LOCAL ASYMPTOTIC PROPERTIES FOR THE GROWTH RATE OF A JUMP-TYPE CIR PROCESS

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Abstract. In this paper, we consider a one-dimensional jump-type Cox-Ingersoll-Ross process driven by a Brownian motion and a subordinator, whose growth rate is a unknown parameter. The Lévy measure of the subordinator is finite or infinite. Considering the process observed continuously or discretely at high frequency, we derive the local asymptotic properties for the growth rate in both ergodic and non-ergodic cases. To do so, three cases are distinguished: subcritical, critical and supercritical. Local asymptotic normality (LAN) is proved in the subcritical case, local asymptotic quadraticity (LAQ) is derived in the critical case, and local asymptotic mixed normality (LAMN) is shown in the supercritical case.

1. Introduction

On a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ which will be specified later on, we consider a one-dimensional jump-type Cox-Ingersoll-Ross (CIR) process $Y^b = (Y_t^b)_{t\in[0,\infty)}$ driven by a subordinator

$$dY_t^b = \left(a - bY_t^b\right) dt + \sigma\sqrt{Y_t^b} dW_t + dJ_t, \tag{1.1}$$

where $Y_0^b = y_0 \in [0,\infty)$ is a given initial condition, $a \in [0,\infty)$, $b \in \mathbb{R}$ and $\sigma \in (0,\infty)$. Here, $W = (W_t)_{t\in[0,\infty)}$ is a one-dimensional standard Brownian motion, and $J = (J_t)_{t\in[0,\infty)}$ is a subordinator (an increasing Lévy process) with zero drift and with Lévy measure m concentrated on $(0,\infty)$ and satisfying the following condition

(A1) $\int_0^\infty z m(dz) \in [0,\infty)$.

Then, the Laplace transform of J is given by

$$\mathrm{E}\left[e^{uJ_t}\right] = \exp\left\{t\int_0^\infty \left(e^{uz} - 1\right) m(dz)\right\}, \tag{1.2}$$

for any $t \in [0,\infty)$ and for any complex number u with $\mathrm{Re}(u) \in (-\infty, 0]$, see, e.g. Sato [42, proof of Theorem 24.11]. We suppose that the processes W and J are independent. Note that the moment condition **(A1)** implies that m is a Lévy measure (since $\min(1, z^2) \leq z$ for

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$z \in (0, \infty)$). Moreover, the subordinator J has sample paths of bounded variation on every compact time interval almost surely, see e.g. Sato [42, Theorem 21.9]. We point out that the assumptions assure that there is a (pathwise) unique strong solution of the stochastic differential equation (SDE) (1.1) with $\mathrm{P}(Y_t^b \in [0, \infty) \text{ for all } t \in [0, \infty)) = 1$ (see Proposition 1.1). In fact, Y^b is a special continuous state and continuous time branching process with immigration (CBI process), see Proposition 1.1 below and [16, 30].

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{R}_-$, $\mathbb{R}_{--}$ and $\mathbb{C}$ denote the sets of real numbers, non-negative real numbers, positive real numbers, non-positive real numbers, negative real numbers and complex numbers, respectively.

In this paper, we will consider the jump-type CIR process Y^b solution to equation (1.1) with known $a \in \mathbb{R}_+$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and Lévy measure m satisfying condition **(A1)**, and we will consider $b \in \mathbb{R}$ as an unknown parameter to be estimated. Let $\{\widehat{\mathcal{F}}_t\}_{t\in\mathbb{R}_+}$ denote the natural filtration generated by two processes W and J. We denote by $\widehat{\mathrm{P}}^b$ the probability measure induced by the process Y^b on the canonical space $(D(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(D(\mathbb{R}_+, \mathbb{R})))$ endowed with the natural filtration $\{\widehat{\mathcal{F}}_t\}_{t\in\mathbb{R}_+}$. Here $D(\mathbb{R}_+, \mathbb{R})$ denotes the set of $\mathbb{R}$-valued càdlàg functions defined on $\mathbb{R}_+$, and $\mathcal{B}(D(\mathbb{R}_+, \mathbb{R}))$ is its Borel σ-algebra. We denote by $\widehat{\mathrm{E}}^b$ the expectation with respect to (w.r.t.) $\widehat{\mathrm{P}}^b$. Let $\xrightarrow{\widehat{\mathrm{P}}^b}$, $\xrightarrow{\mathcal{L}(\widehat{\mathrm{P}}^b)}$, $\widehat{\mathrm{P}}^b$-a.s., $\xrightarrow{\mathrm{P}}$, and $\xrightarrow{\mathcal{L}(\mathrm{P})}$ denote the convergence in $\widehat{\mathrm{P}}^b$-probability, in $\widehat{\mathrm{P}}^b$-law, in $\widehat{\mathrm{P}}^b$-almost surely, in P-probability, and in P-law, respectively. E denotes the expectation w.r.t. P.

Notice that the Lévy-Itô decomposition of J takes the form $J_t = \int_0^t \int_0^\infty z N(ds, dz)$ for any $t \in \mathbb{R}_+$, where $N(dt, dz) := \sum_{0\le s\le t} \mathbf{1}_{\{\Delta J_s \neq 0\}} \delta_{(s, \Delta J_s)}(ds, dz)$ is an interger-valued Poisson random measure in $(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+))$ with intensity measure $m(dz)dt$ associated with J. Here, the jump amplitude of J is defined as $\Delta J_s := J_s - J_{s-}$ for any $s \in \mathbb{R}_{++}$, $\Delta J_0 := 0$, $\delta_{(s,z)}$ denotes the Dirac measure at the point $(s, z) \in \mathbb{R}_+ \times \mathbb{R}_+$, and $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+)$ denotes the Borel σ-algebra on $\mathbb{R}_+ \times \mathbb{R}_+$. Hence, (1.1) can be rewritten in the following integral form

$$\begin{aligned} Y_t^b &= y_0 + \int_0^t \left(a - bY_s^b\right) ds + \sigma \int_0^t \sqrt{Y_s^b} dW_s + J_t \\ &= y_0 + \int_0^t \left(a - bY_s^b\right) ds + \sigma \int_0^t \sqrt{Y_s^b} dW_s + \int_0^t \int_0^\infty z N(ds, dz), \end{aligned} \tag{1.3}$$

for any $t \in \mathbb{R}_+$. Observe that $\mathrm{E}[J_t] = t \int_0^\infty z m(dz)$, for any $t \in \mathbb{R}_+$.

The next proposition is about the existence and uniqueness of a strong solution of the SDE (1.1), and states also that Y^b is a CBI process.

Proposition 1.1. [4, Proposition 2.1] *For all $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and Lévy measure m on $\mathbb{R}_{++}$ satisfying condition* **(A1)***, there is a pathwise unique strong solution $Y^b = (Y_t^b)_{t\in\mathbb{R}_+}$ of the SDE (1.1) such that $\mathrm{P}(Y_0^b = y_0) = 1$ and $\mathrm{P}(Y_t^b \in \mathbb{R}_+ \text{ for all } t \in \mathbb{R}_+) = 1$. Moreover, Y^b is a CBI process having branching mechanism*

$$R(u) = \frac{\sigma^2}{2} u^2 - bu, \qquad u \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}(u) \le 0,$$

and immigration mechanism

$$F(u) = au + \int_0^\infty (e^{uz} - 1) m(dz), \qquad u \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}(u) \le 0.$$

Furthermore, the infinitesimal generator of Y^b *takes the form*

$$(\mathcal{A}f)(y) = (a - by)f'(y) + \frac{\sigma^2}{2} y f''(y) + \int_0^\infty (f(y+z) - f(y))m(dz),$$

where $y \in \mathbb{R}_+$, $f \in C_c^2(\mathbb{R}_+, \mathbb{R})$, f' *and* f'' *denote the first and second order partial derivatives of* f, *and* $C_c^2(\mathbb{R}_+, \mathbb{R})$ *denotes the set of twice continuously differentiable real-valued functions on* $\mathbb{R}_+$ *with compact support.*

If, in addition, $y_0 + a \in \mathbb{R}_{++}$, *then* $\mathrm{P}(\int_0^t Y_s^b ds \in \mathbb{R}_{++}) = 1$ *for all* $t \in \mathbb{R}_{++}$.

Next we present a result on the first moment of Y^b.

Proposition 1.2. [4, Proposition 2.2] *Let* $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ *and let* m *be a Lévy measure on* $\mathbb{R}_{++}$ *satisfying condition* **(A1)**. *Let* Y^b *be the unique strong solution of the SDE* (1.1) *satisfying* $\mathrm{P}(Y_0^b = y_0) = 1$. *Then, for any* $t \in \mathbb{R}_+$,

$$\mathrm{E}[Y_t^b] = \begin{cases} y_0 e^{-bt} + (a + \int_0^\infty z m(dz)) \frac{1-e^{-bt}}{b} & \text{if } \ b \neq 0, \\ y_0 + (a + \int_0^\infty z m(dz)) t & \text{if } \ b = 0. \end{cases}$$

Consequently, if $b \in \mathbb{R}_{++}$, *then*

$$\lim_{t \to \infty} \mathrm{E}[Y_t^b] = (a + \int_0^\infty z m(dz)) \frac{1}{b},$$

if $b = 0$, *then*

$$\lim_{t \to \infty} t^{-1} \mathrm{E}[Y_t^b] = a + \int_0^\infty z m(dz),$$

if $b \in \mathbb{R}_{--}$, *then*

$$\lim_{t \to \infty} e^{bt} \mathrm{E}[Y_t^b] = y_0 - (a + \int_0^\infty z m(dz)) \frac{1}{b}.$$

Based on the asymptotic behavior of the expectations $\mathrm{E}[Y_t^b]$ as $t \to \infty$, one can introduce a classification of the jump-type CIR process (1.1).

Definition 1.3. *Let* $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ *and let* m *be a Lévy measure on* $\mathbb{R}_{++}$ *satisfying condition* **(A1)**. *Let* Y^b *be the unique strong solution of the SDE* (1.1) *satisfying* $\mathrm{P}(Y_0^b = y_0) = 1$. *We call* Y^b *subcritical, critical or supercritical if* $b \in \mathbb{R}_{++}$, $b = 0$ *or* $b \in \mathbb{R}_{--}$, *respectively.*

The CIR processes are extensively used in mathematical finance to model the evolution of short-term interest rates or to describe the stochastic volatility of a price process of an asset in the Heston model. The CIR process appears in the financial literature also as part of the class of affine processes, and a lot of interesting references can be found in this way, e.g. in the work of Teichmann et *al.* [15], Duffie, Filipović and Schachermayer [16], Kallsen [27], Keller-Ressel [29], Keller-Ressel and Mijatović [30] and other authors.

Notice that most existing research works on statistics for CIR processes and Heston models mainly focus on parameter estimation based on continuous observations. In case of the diffusion-type CIR process, Overbeck [38] examined local asymptotic properties for the drift parameter (a, b). It turned out that LAN is valid in the subcritical case when $a \in \left(\frac{\sigma^2}{2}, \infty\right)$. In the critical case LAN is proved for the submodel when $b = 0$ is known, and only LAQ is shown for the submodel when $a \in \left(\frac{\sigma^2}{2}, \infty\right)$ is known, but the asymptotic property of the experiment locally at $(a, 0)$ remained as an open question. In the supercritical case LAMN is proved for the submodel when $a \in \big(0, \infty\big)$ is known. The concept of LAN, LAQ and LAMN can be found, e.g., in Le Cam and Yang [36] or in Subsection 7.1 of Höpfner [21]. Kutoyants [33] investigated statistical inference for one-dimensional ergodic diffusion processes. LAN for the diffusion-type CIR process in the subcritical case also follows from his general result. Later, Ben Alaya and Kebaier [7, 8] show various asymptotic properties of maximum likelihood estimator (MLE) associated to the partial and global drift parameters of the diffusion-type CIR process in both ergodic and non-ergodic cases. More recently, Barczy et *al.* [4] have investigated the asymptotic properties of MLE for the growth rate b of the jump-type CIR process (1.1), which provides the main inspiration for our current work. Recently, Benke and Pap [10] have studied the local asymptotic properties of certain Heston models. Barczy and Pap [6], Barczy et *al.* [3, 5] have studied the asymptotic properties of MLE for Heston models, jump-type Heston models, and stable CIR process, respectively.

In the other direction, Malliavin calculus techniques developed by Gobet [18] are applied to analyze the log-likelihood ratio of the discrete observation of diffusion processes. Concretely, Gobet [18, 19] obtained the LAMN and LAN properties for multidimensional elliptic diffusions and ergodic diffusions. In the presence of jumps, several SDEs have been investigated, see e.g. Kawai [28], Clément et *al.* [13, 14], Kohatsu-Higa et *al.* [31, 32], and Tran [43]. Notice that all these results deal with the SDEs whose coefficients are continuously differentiable and satisfy a global Lipschitz condition. The case where the coefficients of the SDEs do not satisfy these standard assumptions is less investigated. The first contribution in this direction can be found in Ben Alaya et *al.* [9] where the authors have established the local asymptotic properties for the global drift parameters (a, b) of the diffusion-type CIR process.

The LAN, LAMN and LAQ properties for jump-type CIR processes on the basis of both continuous and discrete observations have never been addressed in the literature. Motivated by this fact and inspired by the recent paper [4], the main objective of this paper is to study the local asymptotic properties for the parameter b of the jump-type CIR process (1.1) in both ergodic and non-ergodic cases. That is, LAN will be proved in the subcritical case, LAQ will be derived in the critical case and LAMN will be shown in the supercritical case. Let us mention here that the study of the parameter a needs the asymptotic behavior of $\int_0^t \frac{1}{Y_s^b} ds$ which still remains an open problem.

For our purpose, in the case of continuous observations, we use the Girsanov's theorem (see Proposition 2.1) and the central limit theorem for continuous local martingales (see Lemma 3.14), together with the asymptotic behavior results (2.2)-(2.3), (2.6) and (2.11). Unlike the CIR diffusion, the jump-type CIR process does not have an explicit expression for its transition density and then proving local asymptotic properties based on discrete observations becomes challenging. Therefore, to overcome this difficulty in the case of discrete observations, our strategy is to prove first the existence of a positive transition density, which is of class C^1 w.r.t. b (see Proposition 3.1) and then to use the Malliavin calculus approach developed by Gobet [18, 19] for regular SDEs in order to derive an explicit expression for the logarithm derivative of the transition density in terms of a conditional expectation of a Skorohod integral (see Proposition 3.2, Lemma 3.3 and Corollary 3.4), which allows to obtain an appropriate stochastic expansion of the log-likelihood ratio (see Lemma 4.1, 4.8 and 4.15). Recall that Malliavin calculus for CIR process is established by Alòs and Ewald [1], and Altmayer and Neuenkirch [2]. To treat the main term of the expansion, in the subcritical case, we apply a central limit theorem for triangular arrays of random variables together with the convergence result (2.2)-(2.3) (see Lemma 4.2) whereas in the critical and supercritical cases, the corresponding convergence results (2.6) and (2.11) together with the central limit theorem for continuous local martingales are essentially used (see Lemma 4.9 and 4.16).

The difficulty of the proof is to treat the negligible terms of the expansion. The first difficulty comes from the fact that the conditional expectations are taken under the measure $\widetilde{\mathrm{P}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}$ appearing from the application of the Malliavin calculus whereas the convergence is evaluated under $\widehat{\mathrm{P}}^{b_0}$ with $\widehat{\mathrm{P}}^{b_0} \neq \widetilde{\mathrm{P}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}$ (see e.g. Lemma 4.4). In [18, 19], the authors use a change of transition density functions together with the upper and lower bounds of Gaussian type of the transition density functions. This measures the deviation of the change of transition density functions when the parameters change. For our jump-type CIR process Y^b, the transition density estimates of Gaussian type may not exist since the diffusion coefficient and its derivative are not bounded. To overcome this first difficulty, Girsanov's theorem is essentially used to change the probability measures (see (3.19) and Lemma 3.10). Then a technical Lemma 3.9 is established to measure the deviation of the Girsanov change of measure when the drift parameters change. Furthermore, the second difficulty comes from the jumps of the subordinator appearing in the expansion in the subcritical case due to the rate of convergence $\sqrt{n\Delta_n}$ (see Lemma 4.7). To resolve this problem, we split the jumps of the subordinator into small jumps and big jumps. As in [32, Lemma A.14], we condition on the number of big jumps outside and inside the conditional expectation. In [32], the authors use upper and lower bounds for the transition density conditioned on the jump structure, which is rather complicated since they need to show the estimates for this transition density. However, they study only the case of finite Lévy measure. Here, the cases of finite and infinite Lévy measure are considered. Our approach consists in rewriting the big jumps of the subordinator on each time interval in terms of the increments of the process, the drift and diffusion terms, and the small jumps. Then, conditionally on the events which have no big jump on this interval, the increments of the process can be rewritten in terms of the drift and diffusion terms, and the small jumps. Using all these arguments together with the usual moment estimates instead of density estimates, we get the large deviation type estimates in Lemma 5.3 where the decreasing rate is determined by the intensity of big jumps and the behavior of small jumps of the subordinator. In the critical and supercritical cases, there is no difficulty in treating the jumps of the subordinator appearing in the expansion thanks to the better rates of convergence $n\Delta_n$ and $e^{-b_0\frac{n\Delta_n}{2}}$ (see Lemma 4.14).

Furthermore, some L^p-norm estimation for positive and negative polynomial moments and exponential moment estimates of the jump-type CIR process are needed to show the convergence of the negligible terms (see Lemma 3.5-3.7). For this, condition **(A3)** below, which turns out to be crucial, allows us to obtain the useful moment estimate (3.18) in Lemma 3.8. The lower bound $\frac{15+\sqrt{185}}{4}$ in condition **(A3)** is fixed in an optimal way to get minimal restrictions on the ratio $\frac{a}{\sigma^2}$ (see Subsection 5.7 and Remark 5.1). In addition, our strategy does not require some additional assumptions on the decreasing rate of Δ_n such as $n\Delta_n^p \to 0$ for some $p > 1$.

The paper is organized as follows. In Section 2, we state our main results in Theorem 2.5, 2.7, 2.9, 2.10, 2.11, and 2.12 in the continuous and discrete observation cases as well as in the subcritical, critical and supercritical cases. Furthermore, several examples are given.

Section 3 presents technical results needed for the proof of the main results such as explicit expression for the logarithm derivative of the transition density by means of Malliavin calculus, decomposition of the Skorohod integral, some polynomial and exponential moment estimates for the jump-type CIR process, Girsanov's theorem, a discrete time ergodic theorem, central limit theorems, and a comparison theorem. The proofs of these technical results are postponed to Appendix in order to maintain the flow of the exposition. Finally, we prove our main results in Section 4, which follows the aforementioned strategy.

2. Main results

In this section, we give a statement of our main results which is divided into two cases: continuous and discrete observations.

2.1. **Continuous observations.**

The main result is divided into three cases: subcritical, critical and supercritical.

For all $T \in \mathbb{R}_{++}$, let $\widehat{\mathrm{P}}_T^b := \widehat{\mathrm{P}}^b|_{\widehat{\mathcal{F}}_T}$ be the restriction of $\widehat{\mathrm{P}}^b$ on $\widehat{\mathcal{F}}_T$, and consider the continuous observation $Y^{T,b} := (Y_t^b)_{t\in[0,T]}$ of the process Y^b on the time interval $[0,T]$. The next proposition is about the form of the Radon–Nikodym derivative $\frac{d\widehat{\mathrm{P}}_T^{\widetilde{b}}}{d\widehat{\mathrm{P}}_T^b}$ for $b, \widetilde{b} \in \mathbb{R}$.

Proposition 2.1. [4, Proposition 4.1] *Let $b, \widetilde{b} \in \mathbb{R}$. Then for all $T \in \mathbb{R}_{++}$, the probability measures $\widehat{\mathrm{P}}_T^b$ and $\widehat{\mathrm{P}}_T^{\widetilde{b}}$ are absolutely continuous w.r.t. each other, and*

$$\begin{aligned}\log \frac{d\widehat{\mathrm{P}}_T^{\widetilde{b}}}{d\widehat{\mathrm{P}}_T^b}(Y^{T,b}) &= -\frac{\widetilde{b}-b}{\sigma^2}(Y_T^b - y_0 - aT - J_T) - \frac{\widetilde{b}^2 - b^2}{2\sigma^2}\int_0^T Y_s^b ds \\ &= -\frac{\widetilde{b}-b}{\sigma}\int_0^T \sqrt{Y_s^b}dW_s - \frac{(\widetilde{b}-b)^2}{2\sigma^2}\int_0^T Y_s^b ds.\end{aligned}$$

Moreover, the process $\left(\frac{d\widehat{\mathrm{P}}_T^{\widetilde{b}}}{d\widehat{\mathrm{P}}_T^b}(Y^{T,b})\right)_{T\in\mathbb{R}_+}$ is a martingale.

The martingale property of the process $\left(\frac{d\widehat{\mathrm{P}}_T^{\widetilde{b}}}{d\widehat{\mathrm{P}}_T^b}(Y^{T,b})\right)_{T\in\mathbb{R}_+}$ is a consequence of Theorem 3.4 in Chapter III of Jacod and Shiryaev [23].

In order to investigate the local asymptotic properties of the family

$$(\mathcal{E}_T)_{T\in\mathbb{R}_{++}} := \big(D([0,T],\mathbb{R}), \mathcal{B}(D([0,T],\mathbb{R})), \{\widehat{\mathrm{P}}_T^b : b \in \mathbb{R}\}\big)_{T\in\mathbb{R}_{++}} \tag{2.1}$$

of statistical experiments, we will use the following corollary.

Corollary 2.2. *For any $b \in \mathbb{R}$, $T \in \mathbb{R}_{++}$, $\varphi_T(b) \in \mathbb{R}$ and $u \in \mathbb{R}$, we have the following expansion*

$$\log \frac{d\widehat{\mathrm{P}}_T^{b+\varphi_T(b)u}}{d\widehat{\mathrm{P}}_T^b}(Y^{T,b}) = uU_T(b) - \frac{u^2}{2}I_T(b),$$

where

$$U_T(b) := -\frac{\varphi_T(b)}{\sigma^2}\left(Y_T^b - y_0 - aT + b\int_0^T Y_s^b ds - J_T\right) = -\frac{\varphi_T(b)}{\sigma}\int_0^T \sqrt{Y_s^b}dW_s,$$

$$I_T(b) := \frac{\varphi_T(b)^2}{\sigma^2}\int_0^T Y_s^b ds.$$

Using Proposition 2.1, one can show the unique existence of MLE $\widehat{b}_T$ of the parameter b based on the continuous observations $Y^{T,b}$, see Barczy et *al.* [4, Proposition 4.2].

Proposition 2.3. *Let $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$, and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *If $y_0 + a \in \mathbb{R}_{++}$, then for each $T \in \mathbb{R}_{++}$, there exists a unique MLE $\widehat{b}_T$ of b almost surely having the form*

$$\widehat{b}_T = -\frac{Y_T^b - y_0 - aT - J_T}{\int_0^T Y_s^b ds},$$

provided that $\int_0^T Y_s^b ds \in \mathbb{R}_{++}$ (which holds almost surely due to Proposition 1.1).

In fact, it turned out that for the calculation of the MLE $\widehat{b}_T$ of b, one does not need to know the value of $\sigma \in \mathbb{R}_{++}$ or the measure m. Here, $J_T = \sum_{t\in[0,T]} \Delta Y_t^b$ with the jumps $\Delta Y_t^b := Y_t^b - Y_{t-}^b$ for $t \in \mathbb{R}_{++}$ and $\Delta Y_0^b := 0$ of the process Y^b. Moreover, note that $\widehat{b}_T = b + \varphi_T(b)\frac{U_T(b)}{I_T(b)}$ and $\varphi_T(b)^{-1}(\widehat{b}_T - b) = \frac{U_T(b)}{I_T(b)}$ whenever $I_T(b) \neq 0$ and $\varphi_T(b) \neq 0$.

To simplify our notation, in all what follows, we fix the parameter $b_0 \in \mathbb{R}$.

2.1.1. *Subcritical case.* Let $b \in \mathbb{R}_{++}$. We recall the existence of a unique stationary distribution and the strong law of large numbers of the process Y^b.

Proposition 2.4. [4, Theorem 2.4 and 5.1], [26, Theorem 1.2 and Remark 5.2] *Let $a \in \mathbb{R}_+$, $b \in \mathbb{R}_{++}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *Let Y^b be the unique strong solution of the SDE* (1.1) *satisfying* $\mathrm{P}(Y_0^b = y_0) = 1$. *Then*
(i) *Y^b has a unique stationary distribution denoted by $\pi_b(dy)$ which is given by*

$$\int_0^\infty e^{uy}\pi_b(dy) = \exp\left\{\int_u^0 \frac{F(v)}{R(v)}dv\right\} = \exp\left\{\int_u^0 \frac{av + \int_0^\infty (e^{vz}-1)m(dz)}{\frac{\sigma^2}{2}v^2 - bv}dv\right\},$$

for all $u \in \mathbb{R}_-$. Moreover, $\pi_b(dy)$ has a finite expectation given by

$$\int_0^\infty y\pi_b(dy) = \frac{1}{b}\left(a + \int_0^\infty zm(dz)\right) \in \mathbb{R}_+. \tag{2.2}$$

(ii) *As $t \to \infty$,*

$$\frac{1}{t}\int_0^t Y_s^b ds \xrightarrow{\widehat{\mathrm{P}}^b} \int_0^\infty y\pi_b(dy) = \frac{1}{b}\left(a + \int_0^\infty zm(dz)\right) \in \mathbb{R}_+. \tag{2.3}$$

(iii) *Consider a $\pi_b(dy)$-integrable function $h : \mathbb{R}_{++} \to \mathbb{R}$. Then, as $t \to \infty$,*

$$\frac{1}{t}\int_0^t h(Y_s^b)ds \longrightarrow \int_0^\infty h(y)\pi_b(dy), \quad \widehat{\mathrm{P}}^b\text{-a.s.} \tag{2.4}$$

In [4, Theorem 2.4], Barczy et *al.* show the strong law of large numbers (2.4) for Y^b which is obtained from the exponential ergodicity of Y^b under two conditions on the Lévy measure: **(A1)** and $\int_0^1 z\log(\frac{1}{z})m(dz) < \infty$. More recently, in [26, Theorem 1.2 and Remark 5.1], Jin et *al.* show the validity of (2.4) under the weaker condition: $\int_1^\infty \log z m(dz) < \infty$. This condition relaxes significantly the above two conditions used by Barczy et *al.* in [4, Theorem 2.4]. Notice that condition $\int_1^\infty \log z m(dz) < \infty$ is then satisfied under assumption **(A1)**.

For fixed $b_0 \in \mathbb{R}_{++}$, we consider the continuous observation $Y^{T,b_0} = (Y_t^{b_0})_{t\in[0,T]}$ of the process Y^{b_0} on the time interval $[0,T]$. The first result of this paper is the following LAN property in the subcritical case.

Theorem 2.5. *Let $a \in \mathbb{R}_+$, $b_0 \in \mathbb{R}_{++}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *If $a \in \mathbb{R}_{++}$ or $m \neq 0$, then the family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments given in* (2.1) *is LAN at b_0 with scaling factor $\varphi_T(b_0) := \frac{1}{\sqrt{T}}$ and with*

$$(U_T(b_0), I_T(b_0)) \overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} (U(b_0), I(b_0)), \tag{2.5}$$

as $T \to \infty$, where $U(b_0) = \mathcal{N}(0, I(b_0))$ is a centered $\mathbb{R}$-valued Gaussian random variable with variance

$$I(b_0) := \frac{1}{\sigma^2 b_0}\left(a + \int_0^\infty z m(dz)\right) \in \mathbb{R}_{++}.$$

That is, for all $u \in \mathbb{R}$, as $T \to \infty$,

$$\log \frac{d\widehat{\mathrm{P}}_T^{b_0+\frac{u}{\sqrt{T}}}}{d\widehat{\mathrm{P}}_T^{b_0}}(Y^{T,b_0}) \overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} uU(b_0) - \frac{u^2}{2} I(b_0).$$

2.1.2. *Critical case.* Let $b = 0$. We recall the asymptotic behavior of $Y^0 = (Y_t^0)_{t\in\mathbb{R}_+}$.

Proposition 2.6. [4, Theorem 6.1] *Let $a \in \mathbb{R}_+$, $b = 0$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *Let Y^0 be the unique strong solution of the SDE* (1.1) *satisfying $\mathrm{P}(Y_0^0 = y_0) = 1$. Then, as $t \to \infty$,*

$$\left(\frac{Y_t^0}{t}, \frac{1}{t^2}\int_0^t Y_s^0 ds\right) \overset{\mathcal{L}(\widehat{\mathrm{P}}^0)}{\longrightarrow} \left(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds\right). \tag{2.6}$$

Here, $\mathcal{Y} = (\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ is the unique strong solution of a critical diffusion-type CIR process starting from 0 *defined by*

$$d\mathcal{Y}_t = \left(a + \int_0^\infty z m(dz)\right) dt + \sigma\sqrt{\mathcal{Y}_t} d\mathcal{W}_t, \tag{2.7}$$

where $\mathcal{Y}_0 = 0$, and $\mathcal{W} = (\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a one-dimensional standard Brownian motion. Moreover, the Laplace transform of $(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds)$ takes the form

$$\mathrm{E}\left[e^{u\mathcal{Y}_1 + v\int_0^1 \mathcal{Y}_s ds}\right] = \begin{cases} \left(\cosh\frac{\gamma_v}{2} - \frac{\sigma^2 u}{\gamma_v}\sinh\frac{\gamma_v}{2}\right)^{-\frac{2}{\sigma^2}\left(a+\int_0^\infty z m(dz)\right)} & \text{if } \ v \in \mathbb{R}_{--}, \\ \left(1 - \frac{\sigma^2 u}{2}\right)^{-\frac{2}{\sigma^2}\left(a+\int_0^\infty z m(dz)\right)} & \text{if } \ v = 0, \end{cases}$$

for all $u, v \in \mathbb{R}_-$, where $\gamma_v := \sqrt{-2\sigma^2 v}$.

Consider the diffusion-type CIR process $\mathcal{Y} = (\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ starting from 0 defined by

$$d\mathcal{Y}_t = \left(a + \int_0^\infty z m(dz) - b\mathcal{Y}_t\right) dt + \sigma\sqrt{\mathcal{Y}_t} d\mathcal{W}_t, \tag{2.8}$$

where $\mathcal{Y}_0 = 0$ and $b \in \mathbb{R}$. We denote by $\mathrm{P}_{\mathcal{Y}}^b$ the probability measure induced by the CIR process $\mathcal{Y}$ solution to equation (2.8) on the canonical space $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}))$ endowed with the natural filtration $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ generated by the Brownian motion $\mathcal{W}$. Here $C(\mathbb{R}_+,\mathbb{R})$ denotes the set of $\mathbb{R}$-valued continuous functions defined on $\mathbb{R}_+$, and $\mathcal{B}(C(\mathbb{R}_+,\mathbb{R}))$ is its Borel σ-algebra. For any $T\in\mathbb{R}_{++}$, let $\mathrm{P}^b_{\mathcal{Y},T}$ be the restriction of $\mathrm{P}^b_{\mathcal{Y}}$ on $\mathcal{G}_T$. As a consequence of [4, Proposition 4.1], under condition **(A1)**, for any $b\in\mathbb{R}$, the probability measures $\mathrm{P}^b_{\mathcal{Y},T}$ and $\mathrm{P}^0_{\mathcal{Y},T}$ are absolutely continuous w.r.t. each other and

$$\frac{d\mathrm{P}^b_{\mathcal{Y},T}}{d\mathrm{P}^0_{\mathcal{Y},T}}\big((\mathcal{Y}_s)_{s\in[0,T]}\big)=\exp\left\{-\frac{b}{\sigma}\int_0^T\sqrt{\mathcal{Y}_s}d\mathcal{W}_s-\frac{b^2}{2\sigma^2}\int_0^T\mathcal{Y}_sds\right\}, \tag{2.9}$$

where $\mathcal{Y}=(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ is the CIR process solution to equation (2.8) corresponding to the parameter $b=0$ (i.e., solution to equation (2.7)).

On the other hand, as a consequence of Theorem 3.4 in Chapter III of Jacod and Shiryaev [23], the process $\left(\frac{d\mathrm{P}^b_{\mathcal{Y},T}}{d\mathrm{P}^0_{\mathcal{Y},T}}\big((\mathcal{Y}_s)_{s\in[0,T]}\big)\right)_{T\in\mathbb{R}_+}$ is a martingale w.r.t. the filtration $(\mathcal{G}_T)_{T\in\mathbb{R}_+}$.

For fixed $b_0=0$, we consider the continuous observation $Y^{T,0}=(Y^0_t)_{t\in[0,T]}$ of the process Y^0. The second result of this paper is the following LAQ property in the critical case.

Theorem 2.7. *Let $a\in\mathbb{R}_+$, $b_0=0$, $\sigma\in\mathbb{R}_{++}$, $y_0\in\mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *If $y_0+a\in\mathbb{R}_{++}$, then the family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments is LAQ at $b_0=0$ with scaling factor $\varphi_T(0):=\frac{1}{T}$ and with*

$$(U_T(0),I_T(0))\overset{\mathcal{L}(\widehat{\mathrm{P}}^0)}{\longrightarrow}(U(0),I(0)),$$

as $T\to\infty$, where

$$U(0):=\frac{a+\int_0^\infty zm(dz)-\mathcal{Y}_1}{\sigma^2}=-\frac{1}{\sigma}\int_0^1\sqrt{\mathcal{Y}_s}d\mathcal{W}_s,\qquad I(0):=\frac{1}{\sigma^2}\int_0^1\mathcal{Y}_sds,$$

and $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ is the unique strong solution of the SDE (2.7) with initial condition $\mathcal{Y}_0=0$. That is, for all $u\in\mathbb{R}$, as $T\to\infty$,

$$\log\frac{d\widehat{\mathrm{P}}^{0+\frac{u}{T}}_T}{d\widehat{\mathrm{P}}^0_T}(Y^{T,0})\overset{\mathcal{L}(\widehat{\mathrm{P}}^0)}{\longrightarrow}uU(0)-\frac{u^2}{2}I(0),$$

and $\mathrm{E}\Big[e^{uU(0)-\frac{u^2}{2}I(0)}\Big]=1.$

2.1.3. *Supercritical case.* Let $b\in\mathbb{R}_{--}$. We recall the asymptotic behavior of Y^b.

Proposition 2.8. [4, Theorem 7.1] *Let $a\in\mathbb{R}_+$, $b\in\mathbb{R}_{--}$, $\sigma\in\mathbb{R}_{++}$, $y_0\in\mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *Let Y^b be the unique strong solution of the SDE (1.1) satisfying $\mathrm{P}(Y^b_0=y_0)=1$. Then, there exists a random variable V with $\mathrm{P}(V\in\mathbb{R}_+)=1$ such that as $t\to\infty$,*

$$e^{bt}Y^b_t\longrightarrow V,\quad \widehat{\mathrm{P}}^b\text{-a.s.} \tag{2.10}$$

$$e^{bt}\int_0^tY^b_sds\longrightarrow-\frac{V}{b},\quad \widehat{\mathrm{P}}^b\text{-a.s.} \tag{2.11}$$

Moreover, the Laplace transform of V takes the form

$$\begin{aligned}\mathrm{E}\left[e^{uV}\right] = \exp\left\{\frac{uy_0}{1+\frac{\sigma^2 u}{2b}}\right\}\left(1+\frac{\sigma^2 u}{2b}\right)^{-\frac{2a}{\sigma^2}} \\ \times \exp\left\{\int_0^\infty \left(\int_0^\infty \left(\exp\left\{\frac{zue^{by}}{1+\frac{\sigma^2 u}{2b}e^{by}}\right\}-1\right) m(dz)\right) dy\right\},\end{aligned} \tag{2.12}$$

for all $u \in \mathbb{R}_-$, and consequently $V \stackrel{\mathcal{L}}{=} \widetilde{\mathcal{V}} + \widetilde{\widetilde{\mathcal{V}}}$, where $\widetilde{\mathcal{V}}$ and $\widetilde{\widetilde{\mathcal{V}}}$ are independent random variables such that $e^{bt}\widetilde{\mathcal{Y}}_t \xrightarrow{\text{a.s.}} \widetilde{\mathcal{V}}$ and $e^{bt}\widetilde{\widetilde{\mathcal{Y}}}_t \xrightarrow{\text{a.s.}} \widetilde{\widetilde{\mathcal{V}}}$ as $t \to \infty$, where $\widetilde{\mathcal{Y}} = (\widetilde{\mathcal{Y}}_t)_{t\in\mathbb{R}_+}$ and $\widetilde{\widetilde{\mathcal{Y}}} = (\widetilde{\widetilde{\mathcal{Y}}}_t)_{t\in\mathbb{R}_+}$ are the pathwise unique strong solutions of the supercritical CIR models

$$d\widetilde{\mathcal{Y}}_t = \left(a - b\widetilde{\mathcal{Y}}_t\right) dt + \sigma\sqrt{\widetilde{\mathcal{Y}}_t} d\widetilde{\mathcal{W}}_t, \quad \text{with } \widetilde{\mathcal{Y}}_0 = y_0,$$

and

$$d\widetilde{\widetilde{\mathcal{Y}}}_t = -b\widetilde{\widetilde{\mathcal{Y}}}_t dt + \sigma\sqrt{\widetilde{\widetilde{\mathcal{Y}}}_t} d\widetilde{\widetilde{\mathcal{W}}}_t + dJ_t, \quad \text{with } \widetilde{\widetilde{\mathcal{Y}}}_0 = 0,$$

respectively, where $\widetilde{\mathcal{W}} = (\widetilde{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ and $\widetilde{\widetilde{\mathcal{W}}} = (\widetilde{\widetilde{\mathcal{W}}}_t)_{t\in\mathbb{R}_+}$ are independent one-dimensional standard Brownian motions. Furthermore, $\widetilde{\mathcal{V}} \stackrel{\mathcal{L}}{=} \mathcal{Z}_{-\frac{1}{b}}$, where $\mathcal{Z} = (\mathcal{Z}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the critical CIR model

$$d\mathcal{Z}_t = a dt + \sigma\sqrt{\mathcal{Z}_t} d\mathcal{W}_t, \quad \text{with } \mathcal{Z}_0 = y_0,$$

where $\mathcal{W} = (\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a one-dimensional standard Brownian motion.

If, in addition, $a \in \mathbb{R}_{++}$, then $\mathrm{P}(V \in \mathbb{R}_{++}) = 1$.

For fixed $b_0 \in \mathbb{R}_{--}$, we consider the continuous observation $Y^{T,b_0} = (Y_t^{b_0})_{t\in[0,T]}$ of the process Y^{b_0}. The third result of this paper is the following LAMN property in the supercritical case.

Theorem 2.9. *Let $a \in \mathbb{R}_{++}$, $b_0 \in \mathbb{R}_{--}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *Then, the family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments is LAMN at b_0 with scaling factor $\varphi_T(b_0) := e^{\frac{b_0 T}{2}}$ and with*

$$(U_T(b_0), I_T(b_0)) \xrightarrow{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})} (U(b_0), I(b_0)),$$

as $T \to \infty$, where

$$U(b_0) := \sqrt{-\frac{V}{\sigma^2 b_0}} Z, \qquad I(b_0) := -\frac{V}{\sigma^2 b_0},$$

and V is the positive random variable whose Laplace transform is given by (2.12), and Z is a standard normally distributed random variable, independent of V. That is, for all $u \in \mathbb{R}$, as $T \to \infty$,

$$\log \frac{d\widehat{\mathrm{P}}_T^{b_0 + e^{\frac{b_0 T}{2}} u}}{d\widehat{\mathrm{P}}_T^{b_0}}(Y^{T,b_0}) \xrightarrow{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})} uU(b_0) - \frac{u^2}{2} I(b_0).$$

2.2. **Discrete observations.** The main result is divided into three cases: subcritical, critical and supercritical. For this, let us first add two following assumptions on equation (1.1) we shall work with.

(A2) For any $p > 1$, $\int_1^\infty z^p m(dz) < \infty$.

(A3) $\dfrac{a}{\sigma^2} > \dfrac{15 + \sqrt{185}}{4}$.

Note that conditions **(A1)** and **(A2)** imply that $\int_0^\infty z^p m(dz) < \infty$ for any $p > 1$.

Given the process $Y^b = (Y_t^b)_{t \in \mathbb{R}_+}$ and $n \geq 1$, we consider a discrete observation scheme at deterministic and equidistant times $t_k = k\Delta_n$, $k \in \{0, \ldots, n\}$ of the process Y^b, which is denoted by $Y^{n,b} = (Y_{t_0}^b, Y_{t_1}^b, \ldots, Y_{t_n}^b)$. We assume that the high-frequency and infinite horizon conditions hold. That is, $\Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$. We denote by P_n^b and $p_n(\cdot; b)$ the probability law and the density of the random vector $Y^{n,b}$, respectively.

2.2.1. *Subcritical case.* For fixed $b_0 \in \mathbb{R}_{++}$, we consider a discrete observation $Y^{n,b_0} = (Y_{t_0}^{b_0}, Y_{t_1}^{b_0}, \ldots, Y_{t_n}^{b_0})$ of the process Y^{b_0}.

The next result of this paper is the following LAN property in the subcritical case.

Theorem 2.10. *Assume conditions* **(A1)**-**(A3)**. *Let $b_0 \in \mathbb{R}_{++}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$. Then, the LAN property holds for the likelihood at b_0 with rate of convergence $\sqrt{n\Delta_n}$ and asymptotic Fisher information $I(b_0)$. That is, for all $u \in \mathbb{R}$, as $n \to \infty$,*

$$\log \frac{d\mathrm{P}_n^{b_0 + \frac{u}{\sqrt{n\Delta_n}}}}{d\mathrm{P}_n^{b_0}} \left(Y^{n,b_0}\right) \overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} uU(b_0) - \frac{u^2}{2} I(b_0),$$

where $U(b_0) = \mathcal{N}(0, I(b_0))$ and

$$I(b_0) := \frac{1}{\sigma^2 b_0} \left(a + \int_0^\infty z m(dz) \right).$$

2.2.2. *Critical case.* For fixed $b_0 = 0$, consider a discrete observation $Y^{n,0} = (Y_{t_0}^0, Y_{t_1}^0, \ldots, Y_{t_n}^0)$ of the process Y^0.

The next result of this paper is the following LAQ property in the critical case.

Theorem 2.11. *Assume conditions* **(A1)**-**(A3)**. *Let $b_0 = 0$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$. Then, the LAQ property holds for the likelihood at $b_0 = 0$ with rate of convergence $n\Delta_n$ and random variables $U(0)$ and $I(0)$. That is, for all $u \in \mathbb{R}$, as $n \to \infty$,*

$$\log \frac{d\mathrm{P}_n^{0 + \frac{u}{n\Delta_n}}}{d\mathrm{P}_n^0} \left(Y^{n,0}\right) \overset{\mathcal{L}(\widehat{\mathrm{P}}^0)}{\longrightarrow} uU(0) - \frac{u^2}{2} I(0),$$

with $\mathrm{E}\left[e^{uU(0) - \frac{u^2}{2} I(0)}\right] = 1$, where

$$U(0) := \frac{a + \int_0^\infty z m(dz) - \mathcal{Y}_1}{\sigma^2} = -\frac{1}{\sigma} \int_0^1 \sqrt{\mathcal{Y}_s} d\mathcal{W}_s, \qquad I(0) := \frac{1}{\sigma^2} \int_0^1 \mathcal{Y}_s ds.$$

Here, $\mathcal{Y} = (\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is a critical diffusion-type CIR process starting from 0 defined by (2.7).

2.2.3. *Supercritical case.* For fixed $b_0 \in \mathbb{R}_{--}$, we consider a discrete observation $Y^{n,b_0} = (Y^{b_0}_{t_0}, Y^{b_0}_{t_1}, \ldots, Y^{b_0}_{t_n})$ of the process Y^{b_0}.

The last result of this paper is the following LAMN property in the supercritical case.

Theorem 2.12. *Assume conditions* **(A1)-(A3)**. *Let $b_0 \in \mathbb{R}_{--}$, $\sigma \in \mathbb{R}_{++}$, $y_0 \in \mathbb{R}_+$ and let m be a Lévy measure on $\mathbb{R}_{++}$. Then, the LAMN property holds for the likelihood at b_0 with rate of convergence $e^{-b_0 \frac{n\Delta_n}{2}}$ and asymptotic random Fisher information $I(b_0)$. That is, for all $u \in \mathbb{R}$, as $n \to \infty$,*

$$\log \frac{d\mathrm{P}_n^{b_0 + e^{b_0 \frac{n\Delta_n}{2}} u}}{d\mathrm{P}_n^{b_0}} \left(Y^{n,b_0} \right) \overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} uU(b_0) - \frac{u^2}{2} I(b_0),$$

where

$$U(b_0) := \sqrt{-\frac{V}{\sigma^2 b_0}} Z, \qquad I(b_0) := -\frac{V}{\sigma^2 b_0}.$$

Here, V is a positive random variable whose Laplace transform is given by (2.12), *and Z is a standard normal random variable, independent of V.*

Remark 2.13. *Theorem 2.10, 2.11 and 2.12 extend the results in* [9, Theorem 2.1, 2.2 and 2.3] *when b is the only unknown parameter in the drift coefficient. However, condition* **(A3)** *in* [9] *is less restrictive than that given in this paper (see Remark 5.2).*

Remark 2.14. *When the LAMN property holds, the convolution theorem gives the notion of asymptotically efficient estimators in the sense of Hájek-Le Cam convolution theorem. Moreover, the minimax theorem gives the lower bounds for the asymptotic variance of estimators (see Jeganathan* [24]*).*

Remark 2.15. *From Theorem* 5.2 *and* 7.3 *in* [4] *and the consequence of the LAN and LAMN properties established in Theorem 2.5 and 2.9, the MLE for the growth rate b for the jump-type CIR process* (1.1) *based on continuous time observations, which is proposed in* [4, Proposition 4.2], *is asymptotically efficient in the sense of Hájek-Le Cam convolution theorem.*

Remark 2.16. *Let us mention that condition* **(A3)** *on the ratio of the coefficients $\frac{a}{\sigma^2}$ required in Theorem 2.10, 2.11 and 2.12 is similar to condition* (10) *in* [11, Theorem 2.2] *which is used to prove the strong convergence of the symmetrized Euler scheme applied to CIR process.*

Example 2.17. *It can be checked that the following subordinators satisfy conditions* **(A1)-(A2)**.

1) *J is a Poisson process.*

2) *J is a compound Poisson process with exponentially distributed jump sizes. That is, $m(dz) = C\lambda e^{-\lambda z} \mathbf{1}_{(0,\infty)}(z) dz$, for some constants $C \in (0,\infty)$ and $\lambda \in (0,\infty)$.*

3) *J is a Gamma process with Lévy measure $m(dz) = \gamma z^{-1} e^{-\lambda z} \mathbf{1}_{(0,\infty)}(z) dz$, where γ and λ are positive constants.*

4) *J is a subordinator whose Lévy measure is given by the gamma probability distribution. That is, $m(dz) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} \mathbf{1}_{(0,\infty)}(z) dz$ where $\alpha \in (-1,\infty)$ and λ is a positive constant.*

5) *J is an inverse Gaussian process with $m(dz) = \frac{\delta}{\sqrt{2\pi z^3}} e^{-\frac{\gamma^2 z}{2}} \mathbf{1}_{(0,\infty)}(z) dz$, for a positive constant δ.*

As usual, positive constants will be denoted by C and they will always be independent of time and Δ_n. They may change of value from one line to the next.

3. Technical results

In this section, we introduce some preliminary results needed for the proof of Theorem 2.10, 2.11 and 2.12. Towards this aim, we consider the canonical filtered probability spaces $(\Omega^i, \mathcal{F}^i, \{\mathcal{F}_t^i\}_{t\in\mathbb{R}_+}, \mathrm{P}^i)$, $i \in \{1,\dots,4\}$, associated respectively to each of the four processes W, N, B and M, where $B = (B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, $M(dt,dz)$ is a Poisson random measure with intensity measure $m(dz)dt$ associated with a subordinator $\widetilde{J} = (\widetilde{J}_t)_{t\in\mathbb{R}_+}$, i.e., $\widetilde{J}_t = \int_0^t\int_0^\infty zM(ds,dz)$. The processes W, N, B, M are mutually independent. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, \mathrm{P})$ be the product filtered probability space of these four canonical spaces. We set $\widehat{\Omega} = \Omega^1\times\Omega^2$, $\widehat{\mathcal{F}} = \mathcal{F}^1\otimes\mathcal{F}^2$, $\widehat{\mathrm{P}} = \mathrm{P}^1\otimes\mathrm{P}^2$, $\widehat{\mathcal{F}}_t = \mathcal{F}_t^1\otimes\mathcal{F}_t^2$, $\widetilde{\Omega} = \Omega^3\times\Omega^4$, $\widetilde{\mathcal{F}} = \mathcal{F}^3\otimes\mathcal{F}^4$, $\widetilde{\mathrm{P}} = \mathrm{P}^3\otimes\mathrm{P}^4$, and $\widetilde{\mathcal{F}}_t = \mathcal{F}_t^3\otimes\mathcal{F}_t^4$. Then, $\Omega = \widehat{\Omega}\times\widetilde{\Omega}$, $\mathcal{F} = \widehat{\mathcal{F}}\otimes\widetilde{\mathcal{F}}$, $\mathrm{P} = \widehat{\mathrm{P}}\otimes\widetilde{\mathrm{P}}$, $\mathcal{F}_t = \widehat{\mathcal{F}}_t\otimes\widetilde{\mathcal{F}}_t$, and $\mathrm{E} = \widehat{\mathrm{E}}\otimes\widetilde{\mathrm{E}}$, where E, $\widehat{\mathrm{E}}$, $\widetilde{\mathrm{E}}$ denote the expectation w.r.t. P, $\widehat{\mathrm{P}}$ and $\widetilde{\mathrm{P}}$, respectively.

For any $t > s$, the law of X_t^b conditioned on $X_s^b = x$ admits a positive transition density $p^b(t-s,x,y)$, which is of class C^1 w.r.t. b. To prove this result, we use the affine structure of the jump-type CIR process and the inverse Fourier transform (see Proposition 3.1). In order to deal with the likelihood ratio in Theorem 2.10, we use the Markov property, the chain rule for Radon-Nikodym derivatives, and the mean value theorem on the parameter space to get the following decomposition

$$\begin{aligned}\log\frac{d\mathrm{P}_n^{b_0+\frac{u}{\sqrt{n\Delta_n}}}}{d\mathrm{P}_n^{b_0}}\left(Y^{n,b_0}\right) &= \log\frac{p_n\left(Y^{n,b_0}; b_0+\frac{u}{\sqrt{n\Delta_n}}\right)}{p_n\left(Y^{n,b_0};b_0\right)} = \sum_{k=0}^{n-1}\log\frac{p^{b_0+\frac{u}{\sqrt{n\Delta_n}}}}{p^{b_0}}\left(\Delta_n, Y_{t_k}^{b_0}, Y_{t_{k+1}}^{b_0}\right)\\ &= \sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n}}\int_0^1\frac{\partial_b p^{b_0+\frac{\ell u}{\sqrt{n\Delta_n}}}}{p^{b_0+\frac{\ell u}{\sqrt{n\Delta_n}}}}\left(\Delta_n, Y_{t_k}^{b_0}, Y_{t_{k+1}}^{b_0}\right)d\ell. \end{aligned}\tag{3.1}$$

Next, as in Gobet [18], we apply the Malliavin calculus on each interval $[t_k, t_{k+1}]$ in order to derive an explicit expression for the logarithm derivative of the transition density w.r.t. b. To avoid confusion with the observed process Y^b, we introduce an extra probabilistic representation of Y^b for which the Malliavin calculus can be applied. Explicitly, we consider on the same probability space $(\Omega,\mathcal{F},\mathrm{P})$ the flow $X^b(s,x) = (X_t^b(s,x), t\geq s)$, $x\in\mathbb{R}_{++}$ on the time interval $[s,\infty)$ and with initial condition $X_s^b(s,x) = x$ satisfying

$$X_t^b(s,x) = x + \int_s^t\left(a - bX_u^b(s,x)\right)du + \sigma\int_s^t\sqrt{X_u^b(s,x)}dB_u + \int_s^t\int_0^\infty zM(ds,dz). \tag{3.2}$$

In particular, we write $X_t^b \equiv X_t^b(0,y_0)$, for all $t\in\mathbb{R}_+$. That is,

$$\begin{aligned} X_t^b &= y_0 + \int_0^t\left(a - bX_s^b\right)ds + \sigma\int_0^t\sqrt{X_s^b}dB_s + \int_0^t\int_0^\infty zM(ds,dz)\\ &= y_0 + \int_0^t\left(a - bX_s^b\right)ds + \sigma\int_0^t\sqrt{X_s^b}dB_s + \widetilde{J}_t. \end{aligned}\tag{3.3}$$

We will apply the Malliavin calculus on the Wiener space induced by B. Let D and δ denote the Malliavin derivative and the Skorohod integral w.r.t. B. We denote by $\mathbb{D}^{1,2}$ the Sobolev space of random variables differentiable w.r.t. B in the sense of Malliavin, and by $\mathrm{Dom}\,\delta$ the domain of δ. The Malliavin calculus for CIR process is studied in [1, 2] and the Malliavin calculus for CIR process with jumps is discussed e.g. in [40, Example 1].

For any $k \in \{0, ..., n-1\}$, by definition, the process $(X_t^b(t_k, x), t \in [t_k, t_{k+1}])$ is defined by

$$X_t^b(t_k, x) = x + \int_{t_k}^t \Big(a - bX_u^b(t_k, x)\Big)\, du + \sigma \int_{t_k}^t \sqrt{X_u^b(t_k, x)}dB_u + \int_{t_k}^t \int_0^\infty zM(ds, dz). \quad (3.4)$$

Condition $2a \geq \sigma^2$ and the fact that the subordinator admits only positive jumps imply that the jump-type CIR processes Y^b and $X^b = (X_t^b)_{t \in \mathbb{R}_+}$ never hit 0. Then, by [41, Theorem V.39], the process $(X_t^b(t_k, x), t \in [t_k, t_{k+1}])$ is differentiable w.r.t. x that we denote by $(\partial_x X_t^b(t_k, x), t \in [t_k, t_{k+1}])$. Furthermore, this process admits the derivative w.r.t. b that we denote by $(\partial_b X_t^b(t_k, x), t \in [t_k, t_{k+1}])$ since this problem is similar to the derivative w.r.t. the initial condition (see e.g. [39, Theorem 10.1 page 486]). These processes are solutions to the following equations

$$\partial_x X_t^b(t_k, x) = 1 - b \int_{t_k}^t \partial_x X_u^b(t_k, x) du + \int_{t_k}^t \frac{\sigma \partial_x X_u^b(t_k, x)}{2\sqrt{X_u^b(t_k, x)}} dB_u, \quad (3.5)$$

$$\partial_b X_t^b(t_k, x) = -\int_{t_k}^t \Big(X_u^b(t_k, x) + b\partial_b X_u^b(t_k, x)\Big)\, du + \int_{t_k}^t \frac{\sigma \partial_b X_u^b(t_k, x)}{2\sqrt{X_u^b(t_k, x)}} dB_u. \quad (3.6)$$

Therefore, their explicit solutions are respectively given by

$$\partial_x X_t^b(t_k, x) = \exp\left\{-b(t - t_k) - \frac{\sigma^2}{8}\int_{t_k}^t \frac{du}{X_u^b(t_k, x)} + \frac{\sigma}{2}\int_{t_k}^t \frac{dB_u}{\sqrt{X_u^b(t_k, x)}}\right\}, \quad (3.7)$$

$$\partial_b X_t^b(t_k, x) = -\int_{t_k}^t X_r^b(t_k, x) \exp\left\{ - b(t - r) - \frac{\sigma^2}{8}\int_r^t \frac{du}{X_u^b(t_k, x)} + \frac{\sigma}{2}\int_r^t \frac{dB_u}{\sqrt{X_u^b(t_k, x)}}\right\} dr. \quad (3.8)$$

Observe that from (3.7), (3.8), we can write

$$\partial_b X_t^b(t_k, x) = -\int_{t_k}^t X_r^b(t_k, x)\partial_x X_t^b(t_k, x)(\partial_x X_r^b(t_k, x))^{-1} dr. \quad (3.9)$$

Moreover, under condition $2a \geq \sigma^2$, for any $t \in [t_k, t_{k+1}]$, the random variables $X_t^b(t_k, x)$ and $\partial_x X_t^b(t_k, x)$ belong to $\mathbb{D}^{1,2}$. From (3.4) together with the chain rule of the Malliavin calculus,

$$D_s X_t^b(t_k, x) = \sigma\sqrt{X_s^b(t_k, x)} - b\int_s^t D_s X_u^b(t_k, x) du + \int_s^t \frac{\sigma D_s X_u^b(t_k, x)}{2\sqrt{X_u^b(t_k, x)}} dB_u, \quad (3.10)$$

for $s \leq t$ a.e., and $D_s X_t^b(t_k, x) = 0$ for $s > t$ a.e. Using (3.7) and the chain rule of Malliavin calculus, we have that

$$D_s \Big(\partial_x X_t^b(t_k, x)\Big) = \partial_x X_t^b(t_k, x)\bigg(\frac{\sigma}{2}\frac{1}{\sqrt{X_s^b(t_k, x)}} + \frac{\sigma^2}{8}\int_s^t \frac{1}{(X_u^b(t_k, x))^2} D_s X_u^b(t_k, x) du$$
$$- \frac{\sigma}{4}\int_s^t \frac{1}{(X_u^b(t_k, x))^{\frac{3}{2}}} D_s X_u^b(t_k, x) dB_u\bigg)\mathbf{1}_{[t_k, t]}(s). \quad (3.11)$$

Furthermore, proceeding as [37, (2.59)] by using Itô's formula and the uniqueness of the solution to equation (3.10), we get that

$$D_s X_t^b(t_k, x) = \sigma\sqrt{X_s^b(t_k, x)}\partial_x X_t^b(t_k, x)(\partial_x X_s^b(t_k, x))^{-1}\mathbf{1}_{[t_k, t]}(s). \quad (3.12)$$

Now, for all $k \in \{0, ..., n-1\}$ and $x \in \mathbb{R}_{++}$, we denote by $\widetilde{\mathrm{P}}^b_{t_k,x}$ the probability law of X^b starting at x at time t_k, i.e., $\widetilde{\mathrm{P}}^b_{t_k,x}(A) = \widetilde{\mathrm{E}}[\mathbf{1}_A | X^b_{t_k} = x]$ for all $A \in \widetilde{\mathcal{F}}$, and denote by $\widetilde{\mathrm{E}}^b_{t_k,x}$ the expectation w.r.t. $\widetilde{\mathrm{P}}^b_{t_k,x}$. That is, for all $\widetilde{\mathcal{F}}$-measurable random variables V, we have that $\widetilde{\mathrm{E}}^b_{t_k,x}[V] = \widetilde{\mathrm{E}}[V | X^b_{t_k} = x]$. Similarly, we denote by $\widehat{\mathrm{P}}^b_{t_k,x}$ the probability law of Y^b starting at x at time t_k, i.e., $\widehat{\mathrm{P}}^b_{t_k,x}(A) = \widehat{\mathrm{E}}[\mathbf{1}_A | Y^b_{t_k} = x]$ for all $A \in \widehat{\mathcal{F}}$, and denote by $\widehat{\mathrm{E}}^b_{t_k,x}$ the expectation w.r.t. $\widehat{\mathrm{P}}^b_{t_k,x}$. That is, for all $\widehat{\mathcal{F}}$-measurable random variables V, we have that $\widehat{\mathrm{E}}^b_{t_k,x}[V] = \widehat{\mathrm{E}}[V | Y^b_{t_k} = x]$. Let $\mathrm{P}^b_{t_k,x} := \widehat{\mathrm{P}}^b_{t_k,x} \otimes \widetilde{\mathrm{P}}^b_{t_k,x}$ be the product measure, and $\mathrm{E}^b_{t_k,x} = \widehat{\mathrm{E}}^b_{t_k,x} \otimes \widetilde{\mathrm{E}}^b_{t_k,x}$ denotes the expectation w.r.t. $\mathrm{P}^b_{t_k,x}$.

Now, we prove the existence and the smoothness of the density w.r.t. b.

Proposition 3.1. *Assume condition* **(A1)** *and $2a > \sigma^2$. Then for any $t > 0$ and $b \in \mathbb{R}$, the law of X^b_t admits a strictly positive density function $p^b(t, y_0, y)$. Moreover, $p^b(t, y_0, y)$ is of class C^1 w.r.t. b, for all $b \in \mathbb{R}$.*

As in [18, Proposition 4.1], we have the following explicit expression for the logarithm derivative of the transition density w.r.t. b in terms of a conditional expectation of a Skorohod integral.

Proposition 3.2. *Assume condition* **(A1)** *and $2a > \sigma^2$. Then, for all $k \in \{0, ..., n-1\}$, $b \in \mathbb{R}$, and $x, y \in \mathbb{R}_{++}$,*

$$\frac{\partial_b p^b}{p^b}(\Delta_n, x, y) = \frac{1}{\Delta_n} \widetilde{\mathrm{E}}^b_{t_k,x}\left[\delta\left(\partial_b X^b_{t_{k+1}}(t_k, x) U^b(t_k, x)\right) | X^b_{t_{k+1}} = y\right],$$

where $U^b(t_k, x) := (U^b_t(t_k, x), t \in [t_k, t_{k+1}])$ with $U^b_t(t_k, x) := (D_t X^b_{t_{k+1}}(t_k, x))^{-1}$.

We have the following decomposition of Skorohod integral appearing in Proposition 3.2.

Lemma 3.3. *Assume condition* **(A1)** *and $2a > \sigma^2$. Then, for all $b \in \mathbb{R}$, $k \in \{0, ..., n-1\}$, and $x \in \mathbb{R}_{++}$,*

$$\begin{aligned}\delta\left(\partial_b X^b_{t_{k+1}}(t_k, x) U^b(t_k, x)\right) = &-\frac{\Delta_n}{\sigma^2}\sqrt{\frac{x}{X^b_{t_k}}}\left(X^b_{t_{k+1}} - X^b_{t_k} - (a - bX^b_{t_k})\Delta_n\right) \\ &+ H(X^b) - H_4(X^b) - H_5(X^b) - H_6(X^b),\end{aligned}$$

where

$$\begin{aligned}H(X^b) = &-\Delta_n \frac{x}{\sigma}\int_{t_k}^{t_{k+1}}\left(\frac{\partial_x X^b_s(t_k, x)}{\sqrt{X^b_s(t_k, x)}} - \frac{\partial_x X^b_{t_k}(t_k, x)}{\sqrt{X^b_{t_k}(t_k, x)}}\right) dB_s \\ &+ \int_{t_k}^{t_{k+1}}\left(\frac{\sigma^2}{4}\frac{\partial_b X^b_s(t_k, x)}{X^b_s(t_k, x)\partial_x X^b_s(t_k, x)} - \left(\frac{X^b_s(t_k, x)}{\partial_x X^b_s(t_k, x)} - \frac{X^b_{t_k}(t_k, x)}{\partial_x X^b_{t_k}(t_k, x)}\right)\right) ds \\ &\times \int_{t_k}^{t_{k+1}} \frac{\partial_x X^b_s(t_k, x)}{\sigma\sqrt{X^b_s(t_k, x)}} dB_s - \int_{t_k}^{t_{k+1}} D_s\left(\frac{\partial_b X^b_{t_{k+1}}(t_k, x)}{\partial_x X^b_{t_{k+1}}(t_k, x)}\right)\frac{\partial_x X^b_s(t_k, x)}{\sigma\sqrt{X^b_s(t_k, x)}} ds,\\
H_4(X^b) = &\frac{\Delta_n}{\sigma^2} b\sqrt{\frac{x}{X^b_{t_k}}}\int_{t_k}^{t_{k+1}}\left(X^b_s - X^b_{t_k}\right) ds,\end{aligned}$$

$$H_5(X^b) = -\frac{\Delta_n}{\sigma}\sqrt{\frac{x}{X^b_{t_k}}}\int_{t_k}^{t_{k+1}}\left(\sqrt{X^b_s} - \sqrt{X^b_{t_k}}\right)dB_s,$$

$$H_6(X^b) = -\frac{\Delta_n}{\sigma^2}\sqrt{\frac{x}{X^b_{t_k}}}\int_{t_k}^{t_{k+1}}\int_0^\infty zM(ds,dz).$$

As a consequence of Proposition 3.2 and Lemma 3.3, we have the following explicit expression for the logarithm derivative of the transition density.

Corollary 3.4. *Assume condition* **(A1)** *and* $2a > \sigma^2$. *Then, for all* $b \in \mathbb{R}$, $k \in \{0, ..., n-1\}$, *and* $x, y \in \mathbb{R}_{++}$,

$$\frac{\partial_b p^b}{p^b}(\Delta_n, x, y) = -\frac{1}{\sigma^2}(y - x - (a - bx)\Delta_n) + \frac{1}{\Delta_n}\widetilde{\mathrm{E}}^b_{t_k,x}\left[H(X^b) - H_4(X^b) - H_5(X^b) - H_6(X^b)\big|X^b_{t_{k+1}} = y\right].$$

Let us now give the moment estimates for the jump-type CIR process (1.1).

Lemma 3.5. *Assume condition* **(A1)**.

(i) *For any* $T \in \mathbb{R}_{++}$ *and* $p \in [0, \frac{2a}{\sigma^2} - 1)$,

$$\widehat{\mathrm{E}}^b\left[\sup_{t\in[0,T]}\left(Y^b_t\right)^{-p}\right] < \infty.$$

(ii) *Assume further condition* **(A2)**. *Then for any* $T \in \mathbb{R}_{++}$ *and* $p \geq 1$, *there exists a constant* $C > 0$ *such that*

$$\widehat{\mathrm{E}}^b\left[\sup_{t\in[0,T]}|Y^b_t|^p\right] \leq C(1 + y_0^p).$$

(iii) *Assume further condition* **(A2)**. *Then for any* $t > s \geq 0$ *and* $p \geq 1$, *there exists a constant* $C > 0$ *such that*

$$\widehat{\mathrm{E}}^b\left[\left|Y^b_t - Y^b_s\right|^p\right] \leq C(t-s)^{\frac{p}{2}\wedge 1}(1 + y_0^p + y_0^{\frac{p}{2}}).$$

Moreover, all these statements remain valid for X^b.

Next, we introduce the following exponential moment estimate obtained from [12].

Lemma 3.6. *Assume condition* **(A1)** *and that* $\frac{a}{\sigma^2} > 1$. *Then, for any* $k \in \{0, ..., n-1\}$ *and* $x \in \mathbb{R}_{++}$, *there exists a constant* C *which does not depend on* x *such that for any* $\mu \leq (\frac{2a}{\sigma^2} - 1)^2\frac{\sigma^2}{8}$ *and* $t \in [t_k, t_{k+1}]$,

$$\widetilde{\mathrm{E}}^b_{t_k,x}\left[\exp\left\{\mu\int_{t_k}^t\frac{du}{X^b_u(t_k,x)}\right\}\right] \leq C\left(1 + \frac{1}{x^{\frac{1}{2}(\frac{2a}{\sigma^2}-1)}}\right).$$

We are going to show the following crucial estimates which will be useful in the sequel.

Lemma 3.7. *Assume conditions* **(A1)** *and* **(A2)**. *Then, for any* $b \in \mathbb{R}$, $k \in \{0, ..., n-1\}$ *and* $x \in \mathbb{R}_{++}$, *there exists a constant* $C_p > 0$ *which does not depend on* x *such that for all* $t \in [t_k, t_{k+1}]$,

$$\widetilde{\mathrm{E}}_{t_k,x}^{b}\left[\left|X_t^b(t_k,x)\right|^p\right] \leq C_p\left(1+x^p\right), \tag{3.13}$$

$$\widetilde{\mathrm{E}}_{t_k,x}^{b}\left[\frac{1}{\left|X_t^b(t_k,x)\right|^p}\right] \leq \frac{C_p}{x^p}, \tag{3.14}$$

$$\widetilde{\mathrm{E}}_{t_k,x}^{b}\left[\left|\partial_x X_t^b(t_k,x)\right|^p\right] \leq C_p\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1+p}{2}}}\right), \tag{3.15}$$

$$\widetilde{\mathrm{E}}_{t_k,x}^{b}\left[\left|\partial_b X_t^b(t_k,x)\right|^p\right] \leq C_p\left(1+x^p\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1+p}{2}}}\right), \tag{3.16}$$

where (3.13) *holds for any* $p \geq 1$, (3.14) *holds for any* $p \in [0, \frac{2a}{\sigma^2}-1)$, *and* (3.15), (3.16) *hold for any* $p \geq -\frac{(\frac{2a}{\sigma^2}-1)^2}{2(\frac{2a}{\sigma^2}-\frac{1}{2})}$. *Moreover, all these statements remain valid for* Y^b.

As a consequence of Lemma 3.7, we have the following crucial estimates.

Lemma 3.8. *Assume conditions* **(A1)**-**(A3)**. *Then for any* $b \in \mathbb{R}$, $k \in \{0, ..., n-1\}$ *and* $x \in \mathbb{R}_{++}$, *there exists a constant* $C > 0$ *which does not depend on* x *such that*

$$\widetilde{\mathrm{E}}_{t_k,x}^{b}\left[H(X^b)\right] = 0, \tag{3.17}$$

$$\widetilde{\mathrm{E}}_{t_k,x}^{b}\left[\left(H(X^b)\right)^2\right] \leq C\frac{\Delta_n^{3+\frac{1}{p}}}{x^5}\left(1+x^2\right)\left(1+\frac{1}{x^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta}+\frac{16}{21+\sqrt{185}}+\frac{22}{73+5\sqrt{185}})}}\right), \tag{3.18}$$

where $p = \frac{11+\sqrt{57}}{16}$ *and* $\beta > 1$ *is sufficiently large enough.*

We next recall Girsanov's theorem on each interval $[t_k, t_{k+1}]$. For all $b, b_1 \in \mathbb{R}$, $x \in \mathbb{R}_{++}$ and $k \in \{0, ..., n-1\}$, by [4, Proposition 4.1], the probability measures $\widehat{\mathrm{P}}_{t_k,x}^{b}$ and $\widehat{\mathrm{P}}_{t_k,x}^{b_1}$ are absolutely continuous w.r.t. each other and its Radon-Nikodym derivative is given by

$$\begin{aligned}\frac{d\widehat{\mathrm{P}}_{t_k,x}^{b}}{d\widehat{\mathrm{P}}_{t_k,x}^{b_1}}\left((Y_t^{b_1})_{t\in[t_k,t_{k+1}]}\right) &= \exp\left\{-\frac{b-b_1}{\sigma^2}\int_{t_k}^{t_{k+1}} dY_s^{b_1} - \frac{b^2-b_1^2}{2\sigma^2}\int_{t_k}^{t_{k+1}} Y_s^{b_1} ds\right\} \\ &= \exp\left\{-\frac{b-b_1}{\sigma}\int_{t_k}^{t_{k+1}} \sqrt{Y_s^{b_1}} dW_s - \frac{(b-b_1)^2}{2\sigma^2}\int_{t_k}^{t_{k+1}} Y_s^{b_1} ds\right\}.\end{aligned} \tag{3.19}$$

By Girsanov's theorem, the process $W^{\widehat{\mathrm{P}}_{t_k,x}^{b}} = (W_t^{\widehat{\mathrm{P}}_{t_k,x}^{b}}, t \in [t_k, t_{k+1}])$ is a Brownian motion under $\widehat{\mathrm{P}}_{t_k,x}^{b}$, where for any $t \in [t_k, t_{k+1}]$,

$$W_t^{\widehat{\mathrm{P}}_{t_k,x}^{b}} := W_t + \frac{b-b_1}{\sigma}\int_{t_k}^{t} \sqrt{Y_s^{b_1}} ds.$$

Next, we give the following technical lemma which will be useful in the sequel.

Lemma 3.9. *Assume conditions* **(A1)** *and* **(A2)***. Let* $b_1, b_2 \in \mathbb{R}$ *and* $p, q > 1$ *satisfying that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for any* $k \in \{0, ..., n-1\}$ *and* $x \in \mathbb{R}_{++}$*, there exists a constant* $C > 0$ *which does not depend on* x *such that for any* $\widehat{\mathcal{F}}_{t_{k+1}}$*-measurable random variable* V*,*

$$\left| \widehat{\mathrm{E}}^{b_1}_{t_k,x} \left[V \left(\frac{d\widehat{\mathrm{P}}^{b_2}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}} \left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]} \right) - 1 \right) \right] \right| \leq C\sqrt{\Delta_n} \left(1 + \sqrt{x}\right) \left| \int_{b_1}^{b_2} \left(\widehat{\mathrm{E}}^{b}_{t_k,x} \left[|V|^q \right] \right)^{\frac{1}{q}} db \right|.$$

Lemma 3.10. *Assume condition* **(A1)***. Let* $b_0, b \in \mathbb{R}$*. Then for any* $k \in \{0, ..., n-1\}$ *and* $\widetilde{\mathcal{F}}_{t_{k+1}}$*-measurable random variable* V*,*

$$\widehat{\mathrm{E}}^{b_0} \left[\widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \left[V \middle| X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \right] \middle| \widehat{\mathcal{F}}_{t_k} \right] = \widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \left[V \right].$$

Next, we recall a discrete ergodic theorem.

Lemma 3.11. *Assume that* $b \in \mathbb{R}_{++}$ *and conditions* **(A1)** *and* **(A2)***. Let* $m < \frac{2a}{\sigma^2} - 1$ *and consider a* $\pi_b(dy)$*-integrable function* $h : \mathbb{R}_{++} \to \mathbb{R}$ *satisfying* $|h'(x)| \leq \dfrac{C}{x^m}$*, for some constant* $C > 0$*. Then, as* $n \to \infty$*,*

$$\frac{1}{n} \sum_{k=0}^{n-1} h(Y^b_{t_k}) \xrightarrow{\widehat{\mathrm{P}}^b} \int_0^\infty h(y)\pi_b(dy).$$

We next recall a convergence in probability result and a central limit theorem for triangular arrays of random variables. For each $n \in \mathbb{N}$, let $(\zeta_{k,n})_{k\geq 1}$ be a sequence of random variables defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, \mathrm{P})$, and assume that $\zeta_{k,n}$ are $\mathcal{F}_{t_{k+1}}$-measurable for all k.

Lemma 3.12. [22, Lemma 3.4] *Assume that as* $n \to \infty$*,*

$$\text{(i)} \sum_{k=0}^{n-1} \mathrm{E}\left[\zeta_{k,n} | \mathcal{F}_{t_k}\right] \xrightarrow{\mathrm{P}} 0, \quad \textit{and} \quad \text{(ii)} \sum_{k=0}^{n-1} \mathrm{E}\left[\zeta^2_{k,n} | \mathcal{F}_{t_k}\right] \xrightarrow{\mathrm{P}} 0.$$

Then as $n \to \infty$*,* $\sum_{k=0}^{n-1} \zeta_{k,n} \xrightarrow{\mathrm{P}} 0$.

Lemma 3.13. [22, Lemma 3.6] *Assume that there exist real numbers* M *and* $V > 0$ *such that as* $n \to \infty$*,*

$$\text{(i)} \sum_{k=0}^{n-1} \mathrm{E}\left[\zeta_{k,n} | \mathcal{F}_{t_k}\right] \xrightarrow{\mathrm{P}} M, \qquad \text{(ii)} \sum_{k=0}^{n-1} \left(\mathrm{E}\left[\zeta^2_{k,n} | \mathcal{F}_{t_k}\right] - \left(\mathrm{E}\left[\zeta_{k,n} | \mathcal{F}_{t_k}\right]\right)^2 \right) \xrightarrow{\mathrm{P}} V, \ \textit{and}$$

$$\text{(iii)} \sum_{k=0}^{n-1} \mathrm{E}\left[\zeta^4_{k,n} | \mathcal{F}_{t_k}\right] \xrightarrow{\mathrm{P}} 0.$$

Then as $n \to \infty$*,* $\sum_{k=0}^{n-1} \zeta_{k,n} \xrightarrow{\mathcal{L}(\mathrm{P})} \mathcal{N} + M$*, where* $\mathcal{N}$ *is a centered Gaussian random variable with variance* V*.*

Next, we recall a so called stable central limit theorem for continuous local martingales. We use this limit theorem for studying the asymptotic behavior of the likelihood function connected with the parameter b.

Lemma 3.14. [45, Theorem 4.1] *Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, \mathrm{P}\big)$ be a filtered probability space satisfying the usual conditions. Let $M = (M_t)_{t\in\mathbb{R}_+}$ be a square-integrable continuous local martingale w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ such that $\mathrm{P}(M_0 = 0) = 1$. Suppose that there exists a function $q : [t_0, \infty) \to \mathbb{R}$ with some $t_0 \in \mathbb{R}_+$ such that $q(t) \neq 0$ for all $t \in \mathbb{R}_+$, $\lim_{t\to\infty} q(t) = 0$ and*

$$q(t)^2 \langle M\rangle_t \xrightarrow{\mathrm{P}} \eta^2 \qquad \text{as} \quad t \to \infty, \tag{3.20}$$

where η is a random variable, and $(\langle M\rangle_t)_{t\in\mathbb{R}_+}$ denotes the quadratic variation process of M. Then, for each random variable v defined on $(\Omega, \mathcal{F}, \mathrm{P})$, we have

$$\big(q(t)M_t, v\big) \xrightarrow{\mathcal{L}(\mathrm{P})} (\eta Z, v) \qquad \text{as} \quad t \to \infty, \tag{3.21}$$

where Z is a standard normally distributed random variable independent of (η, v). Moreover,

$$\big(q(t)M_t, q(t)^2\langle M\rangle_t\big) \xrightarrow{\mathcal{L}(\mathrm{P})} (\eta Z, \eta^2) \qquad \text{as} \quad t \to \infty. \tag{3.22}$$

Note that (3.22) follows from (3.21) applied for $v = \eta^2$ and from (3.20) by Theorem 2.7 (iv) of van der Vaart [44].

We finally recall a comparison theorem.

Lemma 3.15. [4, Proposition A.1] *Let $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{++}$, and let m be a Lévy measure on $\mathbb{R}_{++}$ satisfying condition* **(A1)**. *Let η_0 and $\overline{\eta}_0$ be random variables independent of W and J satisfying $\widehat{\mathrm{P}}(\eta_0 \in \mathbb{R}_+) = 1$ and $\widehat{\mathrm{P}}(\overline{\eta}_0 \in \mathbb{R}_+) = 1$. Let $(Y_t^b)_{t\in\mathbb{R}_+}$ be a pathwise unique strong solution of the SDE* (1.1) *such that $\widehat{\mathrm{P}}(Y_0^b = \eta_0) = 1$. Let $(\overline{Y}_t^b)_{t\in\mathbb{R}_+}$ be a pathwise unique strong solution of the SDE*

$$d\overline{Y}_t^b = \Big(a - b\overline{Y}_t^b\Big)\, dt + \sigma\sqrt{\overline{Y}_t^b} dW_t, \tag{3.23}$$

such that $\widehat{\mathrm{P}}(\overline{Y}_0^b = \overline{\eta}_0) = 1$. Then $\widehat{\mathrm{P}}(\eta_0 \geq \overline{\eta}_0) = 1$ implies $\widehat{\mathrm{P}}(Y_t^b \geq \overline{Y}_t^b$ for all $t \in \mathbb{R}_+) = 1$.

4. Proof of main results

This section is devoted to the proof of the main results of this paper. In the case of continuous observations, the proofs of Theorem 2.5, 2.7, and 2.9 will be based on Proposition 2.1, Corollary 2.2, and Lemma 3.14 together with the asymptotic behavior results (2.2)-(2.3), (2.6) and (2.11). In the case of discrete observations, the proofs of Theorem 2.10, 2.11 and 2.12 will be divided into three steps. We begin deriving a stochastic expansion of the log-likelihood ratio using Proposition 3.2, Lemma 3.3 and Corollary 3.4. Next, in the subcritical case, we apply the central limit theorem for triangular arrays together with the convergence results (2.2)-(2.3) in order to show the LAN property. In the critical and supercritical cases, the convergence results (2.6) and (2.11) are respectively used to show the LAQ and LAMN properties. Finally, the last step treats the negligible contributions of the expansion.

4.1. **Proof of Theorem 2.5.**

Proof. By Proposition 2.4 (i), $(Y_t^{b_0})_{t\in\mathbb{R}_+}$ has a unique stationary distribution $\pi_{b_0}(dy)$ with $\int_0^\infty y\pi_{b_0}(dy) = \big(a + \int_0^\infty zm(dz)\big)\frac{1}{b_0} \in \mathbb{R}_{++}$. By Proposition 2.4 (ii), we have $\frac{1}{T}\int_0^T Y_s^{b_0} ds \xrightarrow{\widehat{\mathrm{P}}^{b_0}}$

$\int_0^\infty y\pi_{b_0}(dy)$ as $T \to \infty$. Thus, as $T \to \infty$,

$$I_T(b_0) = \frac{1}{\sigma^2 T}\int_0^T Y_s^{b_0} ds \xrightarrow{\widehat{\mathrm{P}}^{b_0}} \frac{1}{\sigma^2}\int_0^\infty y\pi_{b_0}(dy) = \frac{1}{\sigma^2 b_0}\left(a + \int_0^\infty z m(dz)\right) = I(b_0). \tag{4.1}$$

The quadratic variation process of the square integrable martingale $\left(\int_0^T \sqrt{Y_s^{b_0}} dW_s\right)_{T\in\mathbb{R}_+}$ takes the form $\left(\int_0^T Y_s^{b_0} ds\right)_{T\in\mathbb{R}_+}$. Hence, applying Lemma 3.14 with $\eta := \left(\int_0^\infty y\,\pi_{b_0}(dy)\right)^{1/2}$, we obtain that as $T \to \infty$,

$$\begin{aligned} U_T(b_0) = -\frac{1}{\sigma\sqrt{T}}\int_0^T \sqrt{Y_s^{b_0}} dW_s &\xrightarrow{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})} -\frac{1}{\sigma}\left(\int_0^\infty y\pi_{b_0}(dy)\right)^{1/2}\mathcal{N}(0,1) \\ &= \mathcal{N}\left(0, \frac{1}{\sigma^2}\int_0^\infty y\pi_{b_0}(dy)\right) = \mathcal{N}(0, I(b_0)) = U(b_0). \end{aligned}$$

Consequently, by (4.1) and Theorem 2.7 (v) of van der Vaart [44], we obtain (2.5). Thus, the result follows. □

4.2. **Proof of Theorem 2.7.**

Proof. By strong law of large numbers for the Lévy process $(J_t)_{t\in\mathbb{R}_+}$ (see, e.g., Kyprianou [34, Exercise 7.2]), we have

$$\mathrm{P}\left(\lim_{T\to\infty}\frac{J_T}{T} = \mathrm{E}[J_1] = \int_0^\infty z m(dz)\right) = 1. \tag{4.2}$$

Hence, using (2.6), we obtain that as $T \to \infty$,

$$\begin{aligned} (U_T(0), I_T(0)) &= \left(-\frac{1}{\sigma^2 T}(Y_T^0 - y_0 - aT - J_T), \frac{1}{\sigma^2 T^2}\int_0^T Y_s^0 ds\right) \\ &\xrightarrow{\mathcal{L}(\widehat{\mathrm{P}}^0)} \left(-\frac{1}{\sigma^2}\left(\mathcal{Y}_1 - a - \int_0^\infty z m(dz)\right), \frac{1}{\sigma^2}\int_0^1 \mathcal{Y}_s ds\right) = (U(0), I(0)). \end{aligned}$$

Proposition 1.1 implies $\mathrm{P}(I(0) \in \mathbb{R}_{++}) = \mathrm{P}(\int_0^1 \mathcal{Y}_s ds \in \mathbb{R}_{++}) = 1$.

Finally, using equation (2.7) and (2.9), we have that

$$\begin{aligned} \mathrm{E}\left[e^{uU(0)-\frac{u^2}{2}I(0)}\right] &= \mathrm{E}\left[\exp\left\{\frac{u}{\sigma^2}\left(a + \int_0^\infty z m(dz) - \mathcal{Y}_1\right) - \frac{u^2}{2\sigma^2}\int_0^1 \mathcal{Y}_s ds\right\}\right] \\ &= \mathrm{E}\left[\exp\left\{-\frac{u}{\sigma}\int_0^1 \sqrt{\mathcal{Y}_s} d\mathcal{W}_s - \frac{u^2}{2\sigma^2}\int_0^1 \mathcal{Y}_s ds\right\}\right] = \mathrm{E}\left[\frac{d\mathrm{P}^u_{\mathcal{Y},1}}{d\mathrm{P}^0_{\mathcal{Y},1}}\left((\mathcal{Y}_s)_{s\in[0,1]}\right)\right]. \end{aligned} \tag{4.3}$$

Then, using the fact that the Radon-Nikodym derivative process $\left(\frac{d\mathrm{P}^u_{\mathcal{Y},T}}{d\mathrm{P}^0_{\mathcal{Y},T}}\left((\mathcal{Y}_s)_{s\in[0,T]}\right)\right)_{T\in\mathbb{R}_+}$ is a martingale w.r.t. the filtration $(\mathcal{G}_T)_{T\in\mathbb{R}_+}$, we get that

$$\mathrm{E}\left[\frac{d\mathrm{P}^u_{\mathcal{Y},1}}{d\mathrm{P}^0_{\mathcal{Y},1}}\left((\mathcal{Y}_s)_{s\in[0,1]}\right)\right] = \mathrm{E}\left[\frac{d\mathrm{P}^u_{\mathcal{Y},0}}{d\mathrm{P}^0_{\mathcal{Y},0}}(\mathcal{Y}_0)\right] = 1.$$

This, together with (4.3), concludes $\mathrm{E}\left[e^{uU(0)-\frac{u^2}{2}I(0)}\right] = 1$. Thus, the proof of Theorem 2.7 is now completed. □

4.3. **Proof of Theorem 2.9.**

Proof. By (2.11), $e^{b_0 T}\int_0^T Y_s^{b_0}ds \to -\frac{V}{b_0}$, $\widehat{\mathrm{P}}^{b_0}$-a.s. as $T\to\infty$, and using Lemma 3.14 with $\eta := \sqrt{-\frac{V}{b_0}}$, we obtain that as $T\to\infty$,

$$(U_T(b_0), I_T(b_0)) = \left(-\frac{e^{b_0 T/2}}{\sigma}\int_0^T \sqrt{Y_s^{b_0}}dW_s, \frac{e^{b_0 T}}{\sigma^2}\int_0^T Y_s^{b_0}ds\right) \overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} (U(b_0), I(b_0)).$$

Thus, the result follows. □

4.4. **Proof of Theorem 2.10.**

4.4.1. *Expansion of the log-likelihood ratio.*

Lemma 4.1. *Assume condition* **(A1)**, $b_0\in\mathbb{R}_{++}$ *and* $2a>\sigma^2$. *Then, the log-likelihood ratio at* b_0 *can be expressed as*

$$\begin{aligned}\log\frac{d\mathrm{P}_n^{b_0+\frac{u}{\sqrt{n\Delta_n}}}}{d\mathrm{P}_n^{b_0}}\left(Y^{n,b_0}\right) &= \sum_{k=0}^{n-1}\eta_{k,n} + \sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n^3}}\int_0^1\Big\{H_4(Y^{b_0})+H_5(Y^{b_0})+H_6(Y^{b_0})\\ &+\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b(\ell)}\left[H(X^{b(\ell)})-H_4(X^{b(\ell)})-H_5(X^{b(\ell)})-H_6(X^{b(\ell)})\big|X_{t_{k+1}}^{b(\ell)}=Y_{t_{k+1}}^{b_0}\right]\Big\}d\ell,\end{aligned}\tag{4.4}$$

where $b(\ell):=b_0+\frac{\ell u}{\sqrt{n\Delta_n}}$ *with* $\ell\in[0,1]$, *and*

$$\begin{aligned}\eta_{k,n} &= -\frac{u}{\sigma^2\sqrt{n\Delta_n}}\left(\sigma\sqrt{Y_{t_k}^{b_0}}\left(W_{t_{k+1}}-W_{t_k}\right)+\frac{u\Delta_n}{2\sqrt{n\Delta_n}}Y_{t_k}^{b_0}\right),\\ H_4(Y^{b_0}) &= \frac{\Delta_n}{\sigma^2}b_0\int_{t_k}^{t_{k+1}}\left(Y_s^{b_0}-Y_{t_k}^{b_0}\right)ds,\ H_5(Y^{b_0}) = -\frac{\Delta_n}{\sigma}\int_{t_k}^{t_{k+1}}\left(\sqrt{Y_s^{b_0}}-\sqrt{Y_{t_k}^{b_0}}\right)dW_s,\\ H_6(Y^{b_0}) &= -\frac{\Delta_n}{\sigma^2}\int_{t_k}^{t_{k+1}}\int_0^\infty zN(ds,dz).\end{aligned}$$

Proof. Start from the decomposition (3.1) and use Corollary 3.4, we obtain that

$$\begin{aligned}\log\frac{d\mathrm{P}_n^{b_0+\frac{u}{\sqrt{n\Delta_n}}}}{d\mathrm{P}_n^{b_0}}\left(Y^{n,b_0}\right) &= \sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n}}\int_0^1\frac{\partial_b p^{b(\ell)}}{p^{b(\ell)}}\left(\Delta_n, Y_{t_k}^{b_0}, Y_{t_{k+1}}^{b_0}\right)d\ell\\ &= \sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n}}\int_0^1\Big\{-\frac{1}{\sigma^2}\left(Y_{t_{k+1}}^{b_0}-Y_{t_k}^{b_0}-(a-b(\ell)Y_{t_k}^{b_0})\Delta_n\right)\\ &+\frac{1}{\Delta_n}\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b(\ell)}\left[H(X^{b(\ell)})-H_4(X^{b(\ell)})-H_5(X^{b(\ell)})-H_6(X^{b(\ell)})\big|X_{t_{k+1}}^{b(\ell)}=Y_{t_{k+1}}^{b_0}\right]\Big\}d\ell,\end{aligned}\tag{4.5}$$

where $b(\ell):=b_0+\frac{\ell u}{\sqrt{n\Delta_n}}$. Next using equation (1.1), we have that

$$\begin{aligned}&Y_{t_{k+1}}^{b_0}-Y_{t_k}^{b_0} = (a-b_0Y_{t_k}^{b_0})\Delta_n+\sigma\sqrt{Y_{t_k}^{b_0}}\left(W_{t_{k+1}}-W_{t_k}\right)\\ &-b_0\int_{t_k}^{t_{k+1}}\left(Y_s^{b_0}-Y_{t_k}^{b_0}\right)ds+\sigma\int_{t_k}^{t_{k+1}}\left(\sqrt{Y_s^{b_0}}-\sqrt{Y_{t_k}^{b_0}}\right)dW_s+\int_{t_k}^{t_{k+1}}\int_0^\infty zN(ds,dz),\end{aligned}$$

which, together with (4.5), gives the desired result. □

In the next two subsections, we will show that $\eta_{k,n}$ are the terms that contribute to the limit, and all the others terms are negligible.

4.4.2. *Main contributions: LAN property.*

Lemma 4.2. *Assume conditions* **(A1)**-**(A2)**, $b_0 \in \mathbb{R}_{++}$ *and let* $I(b_0)$ *be defined in Theorem 2.10. Then, as* $n \to \infty$,

$$\sum_{k=0}^{n-1} \eta_{k,n} \overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} u\mathcal{N}\left(0, I(b_0)\right) - \frac{u^2}{2} I(b_0).$$

Proof. It suffices to show that conditions (i), (ii) and (iii) of Lemma 3.13 hold under the measure $\widehat{\mathrm{P}}^{b_0}$ applied to the random variable $\eta_{k,n}$. First, using $\widehat{\mathrm{E}}^{b_0}[W_{t_{k+1}} - W_{t_k} | \widehat{\mathcal{F}}_{t_k}] = 0$ and Lemma 3.11 together with (2.2), we have that as $n \to \infty$,

$$\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^{b_0}\left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k}\right] = -\frac{u^2}{2\sigma^2} \frac{1}{n} \sum_{k=0}^{n-1} Y_{t_k}^{b_0} \overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow} -\frac{u^2}{2\sigma^2} \int_0^\infty y \pi_{b_0}(dy) = -\frac{u^2}{2} I(b_0).$$

This shows Lemma 3.13 (i). Similarly, using $\widehat{\mathrm{E}}^{b_0}[(W_{t_{k+1}} - W_{t_k})^2 | \widehat{\mathcal{F}}_{t_k}] = \Delta_n$, we have that as $n \to \infty$,

$$\sum_{k=0}^{n-1} \left(\widehat{\mathrm{E}}^{b_0}\left[\eta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k}\right] - \left(\widehat{\mathrm{E}}^{b_0}\left[\eta_{k,n} | \widehat{\mathcal{F}}_{t_k}\right]\right)^2 \right) = \frac{u^2}{\sigma^2} \frac{1}{n} \sum_{k=0}^{n-1} Y_{t_k}^{b_0} \overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow} u^2 I(b_0).$$

This shows Lemma 3.13 (ii). Finally, using $\widehat{\mathrm{E}}^{b_0}[(W_{t_{k+1}} - W_{t_k})^4 | \widehat{\mathcal{F}}_{t_k}] = 3\Delta_n^2$, we get that

$$\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^{b_0}\left[\eta_{k,n}^4 | \widehat{\mathcal{F}}_{t_k}\right] \leq C \left(\frac{u^4}{n^2} \sum_{k=0}^{n-1} (Y_{t_k}^{b_0})^2 + \frac{u^8}{n^4} \sum_{k=0}^{n-1} (Y_{t_k}^{b_0})^4 \right),$$

for some constant $C > 0$, which, by Lemma 3.11, converges to zero in $\widehat{\mathrm{P}}^{b_0}$-probability as $n \to \infty$. This shows Lemma 3.13 (iii). Thus, the result follows. □

4.4.3. *Negligible contributions.*

Lemma 4.3. *Assume conditions* **(A1)**-**(A3)** *and* $b_0 \in \mathbb{R}_{++}$. *Then, as* $n \to \infty$,

$$\sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \Big\{ H_4(Y^{b_0}) + H_5(Y^{b_0}) + H_6(Y^{b_0}) \\ + \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^{b_0}}^{b(\ell)} \left[H(X^{b(\ell)}) - H_4(X^{b(\ell)}) - H_5(X^{b(\ell)}) - H_6(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^{b_0} \right] \Big\} d\ell \overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow} 0.$$

Proof. The proof of Lemma 4.3 is completed by combining the four Lemma 4.4-4.7 below. □

Consequently, from Lemma 4.1, 4.2 and 4.3, the proof of Theorem 2.10 is now completed.

Lemma 4.4. *Assume conditions* **(A1)**-**(A3)** *and* $b_0 \in \mathbb{R}_{++}$. *Then, as* $n \to \infty$,

$$\sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^{b_0}}^{b(\ell)} \left[H(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^{b_0} \right] d\ell \overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow} 0.$$

Proof. It suffices to show that conditions (i) and (ii) of Lemma 3.12 hold under the measure $\widehat{\mathrm{P}}^{b_0}$ applied to the random variable

$$\zeta_{k,n} := \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \left[H(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \right] d\ell.$$

We start showing Lemma 3.12 (i). Applying Lemma 3.10 to $b = b(\ell)$ and $V = H(X^{b(\ell)})$, and using the fact that, by (3.17), $\widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}}[H(X^{b(\ell)})] = 0$, we obtain that

$$\begin{aligned}\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^{b_0} \left[\zeta_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] &= \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \widehat{\mathrm{E}}^{b_0} \left[\widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \left[H(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \right] | \widehat{\mathcal{F}}_{t_k} \right] d\ell \\ &= \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n^3}} \int_0^1 \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \left[H(X^{b(\ell)}) \right] d\ell = 0.\end{aligned}$$

Thus, the term appearing in condition (i) of Lemma 3.12 actually equals zero.

Next, applying Jensen's inequality and Lemma 3.10 to $b = b(\ell)$ and $V = (H(X^{b(\ell)}))^2$, and (3.18), we obtain that

$$\begin{aligned}\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^{b_0} \left[\zeta^2_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] &= \sum_{k=0}^{n-1} \frac{u^2}{n\Delta_n^3} \widehat{\mathrm{E}}^{b_0} \left[\left(\int_0^1 \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \left[H(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \right] d\ell \right)^2 | \widehat{\mathcal{F}}_{t_k} \right] \\ &\leq \sum_{k=0}^{n-1} \frac{u^2}{n\Delta_n^3} \int_0^1 \widehat{\mathrm{E}}^{b_0} \left[\widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \left[\left(H(X^{b(\ell)}) \right)^2 | X^{b(\ell)}_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \right] | \widehat{\mathcal{F}}_{t_k} \right] d\ell \\ &= \sum_{k=0}^{n-1} \frac{u^2}{n\Delta_n^3} \int_0^1 \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \left[\left(H(X^{b(\ell)}) \right)^2 \right] d\ell \\ &\leq \frac{Cu^2 \Delta_n^{\frac{1}{p}}}{n} \sum_{k=0}^{n-1} \frac{1}{(Y^{b_0}_{t_k})^5} \left(1 + (Y^{b_0}_{t_k})^2 \right) \left(1 + \frac{1}{(Y^{b_0}_{t_k})^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta} + \frac{16}{21+\sqrt{185}} + \frac{22}{73+5\sqrt{185}})}} \right),\end{aligned}$$

for some constant $C > 0$, where $p = \frac{11+\sqrt{57}}{16}$. Then taking the expectation in both sides, we get that

$$\begin{aligned}\widehat{\mathrm{E}}^{b_0} \left[\left| \sum_{k=0}^{n-1} \widehat{\mathrm{E}}^{b_0} \left[\zeta^2_{k,n} | \widehat{\mathcal{F}}_{t_k} \right] \right| \right] &\leq \frac{Cu^2 \Delta_n^{\frac{1}{p}}}{n} \widehat{\mathrm{E}}^{b_0} \Bigg[\sum_{k=0}^{n-1} \frac{1}{(Y^{b_0}_{t_k})^5} \Bigg(1 + (Y^{b_0}_{t_k})^2 \\ &\quad + \frac{1}{(Y^{b_0}_{t_k})^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta} + \frac{16}{21+\sqrt{185}} + \frac{22}{73+5\sqrt{185}})}} + \frac{1}{(Y^{b_0}_{t_k})^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta} + \frac{16}{21+\sqrt{185}} + \frac{22}{73+5\sqrt{185}})-2}} \Bigg) \Bigg] \\ &\leq Cu^2 \Delta_n^{\frac{1}{p}} \Bigg(\max_{k \in \{0,\dots,n\}} \widehat{\mathrm{E}}^{b_0} \left[\frac{1}{(Y^{b_0}_{t_k})^5} \right] + \max_{k \in \{0,\dots,n\}} \widehat{\mathrm{E}}^{b_0} \left[\frac{1}{(Y^{b_0}_{t_k})^3} \right] \\ &\quad + \max_{k \in \{0,\dots,n\}} \widehat{\mathrm{E}}^{b_0} \left[\frac{1}{(Y^{b_0}_{t_k})^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta} + \frac{16}{21+\sqrt{185}} + \frac{22}{73+5\sqrt{185}})+5}} \right]\end{aligned}$$

$$+ \max_{k\in\{0,\dots,n\}} \widehat{\mathrm{E}}^{b_0}\left[\frac{1}{(Y_{t_k}^{b_0})^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta}+\frac{16}{21+\sqrt{185}}+\frac{22}{73+5\sqrt{185}})+3}}\right]\Bigg) \leq Cu^2\Delta_n^{\frac{1}{p}},$$

for some constant $C>0$, which tends to zero as $n\to\infty$. Here, we have used the boundedness of the negative moment estimates for the jump-type CIR process (1.1) using Lemma 3.5 (i), condition **(A3)** and the fact that $\beta>1$ is sufficiently large enough. Therefore, we have shown that $\sum_{k=0}^{n-1}\widehat{\mathrm{E}}^{b_0}\left[\zeta_{k,n}^2|\widehat{\mathcal{F}}_{t_k}\right]\overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow}0$ as $n\to\infty$. Thus, by Lemma 3.12, the result follows. □

Lemma 4.5. *Assume conditions* **(A1)** *and* **(A2)**. *Then, as* $n\to\infty$,

$$\sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n^3}}\int_0^1\left(H_4(Y^{b_0})-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b(\ell)}\left[H_4(X^{b(\ell)})\big|X_{t_{k+1}}^{b(\ell)}=Y_{t_{k+1}}^{b_0}\right]\right)d\ell\overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow}0.$$

Proof. We rewrite

$$\frac{u}{\sqrt{n\Delta_n^3}}\int_0^1\left(H_4(Y^{b_0})-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b(\ell)}\left[H_4(X^{b(\ell)})\big|X_{t_{k+1}}^{b(\ell)}=Y_{t_{k+1}}^{b_0}\right]\right)d\ell=M_{k,n,1}+M_{k,n,2},$$

where

$$\begin{aligned}M_{k,n,1}&=-\frac{u^2}{2\sigma^2 n\Delta_n}\int_{t_k}^{t_{k+1}}\left(Y_s^{b_0}-Y_{t_k}^{b_0}\right)ds,\\ M_{k,n,2}&=\frac{u}{\sigma^2\sqrt{n\Delta_n}}\int_0^1 b(\ell)\Bigg\{\int_{t_k}^{t_{k+1}}\left(Y_s^{b_0}-Y_{t_k}^{b_0}\right)ds\\ &\quad-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b(\ell)}\left[\int_{t_k}^{t_{k+1}}\left(X_s^{b(\ell)}-X_{t_k}^{b(\ell)}\right)ds\big|X_{t_{k+1}}^{b(\ell)}=Y_{t_{k+1}}^{b_0}\right]\Bigg\}d\ell.\end{aligned}$$

First, using Lemma 3.5 (iii), we get that

$$\widehat{\mathrm{E}}^{b_0}\left[\left|\sum_{k=0}^{n-1}M_{k,n,1}\right|\right]\leq\sum_{k=0}^{n-1}\widehat{\mathrm{E}}^{b_0}\left[|M_{k,n,1}|\right]\leq Cu^2\sqrt{\Delta_n},$$

for some constant $C>0$. Thus, $\sum_{k=0}^{n-1}M_{k,n,1}\overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow}0$ as $n\to\infty$.

Next, to prove $\sum_{k=0}^{n-1}M_{k,n,2}\overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow}0$ as $n\to\infty$, it suffices to show that conditions (i) and (ii) of Lemma 3.12 hold under the measure $\widehat{\mathrm{P}}^{b_0}$. We start showing (i). For this, applying Girsanov's theorem and Lemma 3.10 to $b=b(\ell)$ and $V=\int_{t_k}^{t_{k+1}}(X_s^{b(\ell)}-X_{t_k}^{b(\ell)})ds$, we get that

$$\begin{aligned}&\widehat{\mathrm{E}}^{b_0}\left[\int_{t_k}^{t_{k+1}}\left(Y_s^{b_0}-Y_{t_k}^{b_0}\right)ds-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b(\ell)}\left[\int_{t_k}^{t_{k+1}}\left(X_s^{b(\ell)}-X_{t_k}^{b(\ell)}\right)ds\big|X_{t_{k+1}}^{b(\ell)}=Y_{t_{k+1}}^{b_0}\right]|\widehat{\mathcal{F}}_{t_k}\right]\\ &=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\int_{t_k}^{t_{k+1}}\left(Y_s^{b_0}-Y_{t_k}^{b_0}\right)ds\right]\\ &\quad-\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b(\ell)}\left[\int_{t_k}^{t_{k+1}}\left(X_s^{b(\ell)}-X_{t_k}^{b(\ell)}\right)ds\big|X_{t_{k+1}}^{b(\ell)}=Y_{t_{k+1}}^{b_0}\right]\right]\end{aligned}$$

$$
\begin{aligned}
&= \widehat{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\left(Y^{b(\ell)}_s-Y^{b(\ell)}_{t_k}\right)ds\,\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}}\left((Y^{b(\ell)}_t)_{t\in[t_k,t_{k+1}]}\right)\right]\\
&\qquad-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\left(X^{b(\ell)}_s-X^{b(\ell)}_{t_k}\right)ds\right]\\
&= \widehat{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}(Y^{b(\ell)}_s-Y^{b(\ell)}_{t_k})ds\left(\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}}(Y^{b(\ell)}_t)_{t\in[t_k,t_{k+1}]})-1\right)\right]\\
&\qquad+\widehat{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\left(Y^{b(\ell)}_s-Y^{b(\ell)}_{t_k}\right)ds\right]-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\left(X^{b(\ell)}_s-X^{b(\ell)}_{t_k}\right)ds\right]\\
&= \widehat{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}(Y^{b(\ell)}_s-Y^{b(\ell)}_{t_k})ds\left(\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}}((Y^{b(\ell)}_t)_{t\in[t_k,t_{k+1}]})-1\right)\right],
\end{aligned}
$$

where we have used the fact that $X^{b(\ell)}$ is the independent copy of $Y^{b(\ell)}$. This, together with Lemma 3.9 with $q=2$ and Lemma 3.5 (iii), implies that

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}\widehat{\mathrm{E}}^{b_0}\left[M_{k,n,2}|\widehat{\mathcal{F}}_{t_k}\right]\right|=\left|\sum_{k=0}^{n-1}\frac{u}{\sigma^2\sqrt{n\Delta_n}}\int_0^1 b(\ell)\left\{\widehat{\mathrm{E}}^{b_0}\left[\int_{t_k}^{t_{k+1}}\left(Y^{b_0}_s-Y^{b_0}_{t_k}\right)ds\right.\right.\right.\\
&\qquad\left.\left.\left.-\widetilde{\mathrm{E}}^{b(\ell)}_{Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\left(X^{b(\ell)}_s-X^{b(\ell)}_{t_k}\right)ds|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]|\widehat{\mathcal{F}}_{t_k}\right]\right\}d\ell\right|\\
&=\frac{|u|}{\sigma^2\sqrt{n\Delta_n}}\left|\sum_{k=0}^{n-1}\int_0^1 b(\ell)\widehat{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}(Y^{b(\ell)}_s-Y^{b(\ell)}_{t_k})ds\left(\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}}((Y^{b(\ell)}_t)_{t\in[t_k,t_{k+1}]})-1\right)\right]d\ell\right|\\
&\leq\frac{|u|}{\sigma^2\sqrt{n\Delta_n}}\sum_{k=0}^{n-1}\int_0^1|b(\ell)|\left|\widehat{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}(Y^{b(\ell)}_s-Y^{b(\ell)}_{t_k})ds\left(\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}}((Y^{b(\ell)}_t)_{t\in[t_k,t_{k+1}]})-1\right)\right]\right|d\ell\\
&\leq\frac{C|u|}{\sigma^2\sqrt{n}}\sum_{k=0}^{n-1}\int_0^1|b(\ell)|\left|\int_{b(\ell)}^{b_0}\left(\widehat{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\left(\int_{t_k}^{t_{k+1}}(Y^b_s-Y^b_{t_k})ds\right)^2\right]\right)^{\frac{1}{2}}db\right|d\ell\leq Cu^2\Delta_n,
\end{aligned}
$$

for some constant $C>0$. This tends to zero as $n\to\infty$.

Next, applying Jensen's inequality and Lemma 3.10 to $b=b(\ell)$ and $V=(\int_{t_k}^{t_{k+1}}(X^{b(\ell)}_s-X^{b(\ell)}_{t_k})ds)^2$, and using Lemma 3.5 (iii), we obtain that

$$
\begin{aligned}
\sum_{k=0}^{n-1}\widehat{\mathrm{E}}^{b_0}\left[M^2_{k,n,2}|\widehat{\mathcal{F}}_{t_k}\right]&=\frac{u^2}{\sigma^2 n\Delta_n}\sum_{k=0}^{n-1}\widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\left(\int_0^1 b(\ell)\left\{\int_{t_k}^{t_{k+1}}\left(Y^{b_0}_s-Y^{b_0}_{t_k}\right)ds\right.\right.\right.\\
&\qquad\left.\left.\left.-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\left(X^{b(\ell)}_s-X^{b(\ell)}_{t_k}\right)ds|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right\}d\ell\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{u^2}{\sigma^2 n\Delta_n}\sum_{k=0}^{n-1}\int_0^1 b^2(\ell)\Big\{\widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\left(\int_{t_k}^{t_{k+1}}\left(Y^{b_0}_s-Y^{b_0}_{t_k}\right)ds\right)^2\right]\\
&\qquad+\widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\left(\int_{t_k}^{t_{k+1}}\left(X^{b(\ell)}_s-X^{b(\ell)}_{t_k}\right)ds\right)^2\big|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right]\Big\}d\ell\\
&=\frac{u^2}{\sigma^2 n\Delta_n}\sum_{k=0}^{n-1}\int_0^1 b^2(\ell)\Big\{\widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\left(\int_{t_k}^{t_{k+1}}\left(Y^{b_0}_s-Y^{b_0}_{t_k}\right)ds\right)^2\right]\\
&\qquad+\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[\left(\int_{t_k}^{t_{k+1}}\left(X^{b(\ell)}_s-X^{b(\ell)}_{t_k}\right)ds\right)^2\right]\Big\}d\ell\leq Cu^2\Delta_n^2,
\end{aligned}
$$

for some constant $C>0$, which tends to zero. Thus, by Lemma 3.12, the result follows. □

Lemma 4.6. *Assume conditions* **(A1)** *and* **(A2)***. Then, as* $n\to\infty$,

$$
\sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n^3}}\int_0^1\left(H_5(Y^{b_0})-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[H_5(X^{b(\ell)})\big|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)d\ell\overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow}0.
$$

Proof. We proceed as in the proof of Lemma 4.5. □

Lemma 4.7. *Assume conditions* **(A1)** *and* **(A2)***. Then, as* $n\to\infty$,

$$
\sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n^3}}\int_0^1\left(H_6(Y^{b_0})-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[H_6(X^{b(\ell)})\big|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)d\ell\overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow}0.
$$

Proof. It suffices to show that conditions (i) and (ii) of Lemma 3.12 hold under the measure $\widehat{\mathrm{P}}^{b_0}$ applied to the random variable

$$
\zeta_{k,n}:=\frac{u}{\sqrt{n\Delta_n^3}}\int_0^1\left(H_6(Y^{b_0})-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[H_6(X^{b(\ell)})\big|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)d\ell.
$$

Applying Lemma 3.10 to $b=b(\ell)$ and $V=H_6(X^{b(\ell)})$, we get that

$$
\begin{aligned}
&\widehat{\mathrm{E}}^{b_0}\left[H_6(Y^{b_0})-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[H_6(X^{b(\ell)})\big|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\big|\widehat{\mathcal{F}}_{t_k}\right]\\
&=\widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[H_6(Y^{b_0})\right]-\widehat{\mathrm{E}}^{b_0}\left[\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[H_6(X^{b(\ell)})\big|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\big|\widehat{\mathcal{F}}_{t_k}\right]\\
&=\widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[H_6(Y^{b_0})\right]-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[H_6(X^{b(\ell)})\right]=-\frac{\Delta_n^2}{\sigma^2}\int_0^\infty zm(dz)+\frac{\Delta_n^2}{\sigma^2}\int_0^\infty zm(dz)=0.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\sum_{k=0}^{n-1}\widehat{\mathrm{E}}^{b_0}\left[\zeta_{k,n}|\widehat{\mathcal{F}}_{t_k}\right]&=\sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n^3}}\int_0^1\widehat{\mathrm{E}}^{b_0}\left[H_6(Y^{b_0})-\widetilde{\mathrm{E}}^{b(\ell)}_{t_k,Y^{b_0}_{t_k}}\left[H_6(X^{b(\ell)})\big|X^{b(\ell)}_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\big|\widehat{\mathcal{F}}_{t_k}\right]d\ell\\
&=0.
\end{aligned}
$$

We next show that condition (ii) of Lemma 3.12 holds. Using Cauchy-Schwarz inequality and Lemma 5.3, we obtain that for all $q > 1$,

$$\begin{aligned}
&\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^{b_0}\left[\zeta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k}\right] \leq \frac{u^2}{n\Delta_n^3} \sum_{k=0}^{n-1} \int_0^1 \widehat{\mathrm{E}}^{b_0}\left[\left(H_6(Y^{b_0}) - \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^{b_0}}^{b(\ell)}\left[H_6(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^{b_0}\right]\right)^2 | \widehat{\mathcal{F}}_{t_k}\right] d\ell \\
&\quad = \frac{u^2}{\sigma^4 n\Delta_n} \sum_{k=0}^{n-1} \int_0^1 \widehat{\mathrm{E}}_{t_k, Y_{t_k}^{b_0}}^{b_0}\Bigg[\Bigg(\int_{t_k}^{t_{k+1}} \int_0^\infty zN(ds, dz) \\
&\qquad - \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^{b_0}}^{b(\ell)}\left[\int_{t_k}^{t_{k+1}} \int_0^\infty zM(ds, dz) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^{b_0}\right]\Bigg)^2\Bigg] d\ell \\
&\quad \leq \frac{Cu^2}{\sigma^4 n\Delta_n} \sum_{k=0}^{n-1} \int_0^1 (1 + (Y_{t_k}^{b_0})^2)\Delta_n \left((\lambda_{\upsilon_n}\Delta_n)^{\frac{1}{q}} + \int_{z\leq\upsilon_n} z^2 m(dz) + \Delta_n\Big(\int_{z\leq\upsilon_n} zm(dz)\Big)^2\right) d\ell \\
&\quad = \frac{Cu^2}{\sigma^4}\left((\lambda_{\upsilon_n}\Delta_n)^{\frac{1}{q}} + \int_{z\leq\upsilon_n} z^2 m(dz) + \Delta_n\left(\int_{z\leq\upsilon_n} zm(dz)\right)^2\right)\left(1 + \frac{1}{n}\sum_{k=0}^{n-1}(Y_{t_k}^{b_0})^2\right),
\end{aligned}$$

where $(\upsilon_n)_{n\geq 1}$ defined in Subsection 5.11 is a positive sequence satisfying $\lim_{n\to\infty} \upsilon_n = 0$, and $\lambda_{\upsilon_n} := \int_{z>\upsilon_n} m(dz)$.

When $\int_0^\infty m(dz) < +\infty$, then $\lambda_{\upsilon_n} \leq \int_0^\infty m(dz) < +\infty$. Therefore, $\lambda_{\upsilon_n}\Delta_n \to 0$ as $n \to \infty$.

When $\int_0^\infty m(dz) = +\infty$, then $\lambda_{\upsilon_n} \to \int_0^\infty m(dz) = +\infty$ as $n \to \infty$. Then, there exist $\epsilon \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $\lambda_{\upsilon_n} \leq \Delta_n^{\epsilon-1}$ for all $n \geq n_0$. This implies that $\lambda_{\upsilon_n}\Delta_n \leq \Delta_n^\epsilon$ for all $n \geq n_0$. Therefore, $\lambda_{\upsilon_n}\Delta_n \to 0$ as $n \to \infty$.

By Lebesgue's dominated convergence theorem and conditions **(A1)** and **(A2)** with $p = 2$, we get that $\int_{z\leq\upsilon_n} z^2 m(dz) \to 0$ and $\int_{z\leq\upsilon_n} zm(dz) \to 0$ as $n \to \infty$. Finally, using Lemma 3.11,

$$\frac{1}{n}\sum_{k=0}^{n-1}(Y_{t_k}^{b_0})^2 \overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow} \int_0^\infty y^2 \pi_{b_0}(dy) < +\infty,$$

as $n \to \infty$. Hence, we have shown that $\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^{b_0}\left[\zeta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k}\right] \overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow} 0$ as $n \to \infty$. Thus, the proof is now completed. □

4.5. **Proof of Theorem 2.11.**

4.5.1. *Expansion of the log-likelihood ratio.*

Lemma 4.8. *Assume condition* **(A1)** *and* $2a > \sigma^2$. *Then, the log-likelihood ratio at* 0 *can be expressed as*

$$\begin{aligned}
\log \frac{d\mathrm{P}_n^{0+\frac{u}{n\Delta_n}}}{d\mathrm{P}_n^0}\left(Y^{n,0}\right) &= \sum_{k=0}^{n-1} \eta_{k,n} + \sum_{k=0}^{n-1} \frac{u}{n\Delta_n^2} \int_0^1 \Big\{H_5(Y^0) + H_6(Y^0) \\
&+ \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^0}^{b(\ell)}\left[H(X^{b(\ell)}) - H_4(X^{b(\ell)}) - H_5(X^{b(\ell)}) - H_6(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^0\right]\Big\} d\ell,
\end{aligned} \tag{4.6}$$

where $b(\ell) := 0 + \frac{\ell u}{n\Delta_n}$ *with* $\ell \in [0,1]$, *and*

$$\eta_{k,n} = -\frac{u}{\sigma^2 n\Delta_n}\left(\sigma\sqrt{Y^0_{t_k}}\left(W_{t_{k+1}} - W_{t_k}\right) + \frac{u}{2n}Y^0_{t_k}\right),$$

$$H_5(Y^0) = -\frac{\Delta_n}{\sigma}\int_{t_k}^{t_{k+1}}\left(\sqrt{Y^0_s} - \sqrt{Y^0_{t_k}}\right)dW_s,\ H_6(Y^0) = -\frac{\Delta_n}{\sigma^2}\int_{t_k}^{t_{k+1}}\int_0^\infty zN(ds,dz).$$

Proof. We proceed as in the proof of Lemma 4.1. □

4.5.2. *Main contributions: LAQ property.*

Lemma 4.9. *Assume conditions* **(A1)-(A2)**, *and let* $U(0)$ *and* $I(0)$ *be defined in Theorem 2.11. Then as* $n \to \infty$,

$$\sum_{k=0}^{n-1}\eta_{k,n} \xrightarrow{\mathcal{L}(\widehat{\mathrm{P}}^0)} uU(0) - \frac{u^2}{2}I(0), \tag{4.7}$$

and $\mathrm{E}\left[e^{uU(0)-\frac{u^2}{2}I(0)}\right] = 1$.

Proof. We rewrite

$$\sum_{k=0}^{n-1}\eta_{k,n} = uU_n(0) - \frac{u^2}{2}I_n(0) + H_7(Y^0) + H_8(Y^0), \tag{4.8}$$

where $t_n = n\Delta_n$ and

$$U_n(0) = -\frac{1}{\sigma t_n}\int_0^{t_n}\sqrt{Y^0_s}dW_s,\ I_n(0) = \frac{1}{\sigma^2 t_n^2}\int_0^{t_n}Y^0_s ds,$$

$$H_7(Y^0) = \frac{u}{\sigma n\Delta_n}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left(\sqrt{Y^0_s} - \sqrt{Y^0_{t_k}}\right)dW_s,\ H_8(Y^0) = \frac{u^2}{\sigma^2 n^2\Delta_n^2}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left(Y^0_s - Y^0_{t_k}\right)ds.$$

First, using equation (1.1) with $b = 0$, we get that

$$U_n(0) = -\frac{1}{\sigma^2}\left(\frac{Y^0_{t_n}}{t_n} - \frac{y_0}{t_n} - a - \frac{J_{t_n}}{t_n}\right).$$

On the other hand, by strong law of large numbers for the Lévy process J (see, e.g., Kyprianou [34, Exercise 7.2]), we have that as $n \to \infty$,

$$\frac{J_{t_n}}{t_n} \longrightarrow \mathrm{E}\left[J_1\right] = \int_0^\infty zm(dz), \quad \widehat{\mathrm{P}}^0\text{-a.s.} \tag{4.9}$$

This, together with (2.6), implies that as $n \to \infty$,

$$(U_n(0), I_n(0)) \xrightarrow{\mathcal{L}(\widehat{\mathrm{P}}^0)} (U(0), I(0)). \tag{4.10}$$

Using Lemma 3.12 and 3.5 (ii), conditions **(A1)-(A2)**, and the fact that $n\Delta_n \to \infty$ as $n \to \infty$, it can be checked that as $n \to \infty$,

$$H_7(Y^0) + H_8(Y^0) \xrightarrow{\widehat{\mathrm{P}}^0} 0. \tag{4.11}$$

Therefore, from (4.8), (4.10) and (4.11), we conclude (4.7). Finally, we proceed similarly as in the proof of Theorem 2.7 to get $\mathrm{E}\left[e^{uU(0)-\frac{u^2}{2}I(0)}\right] = 1$. Thus, the result follows. □

4.5.3. *Negligible contributions.*

Lemma 4.10. *Assume conditions* **(A1)-(A3)**. *Then, as $n \to \infty$,*

$$\sum_{k=0}^{n-1} \frac{u}{n\Delta_n^2} \int_0^1 \Big\{ H_5(Y^0) + H_6(Y^0) + \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^0}^{b(\ell)} \Big[H(X^{b(\ell)}) - H_4(X^{b(\ell)}) - H_5(X^{b(\ell)}) - H_6(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^0 \Big] \Big\} d\ell \xrightarrow{\widehat{\mathrm{P}}^0} 0.$$

Proof. The proof of Lemma 4.10 is completed by combining the four Lemma 4.11-4.14 below. □

Consequently, from Lemma 4.8, 4.9 and 4.10, the proof of Theorem 2.11 is now completed.

Lemma 4.11. *Assume conditions* **(A1)-(A3)**. *Then, as $n \to \infty$,*

$$\sum_{k=0}^{n-1} \frac{u}{n\Delta_n^2} \int_0^1 \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^0}^{b(\ell)} \Big[H(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^0 \Big] d\ell \xrightarrow{\widehat{\mathrm{P}}^0} 0.$$

Proof. It suffices to show that conditions (i) and (ii) of Lemma 3.12 hold under the measure $\widehat{\mathrm{P}}^0$ applied to the random variable

$$\zeta_{k,n} := \frac{u}{n\Delta_n^2} \int_0^1 \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^0}^{b(\ell)} \Big[H(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^0 \Big] d\ell.$$

Proceeding as in the proof of Lemma 4.4, we get that $\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^0[\zeta_{k,n} | \widehat{\mathcal{F}}_{t_k}] = 0$.

Next, applying Jensen's inequality and Lemma 3.10 to $b = b(\ell)$ and $V = (H(X^{b(\ell)}))^2$, and (3.18), we obtain that

$$\begin{aligned} \sum_{k=0}^{n-1} \widehat{\mathrm{E}}^0 \Big[\zeta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k} \Big] &= \sum_{k=0}^{n-1} \frac{u^2}{n^2\Delta_n^4} \widehat{\mathrm{E}}^0 \bigg[\bigg(\int_0^1 \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^0}^{b(\ell)} \Big[H(X^{b(\ell)}) \big| X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^0 \Big] d\ell \bigg)^2 | \widehat{\mathcal{F}}_{t_k} \bigg] \\ &\leq \sum_{k=0}^{n-1} \frac{u^2}{n^2\Delta_n^4} \int_0^1 \widehat{\mathrm{E}}^0 \bigg[\widetilde{\mathrm{E}}_{t_k, Y_{t_k}^0}^{b(\ell)} \Big[\Big(H(X^{b(\ell)}) \Big)^2 | X_{t_{k+1}}^{b(\ell)} = Y_{t_{k+1}}^0 \Big] | \widehat{\mathcal{F}}_{t_k} \bigg] d\ell \\ &= \sum_{k=0}^{n-1} \frac{u^2}{n^2\Delta_n^4} \int_0^1 \widetilde{\mathrm{E}}_{t_k, Y_{t_k}^0}^{b(\ell)} \Big[\Big(H(X^{b(\ell)}) \Big)^2 \Big] d\ell \\ &\leq \frac{Cu^2}{(n\Delta_n)^{1-\frac{1}{p}} n^{\frac{1}{p}}} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(Y_{t_k}^0)^5} \left(1 + (Y_{t_k}^0)^2\right) \left(1 + \frac{1}{(Y_{t_k}^0)^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta} + \frac{16}{21+\sqrt{185}} + \frac{22}{73+5\sqrt{185}})}} \right), \end{aligned}$$

for some constant $C > 0$, where $p = \frac{11+\sqrt{57}}{16}$ and $\beta > 1$ is sufficiently large enough. Then taking the expectation in both sides, we get that

$$\widehat{\mathrm{E}}^0 \left[\left| \sum_{k=0}^{n-1} \widehat{\mathrm{E}}^0 \Big[\zeta_{k,n}^2 | \widehat{\mathcal{F}}_{t_k} \Big] \right| \right] \leq \frac{Cu^2}{(n\Delta_n)^{1-\frac{1}{p}} n^{\frac{1}{p}}} \frac{1}{n} \widehat{\mathrm{E}}^0 \Bigg[\sum_{k=0}^{n-1} \frac{1}{(Y_{t_k}^0)^5} \Bigg(1 + (Y_{t_k}^0)^2 + \frac{1}{(Y_{t_k}^0)^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta} + \frac{16}{21+\sqrt{185}} + \frac{22}{73+5\sqrt{185}})}} + \frac{1}{(Y_{t_k}^0)^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta} + \frac{16}{21+\sqrt{185}} + \frac{22}{73+5\sqrt{185}})-2}} \Bigg) \Bigg]$$

$$\leq \frac{Cu^2}{(n\Delta_n)^{1-\frac{1}{p}} n^{\frac{1}{p}}} \Bigg(\max_{k\in\{0,\ldots,n\}} \widehat{\mathrm{E}}^0 \left[\frac{1}{(Y^0_{t_k})^5} \right] + \max_{k\in\{0,\ldots,n\}} \widehat{\mathrm{E}}^0 \left[\frac{1}{(Y^0_{t_k})^3} \right]$$
$$+ \max_{k\in\{0,\ldots,n\}} \widehat{\mathrm{E}}^0 \left[\frac{1}{(Y^0_{t_k})^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta}+\frac{16}{21+\sqrt{185}}+\frac{22}{73+5\sqrt{185}})+5}} \right]$$
$$+ \max_{k\in\{0,\ldots,n\}} \widehat{\mathrm{E}}^0 \left[\frac{1}{(Y^0_{t_k})^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta}+\frac{16}{21+\sqrt{185}}+\frac{22}{73+5\sqrt{185}})+3}} \right] \Bigg) \leq \frac{Cu^2}{(n\Delta_n)^{1-\frac{1}{p}} n^{\frac{1}{p}}},$$

for some constant $C > 0$, which tends to zero as $n \to \infty$ since $n\Delta_n \to \infty$. Here, we have used the boundedness of the negative moment estimates for the jump-type CIR process (1.1) using Lemma 3.5 (i), condition **(A3)** and the fact that $\beta > 1$ is sufficiently large enough. Therefore, we have shown that $\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^0 \left[\zeta^2_{k,n} | \widehat{\mathcal{F}}_{t_k}\right] \xrightarrow{\widehat{\mathrm{P}}^0} 0$ as $n \to \infty$. Thus, by Lemma 3.12, the result follows. □

Lemma 4.12. *Assume conditions* **(A1)** *and* **(A2)**. *Then, as* $n \to \infty$,

$$\sum_{k=0}^{n-1} \frac{u}{n\Delta_n^2} \int_0^1 \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \left[H_4(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \right] d\ell \xrightarrow{\widehat{\mathrm{P}}^0} 0.$$

Proof. We proceed as in the proof of Lemma 4.5. □

Lemma 4.13. *Assume conditions* **(A1)** *and* **(A2)**. *Then, as* $n \to \infty$,

$$\sum_{k=0}^{n-1} \frac{u}{n\Delta_n^2} \int_0^1 \left(H_5(Y^0) - \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \left[H_5(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \right] \right) d\ell \xrightarrow{\widehat{\mathrm{P}}^0} 0.$$

Proof. We proceed as in the proof of Lemma 4.5. □

Lemma 4.14. *Assume conditions* **(A1)** *and* **(A2)** *with* $p = 2$. *Then, as* $n \to \infty$,

$$\sum_{k=0}^{n-1} \frac{u}{n\Delta_n^2} \int_0^1 \left(H_6(Y^0) - \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \left[H_6(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \right] \right) d\ell \xrightarrow{\widehat{\mathrm{P}}^0} 0.$$

Proof. It suffices to show that conditions (i) and (ii) of Lemma 3.12 hold under the measure $\widehat{\mathrm{P}}^0$ applied to the random variable

$$\zeta_{k,n} := \frac{u}{n\Delta_n^2} \int_0^1 \left(H_6(Y^0) - \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \left[H_6(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \right] \right) d\ell.$$

Proceeding as in the proof of Lemma 4.7, we get that $\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^0 [\zeta_{k,n} | \widehat{\mathcal{F}}_{t_k}] = 0$.

We next show Lemma 3.12 (ii). Using Jensen's and Burkholder-David-Gundy's inequalities, and applying Lemma 3.10 to $b = b(\ell)$ and $V = (\int_{t_k}^{t_{k+1}} \int_0^\infty z\widetilde{M}(ds, dz))^2$, we get that

$$\sum_{k=0}^{n-1} \widehat{\mathrm{E}}^0 \left[\zeta^2_{k,n} | \widehat{\mathcal{F}}_{t_k}\right] \leq \frac{u^2}{n^2\Delta_n^4} \sum_{k=0}^{n-1} \int_0^1 \widehat{\mathrm{E}}^0 \left[\left(H_6(Y^0) - \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \left[H_6(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \right] \right)^2 | \widehat{\mathcal{F}}_{t_k} \right] d\ell$$

$$
\begin{aligned}
&= \frac{u^2}{\sigma^4 n^2 \Delta_n^2} \sum_{k=0}^{n-1} \int_0^1 \widehat{\mathrm{E}}^0_{t_k, Y^0_{t_k}} \Bigg[\Bigg(\int_{t_k}^{t_{k+1}} \int_0^\infty z N(ds,dz) \\
&\quad - \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \Bigg[\int_{t_k}^{t_{k+1}} \int_0^\infty z M(ds,dz) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \Bigg] \Bigg)^2 \Bigg] d\ell \\
&= \frac{u^2}{\sigma^4 n^2 \Delta_n^2} \sum_{k=0}^{n-1} \int_0^1 \widehat{\mathrm{E}}^0_{t_k, Y^0_{t_k}} \Bigg[\Bigg(\int_{t_k}^{t_{k+1}} \int_0^\infty z \widetilde{N}(ds,dz) + \Delta_n \int_0^\infty z m(dz) \\
&\quad - \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \Bigg[\int_{t_k}^{t_{k+1}} \int_0^\infty z \widetilde{M}(ds,dz) + \Delta_n \int_0^\infty z m(dz) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \Bigg] \Bigg)^2 \Bigg] d\ell \\
&= \frac{u^2}{\sigma^4 n^2 \Delta_n^2} \sum_{k=0}^{n-1} \int_0^1 \widehat{\mathrm{E}}^0_{t_k, Y^0_{t_k}} \Bigg[\Bigg(\int_{t_k}^{t_{k+1}} \int_0^\infty z \widetilde{N}(ds,dz) \\
&\quad - \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \Bigg[\int_{t_k}^{t_{k+1}} \int_0^\infty z \widetilde{M}(ds,dz) \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \Bigg] \Bigg)^2 \Bigg] d\ell \\
&\leq \frac{Cu^2}{\sigma^4 n^2 \Delta_n^2} \sum_{k=0}^{n-1} \int_0^1 \Bigg\{ \widehat{\mathrm{E}}^0_{t_k, Y^0_{t_k}} \Bigg[\Bigg(\int_{t_k}^{t_{k+1}} \int_0^\infty z \widetilde{N}(ds,dz) \Bigg)^2 \Bigg] \\
&\quad + \widehat{\mathrm{E}}^0_{t_k, Y^0_{t_k}} \Bigg[\widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \Bigg[\Bigg(\int_{t_k}^{t_{k+1}} \int_0^\infty z \widetilde{M}(ds,dz) \Bigg)^2 \big| X^{b(\ell)}_{t_{k+1}} = Y^0_{t_{k+1}} \Bigg] \Bigg] \Bigg\} d\ell \\
&\leq \frac{Cu^2}{\sigma^4 n^2 \Delta_n^2} \sum_{k=0}^{n-1} \int_0^1 \Bigg\{ \Delta_n \int_0^\infty z^2 m(dz) + \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^0_{t_k}} \Bigg[\Bigg(\int_{t_k}^{t_{k+1}} \int_0^\infty z \widetilde{M}(ds,dz) \Bigg)^2 \Bigg] \Bigg\} d\ell \\
&\leq \frac{Cu^2}{\sigma^4 n^2 \Delta_n^2} \sum_{k=0}^{n-1} \int_0^1 \Bigg\{ \Delta_n \int_0^\infty z^2 m(dz) + \Delta_n \int_0^\infty z^2 m(dz) \Bigg\} d\ell \\
&= \frac{Cu^2}{\sigma^4 n \Delta_n} \int_0^\infty z^2 m(dz),
\end{aligned}
$$

which, by condition **(A2)** with $p = 2$, tends to zero as $n \to \infty$ since $n\Delta_n \to \infty$. Here, $\widetilde{N}(dt,dz) := N(dt,dz) - m(dz)dt$ and $\widetilde{M}(dt,dz) := M(dt,dz) - m(dz)dt$ denote two compensated Poisson random measures associated with $N(dt,dz)$ and $M(dt,dz)$, respectively. Thus, the desired proof is now completed. □

4.6. **Proof of Theorem 2.12.**

4.6.1. *Expansion of the log-likelihood ratio.*

Lemma 4.15. *Assume condition* **(A1)**, $b_0 \in \mathbb{R}_{--}$ *and* $2a > \sigma^2$. *Then, the log-likelihood ratio at* b_0 *can be expressed as*

$$
\begin{aligned}
\log \frac{d\mathrm{P}_n^{b_0 + e^{b_0 \frac{n\Delta_n}{2}} u}}{d\mathrm{P}_n^{b_0}} \left(Y^{n,b_0} \right) &= \sum_{k=0}^{n-1} \eta_{k,n} + \sum_{k=0}^{n-1} \frac{e^{b_0 \frac{n\Delta_n}{2}} u}{\Delta_n} \int_0^1 \Bigg\{ H_4(Y^{b_0}) + H_5(Y^{b_0}) + H_6(Y^{b_0}) \\
&\quad + \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \Big[H(X^{b(\ell)}) - H_4(X^{b(\ell)}) - H_5(X^{b(\ell)}) - H_6(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \Big] \Bigg\} d\ell,
\end{aligned}
$$

where $b(\ell) := b_0 + \ell e^{b_0 \frac{n\Delta_n}{2}} u$ *with* $\ell \in [0,1]$*, and*

$$\eta_{k,n} = -\frac{e^{b_0 \frac{n\Delta_n}{2}} u}{\sigma^2} \left(\sigma \sqrt{Y_{t_k}^{b_0}} \left(W_{t_{k+1}} - W_{t_k}\right) + \frac{e^{b_0 \frac{n\Delta_n}{2}} u \Delta_n}{2} Y_{t_k}^{b_0} \right),$$

$$H_4(Y^{b_0}) = \frac{\Delta_n}{\sigma^2} b_0 \int_{t_k}^{t_{k+1}} \left(Y_s^{b_0} - Y_{t_k}^{b_0} \right) ds, \; H_5(Y^{b_0}) = -\frac{\Delta_n}{\sigma} \int_{t_k}^{t_{k+1}} \left(\sqrt{Y_s^{b_0}} - \sqrt{Y_{t_k}^{b_0}} \right) dW_s,$$

$$H_6(Y^{b_0}) = -\frac{\Delta_n}{\sigma^2} \int_{t_k}^{t_{k+1}} \int_0^\infty z N(ds, dz).$$

Proof. We proceed as in the proof of Lemma 4.1. □

4.6.2. *Main contributions: LAMN property.*

Lemma 4.16. *Assume conditions* **(A1)-(A2)** *and* $b_0 \in \mathbb{R}_{--}$*. Let* $U(b_0)$ *and* $I(b_0)$ *be defined in Theorem 2.12. Then, for all* $u \in \mathbb{R}$*, as* $n \to \infty$*,*

$$\sum_{k=0}^{n-1} \eta_{k,n} \overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} uU(b_0) - \frac{u^2}{2} I(b_0). \tag{4.12}$$

Proof. We rewrite

$$\sum_{k=0}^{n-1} \eta_{k,n} = uU_n(b_0) - \frac{u^2}{2} I_n(b_0) + H_7(Y^{b_0}) + H_8(Y^{b_0}), \tag{4.13}$$

where $t_n = n\Delta_n$ and

$$U_n(b_0) = -\frac{1}{\sigma} e^{b_0 \frac{t_n}{2}} \int_0^{t_n} \sqrt{Y_s^{b_0}} dW_s, \; I_n(b_0) = \frac{1}{\sigma^2} e^{b_0 t_n} \int_0^{t_n} Y_s^{b_0} ds,$$

$$H_7(Y^{b_0}) = \frac{e^{b_0 \frac{n\Delta_n}{2}} u}{\sigma} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(\sqrt{Y_s^{b_0}} - \sqrt{Y_{t_k}^{b_0}} \right) dW_s,$$

$$H_8(Y^{b_0}) = \frac{e^{b_0 n\Delta_n} u^2}{2\sigma^2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(Y_s^{b_0} - Y_{t_k}^{b_0} \right) ds.$$

Moreover, using (2.11) and Lemma 3.14, we have that as $n \to \infty$,

$$\begin{aligned} (U_n(b_0), I_n(b_0)) &= \left(-\frac{1}{\sigma} e^{b_0 \frac{t_n}{2}} \int_0^{t_n} \sqrt{Y_s^{b_0}} dW_s, \frac{1}{\sigma^2} e^{b_0 t_n} \int_0^{t_n} Y_s^{b_0} ds \right) \\ &\overset{\mathcal{L}(\widehat{\mathrm{P}}^{b_0})}{\longrightarrow} \left(\sqrt{-\frac{V}{\sigma^2 b_0}} Z, -\frac{V}{\sigma^2 b_0} \right) = (U(b_0), I(b_0)), \end{aligned} \tag{4.14}$$

where Z is a standard normal random variable independent of V.

Finally, using Lemma 3.12 and 3.5 (iii), conditions **(A1)-(A2)** and the fact that $b_0 \in \mathbb{R}_{--}$ and $n\Delta_n \to \infty$ as $n \to \infty$, it can be checked that as $n \to \infty$,

$$H_7(Y^{b_0}) + H_8(Y^{b_0}) \overset{\widehat{\mathrm{P}}^{b_0}}{\longrightarrow} 0. \tag{4.15}$$

Therefore, from (4.13)-(4.15), we conclude (4.12). Thus, the result follows. □

4.6.3. *Negligible contributions.*

Lemma 4.17. *Assume conditions* **(A1)-(A3)** *and* $b_0 \in \mathbb{R}_{--}$. *Then, as* $n \to \infty$,

$$\sum_{k=0}^{n-1} \frac{e^{b_0 \frac{n\Delta_n}{2}} u}{\Delta_n} \int_0^1 \Big\{ H_4(Y^{b_0}) + H_5(Y^{b_0}) + H_6(Y^{b_0}) + \widetilde{\mathrm{E}}^{b(\ell)}_{t_k, Y^{b_0}_{t_k}} \Big[H(X^{b(\ell)}) - H_4(X^{b(\ell)}) - H_5(X^{b(\ell)}) - H_6(X^{b(\ell)}) \big| X^{b(\ell)}_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \Big] \Big\} d\ell \xrightarrow{\widehat{\mathrm{P}}^{b_0}} 0.$$

Proof. We proceed as in the proof of Lemma 4.3 and 4.10. □

Consequently, from Lemma 4.15, 4.16 and 4.17, the proof of Theorem 2.12 is now completed.

5. Appendix

5.1. **Proof of Proposition 3.1.**

Proof. First we recall that since the jump-type CIR process $X^b = (X^b_t)_{t\in\mathbb{R}_+}$ is an affine process, the corresponding characteristic function of $X^b_t = X^b_t(0, y_0)$ is of exponential-affine form (see page 287 and 288 of [25], Section 3 of [26] or Section 4.1 of [17]). That is, for all $(t, u) \in \mathbb{R}_+ \times \mathcal{U}$ with $\mathcal{U} := \{u \in \mathbb{C} : \mathrm{Re}\, u \leq 0\}$,

$$\widetilde{\mathrm{E}}\left[e^{uX^b_t(0,y_0)}\right] = e^{\phi_b(t,u)+y_0\psi_b(t,u)},$$

where Re u denotes the real part of u and the functions $\phi_b(t, u)$ and $\psi_b(t, u)$ are solutions to the generalized Riccati equations

$$\begin{cases} \partial_t \phi_b(t,u) = F(\psi_b(t,u)),\ \phi_b(0,u) = 0, \\ \partial_t \psi_b(t,u) = R(\psi_b(t,u)),\ \psi_b(0,u) = u \in \mathcal{U}, \end{cases}$$

with the functions F and R given by

$$F(u) = au + \int_0^\infty (e^{uz} - 1) m(dz), \qquad R(u) = \frac{\sigma^2 u^2}{2} - bu.$$

Solving the system above, we get the following explicit form

$$\psi_b(t,u) = \frac{ue^{-bt}}{1 - \frac{\sigma^2 u}{2b}(1 - e^{-bt})}.$$

In what follows, the notation constant C will designate a generic constant which can change values from one bound to another. Now, using Lemma C.6 of [17], there exist constants $C > 0$ and $\overline{R} > 0$ such that

$$\left| e^{\phi_b(t,iu)+y_0\psi_b(t,iu)} \right| \leq \frac{C}{(1+|u|)^{\frac{2a}{\sigma^2}}}, \tag{5.1}$$

for all $u \in \mathbb{R}$ with $|u| \geq \overline{R}$. Hence, as $\frac{2a}{\sigma^2} > 1$ we get

$$\int_{\mathbb{R}} \left| \widetilde{\mathrm{E}}\left[e^{iuX^b_t(0,y_0)}\right] \right| du = \int_{|u|\leq \overline{R}} \left| \widetilde{\mathrm{E}}\left[e^{iuX^b_t(0,y_0)}\right] \right| du + \int_{|u|\geq \overline{R}} \left| e^{\phi_b(t,iu)+y_0\psi_b(t,iu)} \right| du < +\infty.$$

Therefore, by the inversion Fourier theorem, we obtain the existence of the density

$$p^b(t,y_0,y)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iyu}\widetilde{\mathrm{E}}\left[e^{iuX_t^b(0,y_0)}\right]du=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iyu}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}du.$$

Next, in order to prove the smoothness of the density w.r.t. b, we are going to show that

$$\int_{\mathbb{R}}\left|e^{-iyu}\frac{\partial}{\partial b}\widetilde{\mathrm{E}}\left[e^{iuX_t^b(0,y_0)}\right]\right|du=\int_{\mathbb{R}}\left|e^{-iyu}\frac{\partial}{\partial b}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}\right|du<+\infty,\ \forall b\in\mathbb{R}. \tag{5.2}$$

First, observe that

$$\left|e^{-iyu}\frac{\partial}{\partial b}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}\right|=\left|\frac{\partial}{\partial b}\phi_b(t,iu)+y_0\frac{\partial}{\partial b}\psi_b(t,iu)\right|\left|e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}\right|, \tag{5.3}$$

and

$$\frac{\partial}{\partial b}\psi_b(t,iu)=\frac{-iute^{-bt}\left(1-\frac{\sigma^2 iu}{2b}(1-e^{-bt})\right)-iue^{-bt}\left(\frac{\sigma^2 iu}{2b^2}(1-e^{-bt})-\frac{\sigma^2 iu}{2b}te^{-bt}\right)}{\left(1-\frac{\sigma^2 iu}{2b}(1-e^{-bt})\right)^2},$$

which is continuous w.r.t. b for all $b\neq 0$ and we can easily see that it is continuous for all $b\in\mathbb{R}$. Therefore, by standard calculations there exists $R'>0$ such that for all $|u|\geq R'$,

$$\left|\frac{\partial}{\partial b}\psi_b(t,iu)\right|\leq\frac{C|u|^2}{1+\frac{\sigma^4u^2}{4b^2}(1-e^{-bt})^2}\leq C. \tag{5.4}$$

Furthermore,

$$\phi_b(t,iu)=a\int_0^t\psi_b(s,iu)ds+\int_0^t\int_0^\infty(e^{\psi_b(s,iu)z}-1)m(dz)ds.$$

Now, for all $|u|\geq R'$, we have $|\frac{\partial}{\partial b}\psi_b(t,iu)|\leq C$ and

$$\left|z\frac{\partial}{\partial b}\psi_b(s,iu)e^{\psi_b(s,iu)z}\right|\leq Cze^{z\mathrm{Re}\ \psi_b(s,iu)}\leq Cz,$$

since

$$\mathrm{Re}\ \psi_b(s,iu)=-\frac{\frac{\sigma^2u^2}{2b}e^{-bt}(1-e^{-bt})}{1+\frac{\sigma^4u^2}{4b^2}(1-e^{-bt})^2}\leq 0.$$

Then, for all $|u|\geq R'$, we have $\int_0^t|\frac{\partial}{\partial b}\psi_b(s,iu)|ds\leq Ct$ and using condition **(A1)**,

$$\int_0^t\int_0^\infty\left|z\frac{\partial}{\partial b}\psi_b(s,iu)e^{\psi_b(s,iu)z}\right|m(dz)ds\leq C\int_0^t\int_0^\infty zm(dz)ds=Ct\int_0^\infty zm(dz)<+\infty.$$

Thus, we have shown that for all $|u|\geq R'$,

$$\frac{\partial}{\partial b}\phi_b(t,iu)=a\int_0^t\frac{\partial}{\partial b}\psi_b(s,iu)ds+\int_0^t\int_0^\infty z\frac{\partial}{\partial b}\psi_b(s,iu)e^{\psi_b(s,iu)z}m(dz)ds,$$

which is also continuous w.r.t. b for all $b\in\mathbb{R}$ and then

$$\left|\frac{\partial}{\partial b}\phi_b(t,iu)\right|\leq Ct. \tag{5.5}$$

Hence, from (5.3), (5.1), (5.4) and (5.5), for all $|u|\geq\overline{R}\vee R'$,

$$\left|e^{-iyu}\frac{\partial}{\partial b}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}\right|\leq\frac{C}{(1+|u|)^{\frac{2a}{\sigma^2}}},$$

which implies that

$$\int_{|u|\geq \overline{R}\vee R'}\left|e^{-iyu}\frac{\partial}{\partial b}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}\right|du\leq \int_{|u|\geq \overline{R}\vee R'}\frac{C}{(1+|u|)^{\frac{2a}{\sigma^2}}}du<+\infty. \tag{5.6}$$

On the other hand, for all $|u|\leq \overline{R}\vee R'$, using (3.16), we get

$$\begin{aligned}\left|e^{-iyu}\frac{\partial}{\partial b}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}\right|&=\left|\frac{\partial}{\partial b}\widetilde{\mathrm{E}}\left[e^{iuX_t^b(0,y_0)}\right]\right|=\left|\widetilde{\mathrm{E}}\left[iu\frac{\partial}{\partial b}X_t^b(0,y_0)e^{iuX_t^b(0,y_0)}\right]\right|\\&\leq |u|\widetilde{\mathrm{E}}\left[\left|\partial_b X_t^b(0,y_0)\right|\right]\leq C\overline{R}\vee R'(1+y_0)\left(1+\frac{1}{y_0^{\frac{a}{\sigma^2}}}\right)\leq C.\end{aligned}$$

This implies that

$$\int_{|u|\leq \overline{R}\vee R'}\left|e^{-iyu}\frac{\partial}{\partial b}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}\right|du\leq C\int_{|u|\leq \overline{R}\vee R'}du<+\infty,\ \forall b\in\mathbb{R}. \tag{5.7}$$

From (5.6) and (5.7), we conclude (5.2), which implies that the $p^b(t,y_0,y)$ is of class C^1 w.r.t. b for all $b\in\mathbb{R}$ and its derivative is given by

$$\begin{aligned}\partial_b p^b(t,y_0,y)&=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iyu}\frac{\partial}{\partial b}e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}du\\&=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iyu}\left(\frac{\partial}{\partial b}\phi_b(t,iu)+y_0\frac{\partial}{\partial b}\psi_b(t,iu)\right)e^{\phi_b(t,iu)+y_0\psi_b(t,iu)}du.\end{aligned}$$

Thus, the result follows. □

5.2. **Proof of Proposition 3.2.**

Proof. Let f be a continuously differentiable function with compact support. The chain rule of the Malliavin calculus implies that $f'(X_{t_{k+1}}^b(t_k,x))=D_t(f(X_{t_{k+1}}^b(t_k,x)))U_t^b(t_k,x)$, for all $b\in\mathbb{R}$ and $t\in[t_k,t_{k+1}]$, where $U_t^b(t_k,x):=(D_tX_{t_{k+1}}^b(t_k,x))^{-1}$. Then, using the Malliavin calculus integration by parts formula on $[t_k,t_{k+1}]$, we get that

$$\begin{aligned}\partial_b\widetilde{\mathrm{E}}\left[f(X_{t_{k+1}}^b(t_k,x))\right]&=\widetilde{\mathrm{E}}\left[f'(X_{t_{k+1}}^b(t_k,x))\partial_bX_{t_{k+1}}^b(t_k,x)\right]\\&=\frac{1}{\Delta_n}\widetilde{\mathrm{E}}\left[\int_{t_k}^{t_{k+1}}f'(X_{t_{k+1}}^b(t_k,x))\partial_bX_{t_{k+1}}^b(t_k,x)dt\right]\\&=\frac{1}{\Delta_n}\widetilde{\mathrm{E}}\left[\int_{t_k}^{t_{k+1}}D_t(f(X_{t_{k+1}}^b(t_k,x)))U_t^b(t_k,x)\partial_bX_{t_{k+1}}^b(t_k,x)dt\right]\\&=\frac{1}{\Delta_n}\widetilde{\mathrm{E}}\left[f(X_{t_{k+1}}^b(t_k,x))\delta\left(\partial_bX_{t_{k+1}}^b(t_k,x)U^b(t_k,x)\right)\right].\end{aligned}$$

Observe that by (3.16), the family $(f'(X_{t_{k+1}}^b(t_k,x))\,\partial_bX_{t_{k+1}}^b(t_k,x),b\in\mathbb{R})$ is uniformly integrable. This justifies that we can interchange ∂_b and $\widetilde{\mathrm{E}}$. Note that $\delta(V)\equiv\delta(V\mathbf{1}_{[t_k,t_{k+1}]}(\cdot))$ for any $V\in\mathrm{Dom}\ \delta$.

Using the fact that $p^b(\Delta_n,x,y)$ and $\partial_bp^b(\Delta_n,x,y)$ are continuous w.r.t. (y,b), and that f is the continuously differentiable function with compact support, the stochastic flow property and the Markov property, we obtain that

$$\partial_b\widetilde{\mathrm{E}}\left[f(X^b_{t_{k+1}}(t_k,x))\right]=\int_0^\infty f(y)\partial_b p^b(\Delta_n,x,y)dy,$$

and

$$\begin{aligned}&\widetilde{\mathrm{E}}\left[f(X^b_{t_{k+1}}(t_k,x))\delta\left(\partial_b X^b_{t_{k+1}}(t_k,x)U^b(t_k,x)\right)\right]\\&=\widetilde{\mathrm{E}}\left[f(X^b_{t_{k+1}})\delta\left(\partial_b X^b_{t_{k+1}}(t_k,x)U^b(t_k,x)\right)|X^b_{t_k}=x\right]\\&=\int_0^\infty f(y)\widetilde{\mathrm{E}}\left[\delta\left(\partial_b X^b_{t_{k+1}}(t_k,x)U^b(t_k,x)\right)|X^b_{t_{k+1}}=y,X^b_{t_k}=x\right]p^b(\Delta_n,x,y)dy,\end{aligned}$$

which gives the desired result. □

5.3. **Proof of Lemma 3.3.**

Proof. From (3.5) and Itô's formula, for any $t\in[t_k,t_{k+1}]$,

$$\begin{aligned}\frac{1}{\partial_x X^b_t(t_k,x)}=1+\int_{t_k}^t\left(\frac{b}{\partial_x X^b_s(t_k,x)}+\frac{\sigma^2}{2X^b_s(t_k,x)\partial_x X^b_s(t_k,x)}\right)ds\\-\int_{t_k}^t\frac{\sigma}{2\sqrt{X^b_s(t_k,x)}\partial_x X^b_s(t_k,x)}dB_s,\end{aligned}\tag{5.8}$$

which, together with (3.6), and Itô's formula, for any $t\in[t_k,t_{k+1}]$,

$$\frac{\partial_b X^b_t(t_k,x)}{\partial_x X^b_t(t_k,x)}=\int_{t_k}^t\left(\frac{\sigma^2}{4}\frac{\partial_b X^b_s(t_k,x)}{X^b_s(t_k,x)\partial_x X^b_s(t_k,x)}-\frac{X^b_s(t_k,x)}{\partial_x X^b_s(t_k,x)}\right)ds.$$

Then, using $U^b_t(t_k,x)=(D_tX^b_{t_{k+1}}(t_k,x))^{-1}=\frac{1}{\sigma\sqrt{X^b_t(t_k,x)}}(\partial_x X^b_{t_{k+1}}(t_k,x))^{-1}\partial_x X^b_t(t_k,x)$, the product rule [37, (1.48)], and the fact that the Skorohod integral and the Itô integral of an adapted process coincide, we obtain that

$$\begin{aligned}\delta\left(\partial_b X^b_{t_{k+1}}(t_k,x)U^b(t_k,x)\right)&=\frac{\partial_b X^b_{t_{k+1}}(t_k,x)}{\partial_x X^b_{t_{k+1}}(t_k,x)}\int_{t_k}^{t_{k+1}}\frac{\partial_x X^b_s(t_k,x)}{\sigma\sqrt{X^b_s(t_k,x)}}dB_s\\&\quad-\int_{t_k}^{t_{k+1}}D_s\left(\frac{\partial_b X^b_{t_{k+1}}(t_k,x)}{\partial_x X^b_{t_{k+1}}(t_k,x)}\right)\frac{\partial_x X^b_s(t_k,x)}{\sigma\sqrt{X^b_s(t_k,x)}}ds\\&=\int_{t_k}^{t_{k+1}}\left(\frac{\sigma^2}{4}\frac{\partial_b X^b_s(t_k,x)}{X^b_s(t_k,x)\partial_x X^b_s(t_k,x)}-\frac{X^b_s(t_k,x)}{\partial_x X^b_s(t_k,x)}\right)ds\int_{t_k}^{t_{k+1}}\frac{\partial_x X^b_s(t_k,x)}{\sigma\sqrt{X^b_s(t_k,x)}}dB_s\\&\quad-\int_{t_k}^{t_{k+1}}D_s\left(\frac{\partial_b X^b_{t_{k+1}}(t_k,x)}{\partial_x X^b_{t_{k+1}}(t_k,x)}\right)\frac{\partial_x X^b_s(t_k,x)}{\sigma\sqrt{X^b_s(t_k,x)}}ds.\end{aligned}$$

We next add and subtract the term $\frac{X^b_{t_k}(t_k,x)}{\partial_x X^b_{t_k}(t_k,x)}$ in the first integral, and the term $\frac{\partial_x X^b_{t_k}(t_k,x)}{\sigma\sqrt{X^b_{t_k}(t_k,x)}}$ in the second integral. This, together with $X^b_{t_k}(t_k,x)=x$, yields

$$\delta\left(\partial_b X^b_{t_{k+1}}(t_k,x)U^b(t_k,x)\right)=-\frac{\Delta_n}{\sigma}\sqrt{x}\left(B_{t_{k+1}}-B_{t_k}\right)+H(X^b).\tag{5.9}$$

On the other hand, equation (3.3) gives

$$B_{t_{k+1}} - B_{t_k} = \frac{1}{\sigma\sqrt{X^b_{t_k}}}\Bigg(X^b_{t_{k+1}} - X^b_{t_k} - (a - bX^b_{t_k})\Delta_n + b\int_{t_k}^{t_{k+1}}\left(X^b_s - X^b_{t_k}\right)ds$$
$$- \sigma\int_{t_k}^{t_{k+1}}\left(\sqrt{X^b_s} - \sqrt{X^b_{t_k}}\right)dB_s - \int_{t_k}^{t_{k+1}}\int_0^\infty zM(ds,dz)\Bigg), \qquad (5.10)$$

which, together with (5.9), gives the desired result. □

5.4. **Proof of Lemma 3.5.**

Proof. (i) From [2, Lemma 2.2] or [12, Lemma A.1], we have that for any $T \in \mathbb{R}_{++}$ and $p \in [0, \frac{2a}{\sigma^2} - 1)$,

$$\widehat{\mathrm{E}}^b\left[\sup_{t\in[0,T]}\left(\overline{Y}^b_t\right)^{-p}\right] < \infty,$$

where $(\overline{Y}^b_t)_{t\in\mathbb{R}_+}$ is the diffusion-type CIR process defined by (3.23). This, together with the fact that by Lemma 3.15, $\widehat{\mathrm{P}}(Y^b_t \geq \overline{Y}^b_t$ for all $t \in \mathbb{R}_+) = 1$, implies that for any $T \in \mathbb{R}_{++}$ and $p \in [0, \frac{2a}{\sigma^2} - 1)$,

$$\widehat{\mathrm{E}}^b\left[\sup_{t\in[0,T]}\left(Y^b_t\right)^{-p}\right] \leq \widehat{\mathrm{E}}^b\left[\sup_{t\in[0,T]}\left(\overline{Y}^b_t\right)^{-p}\right] < \infty.$$

(ii) This result can be easily obtained from equation (1.3) applying Burkholder-David-Gundy's inequality and Gronwall's inequality, together with conditions **(A1)**-**(A2)**.

(iii) From (1.3), we have that for any $t > s \geq 0$,

$$Y^b_t - Y^b_s = \int_s^t\left(a - bY^b_u\right)du + \sigma\int_s^t\sqrt{Y^b_u}dW_u + \int_s^t\int_0^\infty zN(du,dz).$$

Then, applying Burkholder-David-Gundy's and Hölder's inequalities, and Lemma 3.5 (ii), together with conditions **(A1)**-**(A2)**,

$$\widehat{\mathrm{E}}^b\left[\left|Y^b_t - Y^b_s\right|^p\right] \leq 3^{p-1}\Bigg(\widehat{\mathrm{E}}^b\left[\left|\int_s^t\left(a - bY^b_u\right)du\right|^p\right] + \widehat{\mathrm{E}}^b\left[\left|\sigma\int_s^t\sqrt{Y^b_u}dW_u\right|^p\right]$$
$$+ \widehat{\mathrm{E}}^b\left[\left|\int_s^t\int_0^\infty zN(du,dz)\right|^p\right]\Bigg)$$
$$\leq 3^{p-1}\Bigg((t-s)^{p-1}2^{p-1}\int_s^t\left(a^p + |b|^p\widehat{\mathrm{E}}^b\left[|Y^b_u|^p\right]\right)du + C\sigma^p(t-s)^{\frac{p}{2}-1}\int_s^t\widehat{\mathrm{E}}^b\left[|Y^b_u|^{\frac{p}{2}}\right]du$$
$$+ 2^{p-1}\left(\widehat{\mathrm{E}}^b\left[\left|\int_s^t\int_0^\infty z\widetilde{N}(du,dz)\right|^p\right] + \left(\int_s^t\int_0^\infty zm(dz)du\right)^p\right)\Bigg)$$
$$\leq C\Bigg((t-s)^p(1+y_0^p) + (t-s)^{\frac{p}{2}}(1+y_0^{\frac{p}{2}}) + (t-s)\int_0^\infty z^p m(dz) + (t-s)^p\left(\int_0^\infty zm(dz)\right)^p\Bigg)$$
$$\leq C\,(t-s)^{\frac{p}{2}\wedge 1}\left(1 + y_0^p + y_0^{\frac{p}{2}}\right),$$

for a positive constant C. Thus, the result follows. □

5.5. **Proof of Lemma 3.6.**

Proof. We consider on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathrm{P}})$ the diffusion-type CIR process $(\overline{X}_t^b)_{t\in\mathbb{R}_+}$ defined by

$$\overline{X}_t^b = y_0 + \int_0^t \left(a - b\overline{X}_s^b\right) ds + \sigma \int_0^t \sqrt{\overline{X}_s^b} dB_s.$$

We next consider the flow $\overline{X}^b(s,x) = (\overline{X}_t^b(s,x), t \geq s)$, $x \in \mathbb{R}_{++}$ on the time interval $[s, \infty)$ and with initial condition $\overline{X}_s^b(s,x) = x$. That is, for any $t \geq s$,

$$\overline{X}_t^b(s,x) = x + \int_s^t \left(a - b\overline{X}_u^b(s,x)\right) du + \sigma \int_s^t \sqrt{\overline{X}_u^b(s,x)} dB_u.$$

Therefore, applying [12, Lemma 3.1], for any $\frac{a}{\sigma^2} > 1$, $\mu \leq (\frac{2a}{\sigma^2} - 1)^2 \frac{\sigma^2}{8}$ and $t \in [t_k, t_{k+1}]$, we have that

$$\widetilde{\mathrm{E}}_{t_k,x}^b \left[\exp\left\{\mu \int_{t_k}^t \frac{du}{\overline{X}_u^b(t_k,x)}\right\}\right] \leq C \left(1 + \frac{1}{x^{\frac{1}{2}(\frac{2a}{\sigma^2}-1)}}\right).$$

The comparison theorem (see [4, Proposition A.1] or Lemma 3.15) gives

$$\widetilde{\mathrm{P}}\left(X_t^b \geq \overline{X}_t^b, \ \forall t \in \mathbb{R}_+\right) = 1,$$

which implies that

$$\widetilde{\mathrm{P}}\left(X_t^b(t_k,x) \geq \overline{X}_t^b(t_k,x), \ \forall t \geq t_k\right) = 1.$$

Therefore, for any $\frac{a}{\sigma^2} > 1$, $\mu \leq (\frac{2a}{\sigma^2} - 1)^2 \frac{\sigma^2}{8}$ and $t \in [t_k, t_{k+1}]$,

$$\widetilde{\mathrm{E}}_{t_k,x}^b \left[\exp\left\{\mu \int_{t_k}^t \frac{du}{X_u^b(t_k,x)}\right\}\right] \leq \widetilde{\mathrm{E}}_{t_k,x}^b \left[\exp\left\{\mu \int_{t_k}^t \frac{du}{\overline{X}_u^b(t_k,x)}\right\}\right] \leq C \left(1 + \frac{1}{x^{\frac{1}{2}(\frac{2a}{\sigma^2}-1)}}\right).$$

Thus, the result follows. □

5.6. **Proof of Lemma 3.7.**

Proof. Proof of (3.13). This result can be easily obtained from equation (3.4) applying Burkholder-David-Gundy's inequality and Gronwall's inequality together with conditions **(A1)-(A2)**.

Proof of (3.14). From (3.20) of [9, Lemma 3.6], we have that for any $p \in [0, \frac{2a}{\sigma^2} - 1)$,

$$\widetilde{\mathrm{E}}_{t_k,x}^b \left[\frac{1}{\left|\overline{X}_t^b(t_k,x)\right|^p}\right] \leq \frac{C_p}{x^p},$$

for a constant $C_p > 0$. Then, using $\widetilde{\mathrm{P}}\left(X_t^b(t_k,x) \geq \overline{X}_t^b(t_k,x), \ \forall t \geq t_k\right) = 1$, we get

$$\widetilde{\mathrm{E}}_{t_k,x}^b \left[\frac{1}{\left|X_t^b(t_k,x)\right|^p}\right] \leq \widetilde{\mathrm{E}}_{t_k,x}^b \left[\frac{1}{\left|\overline{X}_t^b(t_k,x)\right|^p}\right] \leq \frac{C_p}{x^p}.$$

Proof of (3.15) *and* (3.16). We proceed as in the proof of [9, Lemma 3.6] by applying Lemma 3.6. □

5.7. **Proof of Lemma 3.8.**

Proof. Proof of (3.17). This fact follows easily from (5.9), and properties of the moment of the Skorohod integral and the Brownian motion.

Proof of (3.18). We rewrite $H(X^b) = H_1(X^b) + H_2(X^b) + H_3(X^b)$, where

$$H_1(X^b) = -\Delta_n \frac{x}{\sigma} \int_{t_k}^{t_{k+1}} \left(\frac{\partial_x X_s^b(t_k, x)}{\sqrt{X_s^b(t_k, x)}} - \frac{\partial_x X_{t_k}^b(t_k, x)}{\sqrt{X_{t_k}^b(t_k, x)}} \right) dB_s,$$

$$H_2(X^b) = \int_{t_k}^{t_{k+1}} \left(\frac{\sigma^2}{4} \frac{\partial_b X_s^b(t_k, x)}{X_s^b(t_k, x)\partial_x X_s^b(t_k, x)} - \left(\frac{X_s^b(t_k, x)}{\partial_x X_s^b(t_k, x)} - \frac{X_{t_k}^b(t_k, x)}{\partial_x X_{t_k}^b(t_k, x)} \right) \right) ds$$
$$\times \int_{t_k}^{t_{k+1}} \frac{\partial_x X_s^b(t_k, x)}{\sigma\sqrt{X_s^b(t_k, x)}} dB_s,$$

$$H_3(X^b) = -\int_{t_k}^{t_{k+1}} D_s \left(\frac{\partial_b X_{t_{k+1}}^b(t_k, x)}{\partial_x X_{t_{k+1}}^b(t_k, x)} \right) \frac{\partial_x X_s^b(t_k, x)}{\sigma\sqrt{X_s^b(t_k, x)}} ds.$$

Observe that

$$\widetilde{\mathrm{E}}_{t_k,x}^b \left[\left(H(X^b) \right)^2 \right] = \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left(H_1(X^b) + H_2(X^b) + H_3(X^b) \right)^2 \right]$$
$$\leq 3 \Big(\widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| H_1(X^b) \right|^2 \right] + \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| H_2(X^b) \right|^2 \right] + \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| H_3(X^b) \right|^2 \right] \Big). \tag{5.11}$$

First, we treat the term $H_1(X^b)$. Using Burkholder-David-Gundy's inequality, we have that

$$\widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| H_1(X^b) \right|^2 \right] \leq \Delta_n^2 \frac{x^2}{\sigma^2} \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| \int_{t_k}^{t_{k+1}} \left(\frac{\partial_x X_s^b(t_k, x)}{\sqrt{X_s^b(t_k, x)}} - \frac{\partial_x X_{t_k}^b(t_k, x)}{\sqrt{X_{t_k}^b(t_k, x)}} \right) dB_s \right|^2 \right]$$
$$\leq C\Delta_n^2 \frac{x^2}{\sigma^2} \widetilde{\mathrm{E}}_{t_k,x}^b \left[\int_{t_k}^{t_{k+1}} \left(\frac{\partial_x X_s^b(t_k, x)}{\sqrt{X_s^b(t_k, x)}} - \frac{\partial_x X_{t_k}^b(t_k, x)}{\sqrt{X_{t_k}^b(t_k, x)}} \right)^2 ds \right] \leq C\Delta_n^2 x^2 \int_{t_k}^{t_{k+1}} H_{11}^b ds,$$

where

$$H_{11}^b = \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left(\frac{\partial_x X_s^b(t_k, x)}{\sqrt{X_s^b(t_k, x)}} - \frac{\partial_x X_{t_k}^b(t_k, x)}{\sqrt{X_{t_k}^b(t_k, x)}} \right)^2 \right].$$

By Itô's formula, it can be checked that

$$\frac{\partial_x X_s^b(t_k, x)}{\sqrt{X_s^b(t_k, x)}} - \frac{\partial_x X_{t_k}^b(t_k, x)}{\sqrt{X_{t_k}^b(t_k, x)}} = \int_{t_k}^{s} \partial_x X_u^b(t_k, x) \left(\frac{-\frac{a}{2} + \frac{\sigma^2}{8}}{(X_u^b(t_k, x))^{\frac{3}{2}}} - \frac{b}{2\sqrt{X_u^b(t_k, x)}} \right) du$$
$$+ \int_{t_k}^{s} \int_0^{\infty} \partial_x X_u^b(t_k, x) \left(\frac{1}{\sqrt{X_u^b(t_k, x) + z}} - \frac{1}{\sqrt{X_u^b(t_k, x)}} \right) M(du, dz),$$

which, together with Burkholder-David-Gundy's and Hölder's inequalities with $\frac{1}{p_0}+\frac{1}{q_0}=1$, (3.14) and (3.15), and condition **(A2)**, implies that

$$
\begin{aligned}
H_{11}^b &\leq C\Delta_n \int_{t_k}^s \left\{ \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| \frac{\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^{\frac{3}{2}}} \right|^2 \right] + \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| \frac{\partial_x X_u^b(t_k,x)}{\sqrt{X_u^b(t_k,x)}} \right|^2 \right] \right\} du \\
&\quad + C\int_{t_k}^s \int_0^\infty \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| \partial_x X_u^b(t_k,x) \left(\frac{1}{\sqrt{X_u^b(t_k,x)+z}} - \frac{1}{\sqrt{X_u^b(t_k,x)}} \right) \right|^2 \right] m(dz)du \\
&\leq C\Delta_n \int_{t_k}^s \left\{ \left(\widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| \partial_x X_u^b(t_k,x) \right|^{2p_0} \right] \right)^{\frac{1}{p_0}} \left(\widetilde{\mathrm{E}}_{t_k,x}^b \left[\frac{1}{|X_u^b(t_k,x)|^{3q_0}} \right] \right)^{\frac{1}{q_0}} \right. \\
&\quad \left. + \left(\widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| \partial_x X_u^b(t_k,x) \right|^{2p_0} \right] \right)^{\frac{1}{p_0}} \left(\widetilde{\mathrm{E}}_{t_k,x}^b \left[\frac{1}{|X_u^b(t_k,x)|^{q_0}} \right] \right)^{\frac{1}{q_0}} \right\} du \\
&\quad + C\int_{t_k}^s \int_0^\infty \widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| \frac{\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^{\frac{3}{2}}} \right|^2 \right] z^2 m(dz)du \\
&\leq C\Delta_n \int_{t_k}^s \left\{ \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1+2p_0}{2}}} \right)^{\frac{1}{p_0}} \left(\frac{1}{x^{3q_0}} \right)^{\frac{1}{q_0}} + \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1+2p_0}{2}}} \right)^{\frac{1}{p_0}} \left(\frac{1}{x^{q_0}} \right)^{\frac{1}{q_0}} \right\} du \\
&\quad + C\int_{t_k}^s \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1+2p_0}{2}}} \right)^{\frac{1}{p_0}} \left(\frac{1}{x^{3q_0}} \right)^{\frac{1}{q_0}} du \\
&\leq C\Delta_n \left\{ \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_0}+1}} \right) \frac{1}{x^3} + \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_0}+1}} \right) \frac{1}{x} \right\},
\end{aligned}
$$

where q_0 should be chosen close to 1 in order that $3q_0 < \frac{2a}{\sigma^2}-1$.

Therefore, under condition $\frac{a}{\sigma^2} > 2$, we have shown that

$$
\widetilde{\mathrm{E}}_{t_k,x}^b \left[\left| H_1(X^b) \right|^2 \right] \leq C\Delta_n^4 \left\{ \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_0}+1}} \right) \frac{1}{x} + \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_0}+1}} \right) x \right\}, \tag{5.12}
$$

where $p_0 > 1$ with $\frac{p_0}{p_0-1}$ close to 1.

Next, we treat the term $H_2(X^b)$. From (3.9),

$$
\frac{\partial_b X_s^b(t_k,x)}{\partial_x X_s^b(t_k,x)} = -\int_{t_k}^s \frac{X_r^b(t_k,x)}{\partial_x X_r^b(t_k,x)} dr.
$$

Therefore,

$$
H_2(X^b) = \int_{t_k}^{t_{k+1}} \left(\frac{-\sigma^2}{4X_s^b(t_k,x)} \int_{t_k}^s \frac{X_r^b(t_k,x)}{\partial_x X_r^b(t_k,x)} dr - \left(\frac{X_s^b(t_k,x)}{\partial_x X_s^b(t_k,x)} - \frac{X_{t_k}^b(t_k,x)}{\partial_x X_{t_k}^b(t_k,x)} \right) \right) ds
$$

$$\times \int_{t_k}^{t_{k+1}} \frac{\partial_x X_s^b(t_k,x)}{\sigma\sqrt{X_s^b(t_k,x)}} dB_s,$$

which, together with Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$, implies that

$$\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|H_2(X^b)\right|^2\right] \leq \left(H_{21}^b\right)^{\frac{1}{p}}\left(H_{22}^b\right)^{\frac{1}{q}},$$

where

$$H_{21}^b = \widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\int_{t_k}^{t_{k+1}}\left(\frac{-\sigma^2}{4X_s^b(t_k,x)}\int_{t_k}^s \frac{X_r^b(t_k,x)}{\partial_x X_r^b(t_k,x)}dr - \left(\frac{X_s^b(t_k,x)}{\partial_x X_s^b(t_k,x)} - \frac{X_{t_k}^b(t_k,x)}{\partial_x X_{t_k}^b(t_k,x)}\right)\right)ds\right|^{2p}\right],$$

$$H_{22}^b = \widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\int_{t_k}^{t_{k+1}} \frac{\partial_x X_s^b(t_k,x)}{\sigma\sqrt{X_s^b(t_k,x)}} dB_s\right|^{2q}\right].$$

First, observe that

$$H_{21}^b \leq C\Delta_n^{2p-1}\int_{t_k}^{t_{k+1}}\left(H_{211}^b + H_{212}^b\right)ds,$$

where

$$H_{211}^b = \widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{\sigma^2}{4X_s^b(t_k,x)}\int_{t_k}^s \frac{X_r^b(t_k,x)}{\partial_x X_r^b(t_k,x)}dr\right|^{2p}\right], H_{212}^b = \widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{X_s^b(t_k,x)}{\partial_x X_s^b(t_k,x)} - \frac{X_{t_k}^b(t_k,x)}{\partial_x X_{t_k}^b(t_k,x)}\right|^{2p}\right].$$

We treat H_{211}^b. Using Hölder's inequality with $\frac{1}{\alpha_1}+\frac{1}{\beta_1}=1$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$, (3.14) and (3.15),

$$\begin{aligned}
H_{211}^b &\leq C\Delta_n^{2p-1}\int_{t_k}^s \widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{1}{X_s^b(t_k,x)}\frac{X_r^b(t_k,x)}{\partial_x X_r^b(t_k,x)}\right|^{2p}\right]dr \\
&\leq C\Delta_n^{2p-1}\int_{t_k}^s \left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\frac{1}{|X_s^b(t_k,x)\partial_x X_r^b(t_k,x)|^{2p\alpha_1}}\right]\right)^{\frac{1}{\alpha_1}}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|X_r^b(t_k,x)\right|^{2p\beta_1}\right]\right)^{\frac{1}{\beta_1}}dr \\
&\leq C\Delta_n^{2p-1}\int_{t_k}^s \left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\frac{1}{|X_s^b(t_k,x)|^{2p_1p\alpha_1}}\right]\right)^{\frac{1}{p_1\alpha_1}}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\frac{1}{|\partial_x X_r^b(t_k,x)|^{2q_1p\alpha_1}}\right]\right)^{\frac{1}{q_1\alpha_1}} \\
&\qquad\times\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|X_r^b(t_k,x)\right|^{2p\beta_1}\right]\right)^{\frac{1}{\beta_1}}dr \\
&\leq C\Delta_n^{2p-1}\Delta_n\left(\frac{1}{x^{2p_1p\alpha_1}}\right)^{\frac{1}{p_1\alpha_1}}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1-2q_1p\alpha_1}{2}}}\right)^{\frac{1}{q_1\alpha_1}}\left(1+x^{2p\beta_1}\right)^{\frac{1}{\beta_1}} \\
&\leq C\Delta_n^{2p}\frac{1}{x^{2p}}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1\alpha_1}-p}}\right)\left(1+x^{2p}\right),
\end{aligned}$$

where $\alpha_1>1$ should be chosen close to 1 in order that $2p_1p\alpha_1 < \frac{2a}{\sigma^2}-1$.

Next, we treat H^b_{212}. Using (3.4), (5.8) and Itô's formula, we get that

$$\frac{X^b_s(t_k,x)}{\partial_x X^b_s(t_k,x)} - \frac{X^b_{t_k}(t_k,x)}{\partial_x X^b_{t_k}(t_k,x)} = a\int_{t_k}^s \frac{du}{\partial_x X^b_u(t_k,x)} + \frac{\sigma}{2}\int_{t_k}^s \frac{\sqrt{X^b_u(t_k,x)}}{\partial_x X^b_u(t_k,x)}dB_u + \int_{t_k}^s\int_0^\infty \frac{z}{\partial_x X^b_u(t_k,x)}M(du,dz).$$

Therefore,

$$H^b_{212} \le C\left(H^b_{2121} + H^b_{2122} + H^b_{2123}\right),$$

where

$$H^b_{2121} = \widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\int_{t_k}^s \frac{du}{\partial_x X^b_u(t_k,x)}\right|^{2p}\right],\ H^b_{2122} = \widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\int_{t_k}^s \frac{\sqrt{X^b_u(t_k,x)}}{\partial_x X^b_u(t_k,x)}dB_u\right|^{2p}\right],$$
$$H^b_{2123} = \widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\int_{t_k}^s\int_0^\infty \frac{z}{\partial_x X^b_u(t_k,x)}M(du,dz)\right|^{2p}\right].$$

Using (3.15),

$$H^b_{2121} \le \Delta_n^{2p-1}\int_{t_k}^s \widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|\partial_x X^b_u(t_k,x)|^{2p}}\right]du \le C\Delta_n^{2p}\left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right).$$

Using Burkholder-David-Gundy's and Hölder's inequalities with $\frac{1}{p_1}+\frac{1}{q_1}=1$, (3.13) and (3.15),

$$\begin{aligned}
H^b_{2122} &\le C\Delta_n^{p-1}\int_{t_k}^s \widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\frac{\sqrt{X^b_u(t_k,x)}}{\partial_x X^b_u(t_k,x)}\right|^{2p}\right]du\\
&\le C\Delta_n^{p-1}\int_{t_k}^s \left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|X^b_u(t_k,x)\right|^{p_1p}\right]\right)^{\frac{1}{p_1}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{\left|\partial_x X^b_u(t_k,x)\right|^{2q_1p}}\right]\right)^{\frac{1}{q_1}}du\\
&\le C\Delta_n^p(1+x^p)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1}-p}}\right).
\end{aligned}$$

Finally, using Burkholder-David-Gundy's inequality and condition **(A2)**,

$$H^b_{2123} \le C\int_{t_k}^s\int_0^\infty \widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|\partial_x X^b_u(t_k,x)|^{2p}}\right]z^{2p}m(dz)du \le C\Delta_n\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right).$$

Thus, we have shown that

$$H^b_{212} \le C\Delta_n^{2p}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right) + C\Delta_n^p(1+x^p)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1}-p}}\right) + C\Delta_n\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right),$$

which implies that

$$H^b_{21} \le C\Delta_n^{2p}\Bigg\{\Delta_n^{2p}\frac{1}{x^{2p}}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1\alpha_1}-p}}\right)\left(1+x^{2p}\right) + \Delta_n^{2p}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right)$$

$$+\Delta_n^p\left(1+x^p\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1}-p}}\right)+\Delta_n\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right)\Bigg\}.$$

Next, using Burkholder-David-Gundy's and Hölder's inequalities with $\frac{1}{p_2}+\frac{1}{q_2}=1$, (3.14) and (3.15),

$$\begin{aligned}H_{22}^b&\leq C\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\int_{t_k}^{t_{k+1}}\frac{(\partial_xX_s^b(t_k,x))^2}{X_s^b(t_k,x)}ds\right|^q\right]\leq C\Delta_n^{q-1}\int_{t_k}^{t_{k+1}}\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{\partial_xX_s^b(t_k,x)}{\sqrt{X_s^b(t_k,x)}}\right|^{2q}\right]ds\\&\leq C\Delta_n^{q-1}\int_{t_k}^{t_{k+1}}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\frac{1}{|X_s^b(t_k,x)|^{p_2q}}\right]\right)^{\frac{1}{p_2}}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\partial_xX_s^b(t_k,x)\right|^{2q_2q}\right]\right)^{\frac{1}{q_2}}ds\\&\leq C\Delta_n^q\frac{1}{x^q}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_2}+q}}\right),\end{aligned}$$

where p_2 should be chosen close to 1 in order that $p_2q<\frac{2a}{\sigma^2}-1$.

In order to be able to apply two estimates (3.14) and (3.15) to estimate two terms above H_{21}^b and H_{22}^b, all conditions required here are the following

$$-2q_1p>-\frac{(\frac{2a}{\sigma^2}-1)^2}{2(\frac{2a}{\sigma^2}-\frac{1}{2})},\ 2p_1p<\frac{2a}{\sigma^2}-1,\ q<\frac{2a}{\sigma^2}-1.$$

This implies that

$$\begin{cases}\frac{2a}{\sigma^2}>2q_1p+\sqrt{2q_1p\left(2q_1p+1\right)}+1\\\frac{2a}{\sigma^2}>\frac{2q_1p}{q_1-1}+1\\\frac{2a}{\sigma^2}>\frac{p}{p-1}+1.\end{cases}$$

Here, the optimal choice for p and q_1 corresponds to choose them in a way which gives minimal restrictions on the ratio $\frac{2a}{\sigma^2}$. That is,

$$2q_1p+\sqrt{2q_1p\left(2q_1p+1\right)}=\frac{2q_1p}{q_1-1}=\frac{p}{p-1}.$$

Thus, the unique solution is given by $p=\frac{11+\sqrt{57}}{16}$ and $q_1=\frac{13+\sqrt{57}}{14}$, which implies that $\frac{2a}{\sigma^2}>\frac{9+\sqrt{57}}{2}$. Therefore, under condition $\frac{a}{\sigma^2}>\frac{9+\sqrt{57}}{4}$, we have shown that

$$\begin{aligned}\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|H_2(X^b)\right|^2\right]&\leq C\Bigg(\Delta_n^{2p}\Bigg\{\Delta_n^{2p}\frac{1}{x^{2p}}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1\alpha_1}-p}}\right)\left(1+x^{2p}\right)+\Delta_n^{2p}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right)\\&\quad+\Delta_n^p\left(1+x^p\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1}-p}}\right)+\Delta_n\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2}-p}}\right)\Bigg\}\Bigg)^{\frac{1}{p}}\left(\Delta_n^q\frac{1}{x^q}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_2}+q}}\right)\right)^{\frac{1}{q}}\\&\leq C\Delta_n^2\Bigg\{\Delta_n^2\frac{1}{x^2}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1\alpha_1p}-1}}\right)\left(1+x^2\right)+\Delta_n^2\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p}-1}}\right)\end{aligned}$$

$$
\begin{aligned}
&\quad + \Delta_n (1+x)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1p}-1}}\right) + \Delta_n^{\frac{1}{p}}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p}-1}}\right)\Bigg\}\Delta_n\frac{1}{x}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_2q}+1}}\right)\\
&\le C\frac{\Delta_n^{3+\frac{1}{p}}}{x}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_2q}+1}}\right)\left\{\frac{1}{x^2}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1\alpha_1p}-1}}\right)(1+x^2)+\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p}-1}}\right)\right.\\
&\quad\left. + (1+x)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2q_1p}-1}}\right)\right\}, \qquad (5.13)
\end{aligned}
$$

where $p=\frac{11+\sqrt{57}}{16}$, $q=\frac{p}{p-1}=\frac{7+\sqrt{57}}{2}$, $q_1=\frac{13+\sqrt{57}}{14}$, $q_1p=\frac{25+3\sqrt{57}}{28}$, $q_2>1$ with $\frac{q_2}{q_2-1}$ close to 1, and $\alpha_1>1$ is close to 1.

Finally, we treat the term $H_3(X^b)$. Using Hölder's inequality with $\frac{1}{p_3}+\frac{1}{q_3}=1$,

$$
\begin{aligned}
&\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|H_3(X^b)\right|^2\right]\le\Delta_n\int_{t_k}^{t_{k+1}}\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|D_s\left(\frac{\partial_bX^b_{t_{k+1}}(t_k,x)}{\partial_xX^b_{t_{k+1}}(t_k,x)}\right)\frac{\partial_xX^b_s(t_k,x)}{\sigma\sqrt{X^b_s(t_k,x)}}\right|^2\right]ds\\
&\le\Delta_n\int_{t_k}^{t_{k+1}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|D_s\left(\frac{\partial_bX^b_{t_{k+1}}(t_k,x)}{\partial_xX^b_{t_{k+1}}(t_k,x)}\right)\right|^{2p_3}\right]\right)^{\frac{1}{p_3}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\frac{\partial_xX^b_s(t_k,x)}{\sigma\sqrt{X^b_s(t_k,x)}}\right|^{2q_3}\right]\right)^{\frac{1}{q_3}}ds.
\end{aligned}
$$

From (3.9),

$$
\frac{\partial_bX^b_{t_{k+1}}(t_k,x)}{\partial_xX^b_{t_{k+1}}(t_k,x)}=-\int_{t_k}^{t_{k+1}}\frac{X^b_r(t_k,x)}{\partial_xX^b_r(t_k,x)}dr.
$$

This, together with the chain rule of Malliavin calculus and (3.12), gives

$$
\begin{aligned}
D_s\left(\frac{\partial_bX^b_{t_{k+1}}(t_k,x)}{\partial_xX^b_{t_{k+1}}(t_k,x)}\right)&=-\int_s^{t_{k+1}}\left(\frac{D_sX^b_r(t_k,x)}{\partial_xX^b_r(t_k,x)}-\frac{1}{(\partial_xX^b_r(t_k,x))^2}D_s(\partial_xX^b_r(t_k,x))\right)dr\\
&=-\int_s^{t_{k+1}}\left(\frac{\sigma\sqrt{X^b_s(t_k,x)}}{\partial_xX^b_s(t_k,x)}-\frac{1}{(\partial_xX^b_r(t_k,x))^2}D_s(\partial_xX^b_r(t_k,x))\right)dr.
\end{aligned}\qquad (5.14)
$$

On the other hand, from (3.11) and (3.12), we get that

$$
\begin{aligned}
&\frac{1}{\partial_xX^b_r(t_k,x)}D_s\left(\partial_xX^b_r(t_k,x)\right)=\frac{\sigma}{2}\frac{1}{\sqrt{X^b_s(t_k,x)}}+\frac{\sigma^2}{8}\int_s^r\frac{1}{(X^b_u(t_k,x))^2}D_sX^b_u(t_k,x)du\\
&\quad-\frac{\sigma}{4}\int_s^r\frac{1}{(X^b_u(t_k,x))^{\frac{3}{2}}}D_sX^b_u(t_k,x)dB_u\\
&=\frac{\sigma}{2}\frac{1}{\sqrt{X^b_s(t_k,x)}}+\frac{\sigma^3}{8}\int_s^r\frac{\sqrt{X^b_s(t_k,x)}\partial_xX^b_u(t_k,x)}{(X^b_u(t_k,x))^2\partial_xX^b_s(t_k,x)}du-\frac{\sigma^2}{4}\int_s^r\frac{\sqrt{X^b_s(t_k,x)}\partial_xX^b_u(t_k,x)}{(X^b_u(t_k,x))^{\frac{3}{2}}\partial_xX^b_s(t_k,x)}dB_u,
\end{aligned}
$$

which, together with (5.14), gives

$$
D_s\left(\frac{\partial_bX^b_{t_{k+1}}(t_k,x)}{\partial_xX^b_{t_{k+1}}(t_k,x)}\right)=-\int_s^{t_{k+1}}\left(\frac{\sigma\sqrt{X^b_s(t_k,x)}}{\partial_xX^b_s(t_k,x)}-\frac{1}{\partial_xX^b_r(t_k,x)}\left\{\frac{\sigma}{2}\frac{1}{\sqrt{X^b_s(t_k,x)}}\right.\right.
$$

$$+\frac{\sigma^3}{8}\int_s^r \frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^2\partial_x X_s^b(t_k,x)}du-\frac{\sigma^2}{4}\int_s^r \frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^{\frac{3}{2}}\partial_x X_s^b(t_k,x)}dB_u\bigg\}\bigg)dr.$$

Then, using Burkholder-David-Gundy's and Hölder's inequalities with $\frac{1}{\alpha_2}+\frac{1}{\beta_2}=1$, $\frac{1}{\alpha_3}+\frac{1}{\beta_3}=1$, $\frac{1}{\alpha_4}+\frac{1}{\beta_4}=1$ and $\frac{1}{p_4}+\frac{1}{p_4}+\frac{1}{q_4}=1$, (3.14) and (3.15), we get that

$$\begin{aligned}
&\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|D_s\left(\frac{\partial_b X_{t_{k+1}}^b(t_k,x)}{\partial_x X_{t_{k+1}}^b(t_k,x)}\right)\right|^{2p_3}\right]\leq C\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\Delta_n\frac{\sqrt{X_s^b(t_k,x)}}{\partial_x X_s^b(t_k,x)}\right|^{2p_3}\right]\\
&\quad+C\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\int_s^{t_{k+1}}\frac{dr}{\partial_x X_r^b(t_k,x)}\frac{1}{\sqrt{X_s^b(t_k,x)}}\right|^{2p_3}\right]\\
&\quad+C\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\int_s^{t_{k+1}}\frac{1}{\partial_x X_r^b(t_k,x)}\int_s^r\frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^2\partial_x X_s^b(t_k,x)}dudr\right|^{2p_3}\right]\\
&\quad+C\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\int_s^{t_{k+1}}\frac{1}{\partial_x X_r^b(t_k,x)}\int_s^r\frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^{\frac{3}{2}}\partial_x X_s^b(t_k,x)}dB_udr\right|^{2p_3}\right]\\
&\leq C\Delta_n^{2p_3}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\frac{1}{|\partial_x X_s^b(t_k,x)|^{2p_3\alpha_2}}\right]\right)^{\frac{1}{\alpha_2}}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|X_s^b(t_k,x)\right|^{p_3\beta_2}\right]\right)^{\frac{1}{\beta_2}}\\
&\quad+C\Delta_n^{2p_3-1}\int_s^{t_{k+1}}\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{1}{\partial_x X_r^b(t_k,x)}\frac{1}{\sqrt{X_s^b(t_k,x)}}\right|^{2p_3}\right]dr\\
&\quad+C\Delta_n^{2p_3-1}\int_s^{t_{k+1}}\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{1}{\partial_x X_r^b(t_k,x)}\int_s^r\frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^2\partial_x X_s^b(t_k,x)}du\right|^{2p_3}\right]dr\\
&\quad+C\Delta_n^{2p_3-1}\int_s^{t_{k+1}}\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{1}{\partial_x X_r^b(t_k,x)}\int_s^r\frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^{\frac{3}{2}}\partial_x X_s^b(t_k,x)}dB_u\right|^{2p_3}\right]dr\\
&\leq C\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_2}-p_3}}\right)(1+x^{p_3})\\
&\quad+C\Delta_n^{2p_3-1}\int_s^{t_{k+1}}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\frac{1}{|\partial_x X_r^b(t_k,x)|^{p_3p_4}}\right]\right)^{\frac{2}{p_4}}\left(\widetilde{\mathrm{E}}_{t_k,x}^b\left[\frac{1}{|X_s^b(t_k,x)|^{p_3q_4}}\right]\right)^{\frac{1}{q_4}}dr\\
&\quad+C\Delta_n^{2p_3-1}\Delta_n^{2p_3-1}\int_s^{t_{k+1}}\int_s^r\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{1}{\partial_x X_r^b(t_k,x)}\frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^2\partial_x X_s^b(t_k,x)}\right|^{2p_3}\right]dudr\\
&\quad+C\Delta_n^{2p_3-1}\Delta_n^{p_3-1}\int_s^{t_{k+1}}\int_s^r\widetilde{\mathrm{E}}_{t_k,x}^b\left[\left|\frac{1}{\partial_x X_r^b(t_k,x)}\frac{\sqrt{X_s^b(t_k,x)}\partial_x X_u^b(t_k,x)}{(X_u^b(t_k,x))^{\frac{3}{2}}\partial_x X_s^b(t_k,x)}\right|^{2p_3}\right]dudr
\end{aligned}$$

$$
\begin{aligned}
&\leq C\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_2}-p_3}}\right)(1+x^{p_3})+C\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{p_4}-p_3}}\right)\frac{1}{x^{p_3}}\\
&\quad+C\Delta_n^{4p_3-2}\int_s^{t_{k+1}}\int_s^r\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|\partial_x X^b_r(t_k,x)(X^b_u(t_k,x))^2\partial_x X^b_s(t_k,x)|^{2p_3\alpha_3}}\right]\right)^{\frac{1}{\alpha_3}}\\
&\quad\times\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\sqrt{X^b_s(t_k,x)}\partial_x X^b_u(t_k,x)\right|^{2p_3\beta_3}\right]\right)^{\frac{1}{\beta_3}}dudr\\
&\quad+C\Delta_n^{3p_3-2}\int_s^{t_{k+1}}\int_s^r\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{\left|\partial_x X^b_r(t_k,x)(X^b_u(t_k,x))^{\frac{3}{2}}\partial_x X^b_s(t_k,x)\right|^{2p_3\alpha_3}}\right]\right)^{\frac{1}{\alpha_3}}\\
&\quad\times\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\sqrt{X^b_s(t_k,x)}\partial_x X^b_u(t_k,x)\right|^{2p_3\beta_3}\right]\right)^{\frac{1}{\beta_3}}dudr\\
&\leq C\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_2}-p_3}}\right)(1+x^{p_3})+C\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{p_4}-p_3}}\right)\frac{1}{x^{p_3}}\\
&+C\Delta_n^{4p_3-2}\int_s^{t_{k+1}}\int_s^r\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|\partial_x X^b_r(t_k,x)|^{2p_3\alpha_3p_4}}\right]\right)^{\frac{1}{\alpha_3p_4}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|\partial_x X^b_s(t_k,x)|^{2p_3\alpha_3p_4}}\right]\right)^{\frac{1}{\alpha_3p_4}}\\
&\quad\times\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|X^b_u(t_k,x)|^{4p_3\alpha_3q_4}}\right]\right)^{\frac{1}{\alpha_3q_4}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|X^b_s(t_k,x)\right|^{p_3\beta_3\alpha_4}\right]\right)^{\frac{1}{\beta_3\alpha_4}}\\
&\quad\times\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\partial_x X^b_u(t_k,x)\right|^{2p_3\beta_3\beta_4}\right]\right)^{\frac{1}{\beta_3\beta_4}}dudr\\
&+C\Delta_n^{3p_3-2}\int_s^{t_{k+1}}\int_s^r\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|\partial_x X^b_r(t_k,x)|^{2p_3\alpha_3p_4}}\right]\right)^{\frac{1}{\alpha_3p_4}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|\partial_x X^b_s(t_k,x)|^{2p_3\alpha_3p_4}}\right]\right)^{\frac{1}{\alpha_3p_4}}\\
&\quad\times\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|X^b_u(t_k,x)|^{3p_3\alpha_3q_4}}\right]\right)^{\frac{1}{\alpha_3q_4}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|X^b_s(t_k,x)\right|^{p_3\beta_3\alpha_4}\right]\right)^{\frac{1}{\beta_3\alpha_4}}\\
&\quad\times\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\partial_x X^b_u(t_k,x)\right|^{2p_3\beta_3\beta_4}\right]\right)^{\frac{1}{\beta_3\beta_4}}dudr\\
&\leq C\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_2}-p_3}}\right)(1+x^{p_3})+C\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{p_4}-p_3}}\right)\frac{1}{x^{p_3}}\\
&\quad+C\Delta_n^{4p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\frac{1}{x^{4p_3}}(1+x^{p_3})\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4}+p_3}}\right)
\end{aligned}
$$

$$+ C\Delta_n^{3p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\frac{1}{x^{3p_3}}\left(1+x^{p_3}\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4}+p_3}}\right).$$

Here $\alpha_2>1$ and $\alpha_3>1$ should be chosen close to 1 in order that $-2p_3\alpha_2\geq-\frac{(\frac{2a}{\sigma^2}-1)^2}{2(\frac{2a}{\sigma^2}-\frac{1}{2})}$, $-2p_3\alpha_3p_4\geq-\frac{(\frac{2a}{\sigma^2}-1)^2}{2(\frac{2a}{\sigma^2}-\frac{1}{2})}$ and $4p_3\alpha_3q_4<\frac{2a}{\sigma^2}-1$, and $\beta_4>1$.

Next, using Hölder's inequality with $\frac{1}{p_5}+\frac{1}{q_5}=1$, (3.14) and (3.15),

$$\begin{aligned}\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\frac{\partial_xX^b_s(t_k,x)}{\sigma\sqrt{X^b_s(t_k,x)}}\right|^{2q_3}\right]&\leq C\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|\partial_xX^b_s(t_k,x)\right|^{2q_3p_5}\right]\right)^{\frac{1}{p_5}}\left(\widetilde{\mathrm{E}}^b_{t_k,x}\left[\frac{1}{|X^b_s(t_k,x)|^{q_3q_5}}\right]\right)^{\frac{1}{q_5}}\\&\leq C\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_5}+q_3}}\right)\frac{1}{x^{q_3}},\end{aligned}$$

where q_5 should be chosen close to 1 in order that $q_3q_5<\frac{2a}{\sigma^2}-1$.

In order to be able to apply two estimates (3.14) and (3.15) to estimate the term $H_3(X^b)$, all conditions required here are the following

$$-2p_3p_4>-\frac{(\frac{2a}{\sigma^2}-1)^2}{2(\frac{2a}{\sigma^2}-\frac{1}{2})},\ 4p_3q_4<\frac{2a}{\sigma^2}-1,\ q_3<\frac{2a}{\sigma^2}-1.$$

This implies that

$$\begin{cases}\frac{2a}{\sigma^2}>2p_3p_4+\sqrt{2p_3p_4\left(2p_3p_4+1\right)}+1\\ \frac{2a}{\sigma^2}>\frac{4p_3p_4}{p_4-2}+1\\ \frac{2a}{\sigma^2}>\frac{p_3}{p_3-1}+1.\end{cases}$$

Here, the optimal choice for p_3 and p_4 corresponds to choose them in a way which gives minimal restrictions on the ratio $\frac{2a}{\sigma^2}$. That is,

$$2p_3p_4+\sqrt{2p_3p_4\left(2p_3p_4+1\right)}=\frac{4p_3p_4}{p_4-2}=\frac{p_3}{p_3-1}.$$

Thus, the unique solution is given by $p_3=\frac{21+\sqrt{185}}{32}$ and $p_4=\frac{19+\sqrt{185}}{11}$, which implies that $\frac{2a}{\sigma^2}>\frac{15+\sqrt{185}}{2}$. Therefore, under condition $\frac{a}{\sigma^2}>\frac{15+\sqrt{185}}{4}$, we have shown that

$$\begin{aligned}\widetilde{\mathrm{E}}^b_{t_k,x}\left[\left|H_3(X^b)\right|^2\right]\leq C\Delta_n^2\Bigg\{&\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_2}-p_3}}\right)\left(1+x^{p_3}\right)+\Delta_n^{2p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{p_4}-p_3}}\right)\frac{1}{x^{p_3}}\\&+\Delta_n^{4p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\frac{1}{x^{4p_3}}\left(1+x^{p_3}\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4}+p_3}}\right)\\&+\Delta_n^{3p_3}\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3p_4}-p_3}}\right)\frac{1}{x^{3p_3}}\left(1+x^{p_3}\right)\left(1+\frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4}+p_3}}\right)\Bigg\}^{\frac{1}{p_3}}\end{aligned}$$

$$
\begin{aligned}
&\times \left(\left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_5}+q_3}} \right) \frac{1}{x^{q_3}} \right)^{\frac{1}{q_3}} \\
&\leq C\Delta_n^2 \Bigg\{ \Delta_n^2 \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_2 p_3}-1}} \right) (1+x) + \Delta_n^2 \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{p_3p_4}-1}} \right) \frac{1}{x} \\
&\quad + \Delta_n^4 \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \frac{1}{x^4} (1+x) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4 p_3}+1}} \right) \\
&+ \Delta_n^3 \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \frac{1}{x^3} (1+x) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4 p_3}+1}} \right) \Bigg\} \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_5 q_3}+1}} \right) \frac{1}{x} \\
&\leq C \frac{\Delta_n^4}{x} \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_5 q_3}+1}} \right) \Bigg\{ \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_2 p_3}-1}} \right) (1+x) + \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{p_3p_4}-1}} \right) \frac{1}{x} \\
&\quad + \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \frac{1}{x^4} (1+x) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4 p_3}+1}} \right) \\
&\quad + \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\alpha_3 p_3 p_4}-1}} \right) \frac{1}{x^3} (1+x) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4 p_3}+1}} \right) \Bigg\}, \qquad (5.15)
\end{aligned}
$$

where $p_3 = \frac{21+\sqrt{185}}{32}$, $q_3 = \frac{p_3}{p_3-1} = \frac{416+33\sqrt{185}}{64}$, $p_4 = \frac{19+\sqrt{185}}{11}$, $p_3p_4 = \frac{73+5\sqrt{185}}{44}$, $p_5 > 1$ with $\frac{p_5}{p_5-1}$ close to 1, $\alpha_2 > 1$ and $\alpha_3 > 1$ are close to 1, $\beta_4 > 1$.

From (5.11), (5.12), (5.13) and (5.15), under condition **(A3)**: $\frac{a}{\sigma^2} > \frac{15+\sqrt{185}}{4}$, we obtain that

$$
\begin{aligned}
&\widetilde{\mathrm{E}}^b_{t_k,x} \left[\Big(H(X^b) \Big)^2 \right] \leq C \frac{\Delta_n^{3+\frac{1}{p}}}{x^5} \left(1+x^2\right) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_5 q_3}+1}} \right) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_3}-1}} \right) \\
&\quad \times \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2p_3p_4}-1}} \right) \left(1 + \frac{1}{x^{\frac{\frac{2a}{\sigma^2}-1}{2\beta_3\beta_4 p_3}+1}} \right) \leq C \frac{\Delta_n^{3+\frac{1}{p}}}{x^5} \left(1+x^2\right) \left(1 + \frac{1}{x^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta}+\frac{1}{2p_3}+\frac{1}{2p_3p_4})}} \right) \\
&\quad = C \frac{\Delta_n^{3+\frac{1}{p}}}{x^5} \left(1+x^2\right) \left(1 + \frac{1}{x^{(\frac{2a}{\sigma^2}-1)(\frac{1}{2\beta}+\frac{16}{21+\sqrt{185}}+\frac{22}{73+5\sqrt{185}})}} \right),
\end{aligned}
$$

for some constant $C > 0$, where $p = \frac{11+\sqrt{57}}{16}$, $p_3 = \frac{21+\sqrt{185}}{32}$, $p_3p_4 = \frac{73+5\sqrt{185}}{44}$, and $\beta = \min\{p_5, \beta_3, \beta_4\} > 1$ is sufficiently large enough. Thus, we conclude the estimate (3.18). □

Remark 5.1. *When we use Cauchy-Schwarz's inequality instead of Hölder's inequality to estimate* $(H(X^b))^2$*, in this case the required condition* **(A3)** *will be* $\frac{a}{\sigma^2} > \frac{17+4\sqrt{17}}{2}$ *which is actually bigger than* $\frac{15+\sqrt{185}}{4}$.

Remark 5.2. *In the case of CIR process without jumps studied in* [9] *when the subordinator is degenerate, using the explicit expression for the Malliavin derivative obtained by Alòs and Ewald in* [1, Corollary 4.2] *instead of the expression* (3.12) *in order to estimate the term* $H_3(X^b)$*, condition* **(A3)** *will be* $\frac{a}{\sigma^2} > \frac{5+3\sqrt{2}}{2}$ *(see* [9, Lemma 3.8]*), which is less restrictive than that given in this paper.*

5.8. **Proof of Lemma 3.9.**

Proof. Using (3.19), we have that

$$\begin{aligned}
&\frac{d\widehat{\mathrm{P}}^{b_2}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}\left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]}\right)-1=\frac{d\widehat{\mathrm{P}}^{b_2}_{t_k,x}-d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}\left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]}\right)\\
&=\int_{b_1}^{b_2}\frac{\partial}{\partial b}\left(\frac{d\widehat{\mathrm{P}}^{b}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}\right)\left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]}\right)db\\
&=\int_{b_1}^{b_2}\int_{t_k}^{t_{k+1}}\frac{-\sqrt{Y^{b_1}_s}}{\sigma}\left(dW_s+\frac{b-b_1}{\sigma}\sqrt{Y^{b_1}_s}ds\right)\frac{d\widehat{\mathrm{P}}^{b}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}\left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]}\right)db.
\end{aligned}$$

Then, using Girsanov's theorem, we get that

$$\begin{aligned}
&\widehat{\mathrm{E}}^{b_1}_{t_k,x}\left[V\left(\frac{d\widehat{\mathrm{P}}^{b_2}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}\left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]}\right)-1\right)\right]\\
&=\int_{b_1}^{b_2}\widehat{\mathrm{E}}^{b_1}_{t_k,x}\left[V\int_{t_k}^{t_{k+1}}\frac{-\sqrt{Y^{b_1}_s}}{\sigma}\Big(dW_s+\frac{b-b_1}{\sigma}\sqrt{Y^{b_1}_s}ds\Big)\frac{d\widehat{\mathrm{P}}^{b}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}\left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]}\right)\right]db\\
&=\int_{b_1}^{b_2}\widehat{\mathrm{E}}^{b}_{t_k,x}\left[V\int_{t_k}^{t_{k+1}}\frac{-\sqrt{Y^{b}_s}}{\sigma}dW_s^{\widehat{\mathrm{P}}^b_{t_k,x}}\right]db.
\end{aligned}$$

Here, the process $W^{\widehat{\mathrm{P}}^b_{t_k,x}}=(W_t^{\widehat{\mathrm{P}}^b_{t_k,x}},t\in[t_k,t_{k+1}])$ is a Brownian motion under $\widehat{\mathrm{P}}^b_{t_k,x}$, where for any $t\in[t_k,t_{k+1}]$,

$$W_t^{\widehat{\mathrm{P}}^b_{t_k,x}}:=W_t+\frac{b-b_1}{\sigma}\int_{t_k}^{t}\sqrt{Y^{b_1}_s}ds.$$

Next, using Hölder's and Burkholder-David-Gundy's inequalities and Lemma 3.5 (ii), we get

$$\begin{aligned}
&\left|\widehat{\mathrm{E}}^{b_1}_{t_k,x}\left[V\left(\frac{d\widehat{\mathrm{P}}^{b_2}_{t_k,x}}{d\widehat{\mathrm{P}}^{b_1}_{t_k,x}}\left((Y^{b_1}_t)_{t\in[t_k,t_{k+1}]}\right)-1\right)\right]\right|=\left|\int_{b_1}^{b_2}\widehat{\mathrm{E}}^{b}_{t_k,x}\left[V\int_{t_k}^{t_{k+1}}\frac{-\sqrt{Y^{b}_s}}{\sigma}dW_s^{\widehat{\mathrm{P}}^b_{t_k,x}}\right]db\right|\\
&\leq\left|\int_{b_1}^{b_2}\left|\widehat{\mathrm{E}}^{b}_{t_k,x}\left[V\int_{t_k}^{t_{k+1}}\frac{-\sqrt{Y^{b}_s}}{\sigma}dW_s^{\widehat{\mathrm{P}}^b_{t_k,x}}\right]\right|db\right|\\
&\leq\left|\int_{b_1}^{b_2}\left(\widehat{\mathrm{E}}^{b}_{t_k,x}\left[|V|^q\right]\right)^{\frac{1}{q}}\left(\widehat{\mathrm{E}}^{b}_{t_k,x}\left[\left|\int_{t_k}^{t_{k+1}}\frac{\sqrt{Y^{b}_s}}{\sigma}dW_s^{\widehat{\mathrm{P}}^b_{t_k,x}}\right|^p\right]\right)^{\frac{1}{p}}db\right|\\
&\leq C\left|\int_{b_1}^{b_2}\left(\widehat{\mathrm{E}}^{b}_{t_k,x}\left[|V|^q\right]\right)^{\frac{1}{q}}\left(\Delta_n^{\frac{p}{2}-1}\int_{t_k}^{t_{k+1}}\widehat{\mathrm{E}}^{b}_{t_k,x}\left[\left|\sqrt{Y^{b}_s}\right|^p\right]ds\right)^{\frac{1}{p}}db\right|
\end{aligned}$$

$$\leq C\sqrt{\Delta_n}\left(1+\sqrt{x}\right)\left|\int_{b_1}^{b_2}\left(\widehat{\mathrm{E}}^b_{t_k,x}\left[|V|^q\right]\right)^{\frac{1}{q}}db\right|,$$

for some constant $C>0$, where $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Thus, the result follows. □

5.9. **Proof of Lemma 3.10.**

Proof. For simplicity, we set $g(y):=\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}[V\big|X^b_{t_{k+1}}=y]$ for all $y\in\mathbb{R}_{++}$. Then, applying Girsanov's theorem (see (3.19)), we obtain that

$$\widehat{\mathrm{E}}^{b_0}\left[\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[V\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\big|\widehat{\mathcal{F}}_{t_k}\right]=\widehat{\mathrm{E}}^{b_0}\left[g(Y^{b_0}_{t_{k+1}})\big|Y^{b_0}_{t_k}\right]=\widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[g(Y^{b_0}_{t_{k+1}})\right]$$

$$=\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[g(Y^b_{t_{k+1}})\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}\left((Y^b_t)_{t\in[t_k,t_{k+1}]}\right)\right]$$

$$=\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[g(Y^b_{t_{k+1}})\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}\left((Y^b_t)_{t\in[t_k,t_{k+1}]}\right)\Big|Y^b_{t_{k+1}}\right]\right]$$

$$=\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[g(Y^b_{t_{k+1}})\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}\left((Y^b_t)_{t\in[t_k,t_{k+1}]}\right)\Big|Y^b_{t_{k+1}}\right]\right]$$

$$=\int_0^\infty g(y)\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}((Y^b_t)_{t\in[t_k,t_{k+1}]})\Big|Y^b_{t_{k+1}}=y\right]p^b(\Delta_n,Y^{b_0}_{t_k},y)dy$$

$$=\int_0^\infty \widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[V\big|X^b_{t_{k+1}}=y\right]\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}\left((Y^b_t)_{t\in[t_k,t_{k+1}]}\right)\Big|Y^b_{t_{k+1}}=y\right]p^b(\Delta_n,Y^{b_0}_{t_k},y)dy$$

$$=\int_0^\infty \mathrm{E}^b_{t_k,Y^{b_0}_{t_k}}\left[V\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}\left((Y^b_t)_{t\in[t_k,t_{k+1}]}\right)\Big|Y^b_{t_{k+1}}=y,X^b_{t_{k+1}}=y\right]p^b(\Delta_n,Y^{b_0}_{t_k},y)dy$$

$$=\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[\mathrm{E}^b_{t_k,Y^{b_0}_{t_k}}\left[V\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}((Y^b_t)_{t\in[t_k,t_{k+1}]})\Big|Y^b_{t_{k+1}},X^b_{t_{k+1}}=Y^b_{t_{k+1}}\right]\right]$$

$$=\mathrm{E}^b_{t_k,Y^{b_0}_{t_k}}\left[\mathrm{E}^b_{t_k,Y^{b_0}_{t_k}}\left[V\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}((Y^b_t)_{t\in[t_k,t_{k+1}]})\Big|Y^b_{t_{k+1}},X^b_{t_{k+1}}=Y^b_{t_{k+1}}\right]\right]$$

$$=\mathrm{E}^b_{t_k,Y^{b_0}_{t_k}}\left[V\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}((Y^b_t)_{t\in[t_k,t_{k+1}]})\right]=\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[V\right]\widehat{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\left[\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k,Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^b_{t_k,Y^{b_0}_{t_k}}}\left((Y^b_t)_{t\in[t_k,t_{k+1}]}\right)\right]$$

$$= \widetilde{\mathrm{E}}^b_{t_k, Y^{b_0}_{t_k}} \left[V \right],$$

where we have used that fact that, by definition of $\mathrm{E}^b_{t_k,x}$, for any $\widehat{\mathcal{F}}_{t_{k+1}}$-measurable random variable V_1 and $\widetilde{\mathcal{F}}_{t_{k+1}}$-measurable random variable V_2,

$$\widehat{\mathrm{E}}^b_{t_k,x} \left[V_1 | Y^b_{t_{k+1}} = y \right] \widetilde{\mathrm{E}}^b_{t_k,x} \left[V_2 | X^b_{t_{k+1}} = y \right] = \mathrm{E}^b_{t_k,x} \left[V_1 V_2 | Y^b_{t_{k+1}} = y, X^b_{t_{k+1}} = y \right],$$

and $\widehat{\mathrm{E}}^b_{t_k, Y^{b_0}_{t_k}} \left[\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k, Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^{b}_{t_k, Y^{b_0}_{t_k}}} ((Y^b_t)_{t\in[t_k,t_{k+1}]}) \right] = 1$ together with the independence between V and $\frac{d\widehat{\mathrm{P}}^{b_0}_{t_k, Y^{b_0}_{t_k}}}{d\widehat{\mathrm{P}}^{b}_{t_k, Y^{b_0}_{t_k}}} ((Y^b_t)_{t\in[t_k,t_{k+1}]})$. Thus, the result follows. □

5.10. **Proof of Lemma 3.11.**

Proof. Observe that

$$\widehat{\mathrm{E}}^b \left[\left| \frac{1}{n\Delta_n} \int_0^{n\Delta_n} h(Y^b_s) ds - \frac{1}{n} \sum_{k=0}^{n-1} h(Y^b_{t_k}) \right| \right] \leq \frac{1}{n\Delta_n} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \widehat{\mathrm{E}}^b \left[\left| h(Y^b_s) - h(Y^b_{t_k}) \right| \right] ds. \tag{5.16}$$

Using the mean value theorem, there exists $\alpha \in (0,1)$ such that

$$h(Y^b_s) - h(Y^b_{t_k}) = h'(Y^b_{t_k} + \alpha(Y^b_s - Y^b_{t_k})) \left(Y^b_s - Y^b_{t_k} \right).$$

Then, using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$, Lemma 3.5 (iii) and (i), we get that

$$\begin{aligned} \widehat{\mathrm{E}}^b \left[\left| h(Y^b_s) - h(Y^b_{t_k}) \right| \right] &\leq \left(\widehat{\mathrm{E}}^b \left[\left(h'(Y^b_{t_k} + \alpha(Y^b_s - Y^b_{t_k})) \right)^p \right] \right)^{\frac{1}{p}} \left(\widehat{\mathrm{E}}^b \left[\left(Y^b_s - Y^b_{t_k} \right)^q \right] \right)^{\frac{1}{q}} \\ &\leq C \left(\widehat{\mathrm{E}}^b \left[\frac{1}{(Y^b_{t_k} + \alpha(Y^b_s - Y^b_{t_k}))^{mp}} \right] \right)^{\frac{1}{p}} \left(\Delta_n^{\frac{q}{2}\wedge 1} \right)^{\frac{1}{q}} (1 + y_0^q + y_0^{\frac{q}{2}})^{\frac{1}{q}} \\ &\leq C \left(\widehat{\mathrm{E}}^b \left[\frac{1}{(Y^b_{t_k} \wedge Y^b_s)^{mp}} \right] \right)^{\frac{1}{p}} \Delta_n^{\frac{1}{q}} (1 + y_0^q + y_0^{\frac{q}{2}})^{\frac{1}{q}} \leq C \Delta_n^{\frac{1}{q}} (1 + y_0^q + y_0^{\frac{q}{2}})^{\frac{1}{q}}, \end{aligned}$$

for a constant $C > 0$, where p is chosen close to 1 and q is large enough. This together with (5.16) shows that

$$\widehat{\mathrm{E}}^b \left[\left| \frac{1}{n\Delta_n} \int_0^{n\Delta_n} h(Y^b_s) ds - \frac{1}{n} \sum_{k=0}^{n-1} h(Y^b_{t_k}) \right| \right] \leq C \Delta_n^{\frac{1}{p}}, \tag{5.17}$$

which tends to zero as $n \to \infty$. On the other hand, from Proposition 2.4 (iii), as $n \to \infty$,

$$\frac{1}{n\Delta_n} \int_0^{n\Delta_n} h(Y^b_s) ds \longrightarrow \int_0^{\infty} h(y) \pi_b(dy), \quad \widehat{\mathrm{P}}^b\text{-a.s.} \tag{5.18}$$

Thus, from (5.17) and (5.18), the result follows. □

5.11. **Large deviation type estimates.** Let $(\upsilon_n)_{n\geq 1}$ be a positive sequence satisfying $\lim_{n\to\infty}\upsilon_n = 0$. The process $J^{\upsilon_n} = (J_t^{\upsilon_n})_{t\in\mathbb{R}_+}$ defined by $J_t^{\upsilon_n} = \sum_{0\leq s\leq t}\Delta J_s \mathbf{1}_{\{\Delta J_s>\upsilon_n\}}$ is a compound Poisson process with intensity of big jumps $\lambda_{\upsilon_n} := \int_{z>\upsilon_n} m(dz)$ and distribution of big jumps $\frac{\mathbf{1}_{z>\upsilon_n} m(dz)}{\lambda_{\upsilon_n}}$. Then, we can split the jumps of the subordinator J_t into small jumps and big jumps as follows

$$\int_0^t\int_0^\infty zN(ds,dz) = \int_0^t\int_{z\leq\upsilon_n} z\widetilde{N}(ds,dz) + t\int_{z\leq\upsilon_n} zm(dz) + \int_0^t\int_{z>\upsilon_n} zN(ds,dz).$$

Hence, from (1.3), for any $t\in\mathbb{R}_+$, we can write

$$\begin{aligned} Y_t^b = y_0 + \int_0^t \left(a - bY_s^b\right) ds + \sigma\int_0^t \sqrt{Y_s^b}dW_s + \int_0^t\int_{z\leq\upsilon_n} z\widetilde{N}(ds,dz) \\ + t\int_{z\leq\upsilon_n} zm(dz) + \int_0^t\int_{z>\upsilon_n} zN(ds,dz). \end{aligned} \tag{5.19}$$

Let $N^{\upsilon_n} = (N_t^{\upsilon_n})_{t\in\mathbb{R}_+}$ denote the Poisson process with intensity λ_{υ_n} counting the big jumps of the compound Poisson process J^{υ_n}.

Similarly, the process $\widetilde{J}^{\upsilon_n} = (\widetilde{J}_t^{\upsilon_n})_{t\in\mathbb{R}_+}$ defined by $\widetilde{J}_t^{\upsilon_n} = \sum_{0\leq s\leq t}\Delta\widetilde{J}_s \mathbf{1}_{\{\Delta\widetilde{J}_s>\upsilon_n\}}$ is a compound Poisson process with intensity of big jumps λ_{υ_n} and distribution of big jumps $\frac{\mathbf{1}_{z>\upsilon_n} m(dz)}{\lambda_{\upsilon_n}}$. Then, we can split the jumps of the subordinator $\widetilde{J}_t$ into small jumps and big jumps as follows

$$\int_0^t\int_0^\infty zM(ds,dz) = \int_0^t\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz) + t\int_{z\leq\upsilon_n} zm(dz) + \int_0^t\int_{z>\upsilon_n} zM(ds,dz).$$

Hence, from (3.3), for any $t\in\mathbb{R}_+$, we can write

$$\begin{aligned} X_t^b = y_0 + \int_0^t \left(a - bX_s^b\right) ds + \sigma\int_0^t \sqrt{X_s^b}dB_s + \int_0^t\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz) \\ + t\int_{z\leq\upsilon_n} zm(dz) + \int_0^t\int_{z>\upsilon_n} zM(ds,dz). \end{aligned} \tag{5.20}$$

Let $M^{\upsilon_n} = (M_t^{\upsilon_n})_{t\in\mathbb{R}_+}$ denote the Poisson process with intensity λ_{υ_n} counting the big jumps of the compound Poisson process $\widetilde{J}^{\upsilon_n}$.

Now, for $k\in\{0,...,n-1\}$, we consider the events $\widehat{N}_{0,k}(\upsilon_n) := \{N_{t_{k+1}}^{\upsilon_n} - N_{t_k}^{\upsilon_n} = 0\}$ which have no big jumps of J^{υ_n} in the interval $[t_k, t_{k+1})$ and $\widehat{N}_{\geq 1,k}(\upsilon_n) := \{N_{t_{k+1}}^{\upsilon_n} - N_{t_k}^{\upsilon_n} \geq 1\}$ which have one or more than one big jump of J^{υ_n} in the interval $[t_k, t_{k+1})$. Similarly, we consider the events $\widetilde{N}_{0,k}(\upsilon_n) := \{M_{t_{k+1}}^{\upsilon_n} - M_{t_k}^{\upsilon_n} = 0\}$ which have no big jumps of $\widetilde{J}^{\upsilon_n}$ in the interval $[t_k, t_{k+1})$ and $\widetilde{N}_{\geq 1,k}(\upsilon_n) := \{M_{t_{k+1}}^{\upsilon_n} - M_{t_k}^{\upsilon_n} \geq 1\}$ which have one or more than one big jump of $\widetilde{J}^{\upsilon_n}$ in the interval $[t_k, t_{k+1})$. Next, we obtain the following large deviation type estimates.

Lemma 5.3. *Assume conditions* **(A1)** *and* **(A2)** *with $p = 2$. Then, for any $b\in\mathbb{R}$, there exists a constant $C > 0$ such that for all $q > 1$, and $k\in\{0,...,n-1\}$,*

$$\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\left(\int_{t_k}^{t_{k+1}}\int_0^\infty zN(ds,dz) - \widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\int_{t_k}^{t_{k+1}}\int_0^\infty zM(ds,dz)\big|X_{t_{k+1}}^b = Y_{t_{k+1}}^{b_0}\right]\right)^2\right]$$

$$\leq C\left(1+(Y_{t_k}^{b_0})^2\right)\Delta_n\left((\lambda_{\upsilon_n}\Delta_n)^{\frac{1}{q}}+\int_{z\leq\upsilon_n}z^2m(dz)+\Delta_n\left(\int_{z\leq\upsilon_n}zm(dz)\right)^2\right).$$

Proof. Splitting the Poisson integrals into small jumps and big jumps, we get that

$$\begin{aligned}
&\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\left(\int_{t_k}^{t_{k+1}}\int_0^\infty zN(ds,dz)-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\int_{t_k}^{t_{k+1}}\int_0^\infty zM(ds,dz)\big|X_{t_{k+1}}^b=Y_{t_{k+1}}^{b_0}\right]\right)^2\right]\\
&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\Bigg[\Bigg(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{N}(ds,dz)+\Delta_n\int_{z\leq\upsilon_n}zm(dz)+\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zN(ds,dz)\\
&\quad-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\Bigg[\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{M}(ds,dz)+\Delta_n\int_{z\leq\upsilon_n}zm(dz)\\
&\quad+\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zM(ds,dz)\big|X_{t_{k+1}}^b=Y_{t_{k+1}}^{b_0}\Bigg]\Bigg)^2\Bigg]\leq 3\left(D_{1,k,n}+D_{2,k,n}+D_{3,k,n}\right),
\end{aligned}\tag{5.21}$$

where

$$\begin{aligned}
D_{1,k,n}&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\left(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{N}(ds,dz)\right)^2\right],\\
D_{2,k,n}&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\left(\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{M}(ds,dz)\big|X_{t_{k+1}}^b=Y_{t_{k+1}}^{b_0}\right]\right)^2\right],\\
D_{3,k,n}&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\Bigg[\Bigg(\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zN(ds,dz)\\
&\qquad-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zM(ds,dz)\big|X_{t_{k+1}}^b=Y_{t_{k+1}}^{b_0}\right]\Bigg)^2\Bigg].
\end{aligned}$$

First, using Burkholder-David-Gundy's inequality,

$$D_{1,k,n}\leq C\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z^2m(dz)ds=C\Delta_n\int_{z\leq\upsilon_n}z^2m(dz).\tag{5.22}$$

Next, using Jensen's inequality, Lemma 3.10 and Burkholder-David-Gundy's inequality,

$$\begin{aligned}
D_{2,k,n}&\leq\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\left(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{M}(ds,dz)\right)^2\big|X_{t_{k+1}}^b=Y_{t_{k+1}}^{b_0}\right]\right]\\
&=\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\left(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{M}(ds,dz)\right)^2\right]\leq C\Delta_n\int_{z\leq\upsilon_n}z^2m(dz).
\end{aligned}\tag{5.23}$$

Next, multiplying the random variable outside the conditional expectation of $D_{3,k,n}$ by $\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}+\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}$, we get that

$$\begin{aligned}
D_{3,k,n}&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\Bigg[\left(\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}+\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\right)\Bigg(\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zN(ds,dz)\\
&\quad-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zM(ds,dz)\big|X_{t_{k+1}}^b=Y_{t_{k+1}}^{b_0}\right]\Bigg)^2\Bigg]=M_{0,k,n}^b+M_{\geq 1,k,n}^b,
\end{aligned}\tag{5.24}$$

where

$$
\begin{aligned}
M^b_{0,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n} zN(ds,dz)\\
&\quad - \widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n} zM(ds,dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big],\\
M^b_{\geq 1,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\Big(\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n} zN(ds,dz)\\
&\quad - \widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n} zM(ds,dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big].
\end{aligned}
$$

We start treating $M^b_{0,k,n}$. Multiplying the random variable inside the conditional expectation of $M^b_{0,k,n}$ by $\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}+\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}$ and using equation (5.20), we get that

$$
\begin{aligned}
M^b_{0,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[(\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}+\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)})\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n} zM(ds,dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big]\\
&= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n} zM(ds,dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big]\\
&= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\Big(X^b_{t_{k+1}}-X^b_{t_k}-\int_{t_k}^{t_{k+1}}(a-bX^b_s)ds\\
&\quad -\sigma\int_{t_k}^{t_{k+1}}\sqrt{X^b_s}dB_s-\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz)-\Delta_n\int_{z\leq\upsilon_n} zm(dz)\Big)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big]\\
&\leq 5\sum_{i=1}^{5} M^b_{0,i,k,n},
\end{aligned}
\tag{5.25}
$$

where

$$
\begin{aligned}
M^b_{0,1,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}(X^b_{t_{k+1}}-X^b_{t_k})\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big],\\
M^b_{0,2,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}(a-bX^b_s)ds\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big],\\
M^b_{0,3,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\sigma\int_{t_k}^{t_{k+1}}\sqrt{X^b_s}dB_s\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big],\\
M^b_{0,4,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big],\\
M^b_{0,5,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big(\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\Delta_n\int_{z\leq\upsilon_n} zm(dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big].
\end{aligned}
$$

First, we treat the term $M^b_{0,1,k,n}$. Using equation (5.19) and the fact that there is no big jump of J^{υ_n} in the interval $[t_k,t_{k+1})$, we get that

$$
M^b_{0,1,k,n} = \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)}\Big((Y^{b_0}_{t_{k+1}}-Y^{b_0}_{t_k})\widetilde{\mathrm{E}}^b_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big)^2\Big]
$$

$$
\begin{aligned}
&= \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)} \Big(\Big(\int_{t_k}^{t_{k+1}} (a - b_0 Y^{b_0}_s) ds + \sigma \int_{t_k}^{t_{k+1}} \sqrt{Y^{b_0}_s} dW_s \\
&+ \int_{t_k}^{t_{k+1}} \int_{z \le \upsilon_n} z \widetilde{N}(ds, dz) + \Delta_n \int_{z \le \upsilon_n} z m(dz) \Big) \widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} | X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \big] \Big)^2 \Big] \\
&\le 4 \sum_{i=1}^{4} M^b_{0,1,i,k,n}, \qquad (5.26)
\end{aligned}
$$

where

$$
\begin{aligned}
M^b_{0,1,1,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)} \Big(\int_{t_k}^{t_{k+1}} (a - b_0 Y^{b_0}_s) ds \widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} | X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \big] \Big)^2 \Big], \\
M^b_{0,1,2,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)} \Big(\sigma \int_{t_k}^{t_{k+1}} \sqrt{Y^{b_0}_s} dW_s \widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} | X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \big] \Big)^2 \Big], \\
M^b_{0,1,3,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)} \Big(\int_{t_k}^{t_{k+1}} \int_{z \le \upsilon_n} z \widetilde{N}(ds, dz) \widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} | X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \big] \Big)^2 \Big], \\
M^b_{0,1,4,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\mathbf{1}_{\widehat{N}_{0,k}(\upsilon_n)} \Big(\Delta_n \int_{z \le \upsilon_n} z m(dz) \widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} | X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \big] \Big)^2 \Big].
\end{aligned}
$$

Using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ and Jensen's inequality together with Lemma 3.10,

$$
\begin{aligned}
&M^b_{0,1,1,k,n} \le \left(\widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\Big(\int_{t_k}^{t_{k+1}} (a - b_0 Y^{b_0}_s) ds \Big)^{2p} \Big] \right)^{\frac{1}{p}} \left(\widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} | X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \big] \Big] \right)^{\frac{1}{q}} \\
&\le \left(\Delta_n^{2p-1} \int_{t_k}^{t_{k+1}} \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[(a - b_0 Y^{b_0}_s)^{2p} \Big] ds \right)^{\frac{1}{p}} \left(\widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} \big] \right)^{\frac{1}{q}} \\
&\le C \left(\Delta_n^{2p-1} \int_{t_k}^{t_{k+1}} \big(a^{2p} + b_0^{2p} \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} [(Y^{b_0}_s)^{2p}] \big) ds \right)^{\frac{1}{p}} \left(\widetilde{\mathrm{P}}^{b}_{t_k, Y^{b_0}_{t_k}} \big(\widetilde{N}_{\ge 1,k}(\upsilon_n) \big) \right)^{\frac{1}{q}} \\
&\le C \left(\Delta_n^{2p} \left(1 + (Y^{b_0}_{t_k})^{2p} \right) \right)^{\frac{1}{p}} \left(e^{-\lambda_{\upsilon_n} \Delta_n} \lambda_{\upsilon_n} \Delta_n \right)^{\frac{1}{q}} \le C \left(1 + (Y^{b_0}_{t_k})^2 \right) \Delta_n^2 \left(\lambda_{\upsilon_n} \Delta_n \right)^{\frac{1}{q}}. \qquad (5.27)
\end{aligned}
$$

Next, using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ and Burkholder-David-Gundy's inequality together with Lemma 3.10,

$$
\begin{aligned}
M^b_{0,1,2,k,n} &\le \left(\widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\Big(\sigma \int_{t_k}^{t_{k+1}} \sqrt{Y^{b_0}_s} dW_s \Big)^{2p} \Big] \right)^{\frac{1}{p}} \left(\widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} | X^b_{t_{k+1}} = Y^{b_0}_{t_{k+1}} \big] \Big] \right)^{\frac{1}{q}} \\
&\le C \left(\widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} \Big[\Big(\int_{t_k}^{t_{k+1}} Y^{b_0}_s ds \Big)^{p} \Big] \right)^{\frac{1}{p}} \left(\widetilde{\mathrm{E}}^{b}_{t_k, Y^{b_0}_{t_k}} \big[\mathbf{1}_{\widetilde{N}_{\ge 1,k}(\upsilon_n)} \big] \right)^{\frac{1}{q}} \\
&\le C \left(\Delta_n^{p-1} \int_{t_k}^{t_{k+1}} \widehat{\mathrm{E}}^{b_0}_{t_k, Y^{b_0}_{t_k}} [(Y^{b_0}_s)^{p}] ds \right)^{\frac{1}{p}} \left(\widetilde{\mathrm{P}}^{b}_{t_k, Y^{b_0}_{t_k}} \big(\widetilde{N}_{\ge 1,k}(\upsilon_n) \big) \right)^{\frac{1}{q}} \\
&\le C \left(\Delta_n^{p} \left(1 + (Y^{b_0}_{t_k})^{p} \right) \right)^{\frac{1}{p}} \left(e^{-\lambda_{\upsilon_n} \Delta_n} \lambda_{\upsilon_n} \Delta_n \right)^{\frac{1}{q}} \le C \left(1 + Y^{b_0}_{t_k} \right) \Delta_n \left(\lambda_{\upsilon_n} \Delta_n \right)^{\frac{1}{q}}. \quad (5.28)
\end{aligned}
$$

Using Burkholder-David-Gundy's inequality,

$$M^b_{0,1,3,k,n} \leq \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\Big(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{N}(ds,dz)\Big)^2\Big] \leq C\Delta_n\int_{z\leq\upsilon_n} z^2 m(dz). \tag{5.29}$$

Observe that

$$M^b_{0,1,4,k,n} \leq \left(\Delta_n\int_{z\leq\upsilon_n} zm(dz)\right)^2. \tag{5.30}$$

Therefore, from (5.26)-(5.30), we have shown that

$$M^b_{0,1,k,n} \leq C(1+(Y^{b_0}_{t_k})^2)\Delta_n\left((\lambda_{\upsilon_n}\Delta_n)^{\frac{1}{q}} + \int_{z\leq\upsilon_n} z^2 m(dz) + \Delta_n\left(\int_{z\leq\upsilon_n} zm(dz)\right)^2\right). \tag{5.31}$$

Next, we treat the term $M^b_{0,2,k,n}$. Using Jensen's inequality, Lemma 3.10 and Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
M^b_{0,2,k,n} &\leq \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\Big(\int_{t_k}^{t_{k+1}}(a-bX^b_s)ds\Big)^2\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big]\\
&= \widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\Big(\int_{t_k}^{t_{k+1}}(a-bX^b_s)ds\Big)^2\Big]\\
&\leq \left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\Big(\int_{t_k}^{t_{k+1}}(a-bX^b_s)ds\Big)^{2p}\Big]\right)^{\frac{1}{p}}\left(\widetilde{\mathrm{P}}^{b}_{t_k,Y^{b_0}_{t_k}}\big(\widetilde{N}_{\geq 1,k}(\upsilon_n)\big)\right)^{\frac{1}{q}}\\
&\leq C\left(1+(Y^{b_0}_{t_k})^2\right)\Delta_n^2\left(\lambda_{\upsilon_n}\Delta_n\right)^{\frac{1}{q}}.
\end{aligned} \tag{5.32}$$

Using Jensen's inequality, Lemma 3.10 and Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ together with Burkholder-David-Gundy's inequality,

$$\begin{aligned}
M^b_{0,3,k,n} &\leq \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\Big(\sigma\int_{t_k}^{t_{k+1}}\sqrt{X^b_s}dB_s\Big)^2\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big]\\
&= \widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\Big(\sigma\int_{t_k}^{t_{k+1}}\sqrt{X^b_s}dB_s\Big)^2\Big]\\
&\leq \left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\Big(\sigma\int_{t_k}^{t_{k+1}}\sqrt{X^b_s}dB_s\Big)^{2p}\Big]\right)^{\frac{1}{p}}\left(\widetilde{\mathrm{P}}^{b}_{t_k,Y^{b_0}_{t_k}}\big(\widetilde{N}_{\geq 1,k}(\upsilon_n)\big)\right)^{\frac{1}{q}}\\
&\leq C\left(1+Y^{b_0}_{t_k}\right)\Delta_n\left(\lambda_{\upsilon_n}\Delta_n\right)^{\frac{1}{q}}.
\end{aligned} \tag{5.33}$$

Using Jensen's inequality, Lemma 3.10 and Burkholder-David-Gundy's inequality,

$$\begin{aligned}
M^b_{0,4,k,n} &\leq \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Big[\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\Big(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz)\Big)^2\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Big]\Big]\\
&= \widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Big[\Big(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz)\Big)^2\Big] \leq C\Delta_n\int_{z\leq\upsilon_n} z^2 m(dz).
\end{aligned} \tag{5.34}$$

Finally, observe that

$$M^b_{0,5,k,n} \leq \left(\Delta_n\int_{z\leq\upsilon_n} zm(dz)\right)^2. \tag{5.35}$$

Thus, from (5.25) and (5.31)-(5.35), we have shown that

$$M_{0,k,n}^{b}\leq C\left(1+(Y_{t_k}^{b_0})^2\right)\Delta_n\left((\lambda_{\upsilon_n}\Delta_n)^{\frac{1}{q}}+\int_{z\leq\upsilon_n}z^2m(dz)+\Delta_n\left(\int_{z\leq\upsilon_n}zm(dz)\right)^2\right). \quad (5.36)$$

Finally, we treat $M_{\geq1,k,n}^{b}$. Multiplying the random variable inside the conditional expectation of $M_{\geq1,k,n}^{b}$ by $\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}+\mathbf{1}_{\widetilde{N}_{\geq1,k}(\upsilon_n)}$, we get that

$$\begin{aligned}
M_{\geq1,k,n}^{b}&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\Bigg[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\Bigg(\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zN(ds,dz)\\
&\quad-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\left(\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}+\mathbf{1}_{\widetilde{N}_{\geq1,k}(\upsilon_n)}\right)\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zM(ds,dz)\big|X_{t_{k+1}}^{b}=Y_{t_{k+1}}^{b_0}\right]\Bigg)^2\Bigg]\\
&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\Bigg[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\Bigg(\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zN(ds,dz)\\
&\quad-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\mathbf{1}_{\widetilde{N}_{\geq1,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}\int_{z>\upsilon_n}zM(ds,dz)\big|X_{t_{k+1}}^{b}=Y_{t_{k+1}}^{b_0}\right]\Bigg)^2\Bigg]\\
&=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\Bigg[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\Bigg(Y_{t_{k+1}}^{b_0}-Y_{t_k}^{b_0}-\int_{t_k}^{t_{k+1}}(a-b_0Y_s^{b_0})ds-\sigma\int_{t_k}^{t_{k+1}}\sqrt{Y_s^{b_0}}dW_s\\
&\quad-\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{N}(ds,dz)-\Delta_n\int_{z\leq\upsilon_n}zm(dz)-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\Bigg[\mathbf{1}_{\widetilde{N}_{\geq1,k}(\upsilon_n)}\Bigg(X_{t_{k+1}}^{b}-X_{t_k}^{b}\\
&\quad-\int_{t_k}^{t_{k+1}}(a-bX_s^{b})ds-\sigma\int_{t_k}^{t_{k+1}}\sqrt{X_s^{b}}dB_s-\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{M}(ds,dz)\\
&\quad-\Delta_n\int_{z\leq\upsilon_n}zm(dz)\Bigg)\big|X_{t_{k+1}}^{b}=Y_{t_{k+1}}^{b_0}\Bigg]\Bigg)^2\Bigg]\leq9\sum_{i=1}^{9}M_{\geq1,i,k,n}^{b}, \quad (5.37)
\end{aligned}$$

where

$$M_{\geq1,1,k,n}^{b}=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\left(Y_{t_{k+1}}^{b_0}-Y_{t_k}^{b_0}-\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\mathbf{1}_{\widetilde{N}_{\geq1,k}(\upsilon_n)}(X_{t_{k+1}}^{b}-X_{t_k}^{b})\big|X_{t_{k+1}}^{b}=Y_{t_{k+1}}^{b_0}\right]\right)^2\right],$$

$$M_{\geq1,2,k,n}^{b}=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\left(\int_{t_k}^{t_{k+1}}(a-b_0Y_s^{b_0})ds\right)^2\right],$$

$$M_{\geq1,3,k,n}^{b}=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\left(\sigma\int_{t_k}^{t_{k+1}}\sqrt{Y_s^{b_0}}dW_s\right)^2\right],$$

$$M_{\geq1,4,k,n}^{b}=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\left(\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n}z\widetilde{N}(ds,dz)\right)^2\right],$$

$$M_{\geq1,5,k,n}^{b}=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\left(\Delta_n\int_{z\leq\upsilon_n}zm(dz)\right)^2\right],$$

$$M_{\geq1,6,k,n}^{b}=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\mathbf{1}_{\widetilde{N}_{\geq1,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}(a-bX_s^{b})ds\big|X_{t_{k+1}}^{b}=Y_{t_{k+1}}^{b_0}\right]\right)^2\right],$$

$$M_{\geq1,7,k,n}^{b}=\widehat{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b_0}\left[\mathbf{1}_{\widehat{N}_{\geq1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}_{t_k,Y_{t_k}^{b_0}}^{b}\left[\mathbf{1}_{\widetilde{N}_{\geq1,k}(\upsilon_n)}\sigma\int_{t_k}^{t_{k+1}}\sqrt{X_s^{b}}dB_s\big|X_{t_{k+1}}^{b}=Y_{t_{k+1}}^{b_0}\right]\right)^2\right],$$

$$M^b_{\geq 1,8,k,n} = \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right],$$

$$M^b_{\geq 1,9,k,n} = \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\Delta_n\int_{z\leq\upsilon_n} zm(dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right].$$

First, we treat the term $M^b_{\geq 1,1,k,n}$. Using equation (5.20) and the fact that there is no big jump of $\widetilde{J}^{\upsilon_n}$ in the interval $[t_k,t_{k+1})$, we get that

$$\begin{aligned}
M^b_{\geq 1,1,k,n} &= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\big(Y^{b_0}_{t_{k+1}}-Y^{b_0}_{t_k}-(Y^{b_0}_{t_{k+1}}-Y^{b_0}_{t_k})\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\big[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\big]\big)^2\right]\\
&= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\left(Y^{b_0}_{t_{k+1}}-Y^{b_0}_{t_k}\right)\left(1-\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)\right)^2\right]\\
&= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\left(Y^{b_0}_{t_{k+1}}-Y^{b_0}_{t_k}\right)\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[1-\mathbf{1}_{\widetilde{N}_{\geq 1,k}(\upsilon_n)}\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right]\\
&= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\left(Y^{b_0}_{t_{k+1}}-Y^{b_0}_{t_k}\right)\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right]\\
&= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}(X^b_{t_{k+1}}-X^b_{t_k})\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right]\\
&= \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\Bigg[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\Bigg(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\Bigg[\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}\Bigg(\int_{t_k}^{t_{k+1}}(a-bX^b_s)ds+\sigma\int_{t_k}^{t_{k+1}}\sqrt{X^b_s}dB_s\\
&\qquad +\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz)+\Delta_n\int_{z\leq\upsilon_n} zm(dz)\Bigg)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\Bigg]\Bigg)^2\Bigg]\\
&\leq 4\sum_{i=1}^{4} M^b_{\geq 1,1,i,k,n},
\end{aligned} \tag{5.38}$$

where

$$M^b_{\geq 1,1,1,k,n} = \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}(a-bX^b_s)ds\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right],$$

$$M^b_{\geq 1,1,2,k,n} = \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}\sigma\int_{t_k}^{t_{k+1}}\sqrt{X^b_s}dB_s\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right],$$

$$M^b_{\geq 1,1,3,k,n} = \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}\int_{t_k}^{t_{k+1}}\int_{z\leq\upsilon_n} z\widetilde{M}(ds,dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right],$$

$$M^b_{\geq 1,1,4,k,n} = \widehat{\mathrm{E}}^{b_0}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widehat{N}_{\geq 1,k}(\upsilon_n)}\left(\widetilde{\mathrm{E}}^{b}_{t_k,Y^{b_0}_{t_k}}\left[\mathbf{1}_{\widetilde{N}_{0,k}(\upsilon_n)}\Delta_n\int_{z\leq\upsilon_n} zm(dz)\big|X^b_{t_{k+1}}=Y^{b_0}_{t_{k+1}}\right]\right)^2\right].$$

Proceeding as the terms $M^b_{0,1,1,k,n}$, $M^b_{0,1,2,k,n}$, $M^b_{0,1,3,k,n}$ and $M^b_{0,1,4,k,n}$, we get that

$$\begin{aligned}
&M^b_{\geq 1,1,1,k,n} \leq C\left(1+(Y^{b_0}_{t_k})^2\right)\Delta_n^2\left(\lambda_{\upsilon_n}\Delta_n\right)^{\frac{1}{q}}, \quad M^b_{\geq 1,1,2,k,n} \leq C\left(1+Y^{b_0}_{t_k}\right)\Delta_n\left(\lambda_{\upsilon_n}\Delta_n\right)^{\frac{1}{q}},\\
&M^b_{\geq 1,1,3,k,n} \leq C\Delta_n\int_{z\leq\upsilon_n} z^2m(dz), \quad M^b_{\geq 1,1,4,k,n} \leq \left(\Delta_n\int_{z\leq\upsilon_n} zm(dz)\right)^2.
\end{aligned} \tag{5.39}$$

Thus, from (5.38) and (5.39), we have shown that

$$M^b_{\geq 1,1,k,n} \leq C(1+(Y^{b_0}_{t_k})^2)\Delta_n \left((\lambda_{\upsilon_n}\Delta_n)^{\frac{1}{q}} + \int_{z\leq \upsilon_n} z^2 m(dz) + \Delta_n \Big(\int_{z\leq \upsilon_n} z m(dz) \Big)^2 \right). \quad (5.40)$$

Similarly, we have that

$$M^b_{\geq 1,2,k,n} + M^b_{\geq 1,6,k,n} \leq C \left(1+(Y^{b_0}_{t_k})^2\right) \Delta_n^2 \left(\lambda_{\upsilon_n}\Delta_n\right)^{\frac{1}{q}},$$

$$M^b_{\geq 1,3,k,n} + M^b_{\geq 1,7,k,n} \leq C \left(1+Y^{b_0}_{t_k}\right) \Delta_n \left(\lambda_{\upsilon_n}\Delta_n\right)^{\frac{1}{q}},$$

$$M^b_{\geq 1,4,k,n} + M^b_{\geq 1,8,k,n} \leq C\Delta_n \int_{z\leq \upsilon_n} z^2 m(dz), \quad M^b_{\geq 1,5,k,n} + M^b_{\geq 1,9,k,n} \leq \left(\Delta_n \int_{z\leq \upsilon_n} z m(dz)\right)^2.$$

This, together with (5.37) and (5.40), concludes that

$$M^b_{\geq 1,k,n} \leq C(1+(Y^{b_0}_{t_k})^2)\Delta_n \left((\lambda_{\upsilon_n}\Delta_n)^{\frac{1}{q}} + \int_{z\leq \upsilon_n} z^2 m(dz) + \Delta_n \left(\int_{z\leq \upsilon_n} z m(dz) \right)^2 \right). \quad (5.41)$$

Thus, from (5.21), (5.22), (5.23), (5.24), (5.36) and (5.41), the result follows. □

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