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Metric regularity relative to a cone

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Tribute to Professor Alexander Ioffe on his eighty birthday. With recognition for research achievement and friendship

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Abstract The purpose of this paper is to discuss some of the highlights of the theory of metric regularity relative to a cone. For instance, we establish a slope and some coderivative characterizations of this concept, as well as some stability results.

Keywords Abstract subdifferential · Metric regularity · Directional metric regularity · Metric subregularity · directional Hölder metric subregularity · Coderivative

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1 Introduction and preliminaries

Since the pioneering work of Robinson [37, 38], the study of optimization and complementarity problems, models in game theory, control and design problems, as well as variational inequalities, leads to the study of inclusions of the type:

$$y \in F(x) \quad \text{for} \quad (x,y) \in X \times Y, \tag{1.1}$$

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where, X, Y are metric spaces and $F : X \rightrightarrows Y$ is a set-valued mapping which describes the model under consideration. Undoubtedly, stability of the solutions of (1.1) is an important issue and has been subject over the recent years to a large number of contributions. These works have led to significant progress, both from the theoretical and the numerical point of view. We refer the reader to the monographs [7, 10, 11, 15, 26, 33, 36, 39], to some recent publications [12, 14, 16] and the references therein.

Notation

Before we go any further, let us recall several notions from set-valued analysis. By a set-valued mapping (multifunction) $F : X \rightrightarrows Y$, we mean a mapping from X into the (possibly empty) subsets of Y. For such a mapping, the set $\operatorname{gph} F := \{(x,y) \in X \times Y : y \in F(x)\}$ is the graph of T, the domain of T is $\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\}$, and $F^{-1} : Y \rightrightarrows X$ is the inverse of T defined, for each $y \in Y$, by

$$x \in F^{-1}(y) \iff y \in F(x).$$

In any metric space under consideration, d is the corresponding metric, $\mathbb{B}(x,\rho)$ and $\bar{\mathbb{B}}(x,\rho)$ are the open and the closed ball with radius $\rho > 0$ around $x \in X$, respectively. We also note $\mathbb{B} := \mathbb{B}(0,1)$ and $\bar{\mathbb{B}} := \bar{\mathbb{B}}(0,1)$, the open and closed unit ball when, in addition, the space is a linear vector space. The distance from a point $x \in X$ to a subset Ω of X is $d(x,\Omega) := \inf_{u\in\Omega} d(x,u)$ and $\operatorname{cl}\Omega$ is the closure of Ω. Given a subset V of $X \times Y$ and a point $(x,y) \in X \times Y$, we set

$$V_x := \{z \in Y : \ (x,z) \in V\} \quad \text{and} \quad V_y := \{u \in X : \ (u,y) \in V\}.$$

Let us begin by reminding the notion of metric regularity relative to a set V.

Definition 1.1 (Ioffe [25]) Let X and Y be metric spaces, and let $V \subset X \times Y$. We say that a set-valued mapping $F : X \rightrightarrows Y$ is *metrically regular relatively to V at $(\bar{x},\bar{y}) \in V \cap \operatorname{gph} F$ with a modulus $\tau > 0$*, if there exists $\varepsilon > 0$ such that

$$d\big(x, F^{-1}(y) \cap \operatorname{cl} V_y\big) \leq \tau d(y, F(x)) \tag{1.2}$$

whenever $(x,y) \in \big(\mathbb{B}(\bar{x},\varepsilon) \times \mathbb{B}(\bar{y},\varepsilon)\big) \cap V$ and $d(y,F(x)) < \varepsilon$.

The infinum of all the scalars $\tau > 0$, such that (1.2) holds for some $\varepsilon > 0$, is called the *exact modulus of the metric regularity relative to V for F at* $(\bar{x},\bar{y})$ and is denoted by $\operatorname{reg}_V F(\bar{x},\bar{y})$.

An important subcase of this concept is the notion of directional metric regularity introduced by Arutuynov and Izmailov in [5], and extensively studied in Arutyunov et al [4], Gfrerer [17,18], Ioffe [25] and Ngai-Théra [19]. The idea behind relative metric regularity is that the values of ambient and image variables are not arbitrary points or neighborhood of a certain nominal point of the product space, but are taken from a certain set V. Thus, by choosing among possible V, one may obtain various versions of metric regularity models existing in the literature and central for the analysis of sensitivity and controllability in optimization and control. For instance when $V = X \times Y$, this model subsumes the usual local metric regularity. Some other examples are described in [25, p. 343].

Some other versions of relative metric regularity (equivalently, relative covering) have been initially introduced and studied since 1980 by Dimitruk-Milyutin-Osmolovskii [13] and then by Mordukhovich in [34], [31] and [32]. For instance, let us mention a relative covering property [34, Exercise 3.50], that is taken from [31] and [32]: given set-valued mappings $\Omega : X \rightrightarrows X$ and $F : X \rightrightarrows Y$ between Banach spaces, a real $\kappa > 0$ and $\bar{x} \in \Omega(\bar{x}) \cap \operatorname{dom} F$, we say that F has the covering property around $\bar{x}$ relatively to the mapping Ω with some modulus κ, if there is a neighborhood U of $\bar{x}$ such that $F(x) + \kappa r\mathbb{B} \subset F((x + r\mathbb{B}) \cap \Omega(x))$ whenever $x + r\mathbb{B} \subset U, r > 0$. Observe that if the mapping $\Omega : X \rightrightarrows X$ is a constant mapping: $\Omega(x) := \Omega \subseteq X$, for some given $\Omega \subseteq X$, for all $x \in X$, then the latter definition coincides with Definition 1.1 with $V := \Omega \times Y$. In general, the two definitions are different.

Metric regularity (and related properties) is a powerful tool for dealing with problems related to optimization and variational analysis. For these reasons, it has a long and fascinating history, which goes back to the Banach open mapping theorem. Its most important applications concern the study of stability of variational systems as well as convergence of Newton's type methods (see, e. g., [1–3, 29]). While more recently, due to algorithmic purposes, an increasing attention has been paid to metric regularity concepts, it has been observed that in some classes of particular problems from optimization, variational analysis, control,..., the usual metric regularity property sometimes does not meet practical requirements. Therefore, relative metric regularity and its variants appear to be convenient concepts for different purposes of applications. For instance, in some recent works [30, p. 1155], [6], authors make use of some kinds of relative metric regularity to quantitative convergence analysis of algorithms, such as alternative projection methods, cyclic projections, projected gradients and linear convergence of the Douglas-Rachford algorithm. Some applications in sensitivity analysis of the directional metric regularity are, for instance, also given in [5], [4], [9].

Motivated by the above-mentioned applications, in this contribution we will consider the particular case of relative metric regularity given in Definition 1.1 and called *metric regularity relatively to a cone.* This concept is a natural generalization of directional metric regularity in the sense that we have replaced some direction by some cone.

The paper layout is as follows: after some basic definitions and useful tools given in Section 2, we formulate in Section 3 (Theorem 3.1) a slope criterion for metric regularity relative to a set. In Section 4, we show (Theorem 4.1) that similarly to directional metric regularity studied in [4, 23, 25], metric regularity with respect to a cone is also stable under a suitable Lipschitz perturbation. We finish the paper with Section 5 which contains some coderivative characterization of metric regularity relative to a cone. It is worth mentioning that this coderivative characterization generalizes the one in [23], not only from "direction" to "cone", but also b y the fact that the assumption of pseudo-Lipschitz property of the multifunction under consideration has been removed.

To conclude this introduction, we express the hope that this abstract contribution will serve in a future paper to develop practicable approaches to solving problems arising in concrete applications, in sensitivity analysis, in convergence analysis of algorithms or in Newton 's type methods for generalized equations.

Let Y be a normed linear space; given a cone $C \subseteq Y$ (not nessesarily convex), for any real $\delta > 0$, we define the following two sets:

$$C(\delta) := \{v \in Y : \ d(v,C) \leq \delta \|v\|\}.$$

and

$$V_F(C,\delta) := \{(x,y) \in X \times Y : \ y \in F(x) + C(\delta)\}.$$

Definition 1.2 We will say that F is metrically regular relatively to C, if there exists $\delta > 0$ such that F is metrically regular relatively to $V := V_F(C,\delta)$.

It results from Definition 1.1 that F is metrically regular relatively to a cone C at $(\bar{x},\bar{y}) \in V_F(C,\delta) \cap \operatorname{gph} F$ with a modulus $\tau > 0$, if there exists $\varepsilon > 0$ such that

$$d\big(x, F^{-1}(y) \cap \operatorname{cl} V_{F,y}(C,\delta)\big) \leq \tau d(y, F(x)) \tag{1.3}$$

whenever $(x,y) \in \big(\mathbb{B}(\bar{x},\varepsilon) \times \mathbb{B}(\bar{y},\varepsilon)\big) \cap V_F(C,\delta)$ and $d(y,F(x)) < \varepsilon$.

The organisation of the paper is as follows. We start to establish in Theorem 3.1 a slope characterization of relative metric regularity with respect to a set V. Next we use this result in Theorem 4.1 when we establish the stability of metric regularity with respect to a cone under the perturbation by a Lipschitz continuous function. In Section 5, we give a coderivative characterization of metric regularity relative to a cone. It should be mentioned that this characterization is given by removing the pseudo-Lipschitz property of the multifunction under consideration as it was the case in [19].

2 Basic definitions, notations and basic tools

In this section we recall some necessary notions and results from Variational Analysis used throughout the paper. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function defined on a Banach space X, we denote by $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$, the *domain* of f. The *Fréchet (regular) subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is given as

$$\partial f(\bar{x}) = \left\{ x^* \in X^* : \liminf_{x \to \bar{x},\quad x \neq \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

For reader's convenience, we would like to mention that the terminology *regular subdifferential* alongside of Fréchet subdifferential is also popular due to its use in Rockafellar and Wets [39]. The Fréchet subdifferential is always convex and reduces to the classical subdifferential of convex analysis for the case of convex functions. Note also that this subdifferential obviously satisfies the generalized Fermat rule: $0 \in \partial f(x)$ if x is a local minimizer of f. Every element of the Fréchet subdifferential is termed as a Fréchet (regular) subgradient. If $\bar{x}$ is a point where $f(\bar{x}) = \infty$, then we set $\partial f(\bar{x}) = \emptyset$. In fact one can show that an element x^* is a Fréchet subgradient of f at $\bar{x}$ if and only if

$$f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \quad \text{where} \quad \lim_{x \to \bar{x}} \frac{o(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = 0.$$

Some of the results given in this article will be proved in the context of Asplund spaces. There is a plethora of equivalent characterizations of Asplund spaces and many of them can be found, e.g., in [33] and its bibliography and also in the well written introduction for beginners by Yost [40]. Asplund spaces are Banach spaces for which every convex continuous function is generically Fréchet differentiable. In particular, any space with Fréchet smooth renorming (and hence any reflexive space) is Asplund, as well as each Banach space such that each of its separable subspaces has a separable dual.

It is well known that the Fréchet subdifferential satisfies a fuzzy sum rule on Asplund spaces ([33, Theorem 2.33]). More precisely, if X is an Asplund space and $f_1, f_2 : X \to \mathbb{R} \cup \{\infty\}$ are such that f_1 is Lipschitz continuous around $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_2 is lower semicontinuous around $\bar{x}$, then for any $\gamma > 0$ one has

$$\partial(f_1 + f_2)(\bar{x}) \subset \bigcup \{\partial f_1(x_1) + \partial f_2(x_2) \mid x_i \in \bar{x} + \gamma \overline{B}_X, |f_i(x_i) - f_i(\bar{x})| \leq \gamma, i = 1, 2\} + \gamma B_{X^*}. \tag{2.1}$$

Given a nonempty closed set $C \subseteq X$, the *indicator function* associated to C is the function ι_C defined by $\iota_C(x) = 0$, when $x \in C$ and $\iota_C(x) = \infty$ otherwise. The *Fréchet (regular) normal cone* to C at $\bar{x}$ is the set $N(C, \bar{x}) := \partial \iota_C(\bar{x})$ if $\bar{x} \in C$, and $N(C, \bar{x}) := \partial \iota_C(\bar{x}) = \emptyset$ if $\bar{x} \notin C$. It is a closed and convex cone in X^*.

We will have to use the following fuzzy intersection formula for Fréchet normal cones (see, e.g., [21]).

Lemma 2.1 *Let C_i, $i = 1, \ldots, k$, be nonempty closed subsets of an Asplund space X. For given $\bar{x} \in C := \bigcap_{i=1}^k C_i$, assume that for any sequences $(x_n^i) \in C_i$, $(x_n^{i*}) \subseteq X^*$ with $x_n^{i*} \in N(C_i, x_n^i)$, $x_n^i \to \bar{x}$, $i = 1, \ldots, k$,*

$$\lim_{n \to \infty} \left\| \sum_{i=1}^k x_n^{i*} \right\| = 0 \quad \Longrightarrow \quad \lim_{n \to \infty} \|x_n^{i*}\| = 0, \text{ for all } i = 1, \ldots, k.$$

Then, for any x near $\bar{x}$ and for every $\varepsilon > 0$, one has

$$N(C, x) \subseteq \left\{ \sum_{i=1}^k N(C_i, x^i) + \varepsilon B_{X^*} : \ x^i \in C_i \cap \mathbb{B}(x, \varepsilon), \ i = 1, \ldots, k \right\}.$$

The *limiting subdifferential* (also known as the Mordukhovich subdifferential) is defined as

$$\partial_{\mathscr{M}} f(\bar{x}) = \left\{ x^* \in X^* : \ \exists x_k \to \bar{x}, \ f(x_k) \to f(\bar{x}), \ \text{and} \ \exists x_k^* \in \partial f(x_k), \ x_k^* \xrightarrow{*} x^* \right\}.$$

In other words, the graph of the limiting Fréchet subdifferential is the sequential closure of the graph of the Fréchet subdifferential in the product of the norm topology on X with the weak*- topology on X^*.

The concept of *limiting normal cone* $N_{\mathscr{M}}(C,\bar{x})$ to a closed set C can be defined through the indicator function of the set:

$$N_{\mathscr{M}}(C,\bar{x}) := \partial_{\mathscr{M}}\delta_C(\bar{x}).$$

Given a normal cone $\mathbb{N}$, we can associate with a set-valued mapping $F : X \rightrightarrows Y$ a coderivative $D^*_{\mathbb{N}} : Y^{\star} \rightrightarrows X^*$ through the formula

$$D^*_{\mathbb{N}}F(x,y)(y^*) := \left\{x^* \in X^* \mid (x^*,-y^*) \in \mathbb{N}(\text{gph}\, F,(x,y))\right\}. \tag{2.2}$$

In variational analysis, this notion is recognized to be a powerful tool when applied to problems of optimization and control (e.g., see [28, 33, 35], and the references therein). In what follows, when $\mathbb{N}$ is the Fréchet (regular) normal cone, the coderivative of F will be denoted by $D^*_{\mathscr{F}}F$, while when $\mathbb{N}$ is the limiting normal cone, then we will use the notation by $D^*_{\mathscr{M}}F$. When $\mathbb{N}$ is the normal cone to a convex set C, then all the coderivatives coincide and are simply denoted by D^*.

3 Slope criteria for relative metric regularity

In this section (unless clearly indicated otherwise), we suppose that X is a complete metric space and Y is a metric space. We suppose given a set $V \subset X \times Y$, a real $\beta \in (0,1]$, and a set-valued mapping $F : X \rightrightarrows Y$.

Given $a \in \mathbb{R}$, we set $a_+ = \max\{a,0\}$. Recall from [24], that for an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in X$ with $f(x) < +\infty$, the local and the global strong slope $|\nabla f|(x)$ and $|\Gamma f|(x)$ of f at x are defined by

$$|\nabla f|(x) = \limsup_{x \neq y \to x} \frac{[f(x)-f(y)]_+}{d(x,y)} \quad \text{and} \quad |\Gamma f|(x) = \sup_{y \neq x} \frac{[f(x)-f(y)]}{d(x,y)}. \tag{3.1}$$

If $f(x) = +\infty$, then we set $|\nabla f|(x) = |\Gamma f|(x) = +\infty$.

From now on, P will denote a topological space considered in applications as the space of parameters. The following proposition is a restatement of Theorem 2 and Corollary 1 in [20].

Proposition 3.1 *Let $f : X \times P \to [0,+\infty]$ be a function. For each $p \in P$, set*

$$S(p) = \{x \in X : \quad f(x,p) = 0\}.$$

Suppose that $(\bar{x},\bar{p}) \in X \times P$ is such that $\bar{x} \in S(\bar{p})$, and that, for any p near $\bar{p}$, the function $f(\cdot,p)$ is lower semicontinuous at $\bar{x}$, and $f(\bar{x},\cdot)$ is continuous at $\bar{p}$. Let $\tau > 0$ be given and consider the following statements:
(i) There exist $\gamma > 0$ and a neighborhood $\mathscr{V} \times \mathscr{W}$ of $(\bar{x},\bar{p})$ in $X \times P$ such that for any $p \in \mathscr{W}$, we have $\mathscr{V} \cap S(p) \neq \emptyset$ and

$$d(x,S(p)) \leq \tau f(x,p) \quad \textit{for all} \quad (x,p) \in \mathscr{V} \times \mathscr{W} \ \textit{ with } f(x,p) \in (0,\gamma); \tag{3.2}$$

(ii) There exist a neighborhood $\mathscr{V} \times \mathscr{W}$ of $(\bar{x},\bar{p})$ in $X \times P$ and $\gamma > 0$ such that for each $(x,p) \in \mathscr{V} \times \mathscr{W}$ with $f(x,p) \in (0,\gamma)$ and for any $\varepsilon > 0$, there exists $z \in X$ such that

$$0 < d(x,z) < (\tau+\varepsilon)\big(f(x,p) - f(z,p)\big); \tag{3.3}$$

(iii) There exists a neighborhood $\mathscr{V} \times \mathscr{W}$ of $(\bar{x}, \bar{p})$ in $X \times P$ along with positive reals γ and τ such that $|\nabla f(\cdot, p)|(x) \geq 1/\tau$ for all $(x, p) \in \mathscr{V} \times \mathscr{W}$ with $f(x, p) \in (0, \gamma)$.

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

Note that since f takes non-negative values only, the continuity of $f(\bar{x}, \cdot)$ at $\bar{p}$ is equivalent to the upper semicontinuity at the same point.

Definition 3.1 For each $y \in Y$, the lower semicontinuous envelope relative to V of the function $x \mapsto d(y, F(x))$ is defined by

$$\varphi_{F,V}(x,y) := \begin{cases} \liminf\limits_{\mathrm{cl}\, V_y \times Y \ni (u,v) \to (x,y)} d(v, F(u)) = \liminf\limits_{\mathrm{cl}\, V_y \ni u \to x} d(y, F(u)) & \text{if } x \in \mathrm{cl}\, V_y \\ +\infty & \text{otherwise.} \end{cases} \tag{3.4}$$

Equality in the above definition holds because the function $d(\cdot, F(u))$ is Lipschitz. Observe that $\varphi_{F,V}(x,y) \geq 0$ and $\varphi_{F,V}(x,y) \leq d(y, F(x))$ for every $(x,y) \in \mathrm{cl}\, V_y \times Y$.

Let us start with the following two observations.

Lemma 3.1 *Suppose that the multifunction $F : X \rightrightarrows Y$ has a closed graph. Then*

(i)

$$F^{-1}(y) \cap \mathrm{cl}\, V_y = \{x \in X : \varphi_{F,V}(x,y) = 0\} \quad \textit{whenever} \quad y \in Y;$$

(ii) F is metrically regular relative to V at (x_0, y_0) with a modulus $\tau > 0$, if and only if there exists $\varepsilon > 0$ such that

$$d(x, F^{-1}(y) \cap \mathrm{cl}\, V_y) \leq \tau \varphi_{F,V}(x,y)$$

for all $(x,y) \in (\mathbb{B}(x_0, \varepsilon) \times \mathbb{B}(y_0, \varepsilon)) \cap V$ with $d(y, F(x)) < \varepsilon$.

Proof. (i). Fix any $y \in Y$. If $x \in F^{-1}(y) \cap \mathrm{cl}\, V_y$, then $y \in F(x)$, and so

$$0 = d(y, F(x)) \geq \varphi_{F,V}(x,y) \geq 0.$$

Conversely, if $x \in X$ verifies $\varphi_{F,V}(x,y) = 0$, this means that there is a sequence $(u_n)_{n \in \mathbb{N}}$ in $\mathrm{cl}\, V_y$ converging to x such that $d(y, F(x_n)) \to 0$. This in turn implies the existence of a sequence $(y_n)_{n \in \mathbb{N}}$ converging to y such that $y_n \in F(u_n)$ for every $n \in \mathbb{N}$. As the graph of F is closed, it follows that $x \in F^{-1}(y)$, as claimed.

(ii). The first part of the proof follows from the fact that obviously the estimate $d(x, F^{-1}(y) \cap \mathrm{cl}\, V_y) \leq \tau \varphi_{F,V}(x,y)$ implies that $d(x, F^{-1}(y) \cap \mathrm{cl}\, V_y) \leq \tau d(y, F(x))$ for the same (x,y). Conversely, fix $(x,y) \in (\mathbb{B}(x_0, \varepsilon) \times \mathbb{B}(y_0, \varepsilon)) \cap V$ with $d(y, F(x)) < \varepsilon$ and take a sequence $(x_n) \subset \mathrm{cl}\, V_y$ converging to x such that $d(y, F(x_n))$ tends to $\varphi_{F,V}(x,y)$. Then, for large n, $(x_n, y) \in (\mathbb{B}(x_0, \varepsilon) \times \mathbb{B}(y_0, \varepsilon)) \cap V$ and $d(y, F(x_n)) < \varepsilon$. Thus, by assumption we have $d(x_n, F^{-1}(y) \cap \mathrm{cl}\, V_y) \leq \tau d(y, F(x_n))$. Passing to the limit as n goes to $+\infty$ gives $d(x, F^{-1}(y) \cap \mathrm{cl}\, V_y) \leq \tau \varphi_{F,V}(x,y)$ which completes the proof. □

From Proposition 3.1 and Lemma 3.1, we obtain the following slope characterizations of the relative metric regularity.

Theorem 3.1 *Let X and Y be metric spaces with X complete and $F : X \rightrightarrows Y$ be a set-valued mapping with closed graph. Let $(\bar{x},\bar{y}) \in \operatorname{gph} F \cap V$, $V \subset X \times Y$; $\tau \in (0,+\infty)$ be given. Then, among the following statements,*

(i) F is metrically regular relative to V at $(\bar{x},\bar{y})$;
(ii) There exist $\delta,\gamma > 0$ such that

$$|\Gamma \varphi_{F,V}(\cdot,y)|(x) \geq \tau^{-1} \text{ for all } (x,y) \in \mathbb{B}(\bar{x},\delta)\times\mathbb{B}(\bar{y},\delta) \text{ with } \varphi_{F,V}(x,y) \in (0,\gamma);$$

(iii) There exist $\delta,\gamma > 0$ such that

$$|\nabla \varphi_{F,V}(\cdot,y)|(x) \geq \tau^{-1} \text{ for all } (x,y) \in \mathbb{B}(\bar{x},\delta)\times\mathbb{B}(\bar{y},\delta) \text{ with } \varphi_{F,V}(x,y) \in (0,\gamma);$$

one has $(i) \Leftrightarrow (ii) \Leftarrow (iii)$.

Proof. The implication $(iii) \Rightarrow (ii)$ is obvious. The equivalence between (i) and (ii) follows from the equivalence between (i) and (ii) in Proposition 3.1 by considering the function $f \equiv \varphi_{F,V}$. The proof is complete. □

4 Stability of metric regularity relative to a cone

We establish stability of metric regularity relative to a cone under a sufficiently small Lipschitz perturbation. For a given multifunction $F : X \rightrightarrows Y$ from a complete metric space X to a normed linear space Y, a cone $C \subseteq Y$, and a positive real δ, we remind that

$$V_F(C,\delta) = \{(x,y) : \ y \in F(x) + C(\delta)\}.$$

For $y \in Y$, set

$$V_{F,y}(C,\delta) := \{x \in X : \ y \in F(x) + C(\delta)\}.$$

We note $\varphi_{V_F(C,\delta)}(x,y)$, the lower semicontinuous envelope relative to $V_F(C,\delta)$ of $d(y,F(\cdot))$.

Theorem 4.1 *Let X be a complete metric space and Y be a normed space. Let $C \subseteq Y$ be a nonempty cone in Y. Let $F : X \rightrightarrows Y$ be a closed multifunction and $(x_0,y_0) \in$* gph F. *Suppose that F is metrically regular with a modulus $\tau > 0$ relatively to C, i.e., there exist reals $\varepsilon > 0$ and $\delta > 0$ such that for all $(x,y) \in \mathbb{B}((x_0,y_0),\varepsilon) \cap V_F(C,\delta)$ with $d(y,F(x)) < \varepsilon$. we have:*

$$d(x,F^{-1}(y) \cap \operatorname{cl} V_{F,y}(C,\delta)) \leq \tau d(y,F(x)). \tag{4.1}$$

Let $g : X \to Y$ be locally Lipschitz around x_0 with a Lipschitz constant $L > 0$. Then $F + g$ is metrically regular relatively to C at $(x_0,y_0+g(x_0))$ with modulus

$$\operatorname{reg}_C(F+g)(x_0,y_0+g(x_0)) \leq \left(\frac{1-\alpha}{\tau(1+\alpha)} - L\right)^{-1},$$

provided

$$\alpha \in (0,1), \text{ and } L < \frac{\delta(1-\alpha)\alpha}{\tau(1+\alpha)(1+\delta(1-\alpha))}.$$

Proof. First note that for any $\rho > 0$,

$$\varphi_{V_{F+g}(C,\rho)}(x,y) = \varphi_{V_F(C,\rho)}((x,y-g(x)), \text{ for all } (x,y) \in X \times Y.$$

Let $\varepsilon, \delta, \alpha, L$ be as in the theorem. Reducing ε and δ if necessary, we suppose that $g : X \to Y$ is Lipschitz continuous around x_0 and with constant L on $\mathbb{B}(x_0,\varepsilon)$ and let us fix reals η and ρ such that

$$0 < \rho := \delta(1-\alpha) \quad \& \quad \eta = \min\{\varepsilon/2(L+1), \varepsilon/(8\tau)\}. \tag{4.2}$$

According to Theorem 3.1, it suffices to prove that

$$|\Gamma\varphi_{V_{F+g}(C,\rho)}(\cdot,y)|(x) \geq \left(\frac{1-\alpha}{\tau(1+\alpha)} - L\right), \tag{4.3}$$

whenever

$$(x,y) \in \mathbb{B}((x_0,y_0+g(x_0)),\eta) \text{ satisfies } x \in \operatorname{cl} V_{F+g,y}(\rho) \;\&\; 0 < \varphi_{V_{F+g}}(x,y) < \eta, \tag{4.4}$$

Let x,y be as in (4.4) and take sequences $(\lambda_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ satisfying

$$\lambda_n > 0, \quad z_n \in B_X, \quad (x_n) \to x, \quad d(z_n, C) \leq \rho, \tag{4.5}$$

and

$$y - g(x_n) \in F(x_n) + \lambda_n z_n, \quad \lim_{n\to\infty} d(y, F(x_n) + g(x_n)) = \varphi_{V_{F+g}(C,\rho)}(x,y). \tag{4.6}$$

As $d(z_n,C) \leq \rho$, there is $v_n \in C$ such that $\|z_n - v_n\| \leq \rho + 1/n$. Note that since (x_n) tends to x and $x \in \mathbb{B}(x_0,\eta)$, then for n large ($n \geq n_0$) we have

$$d(y, F(x_n) + g(x_n)) \leq \|\lambda_n z_n\| \leq \lambda_n. \tag{4.7}$$

Setting

$$t_n := \alpha\varphi_{V_{F+g}(C,\rho)}(x_n,y)/(\rho + 1 + 2/n), \tag{4.8}$$

from the triangle inequality

$$\|v_n\| \leq |\|z_n\| + \|z_n - v_n\| \leq 1 + \rho + 1/n$$

we derive that for $n \geq n_0$,

$$t_n\|v_n\| < \alpha\varphi_{V_{F+g}(C,\rho)}(x_n,y)(1+\rho+1/n)/(1+\rho+2/n) < \eta. \tag{4.9}$$

Relations (4.7) and (4.8) yield,

$$t_n(1+\rho+2/n)/\alpha = \varphi_{V_{F+g}(C,\rho)}(x_n,y) \leq d(y, F(x_n)+g(x_n)) \leq \lambda_n.$$

Consequently,

$$t_n/\lambda_n \leq \frac{\alpha}{1+\rho+2/n} \leq \frac{\alpha}{\rho+1} \tag{4.10}$$

Let us observe that

$$\begin{aligned}\|\lambda_n z_n - t_n v_n\| \\ &= \|(\lambda_n - t_n)z_n + t_n(z_n - v_n)\| \\ &\geq \lambda_n - t_n(1+\rho+1/n). \end{aligned} \tag{4.11}$$

From relation (4.10), we deduce that $\lambda_n - t_n \geq \lambda_n \left(1 - \frac{\alpha}{\rho+1}\right)$. Then, observing that $1 - \frac{\alpha}{\rho+1} > 0$, we obtain that $\lambda_n - t_n > 0$, and since C is a cone, necessarily $(\lambda_n - t_n)v_n \in C$. Hence the following estimate holds:

$$\begin{aligned} d(\lambda_n z_n - t_n v_n, C) &\leq \|\lambda_n z_n - t_n v_n - (\lambda_n - t_n)v_n\| \\ &= \lambda_n \|z_n - v_n\| \leq \lambda_n(\rho + 1/n) \\ &= \frac{\lambda_n(\rho+1/n)}{\|\lambda_n z_n - t_n v_n\|}\|\lambda_n z_n - t_n v_n\| \\ &\leq \frac{\lambda_n(\rho+1/n)}{\lambda_n - t_n(1+\rho+1/n)}\|\lambda_n z_n - t_n v_n\| \\ &\leq \frac{\rho+1/n}{1-\alpha(1+\rho+1/n)/(\rho+1)}\|\lambda_n z_n - t_n v_n\| \\ &\leq \frac{\rho+1/n}{1-\dfrac{t_n}{\lambda_n}(1+\rho+1/n)}\|\lambda_n z_n - t_n v_n\| \\ &\leq \frac{\rho\alpha}{1-\dfrac{\alpha}{\rho+1+2/n}(1+\rho+1/n)}\|\lambda_n z_n - t_n v_n\| \\ &\leq \tfrac{\rho}{1-\alpha}\|\lambda_n z_n - t_n v_n\| = \delta\|\lambda_n z_n - t_n v_n\|. \end{aligned}$$

Hence, $d(\lambda_n z_n - t_n v_n, C) \leq \delta\|\lambda_n z_n - t_n v_n\|$ which means that

$$\lambda_n z_n - t_n v_n \in C(\delta). \tag{4.12}$$

Since $y - g(x_n) - t_n v_n = (y - g(x_n) - \lambda_n z_n) + (\lambda_n z_n - t_n v_n)$, combining (4.6) and (4.12) gives

$$y - g(x_n) - t_n v_n \in F(x_n) + C(\delta). \tag{4.13}$$

Moreover,

$$\|y - g(x_n) - t_n v_n - y_0\| \leq \|y - g(x_0) - y_0\| + \|g(x_n) - g(x_0)\| + t_n\|v_n\| < 2\eta(1+L) \leq \varepsilon, \tag{4.14}$$

and combining (4.7) and (4.9) we also have

$$d(y - g(x_n) - t_n v_n, F(x_n)) \leq d(y - g(x_n), F(x_n)) + t_n\|v_n\| < 2\eta \leq \frac{2\varepsilon}{L+2} < \varepsilon. \tag{4.15}$$

From (4.14) and (4.15) we deduce that

$$y - g(x_n) - t_n v_n \in \mathbb{B}(y_0, \varepsilon) \quad \& \quad d(y - g(x_n) - t_n v_n, F(x_n)) < \varepsilon, \tag{4.16}$$

and

$$(x_n, y - g(x_n) - t_n v_n) \in V_F(C, \delta). \tag{4.17}$$

Fact 1 *Replacing y by $y-g(x_n)-t_nv_n$ and x by x_n in the assumptions of (ii) and V by $V_F(C,\delta)$ we know from (4.17) that*

$$(x_n, y-g(x_n)-t_nv_n \in V_F(C,\delta)$$

and from (4.16) that

$$d(y-g(x_n)-t_nv_n, F(x_n)) < \varepsilon \quad \text{and} \quad y-g(x_n)-t_nv_n \in \mathbb{B}(x_0,\varepsilon).$$

Since $\|x_n-x_0\| \le \|x_n-x\|+\|x-x_0\| \le 2\eta \le \frac{\varepsilon}{L+1} \le \varepsilon$, also $x_n \in \mathbb{B}(x_0,\varepsilon)$ and finally all the conditions of Lemma 3.1 are satisfied.

Hence, according to this Lemma we have

$$\begin{aligned} d(x_n, F^{-1}&(y-g(x_n)-t_nv_n)) \\ &< \tau\varphi_{V_{F+g}(C,\delta)}(x_n, y-g(x_n)-t_nv_n) \\ &\le \tau(\varphi_{V_{F+g}(C,\delta)}(x_n,y)+t_n\|v_n\|) \qquad (4.18)\\ &\le \tau t_n \frac{(1+\alpha)(1+\rho)+\frac{2+\alpha}{n}}{\alpha}. \qquad (4.19)\end{aligned}$$

Using the fact that $t_n\|v_n\| < \eta$ and

$$\varphi_{V_{F+g}(C,\delta)}(x_n,y) \le \varphi_{V_{F+g}(C,\rho)}(x_n,y) \le d(y-g(x_n),F(x_n)) < 2\eta \quad (C(\rho)\subset C(\delta),)$$

we obtain

$$d(x_n, F^{-1}(y-g(x_n)-t_nv_n)) < 2\tau\eta.$$

By the choice of η, we derive $d(x_n, F^{-1}(y-g(x_n)-t_nv_n)) < \varepsilon/4$, and therefore for any $r \in (0,1)$, we get the existence of of some $u_n \in F^{-1}(y-g(x_n)-t_nv_n)$ such that for n sufficiently large,

$$d(x_n,u_n) < \tau(1+r)t_n \frac{(1+\alpha)(1+\rho)+\frac{2+\alpha}{n}}{\alpha} \le \varepsilon/4.$$

Since $(x_n) \to x \in \mathbb{B}(x_0,\eta)$, for n sufficiently large we have

$$d(x_n,x_0) \le d(x_n,x)+d(x,x_0) < \varepsilon/2+\eta < \varepsilon,$$

so that $u_n \in \mathbb{B}(x_0,\varepsilon)$. Since

$$u_n \in F^{-1}(y-g(x_n)-t_nv_n) \cap \mathbb{B}(x_0,\varepsilon)$$

and

$$\|g(u_n)-g(x_n)\| \le Ld(u_n,x_n), \text{ (by the Lipschitz property of } g \text{ on } \mathbb{B}(x_0,\varepsilon)),$$

then

$$y \in F(u_n)+g(x_n)+t_nv_n \subseteq F(u_n)+g(u_n)+t_n\Big(v_n+L\frac{d(u_n,x_n)}{t_n}B_Y\Big).$$

Therefore,

$$\varphi_{V_{F+g}(C,\rho)}(u_n,y) \le d(y-g(u_n),F(u_n)) \le t_n\|v_n\| + Ld(x_n,u_n). \tag{4.20}$$

As

$$t_n\|v_n\| \le \alpha\varphi_{V_{F+g}(C,\rho)}(x_n,y)(1+\rho+1/n)/(1+\rho)$$

with $\alpha \in (0,1)$, it follows from (4.20) that

$$\begin{aligned} 0 < \varphi_{V_{F+g}(C,\rho)}&(x,y) \\ &\le \varphi_{V_{F+g}(C,\rho)}(u_n,y) \\ &\le \alpha\varphi_{V_{F+g}(C,\rho)}(x_n,y)(1+\rho+1/n)/(1+\rho) + Ld(x_n,u_n). \end{aligned}$$

When n goes to $+\infty$, we get

$$0 < \varphi_{V_{F+g}(C,\rho)}(x,y) \le \alpha\varphi_{V_{F+g}(C,\rho)}(x,y) + \mathrm{L}\liminf_{n\to\infty} d(x_n,u_n).$$

Thus, $\liminf_{n\to\infty} d(x_n,u_n) > 0$. Observing that

$$\frac{\varphi_{V_{F+g}(C,\rho)}(x,y) - d(y,g(x_n+F(x_n))}{d(x,u_n)} \ge \frac{\varphi_{V_{F+g}(C,\rho)}(x,y) - d(y,g(x_n+F(x_n))}{d(x,x_n)+d(x_n,u_n)},$$

and using the fact that $\lim_{n\to+\infty} d(y,g(x_n+F(x_n)) = \varphi_{V_{F+g}(C,\rho)}(x,y)$, we deduce that

$$\liminf_{n\to\infty} \frac{\varphi_{V_{F+g}(C,\rho)}(x,y) - d(y,g(x_n+F(x_n))}{d(x,u_n)} \ge 0,$$

and therefore,

$$\begin{aligned} &\liminf_{n\to\infty} \frac{\varphi_{V_{F+g}(C,\rho)}(x,y) - \varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x,u_n)} \\ &= \liminf_{n\to\infty} \frac{\varphi_{V_{F+g}(C,\rho)}(x,y) - d(y,g(x_n+F(x_n)) + d(y,g(x_n+F(x_n)) - \varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x,u_n)} \\ &\ge \liminf_{n\to\infty} \frac{d(y,F(x_n)+g(x_n)) - \varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x,u_n).} \end{aligned}$$

Hence,

$$
\begin{aligned}
&\liminf_{n\to\infty}\frac{\varphi_{V_{F+g}(C,\rho)}(x,y)-\varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x,u_n)}\\
&\geq \liminf_{n\to\infty}\frac{d(y,F(x_n)+g(x_n))-\varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x,u_n)}\\
&\geq \liminf_{n\to\infty}\frac{d(y,F(x_n)+g(x_n))-\varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x,x_n)+d(x_n,u_n)} \quad \text{due to } d(x,u_n)\leq d(x,x_n)+d(x_n,u_n)\\
&= \liminf_{n\to\infty}\frac{d(y,F(x_n)+g(x_n))-\varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x_n,u_n)} \quad \text{due to } (x_n)\to x\\
&\geq \liminf_{n\to\infty}\frac{\varphi_{V_{F+g}(C,\rho)}(x_n,y)-\varphi_{V_{F+g}(C,\rho)}(u_n,y)}{d(x_n,u_n)} \quad \text{due to } d(y,F(x_n)+g(x_n))\geq \varphi_{V_{F+g}(C,\rho)}(x_n,y)\\
&\geq \liminf_{n\to\infty}\frac{t_n(1+\rho+2/n)/\alpha-t_n\|v_n\|-Ld(x_n,u_n)}{d(x_n,u_n)}\\
&= \liminf_{n\to\infty}\frac{t_n(1+\rho+2/n)/\alpha-t_n\|v_n\|}{d(x_n,u_n)}-L\\
&\geq \liminf_{n\to\infty}\frac{t_n(1+\rho+2/n)/\alpha-t_n\|v_n\|}{\tau t_n(1+r)[(1+\rho)(1+\alpha)+(2+\alpha)/n]/\alpha}-L\\
&= \liminf_{n\to\infty}\frac{(1+\rho+2/n)/\alpha-(1+\rho+1/n)}{\tau(1+r)[(1+\rho)(1+\alpha)+(2+\alpha)/n]/\alpha}-L\\
&= \frac{1-\alpha}{\tau(1+r)(1+\alpha)}-L,
\end{aligned}
$$

As $r>0$ is arbitrarily small, one obtains

$$|\Gamma\varphi_{V_{F+g}(C,\rho)(\cdot,y)|(x)}\geq \frac{1-\alpha}{\tau(1+\alpha)}-L,$$

which completes the proof. □

5 Coderivative characterizations of relative metric regularity

Sufficient conditions in terms of coderivatives were given for usual metric regularity, by various authors, for instance, in [8, 22, 27, 33]. In this section, we establish a characterization of relative metric regularity using the Fréchet subdifferential in Asplund spaces.

Associated with the multifunction F, for given $\varepsilon>0,\delta_0$ and $(x_0,y_0)\in \operatorname{gph} F$, we define the *localization* of F by

$$F_{(x_0,y_0,\varepsilon)}(x):=\begin{cases} F(x)\cap\bar{\mathbb{B}}(y_0,\delta_0) & \text{if} \quad x\in\bar{\mathbb{B}}(x_0,\varepsilon)\\ \emptyset & \text{otherwise.}\end{cases} \tag{5.1}$$

Note that, by definition, one has

$$D^*_{\mathscr{F}}F(x,y)=D^*_{\mathscr{F}}F_{(x_0,y_0,\varepsilon)}(x,y)\quad \forall (x,y)\in \operatorname{gph} F\cap(\mathbb{B}(x_0,\varepsilon)\times\mathbb{B}(y_0,\varepsilon)). \tag{5.2}$$

The following proposition gives a connection between metric regularity relative to a cone for a multifunction with convex values and metric regularity relative to the same cone for its localizations.

Proposition 5.1 *Suppose given a multifunction $F : X \rightrightarrows Y$ with convex values for x near x_0 and $(x_0, y_0) \in \operatorname{gph} F$. Then, F is metrically regular relatively to a cone $C \subseteq Y$, if and only if, $F_{(x_0,y_0,\varepsilon)}$ is metrically regular relatively to the same cone C for any $\varepsilon > 0$.*

Proposition 5.1 follows immediately from the following lemma.

Lemma 5.1 *Let $F : X \rightrightarrows Y$ be a multifunction with convex values for x near x_0 and $(x_0, y_0) \in \operatorname{gph} F$. Then, for given a nonempty cone $C \subseteq Y$, for any reals $\delta_1, \delta_2 > 0$, there exist reals $\eta, \delta > 0$ such that for all $x \in \mathbb{B}(x_0, \eta)$, one has*

$$\big(F(x) + C(\delta)\big) \cap \mathbb{B}(y_0, \eta) \cap \{y \in Y : \ d(y, F(x)) < \eta\} \subseteq F(x) \cap \mathbb{B}(y_0, \delta_1) + C(\delta_2). \quad (5.3)$$

Proof. For δ_1, δ_2, take $\delta = \delta_2/2$. Let $\eta \in (0, \delta_1/4)$ such that $F(x)$ is convex for all $x \in \mathbb{B}(x_0, \eta)$ and select $x \in \mathbb{B}(x_0, \eta)$ and $y \in (F(x) + C(\delta)) \cap \mathbb{B}(y_0, \eta)$ with $d(y, F(x)) < \eta$. Then, there exist $z, v \in F(x)$ such that

$$y = z + \lambda u, \ \text{ for } \lambda \geq 0, \ u \in Y, \ \ \|u\| = 1; \ \ d(u, C) \leq \delta, \ \ \|y - v\| < \eta.$$

If $z \in \mathbb{B}(y_0, \delta_1)$, then (5.3) holds trivially. Otherwise, one has

$$\lambda = \|y - z\| \geq \|z - y_0\| - \|y - y_0\| \geq \delta_1 - \eta.$$

Setting

$$t := \frac{\eta(1 + \delta_2)}{\delta_2(\delta_1 - \eta)/2 + \eta(1 + \delta_2)}, \ \ w := tz + (1 - t)v \in F(x), \quad (5.4)$$

and by taking η sufficiently small such that $t < 1/2$, one has

$$\|w - y_0\| \leq t\|z - v\| + \|v - y_0\| \leq t\lambda\|u\| + t\|y - v\| + \|v - y_0\| < t(\delta_1 - \eta) + t\eta + 2\eta < \delta_1.$$

and,

$$\begin{aligned} \|y - w\| = \|t\lambda u + (1 - t)(y - v)\| &\geq t\lambda - (1 - t)\eta \\ &= \frac{(\delta_2 + 1)\eta\lambda}{\lambda(\delta_2 - \delta) + \eta(\delta_2 + 1)} - \frac{\lambda\eta(\delta_2 - \delta)}{\lambda(\delta_2 - \delta) + \eta(\delta_2 + 1)} \\ &= \frac{(1 + \delta)\lambda\eta}{\lambda(\delta_2 - \delta) + \eta(\delta_2 + 1)} > 0. \end{aligned}$$

Thus,

$$\begin{aligned} d(y - w, C) = d(t\lambda u + (1 - t)(y - v), C) &\leq (1 - t)\|y - v\| + d(t\lambda u, C) \\ &\leq (1 - t)\eta + t\lambda\delta \leq \frac{(1 - t)\eta + t\lambda\delta}{t\lambda - (1 - t)\eta}\|y - w\| = \delta_2\|y - w\|, \end{aligned}$$

where the last equality follows from the definition of t in (5.4). Hence, $y \in F(x) \cap \mathbb{B}(y_0, \delta_1) + C(\delta_2)$. □

Denote by $\mathbb{S}_{Y^*}$ the unit sphere in the continuous dual Y^* of Y, and by d_* the metric associated with the dual norm on X^*. For given $\bar{y} \in Y$ and $\delta > 0$, let us define the set

$$T(C,\delta) := \{(y_1^*, y_2^*) \in Y^* \times Y^* : \ \exists a \in C \cap \mathbb{S}_{Y^*}, \max\{|\langle y_1^*, a\rangle|, |\langle y_2^*, a\rangle|\} \le \delta, \ \|y_1^* + y_2^*\| = 1\}. \quad (5.5)$$

To a given multifunction $F : X \rightrightarrows Y$, we associate the multifunction $G : X \rightrightarrows Y \times Y$ defined by

$$G(x) = F(x) \times F(x), \quad x \in X.$$

When considering convex-valued multifunctions defined on Asplund spaces, the following theorem establishes a coderivative characterization of relative metric regularity.

Theorem 5.1 *Let X, Y be Asplund spaces and let $F : X \rightrightarrows Y$ be a closed multifunction. Let $(x_0, y_0) \in \operatorname{gph} F$ and a nonempty cone $C \subseteq Y$ be given. Assume that F has convex values around x_0, i.e., $F(x)$ is convex for all x near x_0. If*

$$\liminf_{\substack{(x,y_1,y_2) \xrightarrow{G} (x_0,y_0,y_0) \\ \delta \downarrow 0^+}} d_*(0, D^*_{\mathscr{F}} G(x,y_1,y_2)(T(C,\delta))) > m > 0, \quad (5.6)$$

then F is metrically regular relatively to C with modulus $\tau \le m^{-1}$ at (x_0, y_0).The notation $(x,y_1,y_2) \xrightarrow{G} (x_0,y_0,y_0)$ means that $(x,y_1,y_2) \to (x_0,y_0,y_0)$ with $(x,y_1,y_2) \in \operatorname{gph} G$.

Proof. By the assumption, there is $\delta_0 \in (0,1)$ such that

$$\inf_{(x,y_1,y_2) \in \operatorname{gph} G \cap \mathbb{B}((x_0,y_0,y_0),2\delta_0)} d_*(0, D^*_{\mathscr{F}} G(x,y_1,y_2)(T(\bar{y},\delta_0))) \ge m + \delta_0. \quad (5.7)$$

According to Proposition 5.1 and relation (5.2), by considering the localization $F_{(x_0,y_0,\delta_0)}$ instead of F, without any loss of generality, we may assume that

$$F(x) \subseteq \bar{\mathbb{B}}(y_0, \delta_0) \quad \text{for all } x \in \bar{\mathbb{B}}(x_0, \delta_0). \quad (5.8)$$

Denote by $\varphi_\delta(\cdot, y) := \varphi_{V(C,\delta)}(\cdot, y)$, the lower semicontinuous envelope of $d(y, F(\cdot))$ relative to $V(C,\delta)$. By virtue of Theorem 3.1, it suffices to show that one has $|\nabla \varphi_\delta(\cdot,y)|(x) > m$ for any $(x,y) \in \big(\mathbb{B}(x_0,\delta) \times \mathbb{B}(y_0,\delta)\big)$, $x \in \operatorname{cl} V_y(C,\delta)$ with $\varphi_\delta(x,y) \in (0,\delta)$. Let $(x,y) \in \mathbb{B}(x_0,\delta) \times \mathbb{B}(y_0,\delta)$, $x \in \operatorname{cl} V(\bar{y},\delta)$ with $\varphi_\delta(x,y) \in (0,\delta)$ be given. Set $|\nabla \varphi_\delta(\cdot,y)|(x) := \alpha$. By the definition of the strong slope, for each $\varepsilon \in (0, \min\{\delta, 1/2\})$, there is $\eta \in (0,\varepsilon)$ with

$$2\eta + \varepsilon < \gamma/2, \ \ 2\eta < \varepsilon \varphi_\delta(x,y) \text{ and } 1 - (\alpha + \varepsilon + 2)\eta > 0$$

such that

$$d(y, F(x')) \ge (1-\varepsilon)\varphi_\delta(x,y) \quad \text{for all } x' \in \mathbb{B}(x, 4\eta) \quad (5.9)$$

and

$$\varphi_\delta(x,y) \le \varphi_\delta(x',y) + (\alpha+\varepsilon)\|x'-x\| \quad \text{for all } x' \in \bar{\mathbb{B}}(x,3\eta) \cap \operatorname{cl} V_y(C,\delta). \quad (5.10)$$

Take $u \in \mathbb{B}(x, \eta^2/4) \cap V_y(C, \delta)$, $v \in F(u)$ such that $\|y - v\| \leq \varphi_\delta(x, y) + \eta^2/4$. Then,

$$\|y - v\| \leq d(y, F(x')) + (\alpha + \varepsilon)\|x' - x\| + \eta^2/4 \; \forall x' \in \bar{\mathbb{B}}(u, 2\eta) \cap \mathrm{cl}\, V_y(C, \delta).$$

Consequently, for every $(x', z') \in \big(\bar{\mathbb{B}}(u, 2\eta) \times Y\big) \cap V(C, \delta)$ we have

$$\|y - v\| \leq d(y, F(x')) + (\alpha + \varepsilon)\|x' - u\| + (\alpha + \varepsilon + 1)\eta^2/4. \tag{5.11}$$

Let $z \in C(\delta)$ be such that $y - z \in F(u)$. Then,

$$\|z\| \geq d(y, F(u)) \geq (1 - \varepsilon)\varphi_\delta(x, y)) > \eta/\varepsilon. \tag{5.12}$$

Setting

$$W := \{(x, w_1, w_2, z) \in X \times Y \times Y \times Y : \; (x, w_1, w_2) \in \mathrm{gph}\, G, \; y = w_2 + z, \; z \in C(\delta)\},$$

we derive

$$\begin{aligned} \|y - v\| &\leq \|y - w_1\| + (\alpha + \varepsilon)\|x' - u\| + \iota_W(x', w_1, w_2, z') + (\alpha + \varepsilon + 1)\eta^2/4 \\ &\text{for all} (x', w_1, w_2, z') \in \bar{\mathbb{B}}(u, \eta) \times Y \times Y \times \bar{\mathbb{B}}(z, \eta). \end{aligned}$$

Next, applying the Ekeland variational principle to the function

$$(x', w_1, w_2, z') \mapsto \psi(x', w_1, w_2, z') := \|y - w_1\| + (\alpha + \varepsilon)\|x' - u\| + \iota_W(x', w_1, w_2, z')$$

on $\bar{\mathbb{B}}(u, \eta) \times Y \times Y \times \bar{\mathbb{B}}(z, \eta)$, we select $(u_1, v_1, v_2, z_1) \in (u, v, y - z, z) + \frac{\eta}{4} B_{X \times Y \times Y \times Y}$ with $(u_1, v_1, v_2, z_1) \in W$, such that

$$\|y - v_1\| \leq \|y - v\| (\leq d(y, F(x)) + \eta^2/4) \tag{5.13}$$

and

$$\psi(u_1, v_1, v_2, z_1) \leq \psi(x', w_1, w_2, z') + (\alpha + \varepsilon + 1)\eta \, \|(x', w_1, w_2, z') - (u_1, v_1, v_2, z_1)\|$$

for all $(x', w_1, w_2, z') \in \bar{\mathbb{B}}(u, \eta) \times Y \times Y \times \bar{\mathbb{B}}(z, \eta)$. Thus,

$$0 \in \partial(\psi + (\alpha + \varepsilon + 1)\eta \| \cdot - (u_1, v_1, v_2, z_1)\|)(u_1, v_1, v_2, z_1). \tag{5.14}$$

We need the following claim in order to make use of the fuzzy sum rule.

Claim. For each $(u, w_1, w_2, z_1) \in W$ near $(u, v, y - z, z)$, for every $\varepsilon > 0$, one has

$$N(W, (u, w_1, w_2, z_1)) \subseteq \left\{ \begin{array}{ll} (x^*, w_1^*, w_2^*, z^*) + \varepsilon \mathbb{B}^*_{X \times Y \times Y \times Y} : & (x^*, w_1^*, w_2^*) \in N(\mathrm{gph}\, G, (u', w_1', w_2')), \; z' \in N(C(\delta), z'), \\ & \|w_2^* + z^*\| \leq \varepsilon, \; \|(u', w_1', w_2', z') - (u, w_1, w_2, z_1)\| < \varepsilon. \end{array} \right\}.$$

Proof of the claim. Observe that $W = W_1 \cap W_2 \cap W_3$, where

$$W_1 := \{(x, w_1, w_2, z) \in X \times Y \times Y \times Y : \; w_2 + z = y\};$$

$$W_2 := \{(x, w_1, w_2, z) \in X \times Y \times Y \times Y : \; (x, w_1, w_2) \in \mathrm{gph}\, G\};$$

$$W_3 := \{(x, w_1, w_2, z) \in X \times Y \times Y \times Y : \; z \in C(\delta)\}.$$

It suffices to check that the condition of Lemma 2.1 is satisfied. Indeed, pick any sequences $w_n^i := (u_n^i, w_{1,n}^i, w_{2,n}^i, z_n^i) \in W_i$, converging to $(u, v, y-z, z)$ and $w_n^{i*} := (u_n^{i*}, w_{1,n}^{i*}, w_{2,n}^{i*}, z_n^{i*}) \in N(W_i, (u_n^i, w_{1,n}^i, w_{2,n}^i, z_n^i))$ $(i = 1,2,3)$, such that

$$\|w_n^{1*} + w_n^{2*} + w_n^{3*}\| \to 0.$$

Then, by the definition of W_i, $(i = 1,2,3)$,

$$u_n^{1*} = 0,\ \ w_{1,n}^{1*} = 0,\ \ w_{2,n}^{1*} = -z_n^{1*};$$

$$u_n^{2*} \in D^*_{\mathscr{F}} G((u_n^1, w_{1,n}^2, w_{2,n}^2))(-(w_{1,n}^{2*}, w_{2,n}^{2*}));$$

$$u_n^{3*} = 0,\ \ w_{1,n}^{3*} = 0,\ \ w_{2,n}^{3*} = 0,\ \ z_n^{3*} \in N(C(\delta), z_n^3).$$

Thus,

$$\|u_n^{2*}\| \to 0,\ \ \|w_{1,n}^{2*}\| \to 0,\ \ \text{and}\ \ \|w_{2,n}^{2*} + w_{2,n}^{1*}\| \to 0,\ \|z_n^{1*} + z_n^{3*}\| \to 0,\ \ \text{as } n \to \infty.$$

As $w_{2,n}^{1*} = -z_n^{1*}$, the latter relations imply $\|w_{2,n}^{2*} + z_n^{3*}\| \to 0$ as $n \to \infty$. Since $z_n^{3*} \in N(C(\delta), z_n^3)$, $\langle z_n^{3*}, z_n^3 \rangle = 0$. As $z \neq 0$, $z_n^3 \neq 0$ when n is sufficiently large; therefore for these n there is $a \in C \cap \mathbb{S}_{Y^*}$ such that $\|z_n^3/\|z_n^3\| - a\| \leq (\delta + 1/n) < \delta_0$. Hence for n large,

$$\langle w_{2,n}^{2*}, a\rangle \leq \|w_{2,n}^{2*}\|\delta_0.$$

By (5.8), $(u, v, y-z) \in \mathbb{B}((x_0, y_0, y_0), \delta_0)$; therefore, in view of relation (5.7), the latter relation implies that the sequences (w_n^{i*}) $(i = 1,2,3)$ converge to 0, and the assumption from Lemma 2.1 is satisfied, and the claim is proved.

Now using the claim and applying the fuzzy sum rule to (5.14), we derive the existence of

$$v_3 \in \mathbb{B}(v_1, \eta),\ z_2 \in \mathbb{B}(z, \eta);$$

$$(u_2, w_1, w_2) \in \mathbb{B}(u_1, \eta) \times \mathbb{B}(v_1, \eta) \times \mathbb{B}(v_2, \eta) \cap \text{gph}\, G;$$

$$v_3^* \in \partial\|y - \cdot\|(v_3);\ (u_2^*, -w_1^*, -w_2^*) \in N(\text{gph}\, G, (u_2, w_1, w_2));\ z_2^* \in N(C(\delta), z_2),$$

such that

$$\|v_3^* - w_1^*\| < (\alpha + \varepsilon + 2)\eta;\ \ \|w_2^* + z_2^*\| < (\alpha + \varepsilon + 2)\eta;\ \ \|u_2^*\| \leq \alpha + \varepsilon + (\alpha + \varepsilon + 2)\eta. \quad (5.15)$$

Since $v_3^* \in \partial\|y - \cdot\|(v_3)$ (note that $\|y - v_3\| \geq \|y - v\| - \|v_3 - v\| \geq d(y, F(x)) - \varepsilon - 2\eta > 0$), then $\|v_3^*\| = 1$ and $\langle v_3^*, v_3 - y\rangle = \|y - v_3\|$. Thus, $\|w_1^*\| \leq 1 + (\alpha + \varepsilon + 2)\eta$, and from the first relation of (5.15) it follows that

$$\langle w_1^*, w_1 - y\rangle \geq \langle v_3^*, w_1 - y\rangle - (\alpha + \varepsilon + 2)\eta\|w_1 - y\| \geq (1 - (\alpha + \varepsilon + 2)\eta)\|w_1 - y\| - 2\eta.$$

As

$$\eta \leq \varepsilon d(y, F(x)) \leq \varepsilon d(y, F(u))/(1 - \varepsilon) \qquad \text{for all } u \in \mathbb{B}(x, 4\eta),$$

then $\eta \leq \varepsilon\|w_1 - y\|/(1 - \varepsilon)$, and therefore one obtains

$$\langle w_1^*, w_1 - y\rangle \geq (1 - \varepsilon_1)\|w_1 - y\|, \quad (5.16)$$

where

$$\varepsilon_1 := (\alpha+\varepsilon+2)\eta - 2\varepsilon(1-\varepsilon)^{-1}.$$

Since $w_2 \in \mathbb{B}(v_2,\eta)$ and $v_2 \in \mathbb{B}(y-z,\eta)$, $w_2 \in \mathbb{B}(y-z,2\eta)$. As $F(u_2)$ is convex, $w_2 \in F(u_2)$, and $w_1^* \in -N(F(u_2),w_1)$, one has

$$\langle w_1^*, y-w_2\rangle = \langle w_1^*, y-w_1\rangle + \langle w_1^*, w_1-w_2\rangle \le 0.$$

Therefore,

$$\begin{aligned}\langle w_1^*, z\rangle &= \langle w_1^*, y-w_2\rangle + \langle w_1^*, z-(y-w_2)\rangle \\ &\le 2\eta\|w_1^*\| \le 2\eta[1+(\alpha+\varepsilon+2)\eta],\end{aligned}$$

and by (5.12), $\|z\| \ge \eta/\varepsilon$,

$$\left\langle w_1^*, \frac{z}{\|z\|}\right\rangle \le 2\varepsilon[1+(\alpha+\varepsilon+2)\eta].$$

As $z \in C(\delta)$, there is $d \in C$ such that $\|z/\|z\| - d\| \le 2\delta$, then $\|d\| \ge 1-2\delta$, and from $\|w_1^*\| \le 1+(\alpha+\varepsilon+2)\eta$, one obtains

$$\begin{aligned}\langle w_1^*, d\rangle &\le \langle w_1^*, z/\|z\|\rangle + 2\delta\|w_1^*\| \\ &\le 2\varepsilon[1+(\alpha+\varepsilon+2)\eta] + 2\delta[1+(\alpha+\varepsilon+2)\eta].\end{aligned}$$

Hence for $a := d/\|d\| \in C\cap\mathbb{S}_{Y^*}$, one has

$$\langle w_1^*, a\rangle \le (2\varepsilon[1+(\alpha+\varepsilon+2)\eta] + 2\delta[1+(\alpha+\varepsilon+2)\eta])(1-2\delta)^{-1} := \varepsilon_2. \quad (5.17)$$

As $z_2^* \in N(C(\delta),z_2)$, with $z_2 \ne 0$, then $\langle z_2^*, z_2\rangle = 0$, therefore, from $\|w_2^*+z_2^*\| < (\alpha+\varepsilon+2)\eta$, we have

$$|\langle w_2^*, z_2\rangle| \le (\alpha+\varepsilon+2)\eta\|z_2\|.$$

As $z_2 \in \mathbb{B}(z,\eta)$ and $\|z\| \ge \eta/\varepsilon$, one has $\|z_2\| \le (1+\varepsilon)\|z\|$, and therefore,

$$\begin{aligned}|\langle w_2^*, z\rangle| &\le \langle w_2^*, z_2\rangle + \eta\|w_2^*\| \\ &[(\alpha+\varepsilon+2)\eta(1+\varepsilon) + \varepsilon\|w_2^*\|]\|z\|.\end{aligned}$$

This implies

$$|\langle w_2^*, a\rangle| \le [(\alpha+\varepsilon+2)\eta(1+\varepsilon) + (2\delta+\varepsilon)\|w_2^*\|](1-2\delta)^{-1}. \quad (5.18)$$

We consider the following two cases:
Case 1. $\|w_2^*\| \le 1+2\|w_1^*\| (\le 1+2(1+(\alpha+\varepsilon+2)\eta))$. Then

$$\langle w_2^*, a\rangle| \le (\alpha+\varepsilon+2)\eta(1+\varepsilon) + (2\delta+\varepsilon)(1+2(1+(\alpha+\varepsilon+2)\eta))(1-2\delta)^{-1} := \varepsilon_3. \quad (5.19)$$

Moreover, remind that $\langle z_2^*, z_2\rangle = 0$,

$$|\langle w_2^*, w_2-y\rangle| \le |\langle z_2^*, w_2-(y-z_2)\rangle| + |\langle w_2^*+z_2^*, w_2-y\rangle| \le \varepsilon_4\|w_1-y\|,$$

where

$$\varepsilon_4 := \begin{pmatrix} 3[1+2(1+(\alpha+\varepsilon+2)\eta)) & +(\alpha+\varepsilon+2)\eta]\eta) \\ & +2(\alpha+\varepsilon+2)(\|y_0\|+2\delta_0+2\eta) \end{pmatrix} \varepsilon(1-\varepsilon)^{-1}.$$

The second inequality of the preceding relation follows from (5.15), as well as

$$\eta \le \varepsilon\|w_1 - y\|/(1-\varepsilon);$$

$$\|z_2^*\| \le \|w_2^*\| + \|z_2^* + w_2^*\| \le 1 + 2(1 + (\alpha+\varepsilon+2)\eta)) + (\alpha+\varepsilon+2)\eta;$$
$$\|w_2 - y - z_2\| \le \|w_2 - v_2\| + \|v_2 - (y-z)\| + \|z_2 - z\| \le 3\eta$$

and

$$\|w_2 - y\| \le \|w_2 - v_2\| + \|v_2 - (y-z)\| + \|z\| < 2\eta + 2\delta_0 + \|y_0\|.$$

Hence, using the convexity of $F(u_2)$, and the fact that $w_2^* \in -N(F(u_2), w_2)$ we have

$$\langle w_2^*, w_1 - y\rangle = \langle w_2^*, w_1 - w_2\rangle + \langle w_2^*, w_2 - y\rangle \ge -\varepsilon_4\|w_1 - y\|. \tag{5.20}$$

From relations (5.16) and (5.20), one derives that

$$\langle w_1^* + w_2^*, w_1 - y\rangle \ge (1 - \varepsilon_1 - \varepsilon_4)\|w_1 - y\|. \tag{5.21}$$

Consequently, $\|w_1^* + w_2^*\| \ge 1 - \varepsilon_1 - \varepsilon_4$.

Set

$$y_1^* = \frac{w_1^*}{\|w_1^* + w_2^*\|};\ y_2^* = \frac{w_2^*}{\|w_1^* + w_2^*\|} \text{ and } x^* = \frac{u_2^*}{\|w_1^* + w_2^*\|}.$$

From relations (5.17), (5.18), (5.21), one has

$$\langle y_1^*, a\rangle \le \varepsilon_2(1 - \varepsilon_1 - \varepsilon_4)^{-1};$$
$$|\langle y_2^*, a\rangle| \le \varepsilon_3(1 - \varepsilon_1 - \varepsilon_4)^{-1},$$

and

$$x^* \in D^*_{\mathscr{F}}G(u_2, w_1, w_2)(y_1^*, y_2^*);\ \ \|y_1^* + y_2^*\| = 1.$$

As $\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta, \varepsilon, \eta \to 0$, then $(y_1^*, y_2^* \in T(C, \delta)$. Since $(u_2, w_1, w_2) \in \mathbb{B}((x_0, y_0, y_0), \delta_0)$, according to (5.15), one obtains

$$m + \delta_0 \le \|x^*\| = \|u_2^*\|/\|w_1^* + w_2^*\| \le \frac{\alpha+\varepsilon+(\alpha+\varepsilon+2)\eta}{1 - \varepsilon_1 - \varepsilon_4}. \tag{5.22}$$

As $\varepsilon, \eta, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are arbitrarily small, we obtain $m + \delta_0 \le \alpha$.
Case 2. $\|w_2^*\| > 1 + 2\|w_1^*\|$. For this case,

$$\|w_1^* + w_2^*\| \ge \|w_2^*\| - \|w_1^*\| \ge (\|w_2^*\| + 1)/2 > 1.$$

Therefore,

$$\langle y_1^*, a\rangle \le \varepsilon_2,$$

and by (5.18),

$$|\langle y_2^*, a\rangle| \le [(\alpha+\varepsilon+2)\eta(1+\varepsilon) + (2\delta+\varepsilon)\|w_2^*\|](1-2\delta)^{-1}\|w_1^* + w_2^*\|^{-1}$$
$$\le [(\alpha+\varepsilon+2)\eta(1+\varepsilon) + 2(2\delta+\varepsilon)](1-2\delta)^{-1}.$$

Thus we also get $(y_1^*, y_2^*) \in T(C, \delta_0)$. Similarly to the first case, one has $m + \delta_0 \le \alpha$, and the proof is complete. □

The following proposition shows that condition (5.6) is also a necessary condition for metric regularity relative to a cone in Banach spaces when F is either a multifunction with a convex graph or $F : X \to Y$ is a continuous single-valued mapping.

Proposition 5.2 *Let X,Y be Banach spaces and $C \subseteq Y$ a nonempty cone. Suppose that $F : X \rightrightarrows Y$ is either a closed convex multifunction or $F : X \to Y$ is a continuous single-valued mapping. For a given $(x_0,y_0) \in \operatorname{gph} F$, if F is metrically regular relatively to C at (x_0,y_0), then*

$$\liminf_{\substack{(x,y_1,y_2)\xrightarrow{G}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*_{\mathscr{F}}G(x,y_1,y_2)(T(C,\delta))) > 0.$$

Proof. Assuming that F is metrically regular relatively to C, there exist $\tau > 0, \delta > 0, \varepsilon > \delta$ such that

$$d(x,F^{-1}(y)) \le \tau d(y,F(x)) \quad \text{for all } (x,y) \in \mathbb{B}(x_0,\varepsilon)\times\mathbb{B}(y_0,\varepsilon);\ d(y,F(x)) < \varepsilon;\ y \in F(x)+C(\delta). \tag{5.23}$$

For $\gamma \in (0,\delta)$, $\eta \in (0,\varepsilon-\delta)$, let $(x,y_1,y_2) \in \operatorname{gph} G \cap [\mathbb{B}(x_0,\varepsilon/2)\times\mathbb{B}(y_0,\varepsilon/2)\times\mathbb{B}(y_0,\varepsilon/2)]$, $(y_1^*,y_2^*) \in T(C,\gamma)$ and $x^* \in D^*_{\mathscr{F}}G(x,y_1,y_2)(y_1^*,y_2^*)$.

Case 1. F is a convex multifunction. As $x^* \in D^*_{\mathscr{F}}G(x,y_1,y_2)(y_1^*,y_2^*)$, one has

$$\langle x^*, u-x\rangle + \langle y_1^*, v_1-y_1\rangle - \langle y_2^*, v_2-y_2\rangle \le 0 \tag{5.24}$$

for all $(u,v_1,v_2) \in \operatorname{gph} G$.

For $\delta_1 \in (0,\delta)$, since $(y_1^*,y_2^*) \in T(C,\gamma)$, there are $a \in C \cap \mathbb{S}_{Y^*}$ and $w \in \mathbb{B}_Y$ such that $\langle y_1^*+y_2^*, a+\delta w\rangle \le 2\gamma - \delta_1$. Since (5.23), for $t := \varepsilon-\eta-\delta_1$, then $y_2+t(a+\delta w) \in \mathbb{B}(y_0,\varepsilon)$, $d(y_2+t(a+\delta w),F(x)) \le t(1+\delta)$, and therefore we may find $u \in F^{-1}(y_2+t(a+\delta w))$ such that

$$\|x-u\| \le (1+\alpha)\tau d(y_2+t(a+\delta w),F(x)) \le (1+\alpha)t\|a+\delta w\|.$$

By taking $v_1 = v_2 = y_2+t(\bar{y}+\delta w)$ into account in (5.24), one obtains

$$\begin{aligned}(1+\alpha)\tau\|ta+\delta w\|\|x^*\| & \\ &\ge \langle x^*, x-u\rangle \\ &\ge -\langle y_1^*+y_2^*, v-y_2\rangle - \langle y_1^*, y_2-y_1\rangle \\ &\ge t(\delta_1-\gamma) - 2\eta\|y_1^*\|.\end{aligned} \tag{5.25}$$

As $\alpha > 0$, $\delta_1 \in (0,\delta)$, $\eta \in (0,\varepsilon-\delta)$ are arbitrarily, one has

$$\|x^*\| \ge \frac{\delta-\gamma}{\tau\|a+\delta w\|} \ge \frac{\delta-\gamma}{\tau(1+\delta)}.$$

Thus,

$$\liminf_{\substack{(x,y_1,y_2)\xrightarrow{G}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*_{\mathscr{F}}G(x,y_1,y_2)(T(C,\gamma))) \ge \frac{\delta}{\tau(1+\delta)} > 0.$$

Case 2. $F := f$ is a continuous single-valued maping around x_0. For this case, $y_1 = y_2 = f(x)$, by setting $g := (f,f) : X \to Y \times Y$ and using the usual notation:

$D^*_{\mathscr{F}}g(x)(y^*) := D^*_{\mathscr{F}}f(x, f(x))(y^*)$, one has that for any $\alpha \in (0,1)$, there exists $\beta \in (0, \varepsilon/2)$ such that

$$\langle x^*, u-x\rangle - \langle y_1^* + y_2^*, f(u) - f(x)\rangle \le \alpha(\|u-x\| + \|f(u) - f(x)\|), \tag{5.26}$$

for all $u \in \mathbb{B}(x, \beta)$.

As in the first case, for $\delta_1 \in (0, \delta)$, take $w \in \mathbb{B}_Y$ such that $\langle y_1^* + y_2^*, \bar{y} + \delta w\rangle \le \gamma - \delta_1$. Since (5.23), for all sufficiently small $t > 0$, we may find $u \in f^{-1}(f(x) + t(a + \delta w))$ such that

$$\|x-u\| \le (1+\alpha)\tau\|f(x) + t(\bar{y} + \delta w) - f(x))\| = \tau(1+\alpha)t\|a + \delta w\| < \beta.$$

Therefore, by (5.26), one obtains

$$\begin{aligned}
&(1+\alpha)\tau t\|a+\delta w\|\|x^*\| \\
&\qquad \ge \langle x^*, x-u\rangle \\
&\qquad \ge -\langle y_1^* + y_2^*, f(u) - f(x)\rangle - \alpha(\|u-x\| + \|f(u) - f(x)\|) \\
&\qquad \ge t(\delta_1 - \gamma) - \alpha t\|a + \delta w\|[(1+\alpha)\tau + 1]. \qquad (5.27)
\end{aligned}$$

As $\alpha > 0$, $\delta_1 \in (0, \delta)$ are arbitrary, one has

$$\|x^*\| \ge \frac{\delta - \gamma}{\tau(1+\delta)} \ge \frac{\delta - \gamma}{\tau(1+\delta)}.$$

Thus,

$$\liminf_{\substack{(x,y_1,y_2)\xrightarrow{G}(x_0,y_0,y_0) \\ \gamma \to 0^+}} d_*(0, D^*_{\mathscr{F}}G(x,y_1,y_2)(T(C,\gamma))) \ge \frac{\delta}{\tau(1+\delta)} > 0.$$

The proof is complete. □ □

Remark 5.1 We note that the quantity

$$\liminf_{\substack{(x,y_1,y_2)\xrightarrow{G}(x_0,y_0,y_0) \\ \delta \downarrow 0^+}} d_*(0, D^*_{\mathscr{F}}G(x,y_1,y_2)(T(C,\delta)))$$

depends on the modulus of relative metric regularity, the radius of the neighborhood of the point (x_0, y_0) and the constant δ in the set $C(\delta)$.

Let us now recall the notion of *partial sequential normal compactness* (PSNC, in short, [33, page 76]). A multifunction $F : X \rightrightarrows Y$ with nonempty graph is *partially sequentially normally compact* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, if for any sequence of quadruples $\{(x_k, y_k, x_k^\star, y_k^\star)\}_{n\in\mathbb{N}} \subset \operatorname{gph} F \times X^\star \times Y^\star$ satisfying

$$(x_k, y_k) \to (\bar{x}, \bar{y}), x_k^\star \in D^\star_{\mathscr{F}}F(x_k, y_k)(y_k^\star), y_k^\star \xrightarrow{w^\star} 0, \|x_k^\star\| \to 0,$$

one has $\|y_k^\star\| \to 0$ as $k \to \infty$.

Remark 5.2 Note that condition (PSNC) at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is satisfied if Y is finite dimensional.

The next corollary that follows directly from the preceding corollary, gives a point-based condition for the relative metric regularity.

Corollary 5.1 *Under the assumptions of Theorem 5.1, suppose further that G^{-1} is PSNC at (x_0,y_0,y_0). Then F is metrically regular relatively to C at (x_0,y_0) provided*

$$d_*(0,D^*_{\mathscr{M}}G(x_0,y_0,y_0)(T(C,0))>0.$$

Next, let us consider the special case of $F(x) := f(x)-K$, where, $K \subseteq Y$ is a nonempty closed convex subset, $f : X \to Y$ is a continuous mapping around a given point $x_0 \in X$ with $f(x_0) \in K$. Defining $g := (f,f) : X \to Y \times Y$ and using the usual notation: $D^*_{\mathscr{F}}f(x)(y^*) := D^*_{\mathscr{F}}f(x,f(x))(y^*)$, one has

$$D^*_{\mathscr{F}}G(x,y_1,y_2)((y_1^*,y_2^*)) = \begin{cases} D^*_{\mathscr{F}}g(x)((y_1^*,y_2^*)) & \text{if } f(x)-y_i \in K,\ y_i^* \in N(K,f(x)-y_i),\ i=1,2 \\ \emptyset & \text{otherwise.} \end{cases}$$

From Theorem 5.1 we may deduce the following result.

Corollary 5.2 *Let X,Y be Asplund spaces and $C \subseteq$ be a nonempty cone. Let $K \subseteq Y$ be a nonempty closed convex subset and $f : X \to Y$ be a continuous mapping around $x_0 \in X$ with $k_0 := f(x_0) \in K$. If*

$$\liminf_{\substack{(x,k_1,k_2)\to(x_0,k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0,D^*_{\mathscr{F}}f(x)(T(C,\delta))\cap N(K,k_1)\times N(K,k_2))) > m > 0, \quad (5.28)$$

then the mapping $F(x) := f(x)-K$, $x \in X$ is metrically regular relatively to C with modulus $\tau = m^{-1}$ at x_0.

Remark 5.3 Note that if K is *sequentially normally compact at* $\bar{k}$, i.e., for all sequences $(k_n)_{n\in\mathbb{N}} \subseteq K$, $(k_n^*)_{n\in\mathbb{N}}$ with $k_n^* \in N(K,k_n)$,

$$k_n \to \bar{k} \text{ and } k_n^* \xrightarrow{w^*} 0 \quad \Longleftrightarrow \quad \|k_n^*\| \to 0,$$

then instead of (5.28), the following point-based condition

$$d_*(0,D^*_{\mathscr{L}}f(x_0)[T(C,0)\cap(N(K,k_0)\times N(K,k_0))]) > 0 \quad (5.29)$$

is also sufficient for metric regularity relatively to C of $F(x) := f(x)-K$ at x_0.

Corollary 5.3 *Under the assumptions of Corollary 5.2, suppose further that f is Fréchet differentiable near x_0, and its derivative is continuous at x_0. Then, the mapping $F(x) := f(x)-K$, $x \in X$ is metrically regular relatively to C if and only if*

$$\liminf_{\substack{(k_1,k_2)\to k_0\\ \delta\downarrow 0^+}} d_*(0,f'^*(x_0)[T(C,\delta)\cap(N(K,k_1)\times N(K,k_2))]) > m > 0. \quad (5.30)$$

Here, $f'^*(x)$ stands for the adjoint operator of $f'(x)$. Moreover, if K is sequentially normally compact, then (5.30) is equivalent to

$$d_*(0,f'^*(x_0)[T(C,0)\cap(N(K,k_0)\times N(K,k_0))]) > 0. \quad (5.31)$$

Proof. For the sufficiency part, suppose that

$$\liminf_{\substack{(k_1,k_2)\to(k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0, f'^*(x_0)(T(C,\delta))\cap N(K,k_1)\times N(K,k_2)) > m > 0.$$

Since f' is continuous at x_0, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f'(x) - f'(x_0)\| < \varepsilon \quad \text{for all } x \in \mathbb{B}(x_0,\delta).$$

Therefore, for all $\varepsilon > 0$,

$$\|f'(x)(y_1^*,y_2^*) - f'(x_0)(y_1^*,y_2^*)\| < \varepsilon,$$

$$\text{for all } x \in \mathbb{B}(x_0,\delta),\ k_1,k_2 \in \mathbb{B}(k_0,\varepsilon),\ (y_1^*,y_2^*) \in T(C,\delta)\cap(N(K,k_1)\times N(K,k_2)).$$

Consequently,

$$\begin{aligned}&\liminf_{\substack{(x,k_1,k_2)\to(x_0,k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0, f'^*(x)[T(C,\delta)\cap(N(K,k_1)\times N(K,k_2))])\\ &= \liminf_{\substack{k\to k_0\\ \delta\downarrow 0^+}} d_*(0, f'^*(x_0)[T(C,\delta)\cap(N(K,k_1)\times N(K,k_2))]) > m > 0.\end{aligned}$$

The conclusion follows from Corollary 5.2. For the necessity part, consider the mapping $g : X \to Y$ defined by

$$g(x) := f'(x_0)(x - x_0) + f(x_0) - f(x),\ \ x \in X.$$

Since f is continuously differentiable at x_0, for any $\varepsilon > 0$, there is $\delta > 0$ such that g is Lipschitz with constant ε on $\mathbb{B}(x_0,\delta)$. Hence in view of Theorem 4.1, the metric regularity relative to C of $F := f - K$ around (x_0,y_0) implies the one of

$$(F+g)(x) = f'(x_0)(x - x_0) + f(x_0) - K.$$

As $F+g$ is a convex multifunction, the conclusion of the necessary part follows from Proposition 5.23. The equivalence between (5.30) and (5.31) follows from Remark 5.3. □

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