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On reduced-order linear functional interval observers for nonlinear uncertain time-delay systems with external unknown disturbances

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Abstract In this paper, we consider the problem of designing reduced-order linear functional interval observers for nonlinear uncertain time-delay systems with external unknown disturbances. Given bounds on the uncertainties, we design two reduced-order linear functional state observers in order to compute two estimates, an upper one and a lower one, which bound the unmeasured linear functions of state variables. Conditions for the existence of a pair of reduced-order linear functional observers are presented, and they are translated into a linear programming (LP) problem in which the observers' matrices can be effectively computed. Finally, the effectiveness of the proposed design method is supported by four examples and simulation results.

Keywords Reduced-order observers, interval observers, uncertain models, biological systems

1 Introduction

The control of many practical systems such as life sciences, physics and technology, economics, chemistry and biology has recently attracted much interest (see, for example, [6,16,25,26,39,47,49] and the references therein). However,

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the lack of information of state variables of the systems has limited the application of control theory to these systems. In this case, the estimation of actual states and output feedback control law are very necessary. It is well-known that a large amount of additional information can be obtained through available measurement with the help of state observers. These observers are designed to reconstruct the unmeasured states from the measured ones under a priori knowledge of the method. During this last decades, the problem of designing state observers for dynamical systems has aroused considerable attention in the literature. In particular, state observers have important applications in realisation of state-feedback control, system supervision, fault diagnosis of dynamic processes, and general control and diagnosis issues from available information (see, for example, [7,21,27,45,46,48]). There are various state observer design methods and observer structures available in the literature for linear systems (see, for example, [9,19,20,43] and the references therein) and for nonlinear systems (see, for example, [8,15,18,22,23,28,41,50], and the references therein).

In the literature, when dealing with the problem of state estimation for dynamical systems consisting of 'uncertain' facts such as uncertain biological dynamical system [15], the problem of designing interval observers which derive bounds for the solutions of a differential system at any time instant is much attracted (see, for example, [2,13,24,30,33,42], and the references therein). In particular, in [13], the design of interval observers relies on the positivity notion was reported without mentioning time-delays on the system. In [30], the authors proved that for detectable linear systems there always exists an interval observer which can be constructed via an adequate time-varying change of variable. In [42], the authors provided a different approach for stable uncertain linear positive systems. In [33], based on the decomposition of the nonlinear part of the system as a difference of two monotone functions, the design of interval observers for nonlinear uncertain system was reported. In [24] the problem of positive state-bounding observers for positive linear continuous-time systems with both interval uncertainties and time delay was considered, while the authors of [2] provided conditions for the existence of upper and lower estimates for the instantaneous states of the systems and established the asymptotic convergence of the interval error. It was found that the interval observers had many practical applications, such as for control design [12], for fault detection and isolation (see, for example, [4,5,38]), state estimation in biological systems (see, for example, [15,32,36]), interval observers in automotive domain (see, for example, [11,14]), namely, the problem of air-to-fuel ratio interval estimation and control is consided in [11] and a car positioning problem is solved in [14].

Due to the fact that the interval observers have an enlarged dimension with respect to the system dimension (two times bigger than the system) since the upper and lower estimate of the state interval are generated by an observer (see, for example, [31]), the problem of reduction of an interval observer dimension of time-delay systems has been addressed (see, [10]). On the other hand, in practice, many control processes require only the availability of some linear functions of the system states for the purpose of monitoring and stabilizing independantly of disturbances, failures and attacks. Therefore linear functional observers which estimate linear functions of the state vector without estimating all of the individual states have attracted alot of research attentions (see, for example, [9, 43, 44]). Although there have been abundant outstanding results on linear functional observers for time-delay systems, to the best of our knowledge, little attention has been paid to the problem of designing interval observers for linear functions of state variables, which motivates the present study.

In this paper, we address a new problem of designing interval observers for linear functions of the state vector of nonlinear uncertain time-delay systems with external unknown disturbances. The main contributions of this study are highlighted in the following: (1) We first introduce a pair of reduced-order linear functional observers which constructs interval observers for linear functions of the state vector of the considered system; (2) we then derive new conditions for the existence of such reduced-order linear functional observers; (3) by using these conditions and some auxiliary lemmas, we then propose a computational approach based on linear programming (LP) for the determination of the unknown observer matrices; (4) the effectiveness of the proposed design method is supported by four examples and simulation results.

This paper is organized as follows. In section 2, we provide the problem statement and preliminaries. In section 3, we present the main results and some remarks. In section 4, we provide four examples to demonstrate the effectiveness of our proposed design method. Finally, a conclusion is drawn in Section 5.

Notation: I_n denotes the $n \times n$ identity matrix, $0_{m,n}$ denotes the $m \times n$ zero matrix. $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$. For a real matrix M, M^T denotes the transpose, $M \geq 0$ is called a nonnegative matrix if all of its components are nonnegative (i.e. $m_{ij} \geq 0$ for all i, j). For vectors $x = \begin{bmatrix} x_1 \; x_2 \; \dots \; x_n \end{bmatrix}^T \in \mathbb{R}^n$, $y = \begin{bmatrix} y_1 \; y_2 \; \dots \; y_n \end{bmatrix}^T \in \mathbb{R}^n$, $|x| = \begin{bmatrix} |x_1| \; |x_2| \; \dots \; |x_n| \end{bmatrix}^T$, $||x||$ is the Euclidean norm of vector x. $x < y$ ($x \leq y$) means that $x_i < y_i$ ($x_i \leq y_i$), $\forall i = 1, 2, \dots, n$. $\mathbf{1}_n$ denotes the vector in $\mathbb{R}^n$ with all entries equal one, i.e., $\underbrace{\begin{pmatrix} 1 \; 1 \; \dots \; 1 \end{pmatrix}}_{n}^T$. We denote by diag($\lambda$) the diagonal matrix whose entries are formed by the components of the vector λ. Note that λ=diag(λ)$\mathbf{1}_n$. $\mathcal{B}^s$ denotes the set of all vectors $v \in \mathbb{R}^s$ with the property $||v|| < \infty$. $\mathcal{C}^s_\tau = C([-\tau, 0], \mathbb{R}^s)$ denotes the set of continuous maps from $[-\tau, 0]$ into $\mathbb{R}^s$; $\mathcal{C}^s_{\tau+} = \{\varphi \in \mathcal{C}^s_\tau : \varphi(\zeta) \in \mathbb{R}^s_+, \; \zeta \in ([-\tau, 0]\}$. Given a matrix $M \in \mathbb{R}^{m \times n}$, define $M^+ = \max(0, M)$, $M^- = M^+ - M$ and $|M| = M^+ + M^-$.

2 Problem statement and preliminaries

Consider the following nonlinear system with time-delays

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{q} A_i x(t-\tau_i) + Bu(t) + f(t,\psi(t)) + d(t),\ t \geq 0, \quad (1)$$

$$x(\theta) = \phi(\theta) \in \mathbb{R}^n,\ \forall\theta \in [-\tau_{\max}, 0],\ \tau_{\max} = \max_{1\leq i\leq q} \tau_i, \quad (2)$$

$$y(t) = Cx(t) + \sum_{i=1}^{q} C_i x(t-\rho_i),\ \forall t \geq \rho_{\max},\ \rho_{\max} = \max_{1\leq i\leq q} \rho_i, \quad (3)$$

$$\psi(t) = y(t) + \omega(t), \quad (4)$$

where $x(t) \in \mathbb{R}^n$ is the plan state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the plan output with $y(t)$ being defined as $(C+\sum_{i=1}^{q} C_i)x(t),\ \forall t \in [-\tau_{max}+\rho_{\max}, \rho_{\max}]$, $\psi(t)$ is the measurement for feedback control, $\omega(t) \in \mathbb{R}^p$ is the unknown disturbance in the output satisfying $||\omega|| \leq \Omega$ where Ω is constant and known, $\phi(\theta)$ is a continuous initial function, matrices A, A_i, B, C, C_i $(i = 1, 2, \ldots, q)$ are constant and of appropriate dimensions. The time delays $\tau_i \geq 0, \rho_i \geq 0$, $i = 1, 2, \ldots, q$, are assumed to be known constant. The nonlinearity $f(t,\psi(t)) \in \mathbb{R}_+ \times \mathbb{R}^p$ is assumed to be measured with respect to t and Lipschitz continuous with respect to $\psi(t)$, not well known, functionally bounded, i.e. there exist known maps $f^-(t,\psi(t))$ and $f^+(t,\psi(t))$ (bounded below by $f^-(t,\psi(t))$ and above by $f^+(t,\psi(t))$) such that $f^-(t,\psi(t)) \leq f(t,\psi(t)) \leq f^+(t,\psi(t))$ for all $(t,\cdot) \in \mathbb{R}_+ \times \mathbb{R}^p$. $d(t) \in \mathbb{R}^n$ is the external disturbance satisfying $d^-(t) \leq d(t) \leq d^+(t)$ for all $t \geq 0$, where $d^-(t) \in \mathbb{R}^n$ and $d^+(t) \in \mathbb{R}^n$ are two known bounds.

The following definitions and lemmas will be used in this paper.

Definition 1 The nonlinear time-delay system of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{q} A_i x(t-\tau_i) + g(u(t),\psi(t),d(t)),\ t \geq 0, \quad (5)$$

$$x(\theta) = \phi(\theta),\ \forall\theta \in [-\tau_{\max}, 0], \quad (6)$$

$$y(t) = Cx(t) + \sum_{i=1}^{q} C_i x(t-\rho_i),\ \forall t \geq \rho_{\max},\ \rho_{\max} = \max_{1\leq i\leq q} \rho_i, \quad (7)$$

$$\psi(t) = y(t) + \omega(t) \quad (8)$$

is said to be positive if, for any initial condition $\phi(\theta) \in \mathbb{R}^n_+,\ \forall\theta \in [-\tau, 0],\ \tau = \max\{\tau_{\max}, \rho_{\max}\}$, any input $u(t) \in \mathbb{R}^m_+,\ \forall t \geq 0$, $y(t) \in \mathbb{R}^p$ and $d(t) \in \mathbb{R}^n$, the corresponding trajectory $x(t) \in \mathbb{R}^n_+$ for all $t \geq 0$.

Definition 2 [29] A square real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative, i.e. $m_{ij} \geq 0,\ i \neq j$.

Lemma 1 *[1,13] For a given Metzler matrix A, the following conditions are equivalent.*
(i) The matrix A is Hurwitz stable.
(ii) The matrix A is nonsingular and $A^{-1} < 0$.
(iii) There exists a vector $\lambda > 0$ such that $A\lambda < 0$.

Let $\phi(\theta) \in \mathbb{R}^n_+,\ \forall \theta \in [-\tau, 0]$ and $u(t) \in \mathbb{R}^m_+,\ \forall t \geq 0$. Then from [17] and [44], we obtain the following condition which ensures the positivity of system (5)-(8).

Lemma 2 *System* (5)-(8) *is positive if and only if A is a Metzler matrix, A_i $(i = 1, 2, \dots, q)$ are nonnegative matrices and $g(u(t), \psi(t), d(t)) \geq 0$ for all $t \geq 0$.*

3 Main results

Let $z(t) = Fx(t) \in \mathbb{R}^r$, $1 \leq r \leq n$, be defined as a linear function of the state vector, where $F \geq 0$ is any given $r \times n$ matrix. Our objective in this paper is to design two reduced-order linear functional observers in order to compute two estimates, an upper one $\hat{z}^+(t)$ and a lower one $\hat{z}^-(t)$, which bound the unmeasured linear function $z(t) = Fx(t)$, i.e. $\hat{z}^-(t) \leq z(t) \leq \hat{z}^+(t)$ for all $t \geq 0$ and the estimated error $e(t) = \hat{z}^+(t) - \hat{z}^-(t)$ is bounded. To achieve the objective, we consider the following reduced-order linear functional observers:

$$\begin{aligned}\dot{\hat{z}}^+(t) &= N\hat{z}^+(t) + \sum_{i=1}^{q} N_i \hat{z}^+(t-\tau_i) + J\psi(t) + |J|\mathbf{1}_p \Omega + Ff^+(t, \psi(t)) \\ &\quad + FBu(t) + Fd^+(t),\ t \geq 0, \qquad (9)\\ \hat{z}^+(\theta) &= \phi^+(\theta) \in \mathbb{R}^r,\ \forall \theta \in [-\tau, 0], \qquad (10)\\ \dot{\hat{z}}^-(t) &= N\hat{z}^-(t) + \sum_{i=1}^{q} N_i \hat{z}^-(t-\tau_i) + J\psi(t) - |J|\mathbf{1}_p \Omega + Ff^-(t, \psi(t)) \\ &\quad + FBu(t) + Fd^-(t),\ t \geq 0, \qquad (11)\\ \hat{z}^-(\theta) &= \phi^-(\theta) \in \mathbb{R}^r,\ \forall \theta \in [-\tau, 0], \qquad (12)\end{aligned}$$

where $\hat{z}^+(t) \in \mathbb{R}^r$, $\hat{z}^-(t) \in \mathbb{R}^r$, $\phi^+(\theta)$ and $\phi^-(\theta)$ are continuous initial functions. Matrices $N \in \mathbb{R}^{r\times r}$, $N_i \in \mathbb{R}^{r\times r}$ and $J \in \mathbb{R}^{r\times p}$ are unknown observer parameters.

From now on we assume that $z(t) = Fx(t) \in \mathcal{B}^r$, $1 \leq r \leq n$, where $x(t)$ is the state of the system (1)-(3).

Definition 3 For system (1)-(3), let $f^-(t, \psi(t)) \leq f(t, \psi(t)) \leq f^+(t, \psi(t))$ for all $(t, \cdot) \in \mathbb{R}_+ \times \mathbb{R}^p$ and $d^-(t) \leq d(t) \leq d^+(t)$ for all $t \geq 0$ for some known $f^-(t, \psi(t)) \in \mathbb{R}^n$, $f^+(t, \psi(t)) \in \mathbb{R}^n$, $d^-(t) \in \mathbb{R}^n$ and $d^+(t) \in \mathbb{R}^n$. Then the system (9) and (11) are called a reduced-order linear functional interval observer for (1)-(4) if for any initial conditions, $\phi^+(\theta), \phi^-(\theta) \in \mathbb{R}^r,\ \theta \in [-\tau, 0]$, the solutions of equations (1), (9) and (11) exist, $\hat{z}^-(t), \hat{z}^+(t) \in \mathcal{B}^r$ and

$$\hat{z}^-(t) \le z(t) \le \hat{z}^+(t) \tag{13}$$

for all $t \ge 0$ provided that the relation $\phi^-(\theta) \le F\phi(\theta) \le \phi^+(\theta)$ holds.

Let us next define the upper error $e^+(t)$, the lower error $e^-(t)$ and the total error $e(t)$ as

$$\begin{aligned}
e^+(t) &= \hat{z}^+(t) - z(t),\ t \ge 0,\\
e^+(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0],\\
e^-(t) &= z(t) - \hat{z}^-(t),\ t \ge 0,\\
e^-(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0],\\
e(t) &= e^+(t) + e^-(t) = \hat{z}^+(t) - \hat{z}^-(t),\ t \ge 0,\\
e(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0],
\end{aligned}$$

The following theorem provides conditions which guarantee the existence of a reduced-order linear functional interval observer for (1)-(4).

Theorem 1 *Assume that there exist a vector $\lambda \in \mathbb{R}^r_+$, a Metzler matrix $N \in \mathbb{R}^{r\times r}$, nonnegative matrices $N_i \in \mathbb{R}^{r\times r}_+$, two fixed positive vectors b_1 and b_2 satisfying the following:*

$$(N + \sum_{i=1}^{q} N_i)\lambda < 0, \tag{14}$$

$$NF - FA + JC = 0, \tag{15}$$

$$N_iF - FA_i = 0,\ i = 1, 2, \ldots, q, \tag{16}$$

$$JC_i = 0,\ i = 1, 2, \ldots, q, \tag{17}$$

$$F(f^+(t, \psi(t)) - f^-(t, \psi(t))) \le b_1,\ F(d^+(t) - d^-(t)) \le b_2. \tag{18}$$

Then,

$$\hat{z}^-(t) \le z(t) \le \hat{z}^+(t), \quad \forall t \ge 0 \tag{19}$$

and

$$e(t) \le -(N + \sum_{i=1}^{q} N_i)^{-1}(b_1 + b_2 + 2|J|\mathbf{1}_p\Omega), \quad \forall t \ge 0. \tag{20}$$

Moreover, $-(N + \sum_{i=1}^{q} N_i)^{-1}(b_1 + b_2 + 2|J|\mathbf{1}_p\Omega)$ is the smallest bound of the estimated error.

Proof Regarding (1), (9) and (11), the derivatives of $e^+(t)$ and $e^-(t)$ are given by

$$\begin{aligned}
\dot{e}^+(t) &= \dot{\hat{z}}^+(t) - F\dot{x}(t),\\
&= Ne^+(t) + \sum_{i=1}^{q} N_i e^+(t-\tau_i) + (NF - FA + JC)x(t)\\
&\quad + \sum_{i=1}^{q} JC_i x(t-\rho_i) + \sum_{i=1}^{q}(N_iF - FA_i)x(t-\tau_i)\\
&\quad + F(f^+(t,\psi(t)) - f(t,\psi(t))) + F(d^+(t) - d(t))\\
&\quad + J\omega(t) + |J|\mathbf{1}_p\Omega,\ t \geq 0, \qquad (21)\\
e^+(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0], \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
\dot{e}^-(t) &= F\dot{x}(t) - \dot{\hat{z}}^-(t),\\
&= Ne^-(t) + \sum_{i=1}^{q} N_i e^-(t-\tau_i) + (FA - NF - JC)x(t)\\
&\quad - \sum_{i=1}^{q} JC_i x(t-\rho_i) + \sum_{i=1}^{q}(FA_i - N_iF)x(t-\tau_i)\\
&\quad + F(f(t,\psi(t)) - f^-(t,\psi(t))) + F(d(t) - d^-(t))\\
&\quad - J\omega(t) + |J|\mathbf{1}_p\Omega,\ t \geq 0, \qquad (23)\\
e^-(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0], \qquad (24)
\end{aligned}$$

It is clear from (21) and (23) that if conditions (15)-(16) of Theorem 1 are satisfied, then we obtain the following:

$$\begin{aligned}
\dot{e}^+(t) &= Ne^+(t) + \sum_{i=1}^{q} N_i e^+(t-\tau_i) + F(f^+(t,\psi(t)) - f(t,\psi(t)))\\
&\quad + F(d^+(t) - d(t)) + J\omega(t) + |J|\mathbf{1}_p\Omega,\ t \geq 0, \qquad (25)\\
e^+(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0], \qquad (26)\\
\dot{e}^-(t) &= Ne^-(t) + \sum_{i=1}^{q} N_i e^-(t-\tau_i) + F(f(t,\psi(t)) - f^-(t,\psi(t)))\\
&\quad + F(d(t) - d^-(t)) - J\omega(t) + |J|\mathbf{1}_p\Omega,\ t \geq 0, \qquad (27)\\
e^-(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0], \qquad (28)\\
\dot{e}(t) &= Ne(t) + \sum_{i=1}^{q} N_i e(t-\tau_i) + F(f^+(t,\psi(t)) - f^-(t,\psi(t)))\\
&\quad + F(d^+(t) - d^-(t)) + 2|J|\mathbf{1}_p\Omega,\ t \geq 0, \qquad (29)\\
e(\theta) &= 0 \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0]. \qquad (30)
\end{aligned}$$

Since N is Metzler, $N_i \geq 0$, $F \geq 0$, $d^-(t) \leq d(t) \leq d^+(t)$ for all $t \geq 0$, $f^-(t,\psi(t)) \leq f(t,\psi(t)) \leq f^+(t,\psi(t))$ for all $(t,\cdot) \in \mathbb{R}_+ \times \mathbb{R}^p$, $||\omega|| \leq \Omega$ and $|J| = J^+ + J^-$, $J^+ = \max(0,J)$, $J^- = J^+ - J$, systems (25)-(26), (27)-(28) and (29)-(30) are positive. Hence, we obtain (19) and $e(t) \geq 0$ for all $t \geq 0$. Let us next prove that the errors $e^+(t)$ and $e^-(t)$ are bounded. We will present the proof for $e^+(t)$ (the proof for $e^-(t)$ is the same). Denote $\bar{\Delta}(t) = F(f^+(t,\psi(t)) - f(t,\psi(t))) + F(d^+(t) - d(t)) + J\omega(t) + |J|\mathbf{1}_p\Omega$. By using assumption (18) and $||\omega|| \leq \Omega$, we have $\bar{\Delta}(t) \in \mathcal{B}^r$. Since the matrix N is Metzler and the matrices N_i, $(i = 1,2,\ldots,q)$ are non-negative, it follows from Lemma 3 in [10] that there exist some $p_1, p_2 \in \mathbb{R}^r_+$ ($p_1 > 0$ and $p_2 > 0$) such that

$$p_1^T(N + \sum_{i=1}^{q} N_i) = -p_2^T.$$

Let us consider for equation (25) the Lyapunov functional $V : \mathcal{C}^r_{\tau+} \to \mathbb{R}_+$ defined as

$$V(\varphi) = p_1^T\varphi(0) + \sum_{i=1}^{q}\int_{-\tau_i}^{0} p_1^T N_i \varphi(\zeta) d\zeta.$$

Clearly, for any $\varphi \in \mathcal{C}^r_{\tau+}$ the functional V is positive definite and radially unbounded, its derivative for $e^+(t)$ takes the form

$$\dot{V} = p_1^T\Big[(N + \sum_{i=1}^{q} N_i)e^+(t) + \bar{\Delta}(t)\Big] \leq -p_2^T e^+(t) + p_1^T\bar{\Delta}(t).$$

Therefore, for $\bar{\Delta}(t) = 0$, the system is globally asymptotically stable, and since $\bar{\Delta}(t) \in \mathcal{B}^r$, we have $e^+(t) \in \mathcal{B}^r$ (for further details on the proof, the reader can refer to [37]). The boundedness of $\hat{z}^-(t), \hat{z}^+(t)$ is implied by boundedness of $z(t)$, $e^+(t)$ and $e^-(t)$.

Now, we prove (20). Let us first express $N = \mathcal{N} + N_0$, where $\mathcal{N}$ is Metzler matrix and N_0 is nonnegative matrix. We now denote $\varphi(t, e(t), e(t-\tau_1), \ldots, e(t-\tau_q), \nu(t)) = \sum_{i=0}^{q} N_i e(t-\tau_i) + \nu(t)$, where $\tau_0 = 0$ and $\nu(t) = F(f^+(t,\psi(t)) - f^-(t,\psi(t))) + F(d^+(t) - d^-(t)) + 2|J|\mathbf{1}_p\Omega$. Then equation (29) can be rewritten as follows

$$\dot{e}(t) = \mathcal{N}e(t) + \varphi(t, e(t), e(t-\tau_1), \ldots, e(t-\tau_q), \nu(t)). \tag{31}$$

It follows from (18) that $0 \leq \nu(t) \leq \bar{\nu}$, where $\bar{\nu} = b_1 + b_2 + 2|J|\mathbf{1}_p\Omega$. Since $\mathcal{N}$ is Metzler matrix, $N_i \geq 0$, $i = 0,1,\ldots,q$ and $\mathcal{N} + \sum_{i=0}^{q} N_i = N + \sum_{i=1}^{q} N_i$, from (14) and Lemma 1, we see that matrix $\mathcal{N} + \sum_{i=0}^{q} N_i$ is invertible and $(\mathcal{N} + \sum_{i=0}^{q} N_i)^{-1} < 0$. Hence, from Lemma 3 in the work of [35], we have $\mu(\mathcal{N} + \sum_{i=0}^{q} N_i) < 0$, where $\mu(\mathcal{N} + \sum_{i=0}^{q} N_i)$ stands for the spectral abscissa of matrix $\mathcal{N} + \sum_{i=0}^{q} N_i$. Denote

$$\ell = (N + \sum_{i=1}^{q} N_i)^{-1}(b_1 + b_2 + 2|J|\mathbf{1}_p\Omega). \tag{32}$$

It is easy to see that

$$\ell = (\mathcal{N} + \sum_{i=0}^{q} N_i)^{-1}\bar{\nu}. \tag{33}$$

Therefore, by using Theorem 1 in the work of [35], we have that the box $B(0,\ell) = \{x \in \mathbb{R}^r : |x| \leq \ell\}$, where ℓ defined as in (32) is the smallest box which bounds reachable sets of system (29)-(30), that is, ℓ is the smallest bound of estimated error $e(t)$. The proof is complete.

Remark 1 It is worth noting that for each bound of the uncertainty in the system, the method [15] only gives a bound of the estimated error. In this paper, our method (Theorem 1) gives the smallest bound of the estimated error with respect to each bound of the uncertainty in the system.

Remark 2 From Theorem 1, the design of a pair of reduced-order linear functional observers now rest with determining unknown observer parameters $N \in \mathbb{R}^{r\times r}$, $N_i \in \mathbb{R}^{r\times r}$ such that conditions (14)-(16) of Theorem 1 hold.

For this, we represent equations (15)-(17) into the following form

$$\chi X = Y, \tag{34}$$

where

$$\chi = \begin{bmatrix} N \ N_1 \ N_2 \dots N_q \ J \end{bmatrix}, \ X = \begin{bmatrix} X_1 \ X_2 \\ X_3 \ X_4 \end{bmatrix} \in \mathbb{R}^{[(q+1)r+p]\times(2q+1)n}, \tag{35}$$

$$X_1 = \text{block-diag}\left(F, F, \dots, F\right) \in \mathbb{R}^{(q+1)r\times(q+1)n}, \ X_2 = 0_{(q+1)r,qn}, \tag{36}$$

$$X_3 = \begin{bmatrix} C \ 0 \dots 0 \end{bmatrix} \in \mathbb{R}^{p\times(q+1)n}, \ X_4 = \begin{bmatrix} C_1 \ C_2 \dots C_q \end{bmatrix} \in \mathbb{R}^{p\times qn} \tag{37}$$

$$Y = \begin{bmatrix} FA \ FA_1 \ FA_2 \dots FA_q \ 0 \dots 0 \end{bmatrix} \in \mathbb{R}^{r\times(2q+1)n}. \tag{38}$$

Since X and Y are two known constant matrices, a solution for χ always exists if and only if [3], [40]

$$\text{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = \text{rank}(X). \tag{39}$$

Under condition (39), a general solution for χ is given by

$$\chi = YX^+ + Z\left(I_{(q+1)r+p} - XX^+\right), \tag{40}$$

where $X^+ \in \mathbb{R}^{(2q+1)n\times[(q+1)r+p]}$ is the Moor-Penrose-inverse of X and $Z \in \mathbb{R}^{r\times[(q+1)r+p]}$ is an arbitrary matrix to be determined. Moreover, matrices N, N_i $(i = 1, 2, \dots, q)$ and J can now be extracted from (40) and are expressed as

$$N = \Phi e_N + Z\Psi e_N, \tag{41}$$

$$N_i = \Phi e_{N_i} + Z\Psi e_{N_i}, \tag{42}$$

$$J = \Phi e_J + Z\Psi e_J, \tag{43}$$

where

$$\Phi = YX^{+},\ \Psi = I_{(q+1)r+p} - XX^{+} \tag{44}$$

and $e_N, e_{N_i} \in \mathbb{R}^{[(q+1)r+p]\times r}$, $e_J \in \mathbb{R}^{[(q+1)r+p]\times p}$ are the following

$$e_N = \begin{bmatrix} I_r \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_{N_1} = \begin{bmatrix} 0 \\ I_r \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ e_{N_i} = \begin{bmatrix} 0 \\ 0 \\ \overbrace{I_r}^{(i+1)-th} \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ i = 2, \dots, q-2,$$

$$e_{N_{q-1}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \overbrace{I_r}^{q-th} \\ 0 \\ 0 \end{bmatrix},\ e_{N_q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \overbrace{I_r}^{(q+1)-th} \\ 0 \end{bmatrix},\ e_J = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ I_p \end{bmatrix}. \tag{45}$$

Now, in order to implement reduced-order linear functional observers (9)-(10) and (11)-(12), we will gather conditions (14)-(16) to formulate an LP-based problem for checking the design parameters. By using (41)-(42), condition (14) can be represented as

$$(e_N^T + \sum_{i=1}^{q} e_{N_i}^T)\Phi^T\lambda + (e_N^T + \sum_{i=1}^{q} e_{N_i}^T)\Psi^T\Gamma\mathbf{1}_r < 0. \tag{46}$$

Based the above discussion we obtain the following theorem which provides a computational approach which is based on LP for the determination of the parameters N, N_i $(i = 1, 2, \dots, q)$ of linear functional observers.

Theorem 2 *If the following LP problem with the variables $\lambda \in \mathbb{R}^r$ and $\Gamma \in \mathbb{R}^{[(q+1)r+p]\times r}$ is feasible:*

$$\begin{cases} \lambda > 0, \\ (e_N^T + \sum_{i=1}^{q} e_{N_i}^T)\Phi^T\lambda + (e_N^T + \sum_{i=1}^{q} e_{N_i}^T)\Psi^T\Gamma\mathbf{1}_r < 0, \end{cases} \tag{47}$$

then the observer gains N, N_i $(i = 1, 2, \dots, q)$ are obtained as in (41)-(42) where $Z = (diag(\lambda))^{-1}\Gamma^T$.

We now propose an effective algorithm to obtain the observer parameters for the pair of reduced-order linear functional observers (9)-(10) and (11)-(12).

Algorithm 1
Step 1: For given matrices A, A_i $(i = 1, 2, \dots, q)$, obtain matrices X and Y from (35)-(38). Check the existence condition (39).
Step 2: Compute the matrices Φ and Ψ from (44).
Step 3: Solve the LP problem (47) with respect to Γ and λ.
Step 4: Compute the matrix $Z = (\mathrm{diag}(\lambda))^{-1}\Gamma^T$ where (λ, Γ) is a solution obtained in Step 3.
Step 5: Substitute Z into (41)-(43) to obtain observer gains N, N_i $(i = 1, 2, \dots, q)$ and J.

Remark 3 For the case where $A_i = 0,\ C_i = 0,\ i = 1, 2, \dots, q$, i.e. system (1)-(4) has no time delay, the following result is an extension of the work [15] to the design of reduced-order linear functional interval observers. Note that in [15], only full-order Luenberger-type interval observers for nonlinear systems without inputs, sensor noise as well as external disturbance were considered. Now, we have the following reduced-order linear functional interval observers $[\hat{z}^-(t), \hat{z}^+(t)]$:

$$\begin{aligned}
\dot{\hat{z}}^+(t) &= N\hat{z}^+(t) + Ff^+(t,\psi(t)) + FBu(t) + Fd^+(t) \\
&\quad + J\psi(t) + |J|\mathbf{1}_p\Omega,\ t \geq 0, && (48)\\
\hat{z}^+(0) &= \hat{z}_0^+ \in \mathbb{R}^r, && (49)\\
\dot{\hat{z}}^-(t) &= N\hat{z}^-(t) + Ff^-(t,\psi(t)) + FBu(t) + Fd^-(t) \\
&\quad + J\psi(t) - |J|\mathbf{1}_p\Omega,\ t \geq 0, && (50)\\
\hat{z}^-(0) &= \hat{z}_0^- \in \mathbb{R}^r, && (51)
\end{aligned}$$

where $\hat{z}^+(t) \in \mathbb{R}^r$, $\hat{z}^-(t) \in \mathbb{R}^r$, matrices $N \in \mathbb{R}^{r\times r}$ and $J \in \mathbb{R}^{r\times p}$ are unknown observer parameters. We can obtain conditions to ensure that $[\hat{z}^-(t), \hat{z}^+(t)]$ is an interval observer of the linear function $z(t) = Fx(t)$. The conditions are as given below

$$\begin{cases} N \text{ is Metzler and Hurwitz,} \\ NF - FA + JC = 0, \\ F(f^+(t,\psi(t)) - f^-(t,\psi(t))) \leq b_1,\ F(d^+(t) - d^-(t)) \leq b_2. \end{cases} \tag{52}$$

Hence, N and J can be determined if the following LP problem with the variables $\lambda \in \mathbb{R}^r$ and $\Gamma \in \mathbb{R}^{r\times r}$ is feasible:

$$\begin{cases} \lambda > 0, \\ (e_N^T)\Phi^T\lambda + (e_N^T)\Psi^T\Gamma\mathbf{1}_r < 0. \end{cases} \tag{53}$$

4 Examples

4.1 Example 1 (A numerical example).

Consider a fifth-order time-delay system of the form (1)-(4), where

$$A = \begin{bmatrix} -4 & 2 & 1 & 3 & 2 \\ 1 & -6 & 1 & 4 & 0 \\ 0 & 0 & -9 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 2 & 0 & 0 & 1 & -7 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0.1 & 0.1 & 0.2 & 0.4 & 0.2 \\ 0.3 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0.2 & 0.1 & 0.3 & 0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.4 & 0.1 \\ 0 & 0.2 & 0.1 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \\ 0.1 & 0.2 & 0.4 & 0.3 & 0.5 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \ d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \\ d_4(t) \\ d_5(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+t} + 0.1 \\ \frac{2}{1+t} + 0.2 \\ \frac{1}{t+1} + 1 + |\sin(0.1t)| \\ |\sin(0.05t)| + 1 \\ \frac{1}{t+1} + 0.3 \end{bmatrix},$$

$$f(t, \psi(t)) = \begin{bmatrix} 0 \\ 0 \\ \frac{a_1(t)(y_1(t)+\omega_1(t))}{b_1+(y_2(t)+\omega_2(t))} \\ \frac{a_2(t)(y_2(t)+\omega_2(t))}{b_2+(y_1(t)+\omega_1(t))} \\ -c(t)(y_2(t)+\omega_2(t)) \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Assume that known bounds $Ff^-(t, \psi(t))$, $Ff^+(t, \psi(t))$, $Fd^-(t)$, $Fd^+(t)$ are as

$$Ff^+(t, \psi(t)) = \begin{bmatrix} \frac{a_1^+(t)(y_1(t)+\omega_1(t))}{b_1+(y_2(t)+\omega_2(t))} \\ \frac{a_2^+(t)(y_2(t)+\omega_2(t))}{b_2+(y_1(t)+\omega_1(t))} \end{bmatrix}, \ Ff^-(t, \psi(t)) = \begin{bmatrix} \frac{a_1^-(t)(y_1(t)+\omega_1(t))}{b_1+(y_2(t)+\omega_2(t))} \\ \frac{a_2^-(t)(y_2(t)+\omega_2(t))}{b_2+(y_1(t)+\omega_1(t))} \end{bmatrix},$$

where $a_1(t) = |\sin(0.1t)| + 1$, $a_2(t) = |\sin(0.1t)| + 0.1$, $c(t) = |\sin(0.05t)|$ and $a_1^-(t) \leq a_1(t) \leq a_1^+(t)$, $a_2^-(t) \leq a_2(t) \leq a_2^+(t)$,

$$Fd^-(t) = \left[\frac{1}{t+1} + 0.9 + |\sin(0.1t)| \ |\sin(0.5t)| + 0.6 \right]^T,$$

$$Fd^+(t) = \left[\frac{1}{t+1} + 1.1 + |\sin(0.1t)| \ |\sin(0.5t)| + 1.5 \right]^T, \ \forall t \geq 0.$$

A random measurement disturbance is chosen with $||\omega|| \leq \frac{1}{2} = \Omega$. We aim to use the Algorithm 1 to compute two estimates, an upper one and a lower one, which bound $z_1(t) = x_3(t)$ and $z_2(t) = x_4(t)$. According to Step 1 of Algorithm 1, we obtain matrices X and Y from equations (35)-(38). Since

$$\operatorname{rank}\begin{bmatrix} X \\ Y \end{bmatrix} = 8 = \operatorname{rank}\begin{bmatrix} X \end{bmatrix},$$

condition (39) is satisfied. By setting a constraint

$$\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \leq \lambda \leq \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

the LP problem (47) is feasible with

$$\lambda = \begin{bmatrix} 1.9649 \\ 2.3513 \end{bmatrix} \text{ and } \Gamma^T = \begin{bmatrix} -4.1888\ 0\ 0\ 0\ 0\ 0\ -4.1888\ 0 \\ -4.1888\ 0\ 0\ 0\ 0\ 0\ -4.1888\ 0 \end{bmatrix}. \tag{54}$$

Now, taking (54) into account for Step 4 and Step 5 of Algorithm 1, the observer gains are obtained as

$$N = \begin{bmatrix} -9 & 0 \\ 1 & -4 \end{bmatrix}, \ N_1 = \begin{bmatrix} 0.5 & 0 \\ 0.2 & 0 \end{bmatrix}, \ N_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$J = 0_{2,2}, \ b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ b_2 = \begin{bmatrix} 0.2 \\ 0.9 \end{bmatrix}.$$

With N, N_1, N_2, b_1 and b_2 have been obtained, it is not hard to see that the conditions of Theorem 1 hold. Hence, we obtain a pair of second-order linear functional observers to reconstruct $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix}$ with $\ell = -(N + N_1 + N2)^{-1}(b_1 + b_2 + 2|J|\mathbf{1}_2\Omega) = \begin{bmatrix} 0.1463 \\ 0.5462 \end{bmatrix}$ is the smallest bound of the estimated error $e(t) = \hat{z}^+(t) - \hat{z}^-(t)$. For simulation, let us consider the input $u(t) = e^{0.1t}$, $0 \leq t \leq 30$, $\tau_1 = 0.7s$, $\tau_2 = 0.8s$, $\rho_1 = 0.5s$, $\rho_2 = 0.2s$ and the initial conditions are $x_1(\theta) = 1$, $x_2(\theta) = 2$, $x_3(\theta) = 3$, $x_4(\theta) = 4$, $x_5(\theta) = 5$, $\hat{z}_1^+(\theta) = 4$, $\hat{z}_1^-(\theta) = -2$, $\hat{z}_2^+(\theta) = 5$, $\hat{z}_2^-(\theta) = -3$ for $\theta \in [-0.8, 0]$. Figure 1 shows the responses of $z_1(t) = x_3(t)$, $\hat{z}_1^-(t)$ and $\hat{z}_1^+(t)$, while Figure 2 shows the responses of $z_2(t) = x_4(t)$, $\hat{z}_2^-(t)$ and $\hat{z}_2^+(t)$.

4.2 Example 2 (A practical example).

Consider the following population structured into three stages:

$$\begin{cases} \dot{x}_1(t) = -\alpha_1 x_1(t) - m_1 x_1(t) + r(t, x_3(t)) + d_1(t), \\ \dot{x}_2(t) = \alpha_1 x_1(t-\tau) - \alpha_2 x_2(t) - m_2 x_2(t) + u(t) + d_2(t), \\ \dot{x}_3(t) = \alpha_2 x_2(t-\tau) - m_3 x_3(t) - c(t) x_3(t) + 2u(t) + d_3(t), \\ y(t) = x_3(t), \\ \psi(t) = y(t) + \omega(t) \end{cases} \tag{55}$$

with the initial conditions $x_1(\theta) = x_2(\theta) = x_3(\theta) = 0$ for all $\theta \in [-\tau, 0]$. In the above model $x_1(t)$, $x_2(t)$, $x_3(t)$ are larvae, juveniles, adults, respectively.

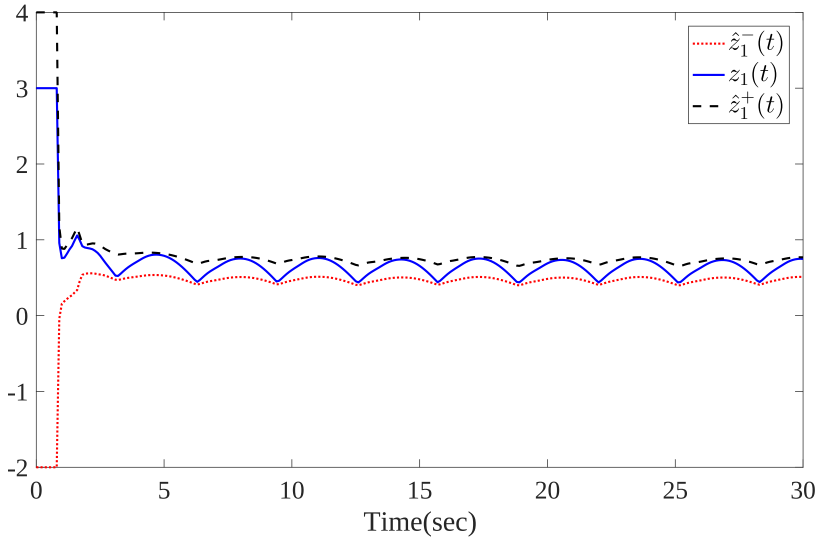

Fig. 1 Responses of $z_1(t) = x_3(t)$, $\hat{z}_1^-(t)$ and $\hat{z}_1^+(t)$

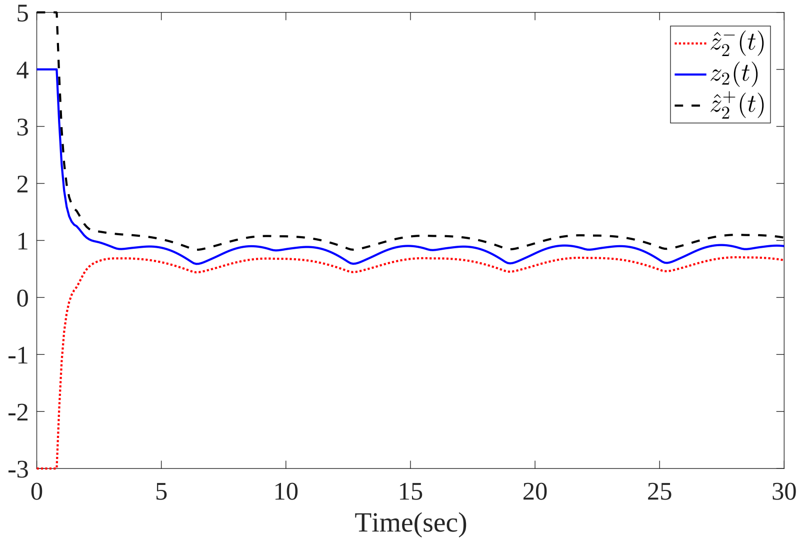

Fig. 2 Responses of $z_2(t) = x_4(t)$, $\hat{z}_2^-(t)$ and $\hat{z}_2^+(t)$

The positive coefficients α_i and m_i represent the growth and mortality rates, respectively. The term $c(t)$ represents a harvesting effort on the adult population. In model (55), we assume that only measurement available observation with a noise concerns the adults stock, i.e. $y(t) = x_3(t)$, $\psi(t) = y(t) + \omega(t)$ where $\omega(t)$ is the unknown disturbance in the output and the births in class $x_1(t)$ are generated only by the adults class $x_3(t)$ with a reproduction law of

Beverton-Holt type [34]:

$$r(t,x_3(t)) = \frac{a(t)x_3(t)}{b+x_3(t)},\ a(t)>0,\ b>0. \tag{56}$$

The parameter $\tau > 0$ exists due to the delay in the increment of larvae and juveniles. The vector $d(t) = \begin{bmatrix} d_1(t)\ d_2(t)\ d_3(t) \end{bmatrix}^T$ is the external disturbance and $u(t)$ is the control input vector, $y(t)$ is the plan output vector. Note that, when $\tau = 0$, $d(t) \equiv 0$, $u(t) \equiv 0$ and $\omega(t) \equiv 0$, the model (55) is reduced to the one considered in [15]. Denoting $\beta_1 = \alpha_1 + m_1$, $\beta_2 = \alpha_2 + m_2$ and $\beta_3 = m_3$. Then (55) can be expressed into the form (1)-(4), where $q = 1$, $\tau_1 = \tau$, $d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \end{bmatrix}$, $A = \begin{bmatrix} -\beta_1 & 0 & 0 \\ 0 & -\beta_2 & 0 \\ 0 & 0 & -\beta_3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $f(t,\psi(t)) = \begin{bmatrix} \frac{a(t)(y(t)+\omega(t))}{b+(y(t)+\omega(t))} \\ 0 \\ -c(t)(y(t)+\omega(t)) \end{bmatrix}$, $C = \begin{bmatrix} 0\ 0\ 1 \end{bmatrix}$, $C_1 = 0_{1,3}$.

Let us now design two reduced-order linear functional observers in order to compute two estimates, an upper one and a lower one, which bound the unmeasured linear function $z(t) = Fx(t)$, where $F = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \end{bmatrix}$. For this, we use the data given in [41] for α_1, α_2, m_1, m_2, $a^+(t)$, $a^-(t)$, b as $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $m_1 = 0$, $m_2 = 0$, $m_3 = 0.3$, $a^+(t) = 0.4$, $a^-(t) = 0.2$, $b = 1$ and let $c(t) = |\cos(0.5t)|$, $d(t) = \begin{bmatrix} \frac{1}{t+1} + 1 + |\sin(0.1t)|\ |\sin(0.5t)| + 1\ |\sin(0.5t)| \end{bmatrix}^T$ and a random measurement disturbance is chosen with $||\omega|| \leq 1 = \Omega$. Then known bounds $Ff^-(t,\psi(t))$, $Ff^+(t,\psi(t))$ are as

$$Ff^+(t,\psi(t)) = \begin{bmatrix} \frac{a^+(t)(y(t)+\omega(t))}{b+(y(t)+\omega(t))} \\ 0 \end{bmatrix},\ Ff^-(t,\psi(t)) = \begin{bmatrix} \frac{a^-(t)(y(t)+\omega(t))}{b+(y(t)+\omega(t))} \\ 0 \end{bmatrix},\ \forall t \geq 0.$$

Assume that known bounds $Fd^-(t)$, $Fd^+(t)$ are as

$$Fd^-(t) = \begin{bmatrix} \frac{1}{t+1} + 0.995 + |\sin(0.1t)|\ |\sin(0.5t)| + 0.95 \end{bmatrix}^T,$$

$$Fd^+(t) = \begin{bmatrix} \frac{1}{t+1} + 1.005 + |\sin(0.1t)|\ |\sin(0.5t)| + 1.01 \end{bmatrix}^T\ \ \forall t \geq 0.$$

According to Step 1 of Algorithm 1, we obtain matrices X and Y from equations (35)-(38). Since

$$\text{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = 5 = \text{rank} \begin{bmatrix} X \end{bmatrix},$$

condition (39) is satisfied. By setting a constraint

$$\begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix} \leq \lambda \leq \begin{bmatrix} 2.1 \\ 3.7 \end{bmatrix},$$

the LP problem (47) is feasible with

$$\lambda = \begin{bmatrix} 1.7748 \\ 1.3135 \end{bmatrix} \text{ and } \Gamma^T = 0_{2,5}. \tag{57}$$

Now, taking (62) into account for Step 4 and Step 5 of Algorithm 1, the observer gains are obtained as

$$N = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.3 \end{bmatrix},\ N_1 = \begin{bmatrix} 0 & 0 \\ 0.3 & 0 \end{bmatrix},\ J = 0_{2,1},\ b_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix},\ b_2 = \begin{bmatrix} 0.01 \\ 0.06 \end{bmatrix}.$$

With N, N_1, b_1 and b_2 have been obtained, it is not hard to see that the conditions of Theorem 1 hold. Hence, we obtain a pair of second-order linear functional observers to reconstruct $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ with $\ell = -(N + N_1)^{-1}(b_1 + b_2 + 2|J|\mathbf{1}_1\Omega) = \begin{bmatrix} 0.7 \\ 0.9 \end{bmatrix}$ is the smallest bound of the estimated error $e(t) = \hat{z}^+(t) - \hat{z}^-(t)$. In order to obtain simulation results, let us consider the input $u(t) = e^{0.1t}$, $0 \le t \le 30$, $\tau = 0.7s$ and the initial conditions are $x_1(\theta) = 1$, $x_2(\theta) = 2$, $x_3(\theta) = 3$, $\hat{z}_1^+(\theta) = 4$, $\hat{z}_1^-(\theta) = -2$, $\hat{z}_2^+(\theta) = 2.5$, $\hat{z}_2^-(\theta) = 1.5$ for $\theta \in [-0.7, 0]$. Figure 3 shows the responses of $z_1(t) = x_1(t)$, $\hat{z}_1^-(t)$ and $\hat{z}_1^+(t)$, while Figure 4 shows the responses of $z_2(t) = x_2(t)$, $\hat{z}_2^-(t)$ and $\hat{z}_2^+(t)$.

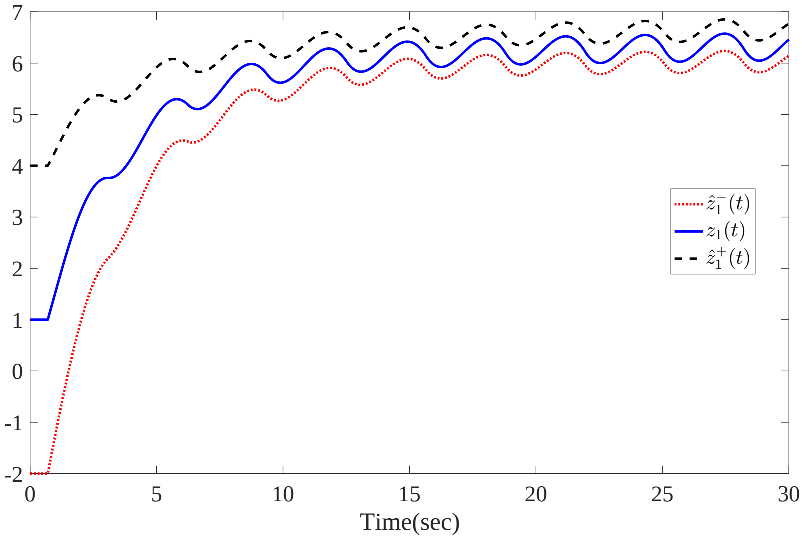

Fig. 3 Responses of $z_1(t) = x_1(t)$, $\hat{z}_1^-(t)$ and $\hat{z}_1^+(t)$

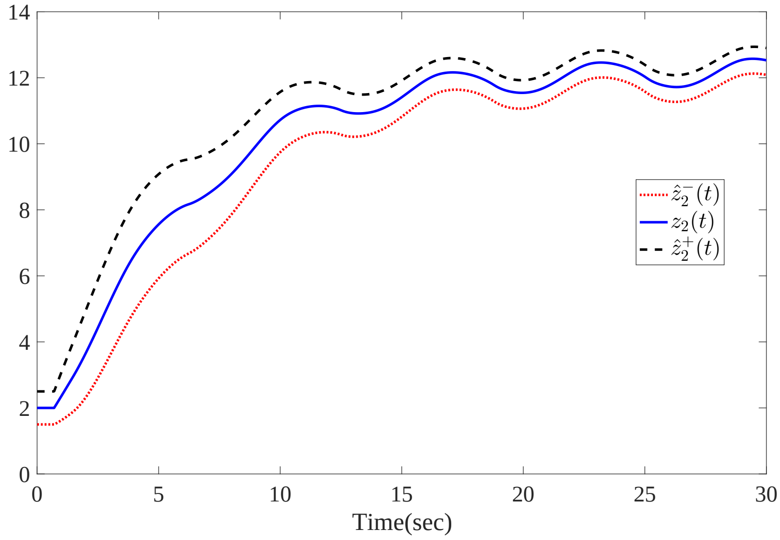

Fig. 4 Responses of $z_2(t) = x_2(t)$, $\hat{z}_2^-(t)$ and $\hat{z}_2^+(t)$

4.3 Example 3

Let us now consider a particular case of the population structured into three stages (55), where time-delay $\tau = 0$, the unknown disturbances $\omega(t) \equiv 0$, $d(t) \equiv 0$ and the control input vector $u(t) \equiv 0$, for all $t \geq 0$. For this case, we can express (55) into the following form

$$\dot{x}(t) = Ax(t) + \phi(t, y(t)),\ t \geq 0, \tag{58}$$
$$y(t) = Cx(t), \tag{59}$$

where

$$A = \begin{bmatrix} -\beta_1 & 0 & 0 \\ \alpha_1 & -\beta_2 & 0 \\ 0 & \alpha_2 & -\beta_3 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \text{ and } \phi(t, y(t)) = \begin{bmatrix} \frac{a(t)y(t)}{b+y(t)} \\ 0 \\ -c(t)y(t) \end{bmatrix}.$$

The authors of the work [15] proposed full-order Luenberger-type interval observers for the state vector $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$. Given bounds on the uncertainties in the model, they provided bounds on the estimation of the variables. However, they did not consider the problem of obtaining the smallest bounds for the estimation of the variables. To demonstrate the advantages and the generalization of our method, we now apply the result presented in the Remark 3 of this paper to design reduced-order linear functional interval observers for the model (58)-(59). For this, we first consider the case where $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, i.e. we aim to design interval observers for $x_1(t)$ and $x_2(t)$. We then consider the case where $F = I_3$, i.e. we aim to design interval observers for the state vector $x(t)$. Note that, the bound of estimated error obtained in this example is the smallest.

Let us consider the following data: $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $m_1 = 0$, $m_2 = 0$, $m_3 = 0.3$, $\beta_1 = \alpha_1 + m_1$, $\beta_2 = \alpha_2 + m_2$ and $\beta_3 = m_3$, $a^+(t) = 0.4$, $a^-(t) = 0.2$, $b = 1$ and let $c(t) = 0.05|\cos(0.5t)|$.

For the case where $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, by setting a constraint

$$\begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix} \leq \lambda \leq \begin{bmatrix} 2.1 \\ 3.1 \end{bmatrix},$$

the LP problem (53) is feasible with

$$\lambda = \begin{bmatrix} 1.6973 \\ 1.3253 \end{bmatrix} \text{ and } \mathit{\Gamma}^T = 0_{2,3}. \tag{60}$$

Now, taking (60) into account for Step 4 and Step 5 of Algorithm 1 (in case $q = 0$), the observer gains are obtained as

$$N = \begin{bmatrix} -0.3 & 0 \\ 0.3 & -0.3 \end{bmatrix}, J = 0_{2,1},\ b_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}.$$

With N and b_1 have been obtained, it is not hard to see that the conditions of (52) hold. Hence, we obtain a pair of second-order linear functional observers to reconstruct $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ with $\ell = -N^{-1}b_1 = \begin{bmatrix} 0.6667 \\ 0.6667 \end{bmatrix}$ is the smallest bound of the estimated error $e(t)$.

Now, for the case where $F = I_3$, by setting a constraint

$$\begin{bmatrix} 0.01 \\ 0.02 \\ 0.03 \end{bmatrix} \leq \lambda \leq \begin{bmatrix} 2.1 \\ 3.1 \\ 4.1 \end{bmatrix},$$

the LP problem (53) is feasible with

$$\lambda = \begin{bmatrix} 2.0527 \\ 1.7144 \\ 1.0006 \end{bmatrix} \text{ and } \mathit{\Gamma}^T = 0_{3,3}. \tag{61}$$

Now, taking (61) into account for Step 4 and Step 5 of Algorithm 1 (in case $q = 0$), the observer gains are obtained as

$$N = \begin{bmatrix} -0.3 & 0 & 0 \\ 0.3 & -0.3 & 0 \\ 0 & 0.3 & -0.3 \end{bmatrix},\ J = 0_{3,1},\ b_1 = \begin{bmatrix} a^+ - a^- \\ 0 \\ (c^+ - c^-)x_3^{\max} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0 \\ 0.015 \end{bmatrix}.$$

With N and b_1 have been obtained, it is not hard to see that the conditions of (52) hold. Hence, we obtain a pair of third-order linear functional observers to reconstruct $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ with $\ell = -N^{-1}b_1 = \begin{bmatrix} 0.6667 \\ 0.6667 \\ 0.7167 \end{bmatrix}$ is the smallest bound of the estimated error $e(t)$.

For simulation, we consider the initial conditions $x_1(0) = 0.1$, $x_2(0) = 0.2$, $x_3(0) = 0.3$, $\hat{z}_1^+(0) = 0.3$, $\hat{z}_1^-(0) = 0$, $\hat{z}_2^+(0) = 0.5$, $\hat{z}_2^-(0) = 0.1$, $\hat{z}_3^+(0) = 0.6$, $\hat{z}_3^-(0) = 0.2$. Figure 5 shows the responses of $z_1(t) = x_1(t)$, $\hat{z}_1^-(t)$ and $\hat{z}_1^+(t)$, Figure 6 shows the responses of $z_2(t) = x_2(t)$, $\hat{z}_2^-(t)$ and $\hat{z}_2^+(t)$ and Figure 7 shows the responses of $z_3(t) = x_3(t)$, $\hat{z}_3^-(t)$ and $\hat{z}_3^+(t)$.

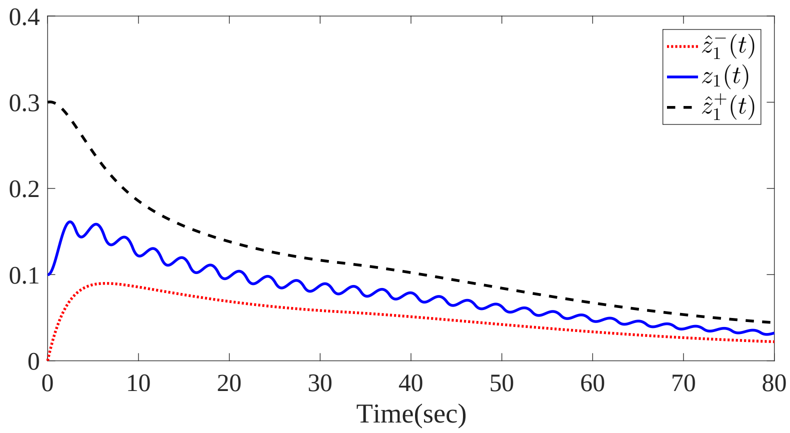

Fig. 5 Responses of $z_1(t) = x_1(t)$, $\hat{z}_1^-(t)$ and $\hat{z}_1^+(t)$

4.4 Example 4

Consider a third-order time-delay system of the form (1)-(4), where

$$A = \begin{bmatrix} -0.5 & 0.1 & 0.01 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \ F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ B^T = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix},$$

known bounds $Ff^-(t, \psi(t))$, $Ff^+(t, \psi(t))$, $Fd^-(t)$ and $Fd^+(t)$ are assumed to satisfy $F(f^+(t, \psi(t)) - f^-(t, \psi(t))) \leq b_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$, $F(d^+(t) - d^-(t)) \leq b_2 = \begin{bmatrix} 0.01 \\ 0.06 \end{bmatrix}$ and a random measurement disturbance is chosen with $||\omega|| \leq 1 = \Omega$.

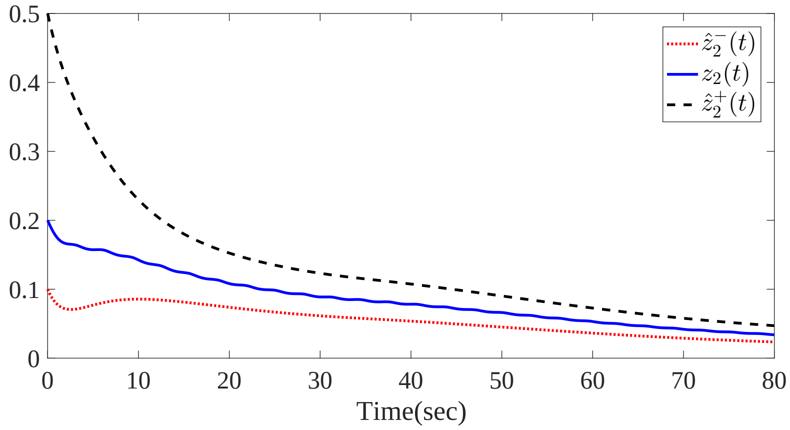

Fig. 6 Responses of $z_2(t) = x_2(t)$, $\hat{z}_2^-(t)$ and $\hat{z}_2^+(t)$

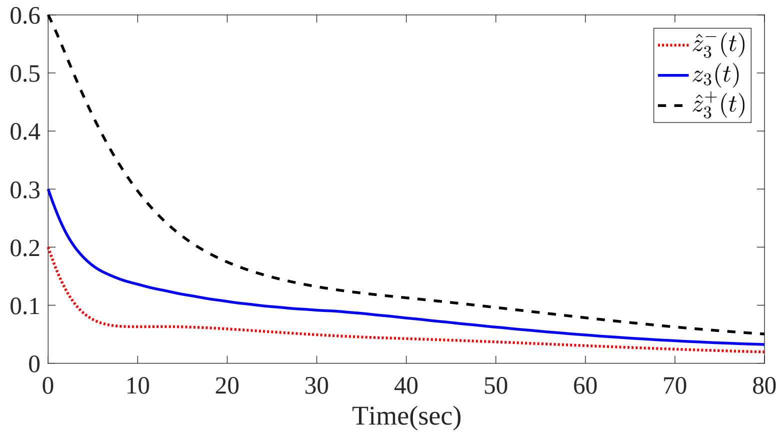

Fig. 7 Responses of $z_3(t) = x_3(t)$, $\hat{z}_3^-(t)$ and $\hat{z}_3^+(t)$

According to Step 1 of Algorithm 1, we obtain matrices X and Y from equations (35)-(38). Since

$$\text{rank}\begin{bmatrix} X \\ Y \end{bmatrix} = 5 = \text{rank}\left[\, X \,\right],$$

condition (39) is satisfied. By setting a constraint

$$\begin{bmatrix} 0.02 \\ 0.05 \end{bmatrix} \leq \lambda \leq \begin{bmatrix} 2.5 \\ 3.6 \end{bmatrix},$$

the LP problem (47) is feasible with

$$\lambda = \begin{bmatrix} 1.8614 \\ 3.5595 \end{bmatrix} \text{ and } \Gamma^T = \begin{bmatrix} -6.3308 \; 0 \; 0 \; 0 \; -6.3308 \\ -6.3308 \; 0 \; 0 \; 0 \; -6.3308 \end{bmatrix}. \tag{62}$$

Now, taking (62) into account for Step 4 and Step 5 of Algorithm 1, the observer gains are obtained as

$$N = \begin{bmatrix} -0.51 & 0.1 \\ 0 & -2 \end{bmatrix}, \; N_1 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \; J = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}.$$

With N, N_1, J have been obtained and b_1 and b_2 are given above, it is not hard to see that the conditions of Theorem 1 hold. Hence, we obtain a pair of second-order linear functional observers to reconstruct $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ with $\ell = -(N+N_1)^{-1}(b_1 + b_2 + 2|J|\mathbf{1}_1\Omega) = \begin{bmatrix} 0.466 \\ 0.0766 \end{bmatrix}$ is the smallest bound of the estimated error $e(t) = \hat{z}^+(t) - \hat{z}^-(t)$.

5 Conclusion

In this paper, a pair of reduced-order linear functional observers for a class of nonlinear uncertain time-delay systems with external unknown disturbances has been proposed. This pair of reduced-order linear functional observers determines upper bound and lower bound of linear functions of the state vector. Conditions for the existence of such obervers and a method based on LP has been provided for determining observer matrices. The effectiveness of the proposed design method is supported by four examples and simulation results. Further work is required to discuss the control laws in terms of computational complexity, physical constraints or limitations in implementation. Also, extending the results of this paper to include unknown time-varying delays would prove to be an interesting problem for future research.

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References

1. M. Ait Rami, F. Tadeo, U. Helmke, Positive observers for linear positive systems, and their implications. Int. J. Control **84**(4), 716–725 (2011)
2. M. Ait Rami, M. Schönlein, J. Jordan, Estimation of linear positive systems with unknown time-varying delays. Eur. J. Control **19**(3), 179–187 (2013)
3. A. Ben-Israel, T.N.E. Greville, *Generalized Inverses Theory and Applications.* (Springer-Verlag, Berlin–Heidelberg, 2003).
4. J. Blesa, V. Puig, Y. Bolea, Fault detection using interval LPV models in an open-flow canal. Control Eng. Practice **18** (5), 460–470 (2010)
5. J. Blesa, D. Rotondo, V. Puig, F. Nejjaria, FDI and FTC of wind turbines using the interval observer approach and virtual actuators/sensors. Control Eng. Practice **24**, 138–155 (2014)
6. S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory.* SIAM Studies in Applied Mathematics, (SIAM, Philadelphia, 1994).
7. J. Chen, R.J. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems.* (Massachusetts: Kluwer Academic Publishers, 1999).
8. Z. Chen, Z. Cao, Q. Huang, S.L. Campbell, Decentralized observer-based reliable control for a class of interconnected Markov Jumped time-delay system subject to actuator saturation and failure. Circuits Syst. Signal Process. (2018). doi: 10.1007/s00034-018-0795-7
9. M. Darouach, Linear functional observers for systems with delays in state variables. IEEE Trans. Autom. Control **46**(3), 491–496 (2011)
10. D. Efimov, W. Perruquetti, J.P. Richard, Interval estimation for uncertain systems with time-varying delays. Int. J. Control. **86**(10), 1777–1787, (2013)
11. D. Efimov, S. Li , Y. Hu, S. Muldoon, H. Javaherian, V.O. Nikiforov, Application of interval observers to estimation and control of air-fuel ratio in a direct injection engine. Proc. ACC, Chicago, 2015
12. D. Efimov, T. Raïssi, Design of interval observers for uncertain dynamical systems. Autom. Remote Control **77**, 191–225 (2016)
13. L. Farina, S. Rinaldi, *Positive Linear Systems: Theory and Applications* (John Wiley & Sons, New York, 2000).
14. G. Goffaux, M. Remy, A.V. Wouwer, Continuous-discrete confidence interval observer–application to vehicle positioning. Inform. Fusion **14**(4), 541–550 (2013)
15. J.L. Gouzé, A. Rapaport, M.Z. Hadj-Sadok, Interval observers for uncertain biological systems. Ecological Modelling **133**(1–2), 45–56 (2000)
16. J. K. Hale, S. M. V. Lunel, *Introduction to Functional Differential Equations.* (Springer, New York, 1993)
17. W. M. Haddad, V. Chellaboina, T. Rajpurohit, Dissipativity theory for nonnegative and compartmental dynamical systems with time delay. IEEE Trans. Autom. Control **49**(5), 747–751 (2004)
18. M.Z. Hadj-Sadok, J.L. Gouzé, Estimation of uncertain models of activated sludge processes with interval observers. J. Process Control **11**(3), 299–310 (2001)
19. M. Hou, P. Zitek, R.J. Patton, An observer design for linear time-delay systems. IEEE Trans. Autom. Control **47**(1), 121–125 (2002)
20. D.C. Huong, M.V. Thuan, State transformations of time-varying delay systems and their applications to state observer design. Discrete Contin. Dyn. Syst. Ser. S, **10**(3), 413-444 (2017)
21. T. Kaczorek, Fractional descriptor observers for fractional descriptor continuous-time linear system. Archive of Control Science **24**, 27–37 (2014)
22. A. Krener, A. Isidori, Linearization by output injection and nonlinear observers. Syst. Control Lett. **3**(1), 47–52 (1983)
23. A. Krener, W. Respondek, Nonlinear observers with Linearization error dynamics. Siam J. Control Optimization **23**(2), 197–216 (1985)
24. P. Li, J. Lam, Positive state-bounding observer for positive interval continuous-time systems with time delay. Int. J. Robust Nonlinear Control **22**(11), 1244–1257 (2011)
25. S. Li, Z. Xiang, Stabilisation of a class of positive switched nonlinear systems under asynchronous switching. Int. J. Syst. Sci. **48**, 1537–1547 (2017)

26. S. Li, Z. Xiang, Stochastic stability analysis and L_∞-gain controller design for positive Markov jump systems with time-varying delays. Nonlinear Anal. Hybrid Syst. **22**, 31–42 (2016)
27. S. Li, Z. Xiang, H. Lin, H.R. Karimi, State estimation on positive Markovian jump systems with time-varying delay and uncertain transition probabilities. Information Sciences **369**, 251–266 (2016)
28. Z. Liu, L. Zhao, H. Xiao, C. Gao, Adaptive H_∞ integral sliding mode control for uncertain singular time-delay systems based on observer. Circuits Syst. Signal Process. **36**(11), 4365–4387 (2017)
29. D. G. Luenberger, *Introduction to Dynamic Systems: Theory, Models and Applications.* (John Wiley & Sons, New York, 1979)
30. F. Mazenc, O. Bernard, Interval observers for linear time-invariant systems with disturbances systems. Automatica **47**, 140–147 (2011)
31. F. Mazenc, S. Niculescu, O. Bernard, Exponentially stable interval observers for linear systems with delay. SIAM J. Control Optim. **50**(1), 286–305 (2012)
32. M. Moisan, O. Bernard, J.L. Gouzé, Near optimal interval observers bundle for uncertain bioreactors. Automatica **45**(1), 291–295 (2009)
33. M. Moisan, O. Bernard, Robust interval observers for global Lipschitz uncertain chaotic systems. Syst. Control Lett. **59**(11), 687–694 (2010)
34. J.D. Murray, *Mathematical Biology.* (Springer-Verlag, 1990)
35. P.T. Nam, H. Trinh, P.N. Pathirana, Reachable set bounding for nonlinear perturbed time-delay systems: The smallest bound. Appl. Math. Lett. **43**, 68–71 (2015)
36. B. Olivier, J.L. Gouzé, Closed loop observers bundle for uncertain biotechnological models. J. Process Control **14**(7), 765–774 (2004)
37. P. Pepe, Z.-P. Jiang, A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. Syst. Control Lett. **55**, 1006–1014, (2006)
38. V. Puig, A. Stancu, T. Escobet, F.Nejjaria, J.Quevedoa, R.J.Patton, Passive robust fault detection using interval observers: Application to the DAMADICS benchmark problem. Control Eng. Practice **14**, 621–633 (2006)
39. L. Qian, Q. Lu, J. Bai, Z. Feng, Dynamics of a prey-dependent digestive model with state-dependent impulsive control. Int. J. Bifurcation Chaos, **22**, 1250092 [11 pages] (2012).
40. C.R. Rao, Calculus of Generalized Inverses of Matrices Part I: General Theory. Sankhyā Ser. A **29**, 317–342 (1967)
41. A. Rapaport, J.L. Gouzé, Practical observers for uncertain affine output injection systems. European Control Conference, CD-Rom, Karlsruhe, 31 August-3 September, 1505–1510 (1999)
42. Z. Shu, J. Lam, H. Gao, B. Du, L. Wu, Positive observers and dynamic outputfeedback controllers for interval positive linear systems. IEEE Trans. Circuits Syst. I: Regular Paper **55**(10), 3209–3222 (2008)
43. H. Trinh, T. Fernando, *Functional Observers for Dynamical Systems.* (Springer-Verlag, Berlin–Heidelberg, 2012).
44. H. Trinh, D. C. Huong, L. V. Hien, S. Nahavandi, Design of reduced-order positive linear functional observers for positive time-delay systems. IEEE Trans. Circuits Syst. II: Exp. Briefs **64**(5), 555–559 (2017)
45. S. Yin, S.X. Ding, X. Xia, H. Luo, A review on basic data-driven approaches for industrial process monitoring. IEEE Trans. Ind. Electronics **61**(11), 6418–6428 (2014)
46. S. Yin, X. Li, H. Gao, O. Kaynak, Data-based techniques focused on modern industry: An overview. IEEE Trans. Ind. Electronics **62**(1), 657–667 (2015)
47. V. D. Yurkevich, Multi-channel control system design for a robot manipulator based on the time-scale method. Optoelectron., Instrum. Data Process **52**, 196–202 (2016)
48. H. Zhang, Y. Shi, J. Wang, On energy-to-peak filtering for nonuniformly sampled nonlinear systems: A Markovian jump system approach. IEEE Trans. Fuzzy Syst. **22**(1), 212–222 (2014)
49. Y. Zhao, Z. Feng, Desynchronization in synchronous multi-coupled chaotic neurons by mix-adaptive feedback control. J. Biol. Dyn. **7**, 1–10 (2013)
50. J. Zheng, B. Cui, State estimation of chaotic Lurie systems via communication channel with transmission delay. Circuits Syst. Signal Process. **37**(10), 4568–4583 (2018)