Triebel-Lizorkin-Morrey Spaces Associated to Hermite Operators

Nguyen Ngoc Trong[a], Le Xuan Truong[b,*], Tran Tri Dung[c], Hanh Nguyen Vo[d]

[a]*Faculty of Mathematics and Computer Science, VUNHCM - University of Science, Ho Chi Minh city, Vietnam*
Department of Primary Education, HoChiMinh City University of Education, Vietnam
[b]*University of Economics Ho Chi Minh City*
[c]*Department of Mathematics, HoChiMinh City University of Education, Vietnam*
[d]*Department of Mathematics, Macquarie University, Sydney, Australia*

Abstract

The aim of this article is to establish molecular decomposition of homogeneous and inhomogeneous Triebel-Lizorkin-Morrey spaces associated to the Hermite operator $\mathbb{H} \equiv -\Delta + |x|^2$ on the Euclidean space $\mathbb{R}^n$. As applications of the molecular decomposition theory, we show the Triebel-Lizorkin-Morrey boundedness of Riesz potential, Bessel potential and spectral multipliers associated to the operator $\mathbb{H}$. These results generalize the corresponding results in [B. T. Anh, D. X. Thinh. Besov and Triebel-Lizorkin Spaces Associated to Hermite Operators. J. Fourier. Anal. Appl **21** (2015) 405–448].

Keywords: Hermite operator, Triebel-Lizorkin-Morrey space, molecular decomposition.

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Contents

1. Introduction

The classical Triebel-Lizorkin spaces, which contain a number of key function spaces, including Sobolev spaces, Bessel-potential spaces, Hardy spaces and BMO spaces, plays an important role in the theory of function spaces because of its wide applications in the theory of partial differential

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*Corresponding author
Email addresses: `trongnn37@gmail.com` (Nguyen Ngoc Trong), `lexuantruong@tdt.edu.vn` (Le Xuan Truong), `dungtt@math.hcmup.edu.vn` (Tran Tri Dung), `hanh.vo@mq.edu.au` (Hanh Nguyen Vo)

equations and harmonic analysis. For example, classical Triebel-Lizorkin spaces can be used in some priori estimates of elliptic differential operators or in constructions of time-local solutions of Navier-Stokes non-linear equations since they measure the oscillatory properties of a distribution more accurately than Sobolev spaces do, while still possessing the same smoothness and scaling properties. It is well-known that the classical Triebel-Lizorkin spaces can be characterized via Laplace operator Δ or its square root on $\mathbb{R}^n$. Recently, the theory of Triebel-Lizorkin spaces associated to operators has been studied intensively by many authors, see for instance [1, 7, 14] and the references therein, since the classical Triebel-Lizorkin spaces (associated to Laplacian) are no longer appropriate for the study of boundedness of a number of operators arising in the theory of partial differential equations and harmonic analysis.

On the other hand, the classical Morrey spaces, which are natural generalizations of $L^p(\mathbb{R}^n)$, were first introduced by Morrey in [6] to investigate the local behavior of solutions to second-order elliptic partial differential equations. Let us recall here the definition of Morrey spaces.

Definition 1.1. (Morrey spaces) For r and p satisfying $0 < p \leq r < \infty$, the Morrey space $\mathbf{M}_p^r$ is defined by

$$\mathbf{M}_p^r \equiv \left\{ f \in L^p_{loc} : \|f\|_{\mathbf{M}_p^r} = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{n/r-n/p} \|f\|_{L^p(B(x_0,R))} < \infty \right\}.$$

Comparing to $L^p(\mathbb{R}^n)$, Morrey spaces generally describe the local regularity more precisely than $L^p(\mathbb{R}^n)$ spaces do and thus can provide subtle improvements in regularity in elliptic boundary value problems and non-linear evolution equations.

In recent years, there have been a number of authors extending the theory of Triebel-Lizorkin spaces to the setting Triebel-Lizorkin-Morrey (TLM in abbreviation) by using Morrey spaces in place of $L^p(\mathbb{R}^n)$ in the definition of Triebel-Lizorkin spaces, as they realized that TLM spaces share a lot of key properties of Triebel-Lizorkin spaces, and represent local oscillations and singularities of functions better than Triebel-Lizorkin spaces do. In 2005, L. Tang and J. Xu ([13]) introduced the inhomogeneous TLM spaces and studied lifting properties, Fourier multiplier theorem and the discrete characterization of inhomogeneous TLM spaces. In 2008, Y. Sawano ([9]) characterized the inhomogeneous TLM spaces in terms of wavelet. After that, in 2009, H. Wang ([15]) established the decomposition of homogeneous TLM in terms of molecules. For more results on TLM spaces, we refer the interested reader to [5, 8, 9, 10, 11, 12, 13, 15, 16]. More recently, in 2015, B. T. Anh and D. X. Thinh ([1]) developed the theory of both homogeneous and inhomogeneous Besov and Triebel-Lizorkin spaces associated to Hermite operators (harmonic oscillator) $\mathbb{H}$ on $\mathbb{R}^n$, which is defined by

$$\mathbb{H} = -\Delta + |x|^2.$$

Motivated by all of the above-mentioned facts, in this paper we aim to develop the molecular decomposition theory for homogeneous and inhomogeneous Hermite-Triebel-Lizorkin-Morrey (HTLM in abbreviation) spaces associated to the Hermite operator $\mathbb{H}$ mentioned above. Let us now introduce the definitions of homogeneous and inhomogeneous HTLM spaces associated to the Hermite operator $\mathbb{H}$ via the semigroup characterization.

Definition 1.2. (Homogeneous HTLM spaces) Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty, p \leq r \leq \infty$ we define the homogeneous HTLM space $\mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H}}$ as any space $\mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H},m}$ for any positive integer m satisfying

$$m > n + \max\{\alpha, 0\} + \operatorname{int}\left[n\left(\frac{1}{\theta_0} - 1\right)\right] + 1,$$

where

- $\theta_0 = \min\{1, p, q\}$,
- $\mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H},m} := \left\{ f \in \mathbf{S}' : \|f\|_{\mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H},m}} = \left\| \left(\int_0^\infty \left(t^{-\alpha} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \right| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{\mathbf{M}_p^r} < \infty \right\}.$

Definition 1.3. (Inhomogeneous HTLM spaces) Let $\alpha \in \mathbb{R}$ and $0 < p \le r \le \infty, 0 < q \le \infty$, we define the inhomogeneous HTLM space $\mathbf{IFM}^{\alpha,\mathbb{H}}_{p,q,r}$ as any space $\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}(\beta,\gamma)$ for any $m \in \mathbb{N}_+, m > \alpha + n$ and $0 < \beta < 1 < \gamma$ with $\gamma - \beta > 1$,

where

$$\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r} := \Bigg\{ f \in \mathbf{S}' : \|f\|_{\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}(\beta,\gamma)} = \int_\beta^\gamma \|e^{-t\sqrt{\mathbb{H}}} f\|_{\mathbf{M}^r_p} \frac{dt}{t} + \left\| \left(\int_0^1 (t^{-\alpha} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f|)^q \frac{dt}{t} \right)^{1/q} \right\|_{\mathbf{M}^r_p} < \infty \Bigg\}.$$

Noticing that these spaces are independent of the choice of m, β, γ (see Theorem 3.3 and Theorem 4.4) and thus are well defined. Then we study the HTLM boundedness of Riesz potential, Bessel potential and spectral multipliers associated to the Hermite operator $\mathbb{H}$, using the molecular decomposition theorey developed. The identification between HTLM spaces and Hermite-Sobolev-Morrey spaces is also investigated.

Remark 1.4. The Hermite-Triebel-Lizorkin spaces in [1] are special cases of our spaces.

It should be pointed out that our approach in this paper for investigating HTLM spaces associated to Hermite operators is to adapt the technique of maximal functions introduced by Fefferman-Stein and Peetre, which has been recently developed further in [1], whereas usual approachs for these types of function spaces are based on Littlewood-Paley decompositions. As a result of this distinct approach, it is possible to extend the theory of the inhomogeneous HTLM spaces to the setting where p and q are below the endpoint 1.

We organize this paper as follows. In Section 2, we recall some notions and known results concerning Hardy-Littlewood maximal functions, Morrey spaces, kernel estimates on Hermite operators and the Calderón reproducing formulas. In Section 3 and 4, we aim to develop the molecular decomposition theory of homogeneous and inhomogeneous HTLM spaces. We then apply the theory of these compositions to establishing the boundedness of singular integrals associated to $\mathbb{H}$ in Section 5. In the last section, we prove the identification between HTLM spaces and Hermite-Sobolev-Morrey spaces. The main results in this paper generalize the corresponding results in [1].

Throughout the paper, the letters C and c are always used to denote positive constants that are independent of the main parameters and whose values may vary from line to line.
Let us denote $A \lesssim B$ if there is a positive constant C so that $A \le CB$ and
$A \sim B$ if $A \lesssim B$ and $B \lesssim A$.
For $a \in \mathbb{R}$, we denote the integer part of a by $\text{int}(a)$.
In addtion, the following notions are also used in the paper.

$$a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}$$

$$\mathbb{N} = \{0, 1, 2, ...\}, \mathbb{N}_+ = \{1, 2, 3, ...\}$$

$$\mathbb{Z}^- = \{-1, -2, ...\}, \mathbb{Z}^-_0 = \{0, -1, -2, ...\}.$$

2. Preliminaries

2.1. Preliminary results

Firstly, let us recall the set of all dyadic cubes $\mathcal{D}$ in $\mathbb{R}^n$

$$\mathcal{D} = \left\{ \prod_{j=1}^n \Big[m_j 2^k, (m_j + 1) 2^k \Big) : m_1, m_2, ..., m_n, k \in \mathbb{Z} \right\}.$$

For a dyadic cube $Q := \prod_{j=1}^n \Big[m_j 2^k, (m_j + 1) 2^k \Big)$, we denote by $\ell(Q)$ and x_Q the length and the center of the dyadic cube Q respectively. It is clear that $\ell(Q) = 2^k$ and $x_Q = \big((m_j + 1/2) 2^k \big)_{j=1}^n$.

For $v \in \mathbb{Z}$, we set

$$\mathcal{D}_v = \{Q \in \mathcal{D} : \ell(Q) = 2^v\}.$$

For $\theta > 0$, the Hardy-Littlewood maximal function $\mathbb{M}_\theta$ is defined by

$$\mathbb{M}_\theta f(x) = \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |f(y)|^\theta dy \right)^{1/\theta}, \ x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. When $\theta = 1$ we just write $\mathbb{M}$.

Let us recall the concept of a molecule associated to the Hermite operator $\mathbb{H}$ in [1].

Definition 2.1. Let $0 < r \leq \infty, \alpha \in \mathbb{R}$, and $N, M \in \mathbb{N}_+$. A function u is said to be an $(\mathbb{H}, M, N, \alpha, r)$ molecule associated to $\mathbb{H}$ if there exist a function b from the domain $(\sqrt{\mathbb{H}})^M$ and a dyadic cube $Q \in \mathcal{D}$ such that

(i) $u = (\sqrt{\mathbb{H}})^M b$;

(ii) $\left|(\sqrt{\mathbb{H}})^k b(x)\right| \leq \ell(Q)^{M-k} |Q|^{\alpha/n - 1/r} \left(1 + \frac{|x - x_Q|}{\ell(Q)}\right)^{-n-N}$ for all $k = 0, ..., 2M$.

We will denote u by m_Q. If a function m_Q satisfies (i) and (ii) for $k = M, ..., 2M$ only then we say that m_Q is an $(\mathbb{H}, M, N, \alpha, r)$-zero level molecule.

In the next sections, we will need the following lemmas.

Lemma 2.2. *[4, p.147] Let $N > 0, \eta, v \in \mathbb{Z}$ and $v \leq \eta$. For any sequence of functions $\{f_Q\}_{Q \in \mathcal{D}_v}$ satisfying*

$$|f_Q(x)| \lesssim \left(1 + 2^{-\eta}|x - x_Q|\right)^{-n-N},$$

then for any $\theta > \frac{n}{n+N}$ and any sequence of numbers $\{s_Q\}_{Q \in \mathcal{D}_v}$, we have

$$\sum_{Q \in \mathcal{D}_v} |s_Q||f_Q(x)| \lesssim 2^{(\eta - v)n/\theta} \mathbb{M}_\theta \left(\sum_{Q \in \mathcal{D}_v} |s_Q| \chi_Q \right)(x).$$

Lemma 2.3. *[1, Lemma 3.6] Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty, p \leq r \leq \infty$. Suppose that $M, N > 0$ and*

$$m > \max\{\alpha, 0\} + N + n.$$

Then for any sequence of $(\mathbb{H}, M, N, \alpha, r)$ molecules $\left\{m_Q : Q \in \mathcal{D}_v, v \in \mathbb{Z}\right\}$, one has

1. $\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)\right| \lesssim |Q|^{\alpha/n - 1/r} \left(\frac{t}{2^v}\right)^{m-N-n} (1 + 2^{-v}|x - x_Q|)^{-n-N}$ *for all* $t \leq 2^v$.
2. $\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)\right| \lesssim |Q|^{\alpha/n - 1/r} \left(\frac{2^v}{t}\right)^{M} \left(1 + \frac{|x - x_Q|}{t}\right)^{-n-N}$ *for all* $2^v \leq t$.

Lemma 2.4. *[1, Lemma 3.6, p. 425-426] Let $x, z \in \mathbb{R}^n, N \in \mathbb{N}_+$ and $t, a > 0$. Then we have*

1. $\int_{\mathbb{R}^n} \left(1 + \frac{|x-y|}{t}\right)^{-n-N} \left(1 + \frac{|z-y|}{a}\right)^{-n-N} dy \lesssim t^n \left(\frac{a}{t}\right)^n \left(1 + \frac{|x-z|}{a}\right)^{-n-N}$ *if* $t \leq a$.
2. $\int_{\mathbb{R}^n} \left(1 + \frac{|x-y|}{t}\right)^{-n-N} \left(1 + \frac{|z-y|}{a}\right)^{-n-N} dy \lesssim t^n \left(1 + \frac{|x-z|}{t}\right)^{-n-N}$ *if* $t \geq a$.

2.2. Morrey space

We recall here some important estimates involving Morrey spaces which are often used in the following sections.

Lemma 2.5. *We have following statements*

1. *For $0 < p \le r < \infty$, we have*
$$\|f\|_{\mathbf{M}_p^r} \sim \sup_{Q\in\mathcal{D}} |Q|^{1/r-1/p}\|f\|_{L^p(Q)}. \tag{2.1}$$
2. *For $0 < p \le r < \infty$ and $\theta > 0$, we have*
$$\|f^\theta\|_{\mathbf{M}_p^r} \lesssim \|f\|^\theta_{\mathbf{M}_{p\theta}^{r\theta}}. \tag{2.2}$$
3. *(Minkowski's inequality) For $0 < p \le r < \infty$ and $0 < q \le p$, we have*
$$\left\| \left(\int_a^b |F(\cdot,t)|^q \frac{dt}{t} \right)^{1/q} \right\|_{\mathbf{M}_p^r} \lesssim \left(\int_a^b \|F(\cdot,t)\|^q_{\mathbf{M}_p^r} \frac{dt}{t} \right)^{1/q}.$$

Proof. By virtue of Remark 1.2 in [15], we include part 1. Next, we use part 1 to get part 2 and part 3 immediately. □

The next lemma is the Feffereman-Stein vector-valued maximal inequality which plays a key role in this paper. The proof of this lemma can be found in [13, Lemma 2.5].

Lemma 2.6. *Let $0 < p \le r \le \infty, 0 < q \le \infty$ and $0 < \theta < p \wedge q$. Then for any sequence of function $\{f_v\}_{v\in\mathbb{Z}}$, we have*
$$\Big\| \Big(\sum_{v\in\mathbb{Z}} |\mathbb{M}_\theta f_v|^q \Big)^{1/q} \Big\|_{\mathbf{M}_p^r} \lesssim \Big\| \Big(\sum_{v\in\mathbb{Z}} |f_v|^q \Big)^{1/q} \Big\|_{\mathbf{M}_p^r}. \tag{2.3}$$

In view of (1.7) in [15, p.776] and (2.3) in [15, p.779] respectively, one has the following two lemmas.

Lemma 2.7. *Let r and p satisfying $0 < p \le r < \infty, 0 < q \le \infty, v \in \mathbb{Z}$, and a sequence $\{s_Q : Q \in \mathcal{D}_v\}$ such that*
$$\left\| \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{1/q} \right\|_{\mathbf{M}_p^r} < \infty.$$

Then, we have
$$\left\| \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{1/q} \right\|_{\mathbf{M}_p^r}$$
$$\sim \sup_{J\in\mathcal{D}} \left\{ \left(\frac{1}{|J|} \right)^{1-p/r} \int_J \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v, Q\subset J} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{p/q} dx \right\}^{1/p}.$$

Lemma 2.8. *Let r and p satisfying $0 < p \le r < \infty, v \in \mathbb{Z}$, and a sequence $\{s_Q : Q \in \mathcal{D}_v\}$ such that*
$$\left\| \sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q \right\|_{\mathbf{M}_p^r} < \infty.$$

Then, we have
$$\left\| \sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q \right\|_{\mathbf{M}_p^r} \sim \left(\sup_{J\in\mathcal{D}, \ell(J)\geqslant 2^v} \left(\frac{1}{|J|} \right)^{1-p/r} \sum_{Q\in\mathcal{D}_v, Q\subset J} |Q|^{1-p/r}|s_Q|^p \right)^{1/p}.$$

2.3. Kernel estimates on Hermite operators

Let us denote by $\mathbf{S}$ and $\mathbf{S}'$ the space of Schwartz functions and the space of tempered distributions on $\mathbb{R}^n$ respectively. We recall that the space of Schwartz functions $\mathbf{S}$ consists of all functions $\phi \in C^\infty(\mathbb{R}^n)$ so that for all multi-indices α and β, we have

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty. \tag{2.4}$$

For $k \geq 0$ and $t > 0$, we denote the kernel associated to $(t\sqrt{\mathbb{H}})^k e^{-t\sqrt{\mathbb{H}}}$ by $p_{t,k}(x,y)$. When $k = 0$, we drop the subscript k to write $p_t(x,y)$. One has the following results (see [1]).

Proposition 2.9. *For all $k \in \mathbb{N}, t > 0$ and $y \in \mathbb{R}^n$ we have $p_{t,k}(\cdot, y) \in \mathbf{S}$.*

Lemma 2.10. *For $k \in \mathbb{N}$, there exist $C > 0$ and $\delta > 0$ so that*

1. $p_t(x,y) \leq C \dfrac{t}{\Big(t + |x-y|\Big)^{n+1}}$.
2. $p_{t,k}(x,y) \leq C \dfrac{t^k}{\Big(t + |x-y|\Big)^{n+k}}$.
3. *If $|h| < t$ then* $|p_{t,k}(x+h,y) - p_{t,k}(x,y)| \leq C\Big(\dfrac{|h|}{t}\Big)^\delta \dfrac{t^k}{\Big(t + |x-y|\Big)^{n+k}}$.

2.4. Calderón reproducing formulas

Here we recall two Calderón reproducing formulas in [1], which play an important role in our study of the molecular decomposition theory of homogenous and inhomogeneous HTLM spaces in the sequent.

Proposition 2.11. *(Homogeneous Calderón reproducing formula) Let $m_1, m_2 \in \mathbb{N}^+$ and $f \in \mathbf{S}'$. Then we have*

$$f = -\frac{1}{2^{m-1}(m-1)!} \int_0^\infty (t\sqrt{\mathbb{H}})^{m_1} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^{m_2} e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t} \ \text{in } \mathbf{S}',$$

where $m = m_1 + m_2$.

Proposition 2.12. *(Inhomogeneous Calderón reproducing formula) Let $m_1, m_2 \in \mathbb{N}^+$ and $f \in \mathbf{S}'$. Then we have*

$$f = -\frac{2^{m-1}}{(m-1)!} \int_0^{1/2} (t\sqrt{\mathbb{H}})^{m_1} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^{m_2} e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t} + \sum_{k=0}^{m-1} \frac{1}{k!} \mathbb{H}^{k/2} e^{-\sqrt{\mathbb{H}}} f \ \text{in } \mathbf{S}',$$

where $m = m_1 + m_2$.

3. Homogeneous HTLM spaces

We obtain the following main results on molecular decompositions of the spaces $\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r}$.

Theorem 3.1. *Let $\alpha \in \mathbb{R}, 0 < p \leq r \leq \infty, 0 < q \leq \infty$ and $\theta_0 = \min\{1, p, q\}$. Suppose $M, N \in \mathbb{N}_+$ and*

$$m > n + \max\{\alpha, 0\} + \operatorname{int}\left[n\left(\frac{1}{\theta_0} - 1\right)\right] + 1.$$

For each $f \in \mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r}$, *then there exists a sequence of* $(\mathbb{H}, M, N, \alpha, r)$ *molecules* $\Big\{m_Q : Q \in \mathcal{D}_v, v \in \mathbb{Z}\Big\}$ *and a sequence of coefficients* $\Big\{s_Q : Q \in \mathcal{D}_v, v \in \mathbb{Z}\Big\}$ *so that*

$$f = \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} s_Q m_Q \ \text{in } \mathbf{S}'.$$

Moreover,

$$\sup_{J\in\mathcal{D}} \left\{ \left(\frac{1}{|J|}\right)^{1-p/r} \int_J \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v, Q\subset J} \Big(|s_Q||Q|^{-1/r}\chi_Q\Big)^q \right]^{p/q} dx \right\}^{1/p} \lesssim \|f\|_{\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q}}.$$

Proof. For $f \in \mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r}$, in the light of Proposition 2.11 we deduce that

$$f = c_{m,M,N} \int_0^\infty (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t},$$

where

$$c_{m,M,N} = -\frac{1}{2^{m+M+N-1}(m+M+N-1)!}$$

and the convergence is in the space of distributions $\mathbf{S}'$. So we have

$$\begin{aligned} f &= c_{m,M,N} \sum_{v\in\mathbb{Z}} \int_{2^v}^{2^{v+1}} (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t} \\ &= c_{m,M,N} \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \int_{2^v}^{2^{v+1}} (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} \Big[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q\Big] \frac{dt}{t}. \end{aligned}$$

For each $v \in \mathbb{Z}$ and $Q \in \mathcal{D}_v$, we set

$$s_Q = 2^{-v(\alpha-n/r)} \sup_{(y,t)\in Q\times[2^v,2^{v+1})} \Big|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)\Big| \tag{3.1}$$

and $m_Q = \mathbb{H}^{M/2} b_Q$, where

$$b_Q = \frac{1}{s_Q} \int_{2^v}^{2^{v+1}} t^M (t\sqrt{\mathbb{H}})^N e^{-t\sqrt{\mathbb{H}}} \Big[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q\Big] \frac{dt}{t}.$$

Then we obtain

$$f = \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} s_Q m_Q \ \text{in } \mathbf{S}'.$$

For $k = 0, ..., 2M$, by Lemma 2.10, for $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |\mathbb{H}^{k/2} b_Q(x)| &= \left| \frac{1}{s_Q} \int_{2^v}^{2^{v+1}} t^{M-k} (t\sqrt{\mathbb{H}})^{N+k} e^{-t\sqrt{\mathbb{H}}} \Big[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q\Big] \frac{dt}{t} \right| \\ &\lesssim \frac{1}{s_Q} \int_{2^v}^{2^{v+1}} t^{M-k} \int_Q |p_{t,N+k}(x,y)| \Big|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)\Big| dy \frac{dt}{t} \\ &\lesssim \frac{1}{s_Q} \sup_{(y,t)\in Q\times[2^v,2^{v+1})} \Big|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)\Big| \int_{2^v}^{2^{v+1}} t^{M-k} \int_Q \frac{t^N}{(t+|x-y|)^{n+N}} dy \frac{dt}{t} \\ &\lesssim 2^{v(\alpha+M-k-n/r)} \left(1 + \frac{|x-x_Q|}{2^v}\right)^{-n-N}. \end{aligned}$$

It follows from these estimates that m_Q is a multiple of an $(\mathbb{H}, M, N, \alpha, r)$ molecule.

In view of Lemma 2.7, we can now conclude that

$$\left\| \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{1/q} \right\|_{\mathbf{M}^r_p} \lesssim \|f\|_{\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q}}.$$

Indeed, it follow from the fact $u(x,t) \equiv \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f(x)$ is a solution of the equation

$$(-u_{tt} - \Delta + |x|^2)u = 0$$

that u is subharmonic. Due to Lemma 5.2 in [2], it is clear to see that

$$\sup_{(y,t)\in\widetilde{Q}} \left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f(y) \right| \lesssim \left(\frac{1}{|\widetilde{Q}|} \int_{\frac{3}{2}\widetilde{Q}} \left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f(z) \right|^\theta dz dt \right)^{1/\theta},$$

where $\widetilde{Q} = Q \times [2^v, 2^{v+1})$ is a cube in $\mathbb{R}^{n+1}$ and $r > 0$.

Observing $|\widetilde{Q}| \sim 2^v|Q|$ and $t \sim 2^v$ whenever $(y,t) \in \widetilde{Q}$ gives us

$$\sup_{(y,t)\in\widetilde{Q}} \left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f(y) \right| \lesssim \left(\frac{1}{|Q|} \int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \int_{\frac{3}{2}Q} \left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f(z) \right|^\theta dz \frac{dt}{t} \right)^{1/\theta}, \tag{3.2}$$

for any $x \in Q$.

Therefore, by Minkowski's inequality, we obtain

$$\begin{aligned} |Q|^{-1/r}|s_Q|\chi_Q(x) &\lesssim 2^{-v\alpha}\chi_Q(x)\frac{1}{|Q|} \int_{\frac{3}{2}Q} \left[\left(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(\left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f(z) \right| \right)^\theta \frac{dt}{t} \right)^{1/\theta} \right] dz \\ &\lesssim \chi_Q(x)\mathbb{M}_\theta \left[\left(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha} \left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f \right| \right)^\theta \frac{dt}{t} \right)^{1/\theta} \right](x). \end{aligned}$$

Then, we have

$$\begin{aligned} &\sum_{Q\in\mathcal{D}_v} \left(|Q|^{-1/r}|s_Q|\chi_Q(x) \right)^q \\ &\lesssim \left(\mathbb{M}_\theta \left[\left(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha} \left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f \right| \right)^\theta \frac{dt}{t} \right)^{1/\theta} \right](x) \right)^q \sum_{Q\in\mathcal{D}_v} \chi_Q(x) \\ &\lesssim \left(\mathbb{M}_\theta \left[\left(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha} \left| \mathbb{H}^{m/2}e^{-t\sqrt{\mathbb{H}}}f \right| \right)^\theta \frac{dt}{t} \right)^{1/\theta} \right](x) \right)^q. \end{aligned}$$

Combining the last estimate above with the Fefferman-Stein vector-valued maximal inequality (2.3), we deduce that

$$\left\| \left\{ \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right\}^{1/q} \right\|_{\mathbf{M}_p^r}$$

$$\lesssim \left\| \left\{ \sum_{v\in\mathbb{Z}} \left(\mathbb{M}_\theta \left[\left(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \right| \right)^\theta \frac{dt}{t} \right)^{1/\theta} \right] \right)^q \right\}^{1/q} \right\|_{\mathbf{M}_p^r}$$

$$\lesssim \left\| \left\{ \sum_{v\in\mathbb{Z}} \left(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \right| \right)^\theta \frac{dt}{t} \right)^{q/\theta} \right\}^{1/q} \right\|_{\mathbf{M}_p^r}.$$

Eventually, it follows from Hölder's inequality that

$$\left\| \left\{ \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right\}^{1/q} \right\|_{\mathbf{M}_p^r}$$

$$\lesssim \left\| \left[\sum_{v\in\mathbb{Z}} \int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \right| \right)^q \frac{dt}{t} \right]^{1/q} \right\|_{\mathbf{M}_p^r}$$

$$\lesssim \left\| \left[\int_0^\infty \left(t^{-\alpha} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \right| \right)^q \frac{dt}{t} \right]^{1/q} \right\|_{\mathbf{M}_p^r} \equiv \|f\|_{\mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H},m}}.$$

□

The follwing can be regarded as the converse to Theorem 3.1.

Theorem 3.2. *Let $\alpha\in\mathbb{R}, 0<q\le\infty, 0<p\le r\le\infty$ and $\theta_0=\min\{1,p,q\}$. If*

$$f=\sum_{v\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_v} s_Q m_Q \text{ in } \mathbf{S}',$$

where $\left\{m_Q : Q\in\mathcal{D}_v, v\in\mathbb{Z}\right\}$ is a sequence of $(\mathbb{H},M,N,\alpha,r)$ molecules and $\left\{s_Q : Q\in\mathcal{D}_v, v\in\mathbb{Z}\right\}$ is a sequence of coefficients satisfying

$$\sup_{J\in\mathcal{D}} \left\{ \left(\frac{1}{|J|}\right)^{1-p/r} \int_J \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v, Q\subset J} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{p/q} dx \right\}^{1/p} < \infty,$$

then $f\in\mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H},m}$ and

$$\|f\|_{\mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H},m}} \lesssim \sup_{J\in\mathcal{D}} \left\{ \left(\frac{1}{|J|}\right)^{1-p/r} \int_J \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v, Q\subset J} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{p/q} dx \right\}^{1/p}$$

provided that

$$m>\max\{\alpha,0\}+N+n, M>\max\{n/\theta_0-\alpha,m\}, N>n(1/\theta_0-1).$$

Proof. Observe that

$$\begin{aligned}
\|f\|^q_{\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r}} &= \left\| \left(\sum_{v\in\mathbb{Z}} \int_{2^v}^{2^{v+1}} \left(t^{-\alpha} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \right| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{\mathbf{M}^r_p} \\
&\lesssim \left\| \left[\sum_{v\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{D}_v} 2^{-v\alpha} \sup_{t\in[2^v,2^{v+1})} |s_Q| \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q \right| \right)^q \right]^{1/q} \right\|_{\mathbf{M}^r_p}.
\end{aligned} \tag{3.3}$$

Let us now choose any $\theta \in (0,\theta_0)$ such that $M > n/\theta - \alpha$ and $N > n(1/\theta - 1)$. Hence, due to $\theta > \dfrac{n}{n+N}$, we then apply Lemmas 2.2 and 2.3 to deduce

$$\sum_{Q\in\mathcal{D}_v} 2^{-v\alpha} \sup_{t\in[2^v,2^{v+1})} |s_Q| \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q \right| \chi_Q(x) \lesssim \mathbb{M}_\theta \left[\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r} |s_Q| \chi_Q \right](x).$$

Combining the above estimate with (3.3) gives us

$$\|f\|^q_{\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r}} \lesssim \left\| \left\{ \sum_{v\in\mathbb{Z}} \left(\mathbb{M}_\theta \left[\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r} |s_Q| \chi_Q \right] \right)^q \right\}^{1/q} \right\|_{\mathbf{M}^r_p}.$$

In the light of the Fefferman-Stein vector-valued maximal inequality (2.3), we then obtain the desired estimates

$$\begin{aligned}
\|f\|^q_{\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r}} &\lesssim \left\| \left[\sum_{v\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r} |s_Q| \chi_Q \right)^q \right]^{1/q} \right\|_{\mathbf{M}^r_p} \\
&\lesssim \left\| \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q| |Q|^{-1/r} \chi_Q \right)^q \right]^{1/q} \right\|_{\mathbf{M}^r_p}.
\end{aligned}$$

Finally, applying Lemma 2.7 completes the proof of the theorem. □

The next result shows the equivalence of the spaces $\mathbf{FM}^{\alpha,\mathbb{H},m}_{p,q,r}$.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $p \le r \le \infty$. Then the spaces $\mathbf{FM}^{\alpha,\mathbb{H},m_1}_{p,q,r}$ and $\mathbf{FM}^{\alpha,\mathbb{H},m_2}_{p,q,r}$ coincide with equivalent norms, provided that m_1 and m_2 are positive integers such that*

$$m_1, m_2 > n + \max\{\alpha, 0\} + \mathrm{int}\left[n\left(\frac{1}{\theta_0} - 1 \right) \right] + 1,$$

where $\theta_0 = \min\{1,p,q\}$.

Proof. Let us take positive integers $N = \mathrm{int}\left[n\left(\frac{1}{\theta_0} - 1 \right) \right] + 1$ and $M > \max\{m_1, m_2, n/\theta_0 - \alpha\}$.

For any $f \in \mathbf{FM}^{\alpha,\mathbb{H},m_1}_{p,q,r}$, in view of Theorem 3.1, there exist a sequence of $(\mathbb{H}, M, N, \alpha, r)$-molecules

$$\left\{ m_Q : Q \in \mathcal{D}_v, v \in \mathbb{Z} \right\}$$

and a sequence of coefficients

$$\left\{ s_Q : Q \in \mathcal{D}_v, v \in \mathbb{Z} \right\}$$

such that

$$f = \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} s_Q m_Q \text{ in } \mathbf{S}'$$

and

$$\left\| \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{1/q} \right\|_{\mathbf{M}_p^r} \lesssim \|f\|_{\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_1}}.$$

Then it follows from Theorem 3.2 that $f \in \mathbf{FM}_{p,q,r}^{\alpha,\mathbb{H},m_2}$ and

$$\|f\|_{\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_2}} \lesssim \left\| \left[\sum_{v\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_v} \left(|s_Q||Q|^{-1/r}\chi_Q \right)^q \right]^{1/q} \right\|_{\mathbf{M}_p^r} \lesssim \|f\|_{\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_1}}.$$

In other words, we obtain

$$\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_1} \subset \mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_2}$$

and

$$\|\cdot\|_{\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_2}} \lesssim \|\cdot\|_{\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_1}}.$$

By a similar argument, it is easy to see that

$$\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_2} \subset \mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_1}$$

and

$$\|\cdot\|_{\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_1}} \lesssim \|\cdot\|_{\mathbf{FM}_{p,q}^{\alpha,\mathbb{H},m_2}}.$$

□

4. Inhomogeneous HTLM spaces

The main goal of this section is to establish molecular decompositions for inhomogeneous HTLM spaces $\mathbf{IFM}_{p,q,r}^{\alpha,\mathbb{H}}(\beta,\gamma)$.

Let us first introduce the definition of inhomogeneous HTLM type spaces $\mathbf{IFM}_{p,q,r}^{\alpha,\mathbb{H},m}(\beta,\gamma)$ via the semigroup characterization.

Definition 4.1. Let $\alpha \in \mathbb{R}$, $m \in \mathbb{N}_+, \alpha + n < m$ and $0 < p \le r \le \infty, 0 < q \le \infty$ and let $0 < \beta < 1 < \gamma$ with $\gamma - \beta > 1$. The inhomogeneous HTLM spaces $\mathbf{IFM}_{p,q,r}^{\alpha,\mathbb{H},m}(\beta,\gamma)$ are defined to be the set of all functions $f \in \mathbf{S}'$ such that

$$\|f\|_{\mathbf{IFM}_{p,q,r}^{\alpha,\mathbb{H},m}(\beta,\gamma)} = \int_\beta^\gamma \|e^{-t\sqrt{L}}f\|_{\mathbf{M}_p^r}\frac{dt}{t} + \left\| \left(\int_0^1 (t^{-\alpha}|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}}f|)^q \frac{dt}{t} \right)^{1/q} \right\|_{\mathbf{M}_p^r} < \infty.$$

Theorem 4.2. *Let $\alpha \in \mathbb{R}, 0 < p < \alpha + n, p \le r \le \infty$ and $0 < q \le \infty$ and let $0 < \beta < 1 < \gamma$ with $\gamma - \beta > 1$. For $M \in \mathbb{N}_+$ and $m > \alpha + n$, if $f \in \mathbf{IFM}_{p,q,r}^{\alpha,\mathbb{H},m}(\beta,\gamma)$ then there exist:*

- *a sequence of $(\mathbb{H}, M, 1, \alpha, r)$ molecules $\{m_Q\}_{Q\in\mathcal{D}_{\nu,\nu\in\mathbb{Z}^-}}$,*
- *a sequence of $(\mathbb{H}, M, 1, \alpha, r)$ zero level molecules $\{m_Q\}_{Q\in\mathcal{D}_0}$,*
- *a sequence of coefficients $\{s_Q\}_{Q\in\mathcal{D}_{\nu,\nu\in\mathbb{Z}_0^-}}$*

such that

$$f = \sum_{\nu\in\mathbb{Z}_0^-}\sum_{Q\in\mathcal{D}_\nu} s_Q m_Q \ \ in\ \mathbf{S}'.$$

Moreover,

$$\sup_{J\in\mathcal{D}}\left\{\left(\frac{1}{|J|}\right)^{1-p/r}\int_J\left[\sum_{v\in\mathbb{Z}_0^-}\sum_{Q\in\mathcal{D}_v, Q\subset J}\Big(|s_Q||Q|^{-1/r}\chi_Q\Big)^q\right]^{p/q}dx\right\}^{1/p} \lesssim \|f\|_{\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}(\beta,\gamma)}.$$

Proof. In the light of Proposition 2.12, we can write

$$f = -\frac{2^{m+M-1}}{(m+M-1)!}\int_0^{1/2}(t\sqrt{\mathbb{H}})^M e^{-t\sqrt{\mathbb{H}}}(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}}f\frac{dt}{t} + \sum_{k=0}^{m+M-1}\frac{1}{k!}(\sqrt{\mathbb{H}})^k e^{-\sqrt{\mathbb{H}}}f \ (\text{in } \mathbf{S}').$$

For each $\nu\in\mathbb{Z}^-$ and $Q\in\mathcal{D}_\nu$, we set

$$s_Q = 2^{-\nu(\alpha-n/r)}\sup_{(y,t)\in Q\times[2^{\nu-1},2^\nu)}|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}}f(y)|$$

and $m_Q = \mathbb{H}^{M/2}b_Q$, where

$$b_Q = \frac{1}{s_Q}\int_{2^{\nu-1}}^{2^\nu} t^M e^{-t\sqrt{\mathbb{H}}}[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}}f.\chi_Q]\frac{dt}{t}.$$

For $Q\in\mathcal{D}_0$ we set

$$s_Q = \sup_{y\in Q}|e^{-\tau_0\sqrt{\mathbb{H}}}f(y)|$$

and

$$m_Q = \frac{1}{s_Q}\sum_{k=0}^{m+M-1}2^k(\sqrt{\mathbb{H}})^k e^{-(1-\tau_0)\sqrt{\mathbb{H}}}(e^{-\tau_0\sqrt{\mathbb{H}}}f.\chi_Q),$$

where $\tau_0 = (1+\beta)/2$.

Using the argument used in the proof of Theorem 3.1 yields that m_Q is an $(\mathbb{H}, M, 1, \alpha, r)$ molecule for $Q\in\mathcal{D}_\nu, \nu\in\mathbb{Z}^-$ and

$$\sup_{J\in\mathcal{D}}\left\{\left(\frac{1}{|J|}\right)^{1-p/r}\int_J\left[\sum_{v\in\mathbb{Z}^-}\sum_{Q\in\mathcal{D}_v, Q\subset J}\Big(|s_Q||Q|^{-1/r}\chi_Q\Big)^q\right]^{p/q}dx\right\}$$
$$\lesssim \left(\int_0^1(t^{-\alpha}\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}}f\|_{\mathbf{M}^r_p})^q\frac{dt}{t}\right)^{1/q}.$$

We now claim that m_Q is an $(\mathbb{H}, M, 1, \alpha, r)$ zero level molecule for $Q\in\mathcal{D}_0$. Indeed, for $k = 0, ..., M$, in view of Lemma 2.10, it is clear that

$$|\mathbb{H}^{k/2}m_Q(x)| \lesssim \frac{1}{s_Q}\int_Q\frac{1}{(1+|x-y|)^{n+1}}|e^{-\tau_0\sqrt{\mathbb{H}}}f(y)|dy \lesssim (1+|x-x_Q|)^{-n-1},$$

which shows that m_Q is an $(\mathbb{H}, M, 1, \alpha, r)$ zero level molecule for $Q\in\mathcal{D}_0$.

Let I be an interval in $(0,\infty)$ such that $|I| = 1/2, \tau_0\in I\subset(\beta,\gamma)$. Then we can find $1<\eta<2$ so that $\eta I\subset(\beta,\gamma)$. We now follow the reasoning used in Theorem 3.1 to deduce that $|e^{-t\sqrt{\mathbb{H}}}f|^2$ is subharmonic.

Therefore, for $Q \in \mathcal{D}_0, 0 < \theta < \min\{1, p, q\}$ and $x \in Q$,

$$|s_Q| \le \sup_{Q\times I} |e^{-\sqrt{\mathbb{H}}} f(y)| \lesssim \left(\frac{1}{\eta|Q| \times \eta|I|} \int_{\eta I} \int_{\eta Q} |e^{-t\sqrt{\mathbb{H}}} f(z)|^\theta dz dt \right)^{1/\theta}$$
$$\lesssim \left[\mathbb{M}\left(\int_\beta^\gamma |e^{-t\sqrt{\mathbb{H}}} f|^\theta \frac{dt}{t} \right)(x) \right]^{1/\theta},$$

which, together with $|Q| = 1$, implies that

$$\sum_{Q\in\mathcal{D}_0} \left(|Q|^{-1/r} |s_Q| \chi_Q(x) \right)^q \lesssim \left[\mathbb{M}\left(\int_\beta^\gamma |e^{-t\sqrt{\mathbb{H}}} f|^\theta \frac{dt}{t} \right)(x) \right]^{q/\theta}.$$

The above estimate, combining with Lemma 2.7, yields that

$$\sup_{J\in\mathcal{D}} \left\{ \left(\frac{1}{|J|} \right)^{1-p/r} \int_J \left[\sum_{Q\in\mathcal{D}_0, Q\subset J} \left(|s_Q| |Q|^{-1/r} \chi_Q \right)^q \right]^{p/q} dx \right\}^{1/p}$$
$$\lesssim \left\| \left[\mathbb{M}\left(\int_\beta^\gamma \left| e^{-t\sqrt{\mathbb{H}}} f \right|^\theta \frac{dt}{t} \right) \right]^{1/\theta} \right\|_{\mathbf{M}_p^r}.$$

By the inequality (2.2) in Lemma 2.5, it is clear to see that

$$\sup_{J\in\mathcal{D}} \left\{ \left(\frac{1}{|J|} \right)^{1-p/r} \int_J \left[\sum_{Q\in\mathcal{D}_0, Q\subset J} \left(|s_Q| |Q|^{-1/r} \chi_Q \right)^q \right]^{p/q} dx \right\}^{1/p}$$
$$\lesssim \left\| \mathbb{M}\left(\int_\beta^\gamma \left| e^{-t\sqrt{\mathbb{H}}} f \right|^\theta \frac{dt}{t} \right) \right\|_{\mathbf{M}_{p/\theta}^{r/\theta}}^{1/\theta}.$$

Finally, using the $\mathbf{M}_{p/\theta}^{r/\theta}-$ boundedness of $\mathbb{M}$, we can then conclude that

$$\sup_{J\in\mathcal{D}} \left\{ \left(\frac{1}{|J|} \right)^{1-p/r} \int_J \left[\sum_{Q\in\mathcal{D}_0, Q\subset J} \left(|s_Q| |Q|^{-1/r} \chi_Q \right)^q \right]^{p/q} dx \right\}^{1/p}$$
$$\lesssim \left\| \int_\beta^\gamma \left| e^{-t\sqrt{\mathbb{H}}} f \right|^\theta \frac{dt}{t} \right\|_{\mathbf{M}_{p/\theta}^{r/\theta}}^{1/\theta} \lesssim \left(\int_\beta^\gamma \left\| \left| e^{-t\sqrt{\mathbb{H}}} f \right|^\theta \right\|_{\mathbf{M}_{p/\theta}^{r/\theta}} \frac{dt}{t} \right)^{1/\theta}$$
$$\lesssim \left(\int_\beta^\gamma \| e^{-t\sqrt{\mathbb{H}}} f \|_{\mathbf{M}_p^r}^\theta \frac{dt}{t} \right)^{1/\theta} \lesssim \int_\beta^\gamma \| e^{-t\sqrt{\mathbb{H}}} f \|_{\mathbf{M}_p^r} \frac{dt}{t},$$

where we used Minkowski's inequality and the inequality (2.2) in the second estimate and the third estimate respectively. This completes the proof. □

Theorem 4.3. *Let* $\frac{n}{n+1} < p \le r \le \infty, \frac{n}{n+1} < q \le \infty, \alpha \in \mathbb{R}$ *and* $m > \alpha + n$*. Assume that*

$$f = \sum_{\nu\in\mathbb{Z}_0^-} \sum_{Q\in\mathcal{D}_\nu} s_Q m_Q \text{ in } \mathbf{S}',$$

where

- $\{m_Q\}_{Q\in\mathcal{D}_{\nu,\nu\in\mathbb{Z}^-}}$ *is a sequence of* $(\mathbb{H}, M, 1, \alpha, r)$ *molecules,*
- $\{m_Q\}_{Q\in\mathcal{D}_0}$ *is a sequence of* $(\mathbb{H}, M, 1, \alpha, r)$ *zero level molecules,*

- $\{s_Q\}_{Q\in\mathcal{D}_{\nu,\nu\in\mathbb{Z}_0^-}}$ *is a sequence of coefficients*

such that

$$\sup_{J\in\mathcal{D}}\left\{\left(\frac{1}{|J|}\right)^{1-p/r}\int_J\left[\sum_{v\in\mathbb{Z}_0^-}\sum_{Q\in\mathcal{D}_v,Q\subset J}\Big(|s_Q||Q|^{-1/r}\chi_Q\Big)^q\right]^{p/q}dx\right\}^{1/p}<\infty.$$

Then $f\in\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}(\beta,\gamma)$ *and*

$$\|f\|_{\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}(\beta,\gamma)}\lesssim\sup_{J\in\mathcal{D}}\left\{\left(\frac{1}{|J|}\right)^{1-p/r}\int_J\left[\sum_{v\in\mathbb{Z}_0^-}\sum_{Q\in\mathcal{D}_v,Q\subset J}\Big(|s_Q||Q|^{-1/r}\chi_Q\Big)^q\right]^{p/q}dx\right\}^{1/p},$$

whenever $M>\max\{n-\alpha,m\}$ *and* $\beta<1<\gamma$ *with* $\gamma-\beta>1$.

Proof. The proof of Theorem 4.3 is similar to that of Theorem 3.2 and we omit details here. □

The following result is a direct consequence of Theorem 4.2 and Theorem 4.3.

Theorem 4.4. *Let* $\alpha\in\mathbb{R}$, $\frac{n}{n+1}<p,q\le\infty$, $0<\beta_1<1<\gamma_1$ *with* $\gamma_1-\beta_1>1$ *and* $0<\beta_2<1<\gamma_2$ *with* $\gamma_2-\beta_2>1$. *Then the spaces* $\mathbf{IFM}^{\alpha,\mathbb{H},m_1}_{p,q,r}(\beta_1,\gamma_1)$ *and* $\mathbf{IFM}^{\alpha,\mathbb{H},m_2}_{p,q,r}(\beta_2,\gamma_2)$ *coincide with equivalent norms, provided that* $m_1,m_2\in\mathbb{N}_+$ *with* $m_1,m_2>\alpha+n$.

As a result of Theorem 4.4, we can define the Triebel-Lizorkin-Morrey space $\mathbf{IFM}^{\alpha,\mathbb{H}}_{p,q,r}$ with $\alpha\in\mathbb{R}$ and $\frac{n}{n+1}<p,q\le\infty$ as the space $\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}(\beta,\gamma)$ for any $m\in\mathbb{N}_+,m>\alpha+n$ and $0<\beta<1<\gamma$ with $\gamma-\beta>1$.

For $\alpha\in\mathbb{R},m\in\mathbb{N}$ and $1\le p,q\le\infty$, we define the (inhomogenenous) HTLM type spaces $\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}$ as follows

$$\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}:=\left\{f\in\mathbf{S}':\|f\|_{\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q}}=\|e^{-\sqrt{\mathbb{H}}}f\|_{\mathbf{M}^r_p}+\left\|\Big(\int_0^1(t^{-\alpha}(t\sqrt{\mathbb{H}})^me^{-t\sqrt{\mathbb{H}}}f)^q\frac{dt}{t}\Big)^{1/q}\right\|_{\mathbf{M}^r_p}<\infty\right\}.$$

It turns out that the spaces $\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q}$ and $\mathbf{IFM}^{\alpha,\mathbb{H}}_{p,q}$ coincide.

Theorem 4.5. *Let* $\alpha\in\mathbb{R}$, $m\in\mathbb{N}_+,m>\alpha+n$ *and* $1\le p,q\le\infty$. *Then the spaces* $\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}$ *and* $\mathbf{IFM}^{\alpha,\mathbb{H}}_{p,q,r}$ *coincide with equivalent norms.*

Proof. The proof of Theorem 4.5 is similar to that of Theorem 4.2. Here we just give a sketch of the proof.

Firstly, we prove that if $f\in\mathbf{IFM}^{\alpha,\mathbb{H},m}_{p,q,r}$ then f has a decomposition as in Theorem 4.2. To this end, by a minor modification of the proof of Proposition 2.12 in [1], one can write

$$f=-\frac{2^{m+M-1}}{(m+M-1)!}\int_0^1(t\sqrt{\mathbb{H}})^Me^{-t\sqrt{\mathbb{H}}}(t\sqrt{\mathbb{H}})^me^{-t\sqrt{\mathbb{H}}}f\frac{dt}{t}+\sum_{k=0}^{m+M-1}\frac{1}{k!}(\sqrt{\mathbb{H}})^ke^{-2\sqrt{\mathbb{H}}}f\text{ in }\mathbf{S}'.$$

For each $\nu\in\mathbb{Z}^-$ and $Q\in\mathcal{D}_\nu$, we set

$$s_Q=2^{-\nu(\alpha-n/r)}\sup_{(y,t)\in Q\times[2^{\nu-1};2^\nu)}|(t\sqrt{\mathbb{H}})^me^{-t\sqrt{\mathbb{H}}}f(y)|$$

and $m_Q=(\sqrt{\mathbb{H}})^Mb_Q$, where

$$b_Q=\frac{1}{s_Q}\int_{2^{\nu-1}}^{2^\nu}t^Me^{-t\sqrt{\mathbb{H}}}[(t\sqrt{\mathbb{H}})^me^{-t\sqrt{\mathbb{H}}}f.\chi_Q]\frac{dt}{t}.$$

For $Q \in \mathcal{D}_0$, we set

$$s_Q = \left(\int_Q |e^{-\sqrt{\mathbb{H}}} f(y)|^p dy \right)^{1/p}$$

and

$$m_Q = \frac{1}{s_Q} \sum_{k=0}^{m+M-1} \frac{2^k}{k!} (\sqrt{\mathbb{H}})^k e^{-\sqrt{\mathbb{H}}} (e^{\sqrt{\mathbb{H}}} f . \chi_Q).$$

Certainly,

$$f = \sum_{\nu \in \mathbb{Z}_0^-} \sum_{Q \in \mathcal{D}_\nu} s_Q m_Q.$$

In addition, it is easy to see that

$$\left(\sup_{J \in \mathcal{D}, \ell(J) \geqslant 1} \left(\frac{1}{|J|} \right)^{1-p/r} \sum_{Q \in \mathcal{D}_0, Q \subset J} |Q|^{1-p/r} |s_Q|^p \right)^{1/p} = \left\| e^{-\sqrt{\mathbb{H}}} f \right\|_{\mathbf{M}_p^r}.$$

At this stage, by applying Lemma 2.10 and Hölder's inequality, we can claim that m_Q are $(\mathbb{H}, M, 1, \alpha, r)$ zero level molecules for $Q \in \mathcal{D}_0$. The remaining of the proof is then similar to that of Theorem 4.2, so we omit details. □

5. Some applications

5.1. Boundedness of Hermite-Riesz and Hermite-Bessel potential on HTLM spaces

In view of Proposition 2.9, for $k \in \mathbb{N}$ and $f \in \mathbf{S}$, we define

$$\left(t\sqrt{\mathbb{H}} \right)^k e^{-t\sqrt{\mathbb{H}}} f(x) = \langle f; p_{t,k}(x, \cdot) \rangle ,$$

where $\langle \cdot ; \cdot \rangle$ is the pair between a linear function in $\mathbf{S}'$ and a function in $\mathbf{S}$.

In the light of [1, Proposition 2.5], it is easy to obtain the following lemma.

Lemma 5.1. *Assume that $\phi \in \mathbf{S}$. Then we have the following statements:*

1. $\mathbb{H}^{-\sigma} \in \mathbf{S}$, *for all* $\sigma > 0$.
2. $\left(t\sqrt{\mathbb{H}} \right)^k e^{-t\sqrt{\mathbb{H}}} \in \mathbf{S}$, *for all* $k \in \mathbb{N}, t > 0$.

Moreover, by adapting the argument used in [1, Proposition 2.5], it can be verified that for $\phi \in \mathbf{S}$,

$$\mathbb{H}^{-\sigma} \phi = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^\sigma e^{-t\mathbb{H}} \phi \frac{dt}{t} \in \mathbf{S}.$$

Definition 5.2. For $\sigma > 0$, thanks to Lemma 5.1, one can define Hermite-Riesz potential $\mathbb{H}^{-\sigma} : \mathbf{S}' \to \mathbf{S}'$ by

$$\langle \mathbb{H}^{-\sigma} f, \phi \rangle = \langle f, \mathbb{H}^{-\sigma} \phi \rangle,$$

for all $f \in \mathbf{S}'$ and $\phi \in \mathbf{S}$.

Let us denote by $K_t(x, y)$ the kernel of the semigroup $e^{-t\mathbb{H}}$. By applying the estimates in Lemma 2.4 of [1], one easily obtains the following lemma.

Lemma 5.3. *For $k \in \mathbb{N}$, there exists $c, C > 0$ such that for all $x, y \in \mathbb{R}^n$, we have*

$$|\partial_x^k K_t(x, y)| \leq \begin{cases} Ct^{-\frac{k+1}{2}} \exp\left(-c\frac{|x-y|^2}{t} \right), & 0 < t \leq 1; \\ e^{-t} e^{-|x-y|^2}, & t > 1. \end{cases}$$

Theorem 5.4. *Let $\alpha \in \mathbb{R}$, $0 < p \le r < \infty$ and $0 < q \le \infty$. Then, for $s > 0$, the operator $\mathbb{H}^{-s}$ is bounded from $\mathbf{IFM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathbf{IFM}^{\alpha+s,\mathbb{H}}_{p,q,r}$.*

Proof. Take a sequence of $(\mathbb{H}, 4M, N, \alpha, r)$ molecules $\left\{m_Q : Q \in \mathcal{D}_v, v \in \mathbb{Z}\right\}$, with $M, N \in \mathbb{N}$ and $M > s + n/2 + N/2$. Then we shall claim that $\mathbb{H}^{-s}(m_Q)$ is an $(H, 2M, N, \alpha + 2s, r)$ molecule associated to the cube Q. To this end, assume that $m_Q = \mathbb{H}^{2M} b_Q$. Then it is clear to see that $\mathbb{H}^{-s} m_Q = \mathbb{H}^M y_Q$, where $y_Q = \mathbb{H}^{-s}\mathbb{H}^M b_Q$. We will prove that

$$|(\sqrt{\mathbb{H}})^k y_Q(x)| \le C\ell(Q)^{2M-k}|Q|^{\alpha/n-1/r}\left(1 + \frac{|x - x_Q|}{\ell(Q)}\right)^{-n-N},$$

for $k = 0, ..., 4M$. Indeed, we can write

$$y_Q(x) = \mathbb{H}^{-s}\mathbb{H}^M b_Q = \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-t\mathbb{H}}\mathbb{H}^M b_Q(x)\frac{dt}{t},$$

which implies that

$$\begin{aligned}(\sqrt{\mathbb{H}})^k y_Q(x) &= \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k} b_Q(x)\frac{dt}{t}\\ &= \frac{1}{\Gamma(s)}\int_0^{4^v} t^s e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k} b_Q(x)\frac{dt}{t}\\ &+ \frac{1}{\Gamma(s)}\int_{4^v}^\infty t^s e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k} b_Q(x)\frac{dt}{t}.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}|(\sqrt{\mathbb{H}})^k y_Q(x)| &\le \frac{1}{\Gamma(s)}\int_0^{4^v} \left|t^s e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k} b_Q(x)\right|\frac{dt}{t}\\ &+ \frac{1}{\Gamma(s)}\int_{4^v}^\infty \left|t^s e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k} b_Q(x)\right|\frac{dt}{t}\\ &:= I_1 + I_2.\end{aligned}$$

Notice that, by Lemma 5.3,

$$K_t(x, y) \le \frac{C}{t^{n/2}}\exp\left(-c\frac{|x - y|^2}{t}\right) \tag{5.1}$$

and that, by Lemma 2.5 in [3],

$$K_{t,k}(x, y) \le \frac{C}{t^{n/2}}\exp\left(-c\frac{|x - y|^2}{t}\right), \tag{5.2}$$

where $K_{t,k}(x, y)$ is the kernel of $(t\mathbb{H})^k e^{-t\mathbb{H}}$ for $k \in \mathbb{N}$.
We are now ready to estimate I_1 and I_2. By applying (5.1), we have

$$\begin{aligned}\left|e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k} b_Q(x)\right| &= \int_{\mathbb{R}^n}\left|K_t(x, y)(\sqrt{\mathbb{H}})^{2M+k} b_Q(y)\right| dy\\ &\lesssim \int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\exp\left(-c\frac{|x - y|^2}{t}\right)|(\sqrt{\mathbb{H}})^{2M+k} b_Q(y)|dy,\end{aligned}$$

which, together with Definition 2.1, gives

$$\begin{aligned}&\left|e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k} b_Q(x)\right|\\ &\lesssim \int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\left(1 + \frac{|x - y|}{\sqrt{t}}\right)^{-n-N}|Q|^{\alpha/n-1/r}2^{v(2M-k)}\left(1 + \frac{|y - x_Q|}{2^v}\right)^{-n-N} dy.\end{aligned}$$

Applying Lemma 2.4 to the above estimate yields

$$\begin{aligned}&\left|e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k}b_Q(x)\right|\\&\lesssim \left(\frac{2^v}{\sqrt{t}}\right)^n |Q|^{\alpha/n-1/r}2^{v(2M-k)}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N},\end{aligned}$$

which implies that

$$\begin{aligned}I_1 &\lesssim 2^{vn}|Q|^{\alpha/n-1/r}2^{v(2M-k)}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N}\int_0^{4^v} t^{s-n/2}\frac{dt}{t}\\&\lesssim |Q|^{(\alpha+2s)/n-1/r}2^{v(2M-k)}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N}.\end{aligned}$$

For the term I_2, we apply (5.2) to obtain

$$\begin{aligned}&\left|\mathbb{H}^M e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^k b_Q(x)\right| = t^{-M}\left|(t\mathbb{H})^M e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^k b_Q(x)\right|\\&\lesssim t^{-M}\int_{\mathbb{R}^n}\left|K_{t,M}(x,y)(\sqrt{\mathbb{H}})^k b_Q(y)\right|dy\\&\lesssim t^{-M}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\exp\left(-c\frac{|x-y|^2}{t}\right)|(\sqrt{\mathbb{H}})^k b_Q(y)|dy,\end{aligned}$$

which combined with Definition 2.1 gives

$$\begin{aligned}&\left|\mathbb{H}^M e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^k b_Q(x)\right|\\&\lesssim t^{-M}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\left(1+\frac{|x-y|}{\sqrt{t}}\right)^{-n-N}|Q|^{\alpha/n-1/r}2^{v(4M-k)}\left(1+\frac{|y-x_Q|}{2^v}\right)^{-n-N}dy.\end{aligned}$$

We then apply Lemma 2.4 to the above estimate, noticing that $t\geq 4^v$, to have

$$\begin{aligned}\left|\mathbb{H}^m e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^k b_Q(x)\right| &\lesssim t^{-M}|Q|^{\alpha/n-1/r}2^{v(M-k)}\left(1+\frac{|x-x_Q|}{\sqrt{t}}\right)^{-n-N}\\&\lesssim \left(\frac{t}{4^v}\right)^{(n+N)/2} t^{-M}|Q|^{\alpha/n-1/r}2^{v(4M-k)}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N},\end{aligned}$$

which finally implies, due to $M>s+n/2+N/2$, that

$$\begin{aligned}I_2 &\lesssim |Q|^{\alpha/n-1/r}2^{v(4M-k-N-n)}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N}\int_{4^v}^{\infty} t^{s+\frac{n+N}{2}-M}\frac{dt}{t}\\&\lesssim |Q|^{\alpha/n-1/r}2^{v(4M-k-N-n)}2^{v(2s+n+N-2M)}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N}\\&\lesssim |Q|^{(\alpha+2s)/n-1/r}2^{v(2M-k)}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N}.\end{aligned}$$

Therefore, we proved that $\mathbb{H}^{-s}(m_Q)$ is an $(H,2M,N,\alpha+2s,r)$ molecule associated to the cube Q.

By the suitable choice of M,N and using Theorem 3.1 and Theorem 3.2, we then deduce the boundedness of $\mathbb{H}^{-s}$ from $\mathbf{IFM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathbf{IFM}^{\alpha+s,\mathbb{H}}_{p,q,r}$ immediately. This completes the proof. □

By adapting the same arguments used in [1, Proposition 2.5], it can be verified that for $\phi\in\mathbf{S}$,

$$(I+\mathbb{H})^{-\sigma}\phi = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^\sigma e^{-t}e^{-t\mathbb{H}}\phi\frac{dt}{t}\in\mathbf{S}.$$

Definition 5.5. For $s > 0$, we define Hermite-Bessel potential $(I+\mathbb{H})^{-s} : \mathbf{S}' \to \mathbf{S}'$ by setting

$$\langle (I+\mathbb{H})^{-s} f, \phi\rangle = \langle f, (I+\mathbb{H})^{-s}\phi\rangle,$$

for all $f \in \mathbf{S}'$ and $\phi \in \mathbf{S}$.

Similarly, we can show the boundedness of the Hermite-Bessel potential $(I+\mathbb{H})^{-s}$ on inhomogeneous HTLM. More precisely, we have the following result.

Theorem 5.6. *Let $s > 0, \alpha \in \mathbb{R}, \frac{n}{n+1} < p \le r < \infty, \frac{n}{n+1} < q \le \infty$. Then $(I+\mathbb{H})^{-s}$ maps continuously from $\mathbf{IFM}^{\alpha,\mathbb{H}}_{p,q,r}$ into $\mathbf{IFM}^{\alpha+2s,\mathbb{H}}_{p,q,r}$.*

Proof. The proof of Theorem 5.6 is analogous to that of Theorem 5.4, so we omit the details. □

5.2. Boundedness of spectral multipliers

In this subsection, we consider the spectral multipliers of Laplace type for the Hermite operators in the following form

$$\mathrm{m}(\mathbb{H}) = \int_0^\infty \phi(t)\mathbb{H}e^{t\mathbb{H}}dt, \tag{5.3}$$

where $\phi \in L^\infty(\mathbb{R})$.
By employing the same arguments as those in [1, Proposition 2.5], one can easily verify that $\mathrm{m}(\mathbb{H})\phi \in \mathbf{S}$ when $\phi \in \mathbf{S}$. Hence, for $f \in \mathbf{S}', \mathrm{m}(\mathbb{H})f$ can be viewed as a functional in $\mathbf{S}'$ by setting

$$\langle \mathrm{m}(\mathbb{H})f, \phi\rangle = \langle f, \mathrm{m}(\mathbb{H})\phi\rangle.$$

As an application of the molecular decomposition theory developed in Section 3, we show boundedness of the spectral multiplier $\mathrm{m}(\mathbb{H})$ on $\mathbf{FM}^{\alpha,\mathbb{H}}_{p,q,r}$.

Theorem 5.7. *Let $\alpha \in \mathbb{R}, 0 < p \le r < \infty$ and $0 < q \le \infty$. Then the spectral multiplier $\mathrm{m}(\mathbb{H})$ is bounded on $\mathbf{FM}^{\alpha,\mathbb{H}}_{p,q,r}$.*

Proof. The proof of this theorem is similar to that of Theorem 5.4. We just give a sketch of the proof here.

With the same notions as in the proof of the Theorem 5.4, it suffices to prove that for any $(\mathbb{H}, 4M, N, \alpha, r)$ molecule $m_Q = \mathbb{H}^{2M} b_Q$, we have $\mathrm{m}(\mathbb{H})m_Q$ is an $(\mathbb{H}, 2M, N, \alpha, r)$ molecule associated to the same dyadic cube $Q \in \mathcal{D}_\nu$ for some $\nu \in \mathbb{Z}$.

To this end, let us first write $\mathrm{m}(\mathbb{H})m_Q = \mathbb{H}^M y_Q$, where

$$y_Q = \int_0^\infty \phi(t)\mathbb{H}e^{-t\mathbb{H}}\mathbb{H}^M b_Q dt.$$

Then, for $k = 0, ..., 4M$, we have

$$(\sqrt{\mathbb{H}})^k y_Q = \int_0^{4^\nu} \phi(t)e^{-t\mathbb{H}}\mathbb{H}^{M+k/2+1} b_Q dt + \int_{4^\nu}^\infty \phi(t)\mathbb{H}^{M+1}e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^k b_Q dt.$$

At this stage, we estimate

$$\begin{aligned}|(\sqrt{\mathbb{H}})^k y_Q(x)| &\le \int_0^{4^\nu} |\phi(t)e^{-t\mathbb{H}}\mathbb{H}^{M+k/2+1} b_Q(x)|dt \\ &\quad + \int_{4^\nu}^\infty |\phi(t)\mathbb{H}^{M+1}e^{-tH}(\sqrt{\mathbb{H}})^k b_Q(x)|dt \\ &\le \int_0^{4^\nu} |e^{-t\mathbb{H}}\mathbb{H}^{M+k/2+1} b_Q(x)|dt + \int_{4^\nu}^\infty |\mathbb{H}^{M+1}e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^k b_Q(x)|dt \\ &= I_1 + I_2.\end{aligned}$$

Finally, by repeating the same arguments as in the proof of Theorem 5.1, we deduce that

$$I_1 + I_2 \lesssim |Q|^{\alpha/n-1/p} 2^{\nu(2M-k)} \left(1 + \frac{|x-x_Q|}{2^\nu}\right)^{-n-N},$$

which completes the proof of Theorem 5.7. □

6. Identification between HTLM spaces and Hermite Sobolev spaces

Let $s \in \mathbb{R}$ and $1 < p < \infty$. We define the homogeneous Sobolev space $\mathbf{W}^{p,r}_{s,\mathbb{H}}$ associated to the Hermite operator $\mathbb{H}$ by

$$\Big\{ f \in \mathbf{S}' : \|f\|_{\mathbf{W}^{p,r}_{s,\mathbb{H}}} := \|(\sqrt{\mathbb{H}})^s\|_{\mathbf{M}^r_p} < \infty \Big\}.$$

For $\beta \in \mathbb{R}_+$, we consider the following square function

$$\mathbf{G}_{\beta,\mathbb{H}} f = \Big(\int_0^\infty |(t\sqrt{\mathbb{H}})^\beta e^{-t\sqrt{\mathbb{H}}} f|^2 \frac{dt}{t} \Big)^{1/2}, f \in L^2(\mathbb{R}^n).$$

One can easily check that $\mathbf{G}_{\beta,\mathbb{H}}$ is bounded on $L^2(\mathbb{R}^n)$, using the spectral theory. In addition, the $\mathbf{M}^r_p$ boundedness of $\mathbf{G}_{\beta,\mathbb{H}}$ will be verified below.

Lemma 6.1. *For $\beta > 0$ and $1 < p \leq r < \infty$, the square function $\mathbf{G}_{\beta,\mathbb{H}}$ is bounded on $\mathbf{M}^r_p$.*

Proof. The proof of this lemma is standard. We just give a sketch of the proof here.
Put $m = \operatorname{int}(\beta) + 1$. For $f \in L^2(\mathbb{R}^n)$, let us write

$$\begin{aligned}
(t\sqrt{\mathbb{H}})^\beta e^{-t\sqrt{\mathbb{H}}} f &= (t\sqrt{\mathbb{H}})^{-(m-\beta)} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \\
&= \frac{1}{\Gamma(m-\beta)} \int_0^\infty \Big(\frac{s}{t}\Big)^{m-\beta} (t\sqrt{\mathbb{H}})^m e^{-(t+s)\sqrt{\mathbb{H}}} f \frac{ds}{s} \\
&= \frac{1}{\Gamma(m-\beta)} \int_0^\infty \Big(\frac{s}{t}\Big)^{m-\beta} \Big(\frac{t}{t+s}\Big)^m [(t+s)\sqrt{\mathbb{H}}]^m e^{-(t+s)\sqrt{\mathbb{H}}} f \frac{ds}{s}.
\end{aligned}$$

Plugging (ii) of Lemma 2.1 into the expression above and by a straightforward calculation we can conclude that

$$p_{t,\beta}(x,y) \lesssim \frac{t^\beta}{(t+|x-y|)^{n+\beta}}.$$

Analogously, we can prove that for $|h| < t$,

$$|p_{t,\beta}(x+h,y) - p_{t,\beta}(x,y)| \lesssim \Big(\frac{|h|}{t}\Big) \frac{t^\beta}{(t+|x-y|)^{n+\beta}}.$$

In the light of the last two estimates, it is well-known from Calderón -Zygmund theory of vector valued singular integrals that the operator $\mathbf{G}_{\beta,\mathbb{H}}$ is bounded on $\mathbf{M}^r_p$. □

We are now in a position to prove the identification between HTLM spaces and Hermite Sobolev spaces.

Theorem 6.2. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then the spaces $\mathbf{W}^{p,r}_{s,\mathbb{H}}$ and $\mathbf{FM}^{s,\mathbb{H}}_{p,2,r}$ coincide with equivalent norms.*

Proof. Assume first that $f \in \mathbf{W}^{p,r}_{s,\mathbb{H}}$. Then for $m > \max\{s, 0\}$ and in view of Lemma 6.1, one has

$$\begin{aligned}
\|f\|_{\mathbf{FM}^{s,\mathbb{H}}_{p,2,r}} &= \Big\| \Big(\int_0^\infty t^{-2s} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{\mathbf{M}^r_p} \\
&\Big\| \Big(\int_0^\infty |(t\sqrt{\mathbb{H}})^{m-s} e^{-t\sqrt{\mathbb{H}}} (H^{s/2} f)|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{\mathbf{M}^r_p} \\
&\Big\| \Big(\int_0^\infty |\mathbf{G}_{m-s,H}[(\sqrt{\mathbb{H}})^s f]|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{\mathbf{M}^r_p} \\
&\lesssim \|(\sqrt{\mathbb{H}})^s f\|_{\mathbf{M}^r_p} = \|f\|_{\mathbf{W}^p_{s,\mathbb{H}}}.
\end{aligned}$$

Conversely, now assume that $f \in \mathbf{FM}^{s,\mathbb{H}}_{p,2,r}$. By Proposition 2.11, we can write

$$\mathbb{H}^{s/2}f(x) = c_m \int_{\mathbb{R}^n} \int_0^\infty (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} \mathbb{H}^{s/2} f(x) \frac{dt}{t}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathbb{H}^{s/2} f(x) g(x) dx \\
&\quad = c_m \int_{\mathbb{R}^n} \int_0^\infty (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} H^{s/2} f(x) g(x) \frac{dt}{t} dx \\
&\quad = c_m \int_{\mathbb{R}^n} \int_0^\infty t^{-s} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(x) (t\sqrt{\mathbb{H}})^{m+s} e^{-t\sqrt{\mathbb{H}}} g(x) \frac{dt}{t} dx.
\end{aligned}$$

Finally, for $g \in \mathbf{S}$ and $m > \max\{s, 0\}$, it follows from Hölder's inequality for Morrey spaces and from Lemma 6.1 that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathbb{H}^{s/2} f(x) g(x) dx \\
&\lesssim \int_{\mathbb{R}^n} \left(\int_0^\infty |t^{-s} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(x)|^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^\infty |(t\sqrt{\mathbb{H}})^{m+s} e^{-t\sqrt{\mathbb{H}}} g(x)|^2 \frac{dt}{t} \right)^{1/2} dx \\
&\lesssim \left\| \left(\int_0^\infty |t^{-s} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{\mathbf{M}^r_p} \left\| \left(\int_0^\infty |(t\sqrt{\mathbb{H}})^{m+s} e^{-t\sqrt{\mathbb{H}}} g(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{\mathbf{M}^{r'}_{p'}} \\
&\lesssim \|f\|_{\mathbf{FM}^{s,\mathbb{H}}_{p,2,r}} \|\mathbf{G}_{m+s,\mathbb{H}} g\|_{\mathbf{M}^{r'}_{p'}} \\
&\lesssim \|f\|_{\mathbf{FM}^{s,\mathbb{H}}_{p,2,r}} \|g\|_{\mathbf{M}^{r'}_{p'}},
\end{aligned}$$

which combined with the fact that $\mathbf{S}$ is dense in $\mathbf{M}^{r'}_{p'}$ implies that

$$\|\mathbb{H}^{s/2} f\|_{\mathbf{M}^r_p} \lesssim \|f\|_{\mathbf{FM}^{s,\mathbb{H}}_{p,2,r}}.$$

This completes our proof. □

Similarly, we can obtain the identification between HTLM and Morrey-Sobolev spaces for inhomogeneous version.

Let $s \in \mathbb{R}$ and $1 < p < \infty$. We define the inhomogeneous Sobolev space $\mathbf{IW}^{p,r}_{s,\mathbb{H}}$ associated to the Hermite operator $\mathbb{H}$ by

$$\left\{ f \in \mathbf{S}' : \|f\|_{\mathbf{IW}^{p,r}_{s,\mathbb{H}}} := \|(I + \mathbb{H})^{s/2} f\|_p < \infty \right\}.$$

Theorem 6.3. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then the spaces $\mathbf{IW}^{p,r}_{s,\mathbb{H}}$ and $\mathbf{IFM}^{s,\mathbb{H}}_{p,2,r}$ coincide with equivalent norms.*

Proof. The proof can be processed using the same arguments as those of Theorem 6.2 and we would like to leave it to the interested reader. □

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