup" in the convex case.

Proposition 13 *Suppose that X is a Banach sapace, F is K-convex K-u.s.c. K-compact-valued and $\bar{x} \in \mathrm{dom} F$ is not a local weak $\preceq_l$-minimizer of F. Then*

$$|\nabla F|(\bar{x}) = \sup_{F(x) \prec_l F(\bar{x})} \frac{-h(F(x), F(\bar{x}))}{d(x; \bar{x})}.$$

Proof It suffices to check the inequality

$$|\nabla F|(\bar{x}) \geq \sup_{F(x) \prec_l F(\bar{x})} \frac{-h(F(x), F(\bar{x}))}{d(x; \bar{x})}$$

because the inverse one is immediate from the definition. Denote $\bar{A} := F(\bar{x})$. By Proposition 10, we have

$$|\nabla F|(\bar{x}) = |\nabla g_{F,\bar{A}}|(\bar{x}).$$

On the other hand, Proposition 6 implies that $g_{F,\bar{A}}$ is convex lsc. Recall that $g_{F,\bar{A}}(\bar{x}) = 0$. Applying Proposition 12 we get

$$|\nabla g_{F,\bar{A}}|(\bar{x}) = \sup_{g_{F,\bar{A}}(x) < g_{F,\bar{A}}(\bar{x})} \frac{-g_{F,\bar{A}}(x)}{d(x; \bar{x})} = \sup_{g_{F,\bar{A}}(x) < 0} \frac{-h(F(x), F(\bar{x}))}{d(x; \bar{x})}.$$

Since $F(x) \prec_l F(\bar{x})$ implies $g_{F,\bar{A}}(x) = h(F(x), F(\bar{x})) < 0$ by Proposition 5, we obtain

$$|\nabla g_{F,\bar{A}}|(\bar{x}) \geq \sup_{F(x)\prec_l F(\bar{x})} \frac{-h(F(x), F(\bar{x}))}{d(x;\bar{x})}$$

and the desired inequality follows. □

Next, we establish a relation between the slope of F and its coderivative. Let $z \in Y$. Consider the function $g_{F,z}$ associated to F and z given by

$$g_{F,z}(x) := \inf_{y\in F(x)} \Delta_{-K}(y - z)$$

and let

$$V_{F,z}(x) := \{y \in F(x) \mid \Delta_{-K}(y - z) = g_{F,z}(x)\}.$$

If F is convex, then Proposition 2.2(c) in [14] yields that that the function $g_{F,z}$ is convex and if F has K-compact values, then Proposition 3.1(i) in [13] yields that $V_{F,z}$ has nonempty values on dom$F$.

When F is convex and closed (i.e. the graph of F is closed), using the exact sum rule for convex functions and arguments similar to that involved in the proof of [13, Theorem 3.3], one can establish the following relation.

Proposition 14 *Assume that X is separable and F is convex closed. Then for any $y_x \in V_{F,z}(x)$, we have*

$$\partial g_{F,z}(x) = \cup_{y^*\in\partial\Delta_{-K}(y_x-z)} D^*F(x, y_x)(y^*).$$

Assume that F is convex and closed. The *subdifferential* of F at $x \in$ dom$F$ is defined by

$$\partial F(x) := \cup_{y\in F(x)} \cup_{y_x\in V_{F,y}(x), y^*\in\partial\Delta_{-K}(y_x-y)} D^*F(x, y_x)(y^*).$$

Proposition 15 *Assume that X is a separable Banach space. Assume further that F is convex closed K-compact-valued and $x \in$ dom$F$. Denote $A = F(x)$. Then*

$$\partial F(x) \subset \partial g_{F,A}(x).$$

Proof Let $x^* \in \partial F(x)$. Let $y \in F(x)$ and $y_x \in V_{F,y}(x)$, $y^* \in \partial\Delta_{-K}(y_x - y)$ such that $x^* \in D^*F(x, y_x)(y^*)$. By the definition, $(x^*, -y^*) \in N((x, y_x); \mathrm{gr}F)$ and therefore, we have

$$\langle x^*, x' - x\rangle - \langle y^*, y' - y_x\rangle \leq 0 \text{ for all } (x', y') \in \mathrm{gr}F.$$

In particular, for all $x' \in$dom$F$ we have

$$\langle x^*, x' - x\rangle \leq \langle y^*, y' - y_x\rangle \text{ for all } y' \in F(x').$$

Further, since $y^* \in \partial\Delta_{-K}(y_x - y)$ and the signed distance function satisfies the triangle inequality, we have

$$\langle y^*, y' - y_x\rangle \leq \Delta_{-K}(y' - y) - \Delta_{-K}(y_x - y) \leq \Delta_{-K}(y' - y_x).$$

Consequently, we get

$$\langle x^*, x' - x\rangle \leq \Delta_{-K}(y' - y_x).$$

As $y' \in F(x')$ is arbitrarily chosen, $y_x \in F(x)$ and $g_{F,A}(x) = 0$, we obtain

$$\begin{aligned}\langle x^*, x' - x\rangle &\leq \sup_{v\in F(x)} \inf_{y'\in F(x')} \Delta_{-K}(y' - v) = h(F(x'), F(x)) \\ &= g_{F,A}(x') = g_{F,A}(x') - g_{F,A}(x).\end{aligned} \tag{1}$$

Therefore, $x^* \in \partial g_{F,A}(x)$. □

Now we are ready to establish a relation between the slope of F and its subdifferential.

Theorem 2 *Assume that F is K-convex F is K-u.s.c. K-compact-valued and $x \in \mathrm{dom}F$. Then*

$$\xi d(0, \partial F(x)) \leq |\nabla F|(x) \leq d(0, \partial F(x)),$$

where $\xi := \sup\{d_{Y\setminus K}(k_o) \mid k_0 \in \mathrm{int}K\ , d_{-K}(k_0) = 1\}$.

Proof Observe that the first inequality

$$|\nabla F|(x) \leq d(0, \partial F(x))$$

follows from Propositions 10, 12 and 15:

$$|\nabla F|(x) = |\nabla g_{F,A}|(x) = d(0, \partial g_{F,A}(x)) \leq d(0, \partial F(x)).$$

We can also establish it directly by using the proof of Proposition 15 as follows. Let $x^* \in \partial F(x)$. It follows from the relations (1) that for all $x' \in \mathrm{dom}F$ we have

$$\langle x^*, x' - x\rangle \leq h(F(x'), F(x))$$

and, hence,

$$-\frac{h(F(x'), F(x))}{\|x' - x\|} \leq -\frac{\langle x^*, x' - x\rangle}{\|x' - x\|} \leq \|x^*\|.$$

Thus, $|\nabla F|(x) \leq \|x^*\|$, as it was to be shown.

Next, we prove that

$$|\nabla F|(x) \geq \xi d(0, \partial F(x)).$$

Let η be a scalar such that

$$\eta < d(0, \cup_{y\in F(x)} \cup_{y_x\in V_{F,y}(x), y^*\in\partial\Delta_{-K}(y_x-y)} D^*F(x, y_x)(y^*)). \tag{2}$$

Let $k_0 \in \mathrm{int}K$ such that $\Delta_{-K}(k_0) = d_{-K}(k_0) = 1$. Consider a set-valued map $G : u \in X \rightrightarrows G(u) := F(u)+\eta\|u-x\|k_0$. We claim that x is not a local weak $\preceq_l$ minimizer of the G. Suppose to the contrary that there exists a neighborhood U of x such that for any $u \in \mathrm{dom}F\cap U$, we have either $G(u)+K = G(x)+K =$

$F(x)+K$ or $G(u)=F(u)+\eta\|u-x\|k_0 \not\preceq_l G(x)=F(x)$. In the first case, Proposition 2 implies $h(G(u),F(x))=0$. In the second case, Lemma 1 yields

$$h(F(u),F(x))+\eta\|x-u\|\geq 0. \tag{3}$$

By the assumption, $F(u)$ is K-compact and Proposition 4 (b) yields the existence of $z\in F(x)$ such that $\inf_{v\in F(u)}\Delta_{-K}(v-z)=h(F(u),F(x))$. Consider the function $g_{F,z}$. Since $z\in F(x)$, we have $g_{F,z}(x)\leq 0$. The inequality (3) then gives $g_{F,z}(u)+\eta\|x-u\|\geq g_{F,z}(x)$ for any $u\in \mathrm{dom}F\cap U$. Thus, x is a local minimizer of the function $g_{F,z}(.)+\eta\|x-.\|$. Recall that the function $g_{F,z}$ also is convex and l.s.c. We have $0\in\partial(g_{F,z}(.)+\eta\|x-.\|)(x)$. The exact sum rule, see [19, Proposition 1, p.200], gives

$$0\in\partial g_{F,z}(x)+\eta\mathbb{B}_{Y^*}=\partial g_{F,z}(x)+\eta\mathbb{B}_{Y^*}.$$

Applying Proposition 14 to the map F and the function $g_{F,z}$, we can find $y_x\in V_{F,z}(x)$ such that

$$\partial g_{F,z}(x)=\cup_{y^*\in\partial\Delta_{-K}(y_x-z)}D^*F(x,y_x)(y^*).$$

Therefore,

$$0\in\cup_{y\in F(x)}\cup_{y_x\in V_{F,y}(x),y^*\in\partial\Delta_{-K}(y_x-y)}\,D^*F(x,y_x)(y^*))+\eta\mathbb{B}_{Y^*},$$

which is a contradiction to (2).

We have showed that x is not a local weak $\preceq_l$ minimizer of G. Then there exists a sequence $\{u_i\}$ with $u_i\in\mathrm{dom}F$ such that $u_i\to x$ and $G(u_i)\prec_l G(x)$ for all i. Hence, we have $F(u_i)+\eta\|u_i-x\|k_0\prec_l F(x)$. Lemma 1 implies $h(F(u_i),F(x))<\eta\|u_i-x\|\Delta_{-K}(-k_0)=-\eta\|u_i-x\|d_{Y\setminus K}(k_0)$. Hence,

$$\frac{-h(F(u_i),F(x)}{\|u_i-x\|}>\eta d_{K^c}(k_0).$$

As $u_i\to x$, we get

$$|\nabla F|(x)\geq\eta d_{K^c}(k_0).$$

Since $k_0\in\mathrm{int}K$ satisfying $d_{-K}(k_0)=1$ and $\eta>0$ satisfying (2) are arbitrarily chosen, we obtain the desired inequality. $\square$

Remark that $\xi\leq 1$ because $d_{Y\setminus K}(k_o)\leq d_{-K}(k_0)=1$. When $K=\mathbb{R}^+_n$, we have $\xi=1/\sqrt{n}$ and in the special case $n=1$, we have $|\nabla F|(x)=d(0,\partial F(x))$.

5 Criteria for error bounds of a lower level set and existence of weak optimal solutions

In the remaining of this section, we will assume that X *is a complete metric space*. For the sake of simplicity, let g_A stand for $g_{F,A}$ and g_A^+ be the function defined by $g_A^+(x)=[g_A(x)]^+$ and we use the notation U^c for the complement of a subset U of X, i.e., $U^c:=X\setminus U$.

In 1994, A. Hamel established the following result for a l.s.c. function [17].

Theorem 3 *Assume that $f : X \to \overline{\mathbb{R}}$ is a proper l.s.c. function and for all $x \in [f \leq \alpha]^c$ there exists $u \in [f \leq \alpha]^c$, $u \neq x$ such that $f(u) + d(x;u) \leq f(x)$. Then $[f \leq \alpha] \neq \emptyset$, and*

$$d(x; [f \leq \alpha]) \leq [f(x) - \alpha]^+, \quad \forall x \in \mathrm{dom} f. \tag{4}$$

It turns out that a set-valued version of Theorem 3 holds.

Theorem 4 *Suppose that F is K-bounded, K-u.s.c., K-compact-valued and $A \subset Y$ is a nonempty K-compact set. If for any $x \in [F \preceq_l A]^c$ there exists $u \in [F \preceq_l A]^c$, $u \neq x$ satisfying*

$$F(u) + \rho d(x,u) k_0 \preceq_l F(x), \tag{5}$$

where $k_0 \in \mathrm{int} K$ with $\Delta_{-K}(-k_0) = -1$ and $\rho > 0$, then $[F \preceq_l A] \neq \emptyset$ and

$$\rho d(x; [F \preceq_l A]) \leq [g_A(x)]^+, \quad \text{for all } x \in \mathrm{dom} F. \tag{6}$$

Remark 4 Theorems 3 and 4 say that the lower level sets $[f \leq \alpha]$ and $[F \preceq_l A]$ have global error bounds. Roughly speaking, the difficulty appeared while one calculates the lower level sets $[f \leq \alpha]$ and $[F \preceq_l A]$ could make the calculation of the distances to these sets from a point outside them difficult or impossible. In such a situation, the existence of error bounds give us an upper bound for these distances through the functions $[f(.) - \alpha]^+$ and $[g_A(.)]^+$ which can be more easily calculated. In the scalar case, error bounds are known to be useful in the study of convergence of numerical algorithms, subdifferential calculus... (see for instance the works [2] and [4]). In the set-valued case, further study of applications of error bounds is needed. Note that (6) reduces to (4) when $Y = \mathbb{R}$, $K = \mathbb{R}^1_+$, F is a single-valued function f, $A = \{\alpha\}$ with $\alpha \in \mathbb{R}$, $k_0 = 1$ and $\Delta_{-K}(y) = d(y; -\mathbb{R}_+)$.

To prove Theorem 4, we need an auxiliary result.

Proposition 16 *Suppose that F is K-u.s.c. K-compact-valued and A is a nonempty K-compact subset of Y. Then*

$$[F \preceq_l A] = [g_A \leq 0]$$

and the lower level set $[F \preceq_l A]$ is closed.

Proof Propositions 6 implies that the function g_A is l.s.c. and hence its lower level set $[g_A \leq 0]$ is closed. Proposition 5 yields that $F(x) \preceq_l A$ iff $h(F(x), A) \leq 0$. Since $g_A(x) = h(F(x), A)$, the assertion follows. □

Let us return to the proof of Theorem 4.

Proof Proposition 6 implies that the function g_A is proper and l.s.c. and Proposition 16 implies that $[F \preceq_l A] = [g_A \leq 0]$. Further, by Lemma 1, the relation (5) gives

$$h(F(u), F(x)) \leq \rho d(x,u) \Delta_{-K}(-k_0) = -\rho d(x,u).$$

Recall that by the triangle inequality (see Proposition 2) we have $h(F(u), A) - h(F(x), A) \leq h(F(u), F(x))$. Therefore,

$$g_A(x) + \rho d(x, u) = h(F(u), A) + \rho d(x, u) \leq h(F(x), A) = g_A(u).$$

Applying Theorem 3 to the function g_A, we get $[g_A \leq 0] \neq \emptyset$, and

$$\rho d(x; [g_A \leq 0]) \leq [g_A(x)]^+, \quad \forall x \in \mathrm{dom} g_A.$$

Since $[F \preceq_l A] = [g_A \leq 0]$, the assertion follows. □

We establish a sufficient condition for the lower sublevel $[F \preceq_l A]$ has an error bound in terms of slope.

Theorem 5 *Suppose that F is K-bounded K-u.s.c. and K-compact-valued. Assume further that*

$$\inf_{x \in [F \preceq_l A]^c} |\nabla F|(x) > 0.$$

Set

$$\sigma_A := \inf_{x \in [F \preceq_l A]^c} \frac{g_A(x)}{d(x; [F \preceq_l A])}.$$

Then for any $\rho \leq \sigma_A$ we have

$$\rho d(x; [F \preceq_l A]) \leq g_A^+(x), \quad \forall x \in X.$$

Proof First, we show that

$$\inf_{x \in [F \preceq_l A]^c} |\nabla F|(x) \leq \sigma_A. \tag{7}$$

Let $\rho > \sigma_A$. By the definition of σ_A, there exists $\bar{x} \in [F \preceq_l A]^c$ such that

$$\rho > \frac{g_A(\bar{x})}{d(\bar{x}; [F \preceq_l A])}.$$

Since $\bar{x} \in [F \preceq_l A]^c$, Proposition 5 implies that $g_A(\bar{x}) > 0$ and hence,

$$g_A^+(\bar{x}) = g_A(\bar{x}) < \rho d(\bar{x}; [F \preceq_l A]).$$

Let $r > 0$ be a scalar such that $g_A^+(\bar{x}) < r\rho < \rho d(\bar{x}; [F \preceq_l A])$. It is clear that $r < d(\bar{x}; [F \preceq_l A])$. Since $g_A^+(x) \geq 0$ for all $x \in X$, we have

$$g_A^+(\bar{x}) < g_A^+(x) + r\rho, \quad \forall x \in X.$$

According to Proposition 6, g_A is l.s.c. and hence, so is the function g_A^+. Applying the Ekeland variational principle established in [9] to the function g_A^+ and $\bar{x}$, we find $u \in X$ such that $d(\bar{x}; u) \leq r$ and

$$g_A^+(u) < g_A^+(x) + \rho d(x, u), \quad \forall x \in X, x \neq u.$$

Again, since $d(\bar{x}; u) \leq r < d(\bar{x}; [F \preceq_l A])$, we deduce that $u \in [F \preceq_l A]^c$ and $g_A(u) > 0$. Note that the set $[F \preceq_l A]^c$ is open and there exists a neighborhood

U of u such that $U \subset [F \preceq_l A]^c = [0 < g_A]$. For any $x \in U$, we have $g_A(x) > 0$. Hence, $g_A^+(x) = g_A(x)$ and

$$g_A(u) < g_A(x) + \rho d(x, u), \quad \forall x \in U, x \neq u.$$

The definition of the strong slope implies $|\nabla g_A|(u) \leq \rho$. Recall that $u \in [F \preceq_l A]^c$. Then we have

$$\inf_{u \in [F \preceq_l A]^c} |\nabla g_A|(u) \leq \rho$$

and since $\rho > \sigma_A$ is arbitrary, (7) follows.

By the assumption, now we have

$$\sigma_A > 0.$$

As $\rho \leq \sigma_A$, we get $\rho d(x; [F \preceq_l A]) \leq g_A(x)$ for all $x \in [F \preceq_l A]^c$ or

$$\rho d(x; [F \preceq_l A]) \leq g_A^+(x), \quad \forall x \in X.$$

□

Remark 5 Let F be the map in Example 2 (a) and $A = \{1\} \times [1, 3]$. Note that $[F \preceq_l A] = [-1, 1]$ and $\inf_{x \in [F \preceq_l A]^c} |\nabla F|(x) = 1 > 0$. Theorem 5 applied to this case yields that F has a global error bound modulus $\sigma_A = 1$ at A.

We conclude the section with a result about the existence of a global weak $\preceq_l$-minimizer under the following Palais-Smale type condition involving slope.

Palais-Smale condition (PS) Any sequence $\{x_i\}$ ($x_i \in \text{dom}F$) satisfying

(i) $\{F(x_i)\}$ is bounded in the sense that there exists K-bounded sets A_1 and A_2 such that

$$A_1 \preceq_l F(x_i) \preceq_l A_2, \quad \forall n$$

(ii) $|\nabla F|(x_i) \to 0$

contains a convergent subsequence.

Theorem 6 *Assume that F is K-bounded K-u.s.c. and K-compact-valued. If F satisfies the Palais-Smale condition (PS), then F has a global weak $\preceq_l$-minimizer.*

To prove this theorem, we need a slope version of the Ekeland variational principle for set-valued maps and an auxiliary result.

Theorem 7 *Suppose that F is K-bounded, K-u.s.c. and K-compact-valued. Then for any $\epsilon > 0$, there exists $\bar{x} \in X$ such that*

$$|\nabla F|(\bar{x}) \leq \epsilon.$$

Proof Let $k_0 \in \mathrm{int}K$ be such that $\Delta_{-K}(k_0) = -1$. Applying the set-valued version of the Ekeland variational principle stated in Theorem 4.1 in [12], we find $\bar{x} \in \mathrm{dom}F$ such that

$$F(x) + \epsilon d(x, \bar{x})k_0 \not\preceq_l F(\bar{x}), \quad \forall x \in \mathrm{dom}F,\ x \neq \bar{x}.$$

Lemma 1 gives

$$-\frac{h(F(x), F\bar{x}))}{d(x, \bar{x})} \leq \epsilon \ \text{ for all } x \in \mathrm{dom}F,\ x \neq \bar{x}$$

and therefore, $|\nabla F|(\bar{x}) \leq \epsilon$. □

Lemma 2 *Suppose that A is a nonempty subset of Y such that $L := [F \preceq_l A] \neq \emptyset$. Let $\bar{x} \in L$ and denote by $F_{|_L} : L \rightrightarrows Y$ the restriction of F to the set L. Then*

(i) *$\bar{x}$ is a local weak $\preceq_l$-minimizer of $F_{|_L}$ on L iff it is a local weak $\preceq_l$-minimizer of F on X.*
(ii) $|\nabla F_{|_L}|(\bar{x}) = |\nabla F|(\bar{x})$.

Proof First, we show that

$$x \in [F \preceq_l A]^c \text{ implies } F(x) \not\prec_l F(u),\ \forall u \in [F \preceq_l A]. \tag{8}$$

Indeed, if $F(x) \prec_l F(u)$, we deduce from $F(u) \subseteq F(x) + \mathrm{int}K$ and $F(u) \preceq_l A$ that

$$A \subseteq F(u) + K \subseteq F(x) + \mathrm{int}K + K \subseteq F(x) + K,$$

which means that $F(x) \preceq_l A$, a contradiction to $x \in [F \preceq_l A]^c$.

Next, we prove the only if part in the assertion (i). Assume that $\bar{x}$ is a local weak $\preceq_l$-minimizer of $F_{|_L}$ on L and let U be the neighborhood of $\bar{x}$ in X such that for all $x \in (\mathrm{dom}F \cap L) \cap U$ either $F(x) \equiv_l F(\bar{x})$ or $F(x) \not\prec_l F(\bar{x})$. Let $x \in (\mathrm{dom}F \cap [F \preceq_l A]^c) \cap U$. Then either $F(x) \equiv_l F(\bar{x})$ or $F(x) \not\prec_l F(\bar{x})$ by (8). Thus, $\bar{x}$ is a local weak $\preceq_l$-minimizer of F on X.

Finally, if $\bar{x}$ is a local weak $\preceq_l$-minimizer of F on X and $F_{|_L}$ on L, then $|\nabla F_{|_L}|(\bar{x}) = |\nabla F|(\bar{x}) = 0$. Otherwise, it follows from Proposition 6 and the implication

$$F(u) \prec_l F(\bar{x}) \text{ and } F(\bar{x}) \preceq_l A \text{ imply } F(u) \prec_l A$$

that

$$\begin{aligned}
|\nabla F|(\bar{x}) &= \limsup_{u \to \bar{x}, F(u) \prec_l F(\bar{x})} \tfrac{-h(F(u), F(\bar{x}))}{d(\bar{x}; u)} \\
&= \limsup_{u \to \bar{x}, F(u) \prec_l F(\bar{x}), F(u) \prec_l A} \tfrac{-h(F(u), F(\bar{x}))}{d(\bar{x}; u)} \\
&= \limsup_{u \to \bar{x}, u \in L, F(u) \prec_l F(\bar{x})} \tfrac{-h(F(u), F(\bar{x}))}{d(\bar{x}; u)} \\
&= |\nabla F_{|_L}(\bar{x}).
\end{aligned}$$

□

We return to the proof of Theorem 6.

Proof Since F is K-bounded, there exists a nonempty K-bounded setA_1 such that for all $x \in \mathrm{dom}F$ we have

$$A_1 \preceq_l F(x).$$

By Lemma 2, we may assume that for all $x \in \mathrm{dom}F$ we have

$$F(x) \preceq_l A_2.$$

with $A_2 := F(\bar{u})$ for an arbitrary $\bar{u} \in \mathrm{dom}F$ because all the assumptions of the theorem remain satisfied if we replace X by the set $L := [F \preceq_l F(\bar{u})]$, which is a complete metric space.

Let $k_o \in \mathrm{int}K$ be such that $\Delta_{-K}(k_0) = 1$. Take a sequence of positive scalars $\{\epsilon_i\}$ such that $\epsilon_i \to 0$ and $\epsilon_i < \epsilon_j$ for $i < j$. Since F is K-bounded, for each i, Proposition 3.1 in [12] implies the existence of an $\epsilon_i k_0$-approximate minimizer $u_i \in \mathrm{dom}F$ of F, which means that

$$F(x) + \epsilon_i k_0 \not\preceq_l F(u_i), \quad \forall x \in \mathrm{dom}F. \tag{9}$$

For each $i = 1, 2, ...$, let $U_i := [F \preceq_l F(u_i)]$ and let $F_{|U_i}$ be the restriction of the map F to the set U_i. The set-valued version of the Ekeland variational principle stated in Theorem 7 applied to $F_{|U_j}$ implies the existence of a sequence $\{x_i\}$ with $x_i \in U_i$ such that for all $i = 1, 2, ...$

$$|\nabla F_{|_{U_i}}|(x_i) \leq \epsilon_i.$$

By Lemma 2, for all $i = 1, 2, ...$ we have

$$|\nabla F|(x_i) = |\nabla F_{|_{U_i}}|(x_i) \leq \epsilon_i.$$

Note that since $x_i \in U_i$, we have $F(x_i) \preceq_l F(u_i)$. One can deduce from (9) that x_i also is an $\epsilon_i k_0$-approximate minimizer of F on X, i.e.,

$$F(x) + \epsilon_i k_0 \not\preceq_l F(x_i), \quad \forall x \in \mathrm{dom}F, x \neq x_i.$$

By Lemma 1, it follows that

$$h(F(x), F(x_i)) > -\epsilon_i \Delta_{-K}(k_0) = -\epsilon_i, \quad \forall x \in \mathrm{dom}F. \tag{10}$$

By the Palais-Smale condition, the sequence $\{x_i\}$ contains a convergent subsequence. Going to a subsequence if necessary, we may assume that $x_i \to \bar{x}$ for some $\bar{x} \in \mathrm{dom}F$. Since F is K-u.s.c. at $\bar{x}$, for $i = 1, 2, ...$ one can find integers k_i such that $k_1 < k_2 < ...$, $i < k_i$ and

$$F(\bar{x}) + \frac{\epsilon_i \mathbb{B}}{\mu_1(\mathbb{B})} \preceq_l F(x_j) \quad \forall j \geq k_i,$$

where $\mu_1(\mathbb{B}) := \sup_{b \in \mathbb{B}} \Delta_{-K}(-b)$. Applying Lemma 1, we get

$$h(F(\bar{x}), F(x_j)) \leq \frac{\epsilon_i}{\mu_1(\mathbb{B})} \mu_1(\mathbb{B}) = \epsilon_i, \quad \forall j \geq k_i. \tag{11}$$

Now, let $x \in \text{dom}F$ be an arbitrary element. We will estimate the quantity $h(F(x), F(\bar{x}))$. For $i = 1, 2, ...$, we deduce from (10) and (11) that

$$h(F(x), F(\bar{x})) \geq h(F(x), F(x_{k_i})) - h(F(\bar{x}), F(x_{k_i})) \geq -\epsilon_{k_i} - \epsilon_i > -2\epsilon_i$$

(recall that since $k_i > i$, we have $\epsilon_{k_i} < \epsilon_i$). Since $\epsilon_i \to 0$, we get $h(F(x), F(\bar{x})) \geq 0$ for all $x \in \text{dom}F$. Proposition 7 implies that $\bar{x}$ is a global weak $\preceq_l$-minimizer of F. □

Remark 6 (i) The map F in Example 2 (a) satisfies all conditions of Theorem 6 and it has a global $\preceq_l$-minimizer $x = 0$.

(ii) Note that the condition that F is K-bounded cannot be dropped as one can see in the case with F being the map in Example 2 (b).

Acknowledgements The work has been carried out partly at the Vietnam Institute for Advanced Study in Mathematics (VIASM). The author would like to thank VIASM for hospitality and financial support.

References

1. Aubin JP, Ekeland E (1983) Applied Nonlinear Analysis. John Wiley & Sons, New York
2. Azé D (2003) A survey on error bounds for lower semicontinuous functions. ESAIM, Proceedings 13, pp 1-17
3. Azé D, Corvellec JN (2002) On the sensitivity analysis of Hoffman constants for systems of linear inequalities. SIAM J. Optim. 12: 913-927
4. Azé D, Corvellec JN (2004) Characterizations of error bounds for lower semicontinuous functions on metric spaces. ESAIM Control Optim. Calc. Var. 10: 409-425
5. Bao TQ, Mordukhovich BS (2010) Set-valued optimization in welfare economics. *Advances in mathematical economics* Volume 13. Adv. Math. Econ., 13, Springer, Tokyo, pp 113-153
6. Bednarczuk EM, Kruger AY (2012) Error bounds for vector-valued functions: necessary and sufficient conditions. Nonlinear Anal. 75: 1124-1140
7. Bednarczuk EM, Kruger AY (2014) Error bounds for vector-valued functions on metric spaces. Vietnam J. Math. 40, no. 2-3: 165-180
8. De Giorgi E, Marino A, Tosques M (1980) Problemi di evoluzione in spazi metrici e curve di massima pendenza (Evolution problems in metric spaces and curves of maximal slope, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 68: 180-187
9. Ekeland I (1974) On the Variational Principle, J. Math.l Anal. Appl. 47: 324-353
10. Fabian MJ, Henrion R, Kruger AY, Outrata JV (2010) Error bounds: necessary and sufficient conditions. Set-valued Var. Anal. 18: 121-149
11. Ha TXD (2001) Existence and density results for proper efficiency in cone compact sets. J. Optim. Theory Appl. 111, no. 1: 173-194
12. Ha TXD (2005) Some variants of the Ekeland variational principle for a set-valued map. J. Optim. Theory Appl. 124: 187-206
13. Ha TXD (2005) Lagrange multipliers for set-valued optimization problems associated with coderivatives. J. Math. Anal. Appl. 311: 647-663
14. Ha TXD (2015) Estimates of error bounds for some sets of efficient solutions of a set-valued optimization problem. Set optimization and applicationsthe state of the art Springer Proc. Math. Stat., 151, Springer, Heidelberg, pp 249-273
15. Ha TXD (2018) Slopes, error bounds and weak sharp Pareto minima of a vector-valued map, J. Optim. Theory Appl. 176: 634-649
16. Ha TXD (2018) A Hausdorff-type distance, a directional derivative of a set-valued map and applications in set optimization, Optimization, 67:1031-1050

17. Hamel A (1994) Remarks to equivalent formulation of Ekeland's variational principle. Optimization 31: 233-238
18. Hiriart-Urruty JB (1979) New concepts in nondifferentiable programming. Bull. Soc. Math. France 60: 57-85
19. Ioffe AD, Tikhomirov VM (1979) Theory of Extremal Problems. North-Holland publishing company, Amsterdam-New York
20. Jahn J (2011) Vector Optimization - Theory, Applications, and Extensions. Second Edition. Springer, Berlin
21. Kuroiwa D (1998) The natural criteria in set-valued optimization. RIMS Kokyuroku 1031, pp 85-90
22. Kuroiwa D, Tanaka T, Ha TXD (1997) On cone convexity of set-valued maps. In: Proceedings of the Second World Congress of Nonlinear Analysts, Part 3, Athens, Nonlinear Anal. 30, , no. 3: 1487-1496
23. Mordukhovich BS (2006) Variational Analysis and Generalized differentiation, I: Basic Theory. Grundlehren Series (Fundamental Principles of Mathematical Sciences) 330, Springer, Berlin
24. Luc DT (1989) Theory of Vector Optimization. Springer-Verlag, Berlin
25. Ngai HV, Kruger AY, Théra M (2012) Slopes of multifunctions and extensions of metric regularity. Vietnam J. Math. 40: 355-369
26. Zaffaroni A (2003) Degrees of efficiency and degrees of minimality. SIAM J. Control Optim. 42:1071-1086.