SYMMETRY AND NONEXISTENCE RESULTS FOR A FRACTIONAL CHOQUARD EQUATION WITH WEIGHTS

ANH TUAN DUONG, PHUONG LE, AND NHU THANG NGUYEN

Abstract. Let $u \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ be a nonnegative solution of the fractional order equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \left(\frac{1}{|x|^{n-\beta}} * |x|^a u^p\right) |x|^a u^{p-1} \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

where $0 < \alpha < 2$, $0 < \beta < n$ and $a > \max\{-\alpha, -\frac{\alpha+\beta}{2}\}$. By exploiting the method of scaling spheres and moving planes in integral forms, we show that u must be zero if $1 \le p < \frac{n+\beta+2a}{n-\alpha}$ and must be radially symmetric about the origin if $a < 0$ and $\frac{n+\beta+2a}{n-\alpha} \le p \le \frac{n+\beta+a}{n-\alpha}$.

1. Introduction

In this paper, we study the fractional Choquard type equation with weights

$$(-\Delta)^{\frac{\alpha}{2}} u = \left(\frac{1}{|x|^{n-\beta}} * |x|^a u^p\right) |x|^a u^{p-1} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{1}$$

where $0 < \alpha < 2$, $0 < \beta < n$, $a > \max\{-\alpha, -\frac{\alpha+\beta}{2}\}$, $p \ge 1$. Here, the convolution of two functions f and g is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy.$$

We also remind that the fractional Laplacian in $\mathbb{R}^n$ is defined as a nonlocal pseudo-differential operator

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}} dy = C_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}} dy,$$

where $C_{n,\alpha}$ is a normalization constant, $B_\varepsilon(x)$ is the ball of radius ε and center $x \in \mathbb{R}^n$, and PV stands for the Cauchy principle value. This operator is well defined in the Schwartz space of rapidly decreasing continuously differentiable functions in $\mathbb{R}^n$. In this space, the fractional Laplacian can also be defined by the Fourier transform

$$\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u](\xi) = |\xi|^\alpha \mathcal{F} u(\xi),$$

where $\mathcal{F}u$ is the Fourier transform of u. One can extend this operator to the distributions u in the space L_α by

$$\langle (-\Delta)^{\frac{\alpha}{2}} u, \varphi \rangle = \int_{\mathbb{R}^n} u (-\Delta)^{\frac{\alpha}{2}} \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n),$$

2010 *Mathematics Subject Classification.* 35R11, 35B06, 35B53, 45G10.

Key words and phrases. fractional Choquard equations with weights, nonexistence of solutions, symmetry of solutions.

where

$$L_\alpha = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+\alpha}} dx < \infty \right\}.$$

It is also clear that if $u \in L_\alpha \cap C^{1,1}_{\text{loc}}(\Omega)$, where Ω is an open domain of $\mathbb{R}^n$, then $(-\Delta)^{\frac{\alpha}{2}} u(x)$ is well defined for all $x \in \Omega$. Throughout this paper, we study solutions of (1) in the classical sense. That is, we call u a nonnegative solution of (1) if u is a nonnegative function in $L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ and u verifies (1) for all $x \in \mathbb{R}^n \setminus \{0\}$.

In the last decades, the fractional Laplacian has been widely used to model various physical phenomena, such as the turbulence, water waves, anomalous diffusion, phase transitions, flame propagation and quasi-geostrophic flows (see [4, 6, 9] and the references therein). It also has several applications in probability, optimization and finance. In particular, the fractional Laplacian can be seen as the infinitesimal generator of a stable Lévy process (see [1, 3, 5]).

One may observe that the right hand side of (1) is also a nonlocal term. This phenomenon causes some mathematical difficulties which make the study of such problem particularly interesting. Moreover, problem of type (1) has a strong physical motivation. In fact, it is an analog of the nonlinear stationary Choquard equation

$$-\Delta u + V(x)u = 2\left(\frac{1}{|x|^{n-2}} * u^2\right) u \quad \text{ in } \mathbb{R}^3,$$

which arises naturally in a variety of applications, for instance, the physics of multiple particle systems, quantum mechanics, Hartree-Fock theory, physics of laser beams and so on, which we refer to [19, 23].

Elliptic problems of Choquard type have been studied extensively by several authors in recent years. An introduction to mathematical treatment of Choquard type equations can be found in the review paper [22] by Moroz and Schaftingen. Without any intention to provide a survey about the subject, we would like to refer the reader to the papers [2, 12, 20, 25] and the references therein, where some recent existence and multiplicity results for Choquard equations and related problems can be found. In this paper, we focus to problem (1), which is usually called the fractional stationary Choquard equation with Hénon-Hardy weights and vanishing potential. This problem was studied by various authors and some optimal Liouville theorems as well as classification results were established recently.

First, we consider the case $a = 0$. One may show that, if $a = 0$ and $0 < \alpha, \beta \leq 2$, then (1) is equivalent to the elliptic system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = v u^{p-1} & \text{in } \mathbb{R}^n \setminus \{0\}, \\ (-\Delta)^{\frac{\beta}{2}} v = u^p & \text{in } \mathbb{R}^n \setminus \{0\}, \end{cases} \tag{2}$$

up to a suitable scaling. By applying the method of moving planes for this system, Lei [17] classified all positive solutions to (1) when $\alpha = \beta = 2$ and $p = \frac{n+2}{n-2}$. He also derived Liouville theorem in the subcritical case $1 \leq p < \frac{n+2}{n-2}$. The more general cases $0 < \alpha, \beta < 2$ were studied by Ma and Zhang in [21], where they established the symmetry and nonexistence of positive solutions of (1) in the critical case $p = \frac{n+\beta}{n-\alpha}$ and subcritical case $\frac{n}{n-\alpha} \leq p < \frac{n+\beta}{n-\alpha}$, respectively. The method used in [21] is the direct method of moving planes, which was first introduced by Chen, Li, Li [7], for the fractional system (2). Note that the assumption $0 < \beta \leq 2$ is compulsory in this approach. Later, Dai, Fang and Qin [10] found a way to apply the method to equation (1) directly, and they successfully classified all positive solutions to (1) when $\beta = n - 2\alpha$, $\alpha \in (0, \min\{2, \frac{n}{2}\})$ and $p = 2$. Very recently, these results were extended to the full range $0 < \beta < n$ by Le in [15]. Indeed, it is proved in [15] that, if $0 < \beta < n$ and $u \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative solution of (1), then $u \equiv 0$ provided that $1 \le p < \frac{n+\beta}{n-\alpha}$ and u must assume the form $u(x) = c\left(\frac{t}{t^2+|x-x^0|^2}\right)^{\frac{n-\alpha}{2}}$ provided that $p = \frac{n+\beta}{n-\alpha}$.

Now we turn our attention to the weighted case, i.e., $a \neq 0$. When $a < 0$, the study of equation (1) is motivated by the doubly weighted Hardy-Littlewood-Sobolev inequality

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^{\frac{n+\beta+2a}{n-\alpha}}|u(y)|^{\frac{n+\beta+2a}{n-\alpha}}}{|x|^{-a}|x-y|^{n-\alpha}|y|^{-a}}dxdy\right)^{\frac{n-\alpha}{2(n+\beta+2a)}} \le C\|(-\Delta)^{\frac{\alpha}{4}}u\|_{L^2(\mathbb{R}^n)},$$

see [18, 24] for more details. In [13], Du, Gao and Yang proved that, if $\alpha = 2$, $\frac{n-\beta-\min\{4,n\}}{2} \le a \le 0$ and $p = \frac{n+\beta+2a}{n-2}$, then every $D^{1,2}(\mathbb{R}^n)$ positive solutions of (1) must be radially symmetric about the origin. There is also a radial symmetry result for positive solutions u of the Choquard type equation involving fractional p-Laplacian

$$(-\Delta)_p^{\frac{\alpha}{2}}u = \left(\frac{1}{|x|^{n-\beta}} * |x|^a u^p\right)|x|^a u^{p-1} \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

where $a < 0$ and $p > 2$, see Le [16]. However, a priori asymptotic assumptions on u is required in [16].

The main purpose of this paper is to extend the result in [15] to the case $a \neq 0$. Our results will cover the full range $0 < \alpha < 2$, $0 < \beta < n$ and we do not assume any integrability or asymptotic behavior of solutions.

In our first result is the following Liouville theorem.

Theorem 1. *Assume $0 < \alpha < 2$, $0 < \beta < n$, $a > \max\{-\alpha, -\frac{\alpha+\beta}{2}\}$ and $1 \le p < \frac{n+\beta+2a}{n-\alpha}$. If u is a nonnegative solution of equation (1), then $u \equiv 0$.*

Remark 1. The case $a = 0$ was already established in [15] via the direct method of moving planes. However, that method cannot be applied to the case $a \neq 0$ to get Liouville theorem. If one use that method on equation (1) with $a \neq 0$, one can only obtain the radial symmetry of nonnegative solutions in the range $1 \le p \le \frac{n+\beta+a}{n-\alpha}$. To obtain Liouville theorem in the full range $1 \le p < \frac{n+\beta+2a}{n-\alpha}$, we will take another approach: the method of scaling spheres. This method was introduced by Dai and Qin recently in [11] and was successfully used to establish the optimal Liouville theorem for nonnegative solutions of the fractional Hénon-Hardy equation

$$(-\Delta)^{\frac{\alpha}{2}}u = |x|^a u^{p-1} \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

In order to prove Theorem (1), we will turn equation (1) into an equivalent integral system, then we extend the method of scaling spheres to this system to get the desired result.

Our second result concerns with the radial symmetry of nonnegative solutions when $p \ge \frac{n+\beta+2a}{n-\alpha}$.

Theorem 2. *Assume* $0 < \alpha < 2$, $0 < \beta < n$, $\max\{-\alpha, -\frac{\alpha+\beta}{2}\} < a < 0$ *and* $\frac{n+\beta+2a}{n-\alpha} \le p \le \frac{n+\beta+a}{n-\alpha}$. *If* u *is a nonnegative solution of equation* (1)*, then* u *must be radially symmetric about the origin.*

Remark 2. Clearly, if $\frac{n+\beta+2a}{n-\alpha} \le p \le \frac{n+\beta+a}{n-\alpha}$, then $a \le 0$. Furthermore, when $a = 0$, the assumption $\frac{n+\beta+2a}{n-\alpha} \le p \le \frac{n+\beta+a}{n-\alpha}$ reduces to $p = \frac{n+\beta}{n-\alpha}$. This case was already studied in [15], where it is proved that every nonnegative solutions must be radially symmetric about some point in $\mathbb{R}^n$ and therefore assume an explicit form. Such an explicit form is, however, not available for equation (1) in the case $a < 0$ to the best of our knowledge.

Note that Theorem (2) can be proved by the direct method of moving planes as in [15]. However, in this paper, we will utilize the method of moving planes in integral forms instead, see [8]. This gives us a new proof which is simpler than that in [15] and more consistent with the proof of Theorem (1).

The remainder of this paper is organized as follows. In Section 2, we use maximum principle and Liouville theorem for α-harmonic functions to transform equation (1) into an equivalent integral system. Then we establish the method of scaling spheres for this system to prove Theorem 1 in Section 2. The last section is devoted to the proof of the radial symmetry of solutions, namely, Theorem (2).

Throughout this paper, we use C to denote some positive constant which may change from line to line or even in the same line. At times, we append subscripts to C to specify its dependence on the subscript parameters. For the sake of simplicity, we also denote by B_R the ball of center 0 and radius $R > 0$.

2. Preliminaries

It is not easy to investigate the qualitative properties of solutions to (1) directly due to the presence of the convolution term in the right hand side. To overcome this difficulty, throughout this paper, we denote

$$v = \frac{1}{|x|^{n-\beta}} * |x|^a u^p,$$

then we transform (1) into an equivalent integral system. More precisely, we have

Theorem 3. *If* u *is a nonnegative solution of* (1)*, then* (u, v) *is a solution of the integral system*

$$\begin{cases} u(x) = C_{n,\alpha} \displaystyle\int_{\mathbb{R}^n} \frac{|y|^a u^{p-1}(y) v(y)}{|x-y|^{n-\alpha}} dy, & x \in \mathbb{R}^n, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x-y|^{n-\beta}} dy, & x \in \mathbb{R}^n. \end{cases} \tag{3}$$

In order to prove Theorem 3, we need the following lemmas.

Lemma 4 (Lemma 2.2 in [11])**.** *Assume that* $n \ge 2$ *and* $0 < \alpha < 2$. *If* w *is* α*-harmonic in* $B_R \setminus \{0\}$ *and satisfies* $w(x) = o(|x|^{\alpha-n})$ *as* $|x| \to 0$*, then* w *can be defined at* 0 *so that it is* α*-harmonic in* B_R.

Lemma 5 (Maximum principle [7])**.** *Let* $n \ge 2$, $0 < \alpha < 2$ *and* Ω *be a bounded domain in* $\mathbb{R}^n$. *Assume that* $w \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ *and is lower semi-continuous on* $\overline{\Omega}$. *If*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w(x) \ge 0, & x \in \Omega, \\ w(x) \ge 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

then $w(x) \geq 0$ for all $x \in \mathbb{R}^n$. Moreover, if $w = 0$ at some point in Ω, then $w(x) = 0$ almost everywhere in $\mathbb{R}^n$. These conclusions hold for unbounded region Ω if we further assume that $\liminf_{|x|\to\infty} w(x) \geq 0$.

Lemma 6 (Liouville theorem [26])**.** *Assume that $n \geq 2$, $0 < \alpha < 2$ and w is a strong solution of*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w(x) = 0, & x \in \mathbb{R}^n, \\ w(x) \geq 0, & x \in \mathbb{R}^n. \end{cases}$$

Then, $u \equiv C$ for some constant $C \geq 0$.

Proof of Theorem 3. Assume that u is a nonnegative solution of (1). For arbitrary $R > 0$, let

$$u_R(x) = \int_{B_R} G_R^\alpha(x,y)|y|^a u^{p-1}(y)v(y)dy,$$

where G_R^α is the Green's functions on B_R for $(-\Delta)^{\frac{\alpha}{2}}$. That is, G_R^α is given by

$$G_R^\alpha(x,y) = \begin{cases} \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} \displaystyle\int_0^{\frac{t_R}{s_R}} \frac{b^{\frac{\alpha}{2}-1}}{(1+b)^{\frac{n}{2}}} db, & \text{if } x, y \in B_R(0), \\ 0, & \text{if } x \text{ or } y \ \in \mathbb{R}^n \setminus B_R(0), \end{cases}$$

where $s_R = \frac{|x-y|^2}{R^2}$ and $t_R = \left(1 - \frac{|x|^2}{R^2}\right)\left(1 - \frac{|y|^2}{R^2}\right)$ (see [14]). Then $u_R \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ and satisfies

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u_R(x) = |x|^a u^{p-1}(x)v(x), & x \in B_R \setminus \{0\}, \\ u_R(x) = 0, & x \in \mathbb{R}^n \setminus B_R. \end{cases} \tag{4}$$

Let $\overline{u}_R(x) = u(x) - u_R(x)$, then $(-\Delta)^{\frac{\alpha}{2}}\overline{u}_R = 0$ in $B_R \setminus \{0\}$. By Lemma 4, we have $\overline{u}_R \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ and

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \overline{u}_R(x) = 0, & x \in B_R, \\ \overline{u}_R(x) \geq 0, & x \in \mathbb{R}^n \setminus B_R. \end{cases}$$

By Lemma 5, we derive

$$\overline{u}_R(x) \geq 0, \quad x \in \mathbb{R}^n.$$

Therefore, letting $R \to \infty$, we have

$$u(x) \geq u_\infty(x) := C_{n,\alpha} \int_{\mathbb{R}^n} \frac{|y|^a u^{p-1}(y)v(y)}{|x-y|^{n-\alpha}} dy,$$

and $u_\infty \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ is a solution of

$$(-\Delta)^{\frac{\alpha}{2}} u_\infty(x) = |x|^a u^{p-1}(x)v(x), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Now let $\overline{u}_\infty(x) = u(x) - u_\infty(x)$, then by Lemma 4, we have $\overline{u}_\infty \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ and satisfies

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \overline{u}_\infty(x) = 0, & x \in \mathbb{R}^n, \\ \overline{u}_\infty(x) \geq 0, & x \in \mathbb{R}^n. \end{cases}$$

From Lemma 6, we get $\overline{u}_\infty \equiv C_1 \geq 0$, which indicates $u \geq C_1$. Thus,

$$v(x) = \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x-y|^{n-\beta}} dy \geq C_1^p \int_{\mathbb{R}^n} \frac{|y|^a}{|x-y|^{n-\beta}} dy = C_1^p C_2$$

and

$$u(x) = C_1 + C_{n,\alpha}\int_{\mathbb{R}^n}\frac{|y|^a u^{p-1}(y)v(y)}{|x-y|^{n-\alpha}}dy \geq C_1 + C_{n,\alpha}C_1^{2p-1}C_2\int_{\mathbb{R}^n}\frac{|y|^a}{|x-y|^{n-\alpha}}dy.$$

If we choose $x = 0$, then the above inequality implies $C_1 = 0$. Therefore, (u, v) satisfies integral system (3).

Conversely, assume that (u, v) is a nonnegative solution of integral system (3), then

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}u(x) &= \int_{\mathbb{R}^n}(-\Delta)^{\frac{\alpha}{2}}\left(\frac{C_{n,\alpha}}{|x-y|^{n-\alpha}}\right)|y|^a u^{p-1}(y)v(y)dy\\ &= \int_{\mathbb{R}^n}\delta_x(y)|y|^a u^{p-1}(y)v(y)dy = |x|^a u^{p-1}(x)v(x).\end{aligned}$$

That is, u satisfies equation (1). This completes the proof of Theorem 3. □

For any $\lambda > 0$, we denote by $S_\lambda = \partial B_\lambda$ the sphere of center 0 and radius λ. We also denote by

$$x^\lambda = \frac{\lambda^2 x}{|x|^2}$$

the inversion of $x \in \mathbb{R}^n \setminus \{0\}$ about the sphere S_λ. We then define the Kelvin transform of u and v with respect to S_λ by

$$u_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-\alpha}u(x^\lambda) \quad \text{and} \quad v_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-\beta}v(x^\lambda).$$

One may check that if (u, v) is a solution of (3), then (u_λ, v_λ) satisfies the integral system

$$\begin{cases} u_\lambda(x) = C_{n,\alpha}\displaystyle\int_{\mathbb{R}^n}\frac{|y|^a}{|x-y|^{n-\alpha}}\left(\frac{\lambda}{|y|}\right)^{\tau}u_\lambda^{p-1}(y)v_\lambda(y)dy, & x \in \mathbb{R}^n \setminus \{0\},\\ v_\lambda(x) = \displaystyle\int_{\mathbb{R}^n}\frac{|y|^a}{|x-y|^{n-\beta}}\left(\frac{\lambda}{|y|}\right)^{\tau}u_\lambda^{p}(y)dy, & x \in \mathbb{R}^n \setminus \{0\},\end{cases} \tag{5}$$

where

$$\tau = n + \beta + 2a - p(n-\alpha). \tag{6}$$

From Theorem 3, we deduce that, if u is a nonnegative solution of (1), then u, v are both positive or $u \equiv v \equiv 0$. Therefore, in the remainder of this paper, we may assume that (u, v) is a positive solution of (3). Then we try to reach a contradiction in the case $1 \leq p < \frac{n+\beta+2a}{n-\alpha}$ and the symmetry of u in the case $\frac{n+\beta+2a}{n-\alpha} \leq p \leq \frac{n+\beta+a}{n-\alpha}$.

3. Nonexistence of positive solutions

In this section, we utilize the direct method of scaling spheres to prove the Theorem 1. For that purpose, let (u, v) be a positive solution of (3) and define

$$U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).$$

From the first equation in (3), we have

$$\begin{aligned} u(x) = C_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{|x-y|^{n-\alpha}} u^{p-1}(y) v(y) dy \\ + C_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{\left| \frac{|y|}{\lambda} x - \frac{\lambda}{|y|} y \right|^{n-\alpha}} \left(\frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) v_\lambda(y) dy. \end{aligned} \tag{7}$$

From the first equation in (5), we deduce

$$\begin{aligned} u_\lambda(x) = C_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{\left| \frac{|y|}{\lambda} x - \frac{\lambda}{|y|} y \right|^{n-\alpha}} u^{p-1}(y) v(y) dy \\ + C_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{|x-y|^{n-\alpha}} \left(\frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) v_\lambda(y) dy. \end{aligned} \tag{8}$$

By combining (7) and (8), one can derive that, for any $x \in B_\lambda \setminus \{0\}$,

$$\begin{aligned} U_\lambda(x) &= C_{n,\alpha} \int_{B_\lambda} \left(\frac{|y|^a}{|x-y|^{n-\alpha}} - \frac{|y|^a}{\left| \frac{|y|}{\lambda} x - \frac{\lambda}{|y|} y \right|^{n-\alpha}} \right) \\ &\qquad \times \left(\left(\frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \right) dy \\ &\geq C_{n,\alpha} \int_{B_\lambda} \left(\frac{|y|^a}{|x-y|^{n-\alpha}} - \frac{|y|^a}{\left| \frac{|y|}{\lambda} x - \frac{\lambda}{|y|} y \right|^{n-\alpha}} \right) \\ &\qquad \times \left(u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \right) dy. \end{aligned} \tag{9}$$

In the last inequality, we have used the fact

$$\left| \frac{|y|}{\lambda} x - \frac{\lambda}{|y|} y \right|^2 - |x-y|^2 = \frac{(|x|^2 - \lambda^2)(|y|^2 - \lambda^2)}{\lambda^2} > 0 \quad \text{for } x, y \in B_\lambda \setminus \{0\}.$$

By the mean value theorem, one may show that, for any $0 < a \leq b$ and $q \geq 0$,

$$a^q - b^q \geq \max\{q, 1\} b^{q-1} (a - b). \tag{10}$$

Using (10), for each $y \in B_\lambda$, there are four possible cases:

- If $u_\lambda(y) \geq u(y)$ and $v_\lambda(y) \geq v(y)$, then
$$u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \geq 0.$$
- If $u_\lambda(y) \geq u(y)$ and $v_\lambda(y) < v(y)$, then
$$u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \geq u^{p-1}(y) V_\lambda(y).$$
- If $u_\lambda(y) < u(y)$ and $v_\lambda(y) \geq v(y)$, then
$$\begin{aligned} u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) &\geq \left(u_\lambda^{p-1}(y) - u^{p-1}(y) \right) v(y) \\ &\geq \max\{p-1, 1\} u^{p-2}(y) v(y) U_\lambda(y). \end{aligned}$$

- If $u_\lambda(y) < u(y)$ and $v_\lambda(y) < v(y)$, then

$$u_\lambda^{p-1}(y)v_\lambda(y) - u^{p-1}(y)v(y) = \left(u_\lambda^{p-1}(y) - u^{p-1}(y)\right) v_\lambda(y) + u^{p-1}(y)\left(v_\lambda(y) - v(y)\right)$$
$$\geq \max\{p-1,1\}u^{p-2}(y)v(y)U_\lambda(y) + u^{p-1}(y)V_\lambda(y).$$

Therefore, in all cases, we have

$$u_\lambda^{p-1}(y)v_\lambda(y) - u^{p-1}(y)v(y) \geq \max\{p-1,1\}u^{p-2}(y)v(y)U_\lambda^-(y) + u^{p-1}(y)V_\lambda^-(y), \tag{11}$$

where, as usual, we denote $a^- = \min\{0, a\}$ for any $a \in \mathbb{R}$.

We also denote

$$B_\lambda^u = \{x \in B_\lambda \setminus \{0\} \mid U_\lambda(x) < 0\} \quad \text{and} \quad B_\lambda^v = \{x \in B_\lambda \setminus \{0\} \mid V_\lambda(x) < 0\}.$$

From (9) and (11), we have

$$\begin{aligned} U_\lambda(x) = C_{n,\alpha} \int_{B_\lambda} &\left(\frac{|y|^a}{|x-y|^{n-\alpha}} - \frac{|y|^a}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\alpha}} \right) \\ &\times \left(\max\{p-1,1\}u^{p-2}(y)v(y)U_\lambda^-(y) + u^{p-1}(y)V_\lambda^-(y)\right) dy \\ \geq C \int_{B_\lambda^u} &\frac{|y|^a u^{p-2}(y)v(y)U_\lambda(y)}{|x-y|^{n-\alpha}} dy + C \int_{B_\lambda^v} \frac{|y|^a u^{p-1}(y)V_\lambda(y)}{|x-y|^{n-\alpha}} dy. \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} V_\lambda(x) &= \int_{B_\lambda} \left(\frac{|y|^a}{|x-y|^{n-\beta}} - \frac{|y|^a}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\beta}} \right) \left(\left(\frac{\lambda}{|y|}\right)^\tau u_\lambda^p(y) - u^p(y) \right) dy \\ &\geq \int_{B_\lambda} \left(\frac{|y|^a}{|x-y|^{n-\beta}} - \frac{|y|^a}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\beta}} \right) \left(u_\lambda^p(y) - u^p(y)\right) dy, \end{aligned} \tag{13}$$

which leads to

$$V_\lambda(x) \geq p \int_{B_\lambda^u} \frac{|y|^a u^{p-1}(y)U_\lambda(y)}{|x-y|^{n-\beta}} dy. \tag{14}$$

We also recall the following Hardy-Littlewood-Sobolev inequality.

Lemma 7 (Hardy-Littlewood-Sobolev inequality [18, 24]). *Let $0 < \alpha < n$ and $p, q > 1$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then we have*

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right\|_{L^q(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$.

Let $s > \frac{n}{n-\alpha}$ and $t > \frac{n}{n-\beta}$ be such that

$$\frac{1}{s} - \frac{1}{t} = \frac{\beta - \alpha}{2n}.$$

Applying Hardy-Littlewood-Sobolev and Hölder's inequality, from (12) we have

$$\begin{aligned}\|U_\lambda\|_{L^s(B^u_\lambda)} &\le C\left\||x|^a u^{p-2} v U_\lambda\right\|_{L^{\frac{ns}{n+\alpha s}}(B^u_\lambda)} + C\left\||x|^a u^{p-1} V_\lambda\right\|_{L^{\frac{ns}{n+\alpha s}}(B^v_\lambda)}\\ &\le C\left\||x|^a u^{p-2} v\right\|_{L^{\frac{n}{\alpha}}(B^u_\lambda)}\|U_\lambda\|_{L^s(B^u_\lambda)}\\ &\qquad + C\left\||x|^a u^{p-1}\right\|_{L^{\frac{2n}{\alpha+\beta}}(B^v_\lambda)}\|V_\lambda\|_{L^t(B^v_\lambda)}.\end{aligned}\tag{15}$$

Similarly, from (14), we have

$$\begin{aligned}\|V_\lambda\|_{L^t(B^v_\lambda)} &\le C\||x|^a u^{p-1} U_\lambda\|_{L^{\frac{nt}{n+\beta t}}(B^u_\lambda)}\\ &\le C\||x|^a u^{p-1}\|_{L^{\frac{2n}{\alpha+\beta}}(B^u_\lambda)}\|U_\lambda\|_{L^s(B^u_\lambda)}.\end{aligned}\tag{16}$$

From (15) and (16), we deduce

$$\begin{aligned}\|U_\lambda\|_{L^s(B^u_\lambda)} \le C\left\{\left\||x|^a u^{p-2} v\right\|_{L^{\frac{n}{\alpha}}(B^u_\lambda)} + \||x|^a u^{p-1}\|_{L^{\frac{2n}{\alpha+\beta}}(B^u_\lambda)}\||x|^a u^{p-1}\|_{L^{\frac{2n}{\alpha+\beta}}(B^v_\lambda)}\right\}\\ \times\|U_\lambda\|_{L^s(B^u_\lambda)},\end{aligned}\tag{17}$$

where the constant C is independent of λ.

Proof of Theorem 1. The proof is divided into three steps.

Step 1. (Start dilating the sphere from near $\lambda = 0$)

In this step, we will prove that for $\lambda > 0$ sufficiently small,

$$U_\lambda \ge 0 \quad \text{ in } B_\lambda \setminus \{0\}.\tag{18}$$

Indeed, since $a > \max\{-\alpha, -\frac{\alpha+\beta}{2}\}$, there exists $\varepsilon_0 > 0$ small enough, such that

$$\left\||x|^a u^{p-2} v\right\|_{L^{\frac{n}{\alpha}}(B_\lambda)} + \||x|^a u^{p-1}\|^2_{L^{\frac{2n}{\alpha+\beta}}(B_\lambda)} \le \frac{1}{2C}$$

for all $0 < \lambda \le \varepsilon_0$, where the constant C is the same as in (17). Hence, (17) indicates $\|U_\lambda\|_{L^s(B^u_\lambda)} = 0$, which means $B^u_\lambda = \emptyset$. Therefore, (18) holds for all $\lambda < \varepsilon_0$. This completes Step 1.

Step 2. (Dilate the sphere S_λ outward until $\lambda = \infty$)

Step 1 provides us a starting point to dilate the sphere S_λ from near $\lambda = 0$. Now we dilate the sphere S_λ outward as long as (18) holds. Let

$$\lambda_0 = \sup\{\lambda > 0 \mid U_\mu \ge 0 \text{ in } B_\mu \setminus \{0\} \text{ for all } \mu \in (0, \lambda]\}.$$

In this step, we show that

$$\lambda_0 = \infty.\tag{19}$$

Suppose on contrary that $0 < \lambda_0 < \infty$. Since U_λ is continuous with respect to λ, we already have $U_{\lambda_0} \ge 0$ in $B_{\lambda_0} \setminus \{0\}$. From (13), we deduce $V_{\lambda_0} > 0$ in $B_{\lambda_0} \setminus \{0\}$. Then (9) implies $U_{\lambda_0} > 0$ in $B_{\lambda_0} \setminus \{0\}$.

Now we claim that, there exists $C > 0$ and $\eta > 0$ such that

$$U_{\lambda_0}, V_{\lambda_0} \ge C \quad \text{ in } B_\eta \setminus \{0\}.\tag{20}$$

Indeed, from (9), we can derive that, for any $x \in B_{\lambda_0} \setminus \{0\}$,

$$U_{\lambda_0}(x) \ge C_{n,\alpha}\int_{B_{\lambda_0}}\left(\frac{|y|^a}{|x-y|^{n-\alpha}} - \frac{|y|^a}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}}\right)\tag{21}$$

$$
\begin{aligned}
&\times \left(u_{\lambda_0}^{p-1}(y)v_{\lambda_0}(y) - u^{p-1}(y)v(y)\right) dy \\
&\geq C_{n,\alpha} \int_{B_{\lambda_0}} \left(\frac{|y|^a}{|x-y|^{n-\alpha}} - \frac{|y|^a}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}} \right) u^{p-1}(y)V_{\lambda_0}(y)dy. \qquad (22)
\end{aligned}
$$

Note that if $|y| < \frac{\lambda_0}{2}$, then

$$
\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}} \to \frac{1}{|y|^{n-\alpha}} - \frac{1}{\lambda_0^{n-\alpha}} > \frac{2^{n-\alpha}-1}{\lambda_0^{n-\alpha}} \qquad \text{as } x \to 0.
$$

Hence for $x \in B_\eta \setminus \{0\}$, where η is sufficiently small, we deduce from (22),

$$
U_{\lambda_0}(x) \geq C \int_{B_{\frac{\lambda_0}{2}}} u^{p-1}(y)V_{\lambda_0}(y)dy > 0
$$

for all $x \in B_\eta \setminus \{0\}$.

Similarly, from (13), we can derive

$$
\begin{aligned}
V_{\lambda_0}(x) &\geq \int_{B_{\lambda_0}} \left(\frac{|y|^a}{|x-y|^{n-\beta}} - \frac{|y|^a}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\beta}} \right) \left(u_{\lambda_0}^p(y) - u^p(y)\right) dy \\
&\geq C \int_{B_{\frac{\lambda_0}{2}}} u^{p-1}(y)U_{\lambda_0}(y)dy > 0
\end{aligned}
$$

for all $x \in B_\eta \setminus \{0\}$, where η is chosen smaller if necessary. This proves (20).

Now we fix $0 < r_0 < \frac{\lambda_0}{2}$ small enough, such that

$$
\left\| |x|^a u^{p-2} v \right\|_{L^{\frac{n}{\alpha}}(B_{\lambda_0+r_0} \setminus B_{\lambda_0-r_0})} + \left\| |x|^a u^{p-1} \right\|^2_{L^{\frac{2n}{\alpha+\beta}}(B_{\lambda_0+r_0} \setminus B_{\lambda_0-r_0})} \leq \frac{1}{2C}, \qquad (23)
$$

where the constant C is the same as in (17).

It follows from (20), the continuity and positivity of U_{λ_0} and V_{λ_0} that, there exists a constant $C > 0$ such that

$$
U_{\lambda_0}, V_{\lambda_0} \geq C \quad \text{in } B_{\lambda_0 - r_0} \setminus \{0\}.
$$

Since u and v are uniformly continuous on arbitrary compact set, there exists $\rho_0 \in (0, r_0)$ such that, for any $\lambda \in (\lambda_0, \lambda_0 + \rho_0)$,

$$
U_\lambda, V_\lambda \geq \frac{C}{2} > 0 \quad \text{in } B_{\lambda_0 - r_0} \setminus \{0\}. \qquad (24)
$$

Therefore, for any $\lambda \in (\lambda_0, \lambda_0 + \rho_0)$,

$$
B_\lambda^u, B_\lambda^v \subset B_{\lambda_0+r_0} \setminus B_{\lambda_0-r_0}.
$$

Hence, estimates (17) and (23) yield $\|U_\lambda\|_{L^s(B_\lambda^u)} = 0$, which means $B_\lambda^u = \emptyset$. Thus, for any $\lambda \in (\lambda_0, \lambda_0 + \rho_0)$,

$$
U_\lambda \geq 0 \quad \text{in } B_\lambda \setminus \{0\}.
$$

However, this contradicts the definition of λ_0 and (19) is proved.

Step 3. (Derive lower bound estimates on u and v)

Since $\lambda_0 = \infty$, we have $U_\lambda \geq 0$ in $B_\lambda \setminus \{0\}$ for all $\lambda > 0$, that is,

$$u(x) \geq \left(\frac{\lambda}{|x|}\right)^{n-\alpha} u\left(\frac{\lambda^2 x}{|x|^2}\right) \quad \text{for all } |x| \geq \lambda \text{ and } \lambda > 0.$$

Choose $\lambda = \sqrt{|x|}$, we have

$$u(x) \geq \frac{1}{|x|^{\frac{n-\alpha}{2}}} u\left(\frac{x}{|x|}\right) \quad \text{for all } |x| \geq 1.$$

Hence, we arrive at the following lower bound estimate

$$u(x) \geq \frac{\min_{S_1} u}{|x|^{\frac{n-\alpha}{2}}} = \frac{C}{|x|^{\frac{n-\alpha}{2}}} \quad \text{for all } |x| \geq 1.$$

That is,

$$u(x) \geq \frac{C}{|x|^{\tau_0}} \quad \text{for all } |x| \geq 1,$$

where $\tau_0 = \frac{n-\alpha}{2}$.

Now we have for $|x| \geq 1$,

$$v(x) = \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x-y|^{n-\beta}} dy \geq \frac{C}{|x|^{n-\beta}} \int_{2|x| \leq |y| \leq 3|x|} \frac{dy}{|y|^{p\tau_0 - a}} = \frac{C}{|x|^{p\tau_0 - (a+\beta)}}$$

and hence

$$\begin{aligned} u(x) &= C_{n,\alpha} \int_{\mathbb{R}^n} \frac{|y|^a u^{p-1}(y) v(y)}{|x-y|^{n-\alpha}} dy \\ &\geq \frac{C}{|x|^{n-\alpha}} \int_{2|x| \leq |y| \leq 3|x|} \frac{dy}{|y|^{(p-1)\tau_0 - a + p\tau_0 - (a+\beta)}} \\ &= \frac{C}{|x|^{(2p-1)\tau_0 - (2a+\alpha+\beta)}}. \end{aligned}$$

That is,

$$u(x) \geq \frac{C}{|x|^{\tau_1}} \quad \text{for all } |x| \geq 1,$$

where $\tau_1 = (2p-1)\tau_0 - (2a+\alpha+\beta)$.

Continuing the above iteration process, we have the following lower bound estimates for every $k \in \mathbb{N}$,

$$u(x) \geq \frac{C}{|x|^{\tau_k}} \quad \text{for all } |x| \geq 1, \tag{25}$$

where

$$\tau_{k+1} = (2p-1)\tau_k - (2a+\alpha+\beta).$$

If $p = 1$, then

$$\tau_k = \tau_0 - k(2a+\alpha+\beta) \to -\infty \quad \text{as } k \to \infty.$$

If $p > 1$, then

$$\tau_k = (2p-1)^k \left(\frac{n-\alpha}{2} - \frac{2a+\alpha+\beta}{2(p-1)}\right) + \frac{2a+\alpha+\beta}{2(p-1)}.$$

This indicates

$$\tau_k \to -\infty \quad \text{as} \quad k \to \infty \quad \text{if} \quad 1 < p < \frac{n+\beta+2a}{n-\alpha}.$$

Therefore, in all cases, $\tau_k \to -\infty$ as $k \to \infty$. Combining this fact with (25), we arrive

$$u(x) \geq C \quad \text{for all} \quad |x| \geq 1.$$

Therefore, we have

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x-y|^{n-\beta}} dy \geq \frac{C}{|x|^{n-\beta}} \int_{2|x|\leq|y|\leq 3|x|} \frac{dy}{|y|^{-a}} \\ &= \frac{C}{|x|^{-(a+\beta)}} > C \quad \text{for all} \quad |x| \geq 1. \end{aligned}$$

Then

$$\infty > u(0) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{|y|^a u^{p-1}(y) v(y)}{|y|^{n-\alpha}} dy \geq C \int_{\mathbb{R}^n \setminus B_1} \frac{dy}{|y|^{n-a-\alpha}}.$$

However, the last integral would be infinity since $a + \alpha > 0$. We reach a contradiction. Therefore, system (3) has no positive solution. This completes the proof of Theorem 1. □

4. Symmetry of positive solutions

Let (u, v) be a positive solution of (3) and denote by $\overline{u}$ and $\overline{v}$ the Kelvin transform of u and v with respect to S_1. That is,

$$\overline{u}(x) = \frac{1}{|x|^{n-\alpha}} u\left(\frac{x}{|x|^2}\right) \quad \text{and} \quad \overline{v}(x) = \frac{1}{|x|^{n-\beta}} v\left(\frac{x}{|x|^2}\right).$$

From (5), we see that $(\overline{u}, \overline{v})$ satisfies the integral system

$$\begin{cases} \overline{u}(x) = C_{n,\alpha} \displaystyle\int_{\mathbb{R}^n} \frac{\overline{u}^{p-1}(y)\overline{v}(y)}{|x-y|^{n-\alpha}|y|^\gamma} dy, & x \in \mathbb{R}^n \setminus \{0\}, \\ \overline{v}(x) = \displaystyle\int_{\mathbb{R}^n} \frac{\overline{u}^p(y)}{|x-y|^{n-\beta}|y|^\gamma} dy, & x \in \mathbb{R}^n \setminus \{0\}, \end{cases} \tag{26}$$

where

$$\gamma = n + \beta + a - p(n-\alpha) \geq 0.$$

Moreover, we have

$$\overline{u}(x) \sim \frac{1}{|x|^{n-\alpha}} \quad \text{and} \quad \overline{v}(x) \sim \frac{1}{|x|^{n-\beta}} \quad \text{as } |x| \to \infty. \tag{27}$$

To prove Theorem 2, we exploit the method of moving planes in integral forms. For arbitrary $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}$$

be the moving plane,

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

be the region to the left of the plane.

In this section, we redefine x^λ, U_λ and V_λ as follows

- $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ is the reflection of the point $x = (x_1, x_2, \ldots, x_n)$ about the plane T_λ,
- $\overline{u}_\lambda(x) = \overline{u}(x^\lambda)$ and $\overline{v}_\lambda(x) = \overline{v}(x^\lambda)$,
- $U_\lambda(x) = \overline{u}_\lambda(x) - \overline{u}(x)$ and $V_\lambda(x) = \overline{v}_\lambda(x) - \overline{v}(x)$.

We also denote

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda \setminus \{0\} \mid U_\lambda(x) < 0\} \quad \text{and} \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda \setminus \{0\} \mid V_\lambda(x) < 0\}.$$

One can observe from Lemma 26 that, for any $x \in \Sigma_\lambda$ with $\lambda < 0$,

$$\overline{u}(x) = C_{n,\alpha} \int_{\Sigma_\lambda} \frac{\overline{u}^{p-1}(y)\overline{v}(y)}{|x-y|^{n-\alpha}|y|^\gamma} dy + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{\overline{u}_\lambda^{p-1}(y)\overline{v}_\lambda(y)}{|x-y^\lambda|^{n-\alpha}|y^\lambda|^\gamma} dy \tag{28}$$

and

$$\overline{u}_\lambda(x) = C_{n,\alpha} \int_{\Sigma_\lambda} \frac{\overline{u}^{p-1}(y)\overline{v}(y)}{|x^\lambda-y|^{n-\alpha}|y|^\gamma} dy + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{\overline{u}_\lambda^{p-1}(y)\overline{v}_\lambda(y)}{|x^\lambda-y^\lambda|^{n-\alpha}|y^\lambda|^\gamma} dy. \tag{29}$$

Since $|x-y^\lambda| = |x^\lambda - y|$ and $|x-y| = |x^\lambda - y^\lambda|$, from (28) and (29), we obtain

$$\begin{aligned} U_\lambda(x) &= C_{n,\alpha} \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda-y|^{n-\alpha}} \right) \left(\frac{\overline{u}_\lambda^{p-1}(y)\overline{v}_\lambda(y)}{|y^\lambda|^\gamma} - \frac{\overline{u}^{p-1}(y)\overline{v}(y)}{|y|^\gamma} \right) dy \\ &\geq C_{n,\alpha} \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda-y|^{n-\alpha}} \right) \frac{\overline{u}_\lambda^{p-1}(y)\overline{v}_\lambda(y) - \overline{u}^{p-1}(y)\overline{v}(y)}{|y|^\gamma} dy. \end{aligned} \tag{30}$$

Arguing as in (11), we have

$$\overline{u}_\lambda^{p-1}(y)\overline{v}_\lambda(y) - \overline{u}^{p-1}(y)\overline{v}(y) \geq \max\{p-1,1\}\overline{u}^{p-2}(y)\overline{v}(y)U_\lambda^-(y) + \overline{u}^{p-1}(y)V_\lambda^-(y). \tag{31}$$

From (30) and (31), we have

$$U_\lambda(x) \geq C \int_{\Sigma_\lambda^u} \frac{\overline{u}^{p-2}(y)\overline{v}(y)U_\lambda(y)}{|x-y|^{n-\alpha}|y|^\gamma} dy + C \int_{\Sigma_\lambda^v} \frac{\overline{u}^{p-1}(y)V_\lambda(y)}{|x-y|^{n-\alpha}|y|^\gamma} dy. \tag{32}$$

Similarly,

$$\begin{aligned} V_\lambda(x) &= \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^\lambda-y|^{n-\beta}} \right) \left(\frac{\overline{u}_\lambda^p(y)}{|y^\lambda|^\gamma} - \frac{\overline{u}^p(y)}{|y|^\gamma} \right) dy \\ &\geq \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^\lambda-y|^{n-\beta}} \right) \frac{\overline{u}_\lambda^p(y) - \overline{u}^p(y)}{|y|^\gamma} dy, \end{aligned} \tag{33}$$

which leads to

$$V_\lambda(x) \geq p \int_{B_\lambda^u} \frac{\overline{u}^{p-1}(y)U_\lambda(y)}{|x-y|^{n-\beta}|y|^\gamma} dy. \tag{34}$$

As in the previous section, we let $s > \frac{n}{n-\alpha}$ and $t > \frac{n}{n-\beta}$ be such that

$$\frac{1}{s} - \frac{1}{t} = \frac{\beta-\alpha}{2n}.$$

Then using (32), (34) and arguing as in Section 3, we derive

$$\begin{aligned} \|U_\lambda\|_{L^s(\Sigma_\lambda^u)} \leq C \left\{ \left\| \frac{\overline{u}^{p-2}\overline{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^u)} + \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma_\lambda^u)} \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma_\lambda^v)} \right\} \\ \times \|U_\lambda\|_{L^s(\Sigma_\lambda^u)}, \end{aligned} \tag{35}$$

where the constant C is independent of λ.

From $a > \max\{-\alpha, -\frac{\alpha+\beta}{2}\}$ and (27), it is easy to check that

$$\frac{\overline{u}^{p-2}\overline{v}}{|x|^\gamma} \in L^{\frac{n}{\alpha}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{and} \quad \frac{\overline{u}^{p-1}}{|x|^\gamma} \in L^{\frac{2n}{\alpha+\beta}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{for all} \quad \varepsilon > 0. \tag{36}$$

Proof of Theorem 2. We consider two cases.

Case 1: The subscritical case $p < \frac{n+\beta+a}{n-\alpha}$. In this case, $\gamma > 0$.

We start moving the plane T_λ from near $\lambda = -\infty$ to the right until it reaches the limiting position in order to derive symmetry. This procedure contains two steps.

Step 1. We show that, for λ sufficiently negative,

$$U_\lambda \geq 0 \quad \text{in } \Sigma_\lambda. \tag{37}$$

Indeed, from (36), we can choose $R_0 > 0$ sufficiently large, such that for $\lambda \leq -R_0$, we have

$$\left\| \frac{\overline{u}^{p-2}\overline{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{\alpha}}(\Sigma^u_\lambda)} + \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma^u_\lambda)} \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma^v_\lambda)} \leq \frac{1}{2C}, \tag{38}$$

where the constant C is the same as in (35).

Therefore, (35) and (38) imply that $\|U_\lambda\|_{L^s(\Sigma^u_\lambda)} = 0$ and hence $|\Sigma^u_\lambda| = 0$ for $\lambda \leq -R_0$. Thus, (37) holds for $\lambda \leq -R_0$. This completes Step 1.

Step 2. Let

$$\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu \leq \lambda\}. \tag{39}$$

In this step, we show that

$$\lambda_0 = 0. \tag{40}$$

Suppose on contrary that $\lambda_0 < 0$. By continuity, we have $U_{\lambda_0} \geq 0$. Hence, it follows from (33) that

$$V_{\lambda_0}(x) \geq \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^{\lambda_0}-y|^{n-\beta}} \right) \left(\frac{1}{|y^{\lambda_0}|^\gamma} - \frac{1}{|y|^\gamma} \right) \overline{u}^p(y)dy > 0$$

and hence (30) implies

$$U_{\lambda_0}(x) \geq \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{n-\alpha}} \right) \left(\frac{1}{|y^{\lambda_0}|^\gamma} - \frac{1}{|y|^\gamma} \right) \overline{u}^{p-1}(y)\overline{v}(y)dy > 0.$$

That is, $U_{\lambda_0} > 0$ in Σ_{λ_0}. We will obtain a contradiction with (39) by showing the existence of an $\varepsilon > 0$ small enough such that $U_\lambda \geq 0$ in Σ_λ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$.

It can be clearly seen from (35) that, our primary task is to prove that, one can choose $\varepsilon > 0$ sufficiently small such that

$$\left\| \frac{\overline{u}^{p-2}\overline{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{\alpha}}(\Sigma^u_\lambda)} + \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma^u_\lambda)} \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma^v_\lambda)} \leq \frac{1}{2C} \tag{41}$$

for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$, where the constant C is the same as in (35).

From (36), there exists $R > 0$ large enough such that

$$\left\| \frac{\overline{u}^{p-2}\overline{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{\alpha}}(\Sigma^u_\lambda \setminus B_R)} + \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma^u_\lambda \setminus B_R)} \left\| \frac{\overline{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+\beta}}(\Sigma^v_\lambda \setminus B_R)} \leq \frac{1}{2C}, \tag{42}$$

Now fix this R, in order to derive (41), we only need to show that

$$\lim_{\lambda \to \lambda_0^+} |\Sigma^u_\lambda \cap B_R| = \lim_{\lambda \to \lambda_0^+} |\Sigma^v_\lambda \cap B_R| = 0. \tag{43}$$

To prove this, we define $E_\delta = \{x \in \Sigma_{\lambda_0} \cap B_R(0) \mid U_{\lambda_0}(x) > \delta\}$ and $F_\delta = \Sigma_{\lambda_0} \cap B_R(0) \setminus E_\delta$ for any $\delta > 0$, and let $D_\lambda = (\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \cap B_R(0)$ for any $\lambda > \lambda_0$. Then

$$\lim_{\delta \to 0^+} |F_\delta| = 0, \quad \lim_{\lambda \to \lambda_0^+} |D_\lambda| = 0 \tag{44}$$

and

$$\Sigma_\lambda^u \cap B_R(0) \subset \Sigma_\lambda^u \cap (E_\delta \cup F_\delta \cup D_\lambda) \subset (\Sigma_\lambda^u \cap E_\delta) \cup F_\delta \cup D_\lambda. \tag{45}$$

Therefore, for an arbitrarily fixed $\eta > 0$, one can choose $\delta > 0$ small enough such that $|F_\delta| \leq \eta$. For this fixed δ, we will point out that

$$\lim_{\lambda \to \lambda_0^+} |\Sigma_\lambda^u \cap E_\delta| = 0. \tag{46}$$

Indeed, for all $x \in \Sigma_\lambda^u \cap E_\delta$, we have $\overline{u}(x^{\lambda_0}) - \overline{u}(x^\lambda) = U_{\lambda_0}(x) - U_\lambda(x) > \delta$. It follows that $\Sigma_\lambda^u \cap E_\delta \subset G_\delta^\lambda := \{x \in B_R(0) \mid \overline{u}(x^{\lambda_0}) - \overline{u}(x^\lambda) > \delta\}$. By Chebyshev inequality, we get

$$\begin{aligned} |G_\delta^\lambda| &\leq \frac{1}{\delta} \int_{G_\delta^\lambda} \left| \overline{u}(x^{\lambda_0}) - \overline{u}(x^\lambda) \right| dx \\ &= \frac{1}{\delta} \int_{B_R(2\lambda_0 e_1)} |\overline{u}(x) - \overline{u}(x + 2(\lambda_0 - \lambda)e_1)| \, dx, \end{aligned}$$

where $e_1 = (1, 0, \ldots, 0)$. Hence $\lim\limits_{\lambda \to \lambda_0^+} |G_\delta^\lambda| = 0$, from which (46) follows.

Therefore, by (44), (45) and (46), we have

$$\lim_{\lambda \to \lambda_0^+} |\Sigma_\lambda^u \cap B_R(0)| \leq |F_\delta| \leq \eta.$$

This implies the first claim in (43) since $\eta > 0$ is arbitrarily chosen. The second one can be obtain by similar reasoning. From (42) and (43), we arrive at (41).

Now we deduce from (35) and (41) that, there exists an $\varepsilon > 0$ sufficiently small such that $|\Sigma_\lambda^u| = 0$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$. Hence $U_\lambda \geq 0$ in Σ_λ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$. This contradicts with the definition of λ_0 in (39). Therefore, (40) must hold and hence $U_0(x) \leq 0$ in Σ_0.

This completes Step 2.

Similar to the previous steps, one can move the plane T_λ from $+\infty$ to the left to get that $U_0(x) \leq 0$ in Σ_0. Hence, $U_0 \equiv 0$ and $\overline{u}$ is symmetric about T_0. Since we can repeat the previous arguments to any direction, we deduce that $\overline{u}$ is radially symmetric about 0. So is u.

Case 2: The critical case $p = \frac{n+\beta+a}{n-\alpha}$. In this case, $\gamma = 0$.

By contradiction, assume that (3) has a positive solution (u, v) such that u is not radially symmetric about the origin. Then there exists a hyperplane H passing through the origin such that u is not symmetric about H. Without loss of generality, we may assume $H = T_0$.

Step 1 is entirely the same as that in the subcritical case, that is, we can show that for λ sufficiently negative,

$$U_\lambda \geq 0 \quad \text{ in } \Sigma_\lambda. \tag{47}$$

Let

$$\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu \leq \lambda\}.$$

We show that

$$\lambda_0 = 0.$$

Suppose on contrary that $\lambda_0 < 0$. We consider 2 possibilities.

Possibility (i): $U_{\lambda_0} = 0$ in Σ_{λ_0}. In this case, 0 is not a singular point of $\overline{u}$ and $\overline{v}$. Hence

$$u(x) \sim \frac{1}{|x|^{n-\alpha}} \quad \text{ and } \quad v(x) \sim \frac{1}{|x|^{n-\beta}} \quad \text{ as } |x| \to \infty.$$

Therefore,

$$|x|^a u^{p-2} v \in L^{\frac{n}{\alpha}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{and} \quad |x|^a u^{p-1} \in L^{\frac{2n}{\alpha+\beta}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{for all} \quad \varepsilon > 0.$$

This enables us to apply the method of moving plane to integral system (3) directly and show that u is symmetric about the origin, which is a contradiction. The proof is very similar to that of Case 1. The only difference is that we deal with $u, v, -a$ instead of $\overline{u}, \overline{v}, \gamma$.

Possibility (ii): $U_{\lambda_0} \geq 0$, but $U_{\lambda_0} \not\equiv 0$ in Σ_{λ_0}. It follows from (33) that

$$V_{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^{\lambda_0} - y|^{n-\beta}} \right) \left(\overline{u}^p_{\lambda_0}(y) - \overline{u}^p(y) \right) dy > 0$$

and hence (30) implies

$$\begin{aligned} U_{\lambda_0}(x) \geq & \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0} - y|^{n-\alpha}} \right) \left(\overline{u}^{p-1}_{\lambda_0}(y) \overline{v}_{\lambda_0}(y) - \overline{u}^{p-1}(y) \overline{v}(y) \right) dy \\ & > 0. \end{aligned}$$

Similar to the subcritical case, one can show that the plane T_λ can move a little bit to the right such that (47) still holds. This contradicts the definition of λ_0.

Therefore, $\lambda_0 = 0$. Similarly, one can move the plane T_λ from $+\infty$ to the left to finally get that u is symmetric about T_0, which is a contradiction.

This completes the proof of Theorem 2. □

Acknowledgments

This work was done while the authors were visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2019. They are grateful to the staff of the institute for their kind support and hospitality.

References

[1] Applebaum, D. *Lévy processes and stochastic calculus*, second ed., vol. 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2009. 2

[2] Belchior, P., Bueno, H., Miyagaki, O. H., and Pereira, G. A. Remarks about a fractional Choquard equation: ground state, regularity and polynomial decay. *Nonlinear Anal. 164* (2017), 38–53. 2

[3] Bertoin, J. *Lévy processes*, vol. 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. 2

[4] Bouchaud, J.-P., and Georges, A. Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications. *Phys. Rep. 195*, 4-5 (1990), 127–293. 2

[5] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations 32*, 7-9 (2007), 1245–1260. 2

[6] Caffarelli, L. A., and Vasseur, A. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math. (2) 171*, 3 (2010), 1903–1930. 2

[7] Chen, W., Li, C., and Li, Y. A direct method of moving planes for the fractional Laplacian. *Adv. Math. 308* (2017), 404–437. 2, 4

[8] Chen, W., Li, C., and Ou, B. Classification of solutions for an integral equation. *Comm. Pure Appl. Math. 59*, 3 (2006), 330–343. 4

[9] Constantin, P. Euler equations, Navier-Stokes equations and turbulence. In *Mathematical foundation of turbulent viscous flows*, vol. 1871 of *Lecture Notes in Math.* Springer, Berlin, 2006, pp. 1–43. 2

[10] Dai, W., Fang, Y., and Qin, G. Classification of positive solutions to fractional order Hartree equations via a direct method of moving planes. *J. Differential Equations 265*, 5 (2018), 2044–2063. 3

[11] Dai, W., and Qin, G. Liouville type theorems for fractional and higher order Hénon-Hardy type equations via the method of scaling spheres. *arXiv e-prints* (Oct 2018), arXiv:1810.02752. 3, 4
[12] d'Avenia, P., Siciliano, G., and Squassina, M. On fractional Choquard equations. *Math. Models Methods Appl. Sci. 25*, 8 (2015), 1447–1476. 2
[13] Du, L., Gao, F., and Yang, M. Existence and qualitative analysis for nonlinear weighted Choquard equations. *arXiv e-prints* (Oct 2018), arXiv:1810.11759. 3
[14] Kulczycki, T. Properties of Green function of symmetric stable processes. *Probab. Math. Statist. 17*, 2, Acta Univ. Wratislav. No. 2029 (1997), 339–364. 5
[15] Le, P. Liouville theorem and classification of positive solutions for a fractional Choquard type equation. *Nonlinear Anal. 185* (2019), 123–141. 3, 4
[16] Le, P. Symmetry of singular solutions for a weighted Choquard equation involving the fractional p-Laplacian. *Commun. Pure Appl. Anal.* (in press). 3
[17] Lei, Y. Liouville theorems and classification results for a nonlocal Schrödinger equation. *Discrete Contin. Dyn. Syst. 38*, 11 (2018), 5351–5377. 2
[18] Lieb, E. H. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2) 118*, 2 (1983), 349–374. 3, 8
[19] Lieb, E. H., and Simon, B. The Hartree-Fock theory for Coulomb systems. *Comm. Math. Phys. 53*, 3 (1977), 185–194. 2
[20] Ma, P., and Zhang, J. Existence and multiplicity of solutions for fractional Choquard equations. *Nonlinear Anal. 164* (2017), 100–117. 2
[21] Ma, P., and Zhang, J. Symmetry and Nonexistence of Positive Solutions for Fractional Choquard Equations. *arXiv e-prints* (Apr 2017), arXiv:1704.02190. 2
[22] Moroz, V., and Van Schaftingen, J. A guide to the Choquard equation. *J. Fixed Point Theory Appl. 19*, 1 (2017), 773–813. 2
[23] Nazin, G. I. Limit distribution functions of systems with many-particle interactions in classical statistical physics. *Teoret. Mat. Fiz. 25*, 1 (1975), 132–140. 2
[24] Stein, E. M. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970. 3, 8
[25] Zhang, W., and Wu, X. Nodal solutions for a fractional Choquard equation. *J. Math. Anal. Appl. 464*, 2 (2018), 1167–1183. 2
[26] Zhuo, R., Chen, W., Cui, X., and Yuan, Z. Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian. *Discrete Contin. Dyn. Syst. 36*, 2 (2016), 1125–1141. 5

Anh Tuan Duong
Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Ha noi, Viet Nam
Email address: tuanda@hnue.edu.vn

Phuong Le
Department of Mathematical Economics, Banking University of Ho Chi Minh City, Ho Chi Minh City, Vietnam
Email address: phuongl@buh.edu.vn

Nhu Thang Nguyen
Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Ha noi, Viet Nam
Email address: thangnn@hnue.edu.vn