Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

SYMMETRY OF SINGULAR SOLUTIONS FOR A WEIGHTED CHOQUARD EQUATION INVOLVING THE FRACTIONAL p-LAPLACIAN

Phuong Le

Division of Computational Mathematics and Engineering, Institute for Computational Science,
Ton Duc Thang University, Ho Chi Minh City, Vietnam;
Faculty of Mathematics and Statistics,
Ton Duc Thang University, Ho Chi Minh City, Vietnam

(Communicated by the associate editor name)

Abstract. Let $u \in L_{sp} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ be a positive solution, which may blow up at zero, of the equation

$$(-\Delta)^s_p u = \left(\frac{1}{|x|^{n-\beta}} * \frac{u^q}{|x|^\alpha} \right) \frac{u^{q-1}}{|x|^\alpha} \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

where $0 < s < 1$, $0 < \beta < n$, $p > 2$, $q \geq 1$ and $\alpha > 0$. We prove that if u satisfies some suitable asymptotic properties, then u must be radially symmetric and monotone decreasing about the origin. In stead of using equivalent fractional systems, we exploit a direct method of moving planes for the weighted Choquard nonlinearity. This method allows us to cover the full range $0 < \beta < n$ in our results.

1. **Introduction.** In this paper, we establish the symmetry and monotonicity of positive solutions to the following nonlocal equation

$$(-\Delta)^s_p u = \left(\frac{1}{|x|^{n-\beta}} * \frac{u^q}{|x|^\alpha} \right) \frac{u^{q-1}}{|x|^\alpha} \quad \text{in } \mathbb{R}^n \setminus \{0\} \tag{1}$$

where $0 < s < 1$, $0 < \beta < n$, $p > 2$, $q \geq 1$, $\alpha > 0$ and the convolution of two functions f and g is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy.$$

Let us recall that $(-\Delta)^s_p$ is the fractional p-Laplacian in $\mathbb{R}^n$ which is defined as a nonlocal pseudo-differential operator

$$(-\Delta)^s_p u(x) = C_{n,sp} PV \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)]}{|x - y|^{n+sp}} dy.$$

Here, PV stands for the Cauchy principal value and $C_{n,sp}$ is a normalization constant. Such operator is well-defined for all $x \in \mathbb{R}^n \setminus \{0\}$ if $u \in L_{sp} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$,

2010 *Mathematics Subject Classification.* 35R11, 35J92, 35J75, 35B06.

Key words and phrases. Choquard equations, fractional p-Laplacian, symmetry of solutions, positive solutions, singular solutions.

This work was done while the author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2019. He wish to thank the institute for their hospitality and support.

where

$$L_{sp} = \left\{ u \in L^{p-1}_{\text{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|1+u(x)|^{p-1}}{1+|x|^{n+sp}} dx < \infty \right\},$$

see [6] for more details.

The fractional p-Laplacian appears in some mathematical models such as the non-local "Tug-of-War" game (see [3, 4]). When $p = 2$, this operator become the well-known fractional Laplacian, which has been studied intensively by various authors in the last decade. Several methods have been proposed to overcome the difficulty caused by the nonlocal nature of this operator. One may use Caffarelli-Silvestre's *extension method* [5] to reduce the nonlocal problem into a local one in higher dimensions. Another useful tool to handle the fractional Laplacian is the *integral equation method* [8], in which a given pseudo-differential equation was transformed into their equivalent integral one. Later, Chen-Li-Li [7] introduced the *direct method of moving planes* for the fractional Laplacian which allows one to study the symmetry and nonexistence of positive $L_{2s} \cap C^{1,1}$ solutions of various nonlocal equations and systems. However, none of aforementioned methods can be applied to the fractional p-Laplacian. Very recently, Chen-Li [6] introduced some new ideas which allow them to establish the symmetry properties of positive solutions for the fractional p-Laplacian equation

$$(-\Delta)^s_p u = f(u) \quad \text{ in } \mathbb{R}^n \text{ or } B_1(0).$$

After the direct method of moving planes was introduced, a series of fruitful results have been obtained. Many of them improve previous ones established by using other methods, we refer to [9, 10, 12, 14, 16, 18, 20, 24] and references therein.

One may observe that the right hand side of (1) is also a nonlocal term. This phenomenon causes some mathematical difficulties which make the study of such problem particularly interesting. Furthermore, problem (1) is an analog of the Choquard equation

$$-\Delta u + V(x)u = \left(\frac{1}{|x|^{n-2}} * u^2 \right) u \quad \text{ in } \mathbb{R}^3,$$

which arises naturally in a variety of applications, for instance, the physics of multiple particle systems, quantum mechanics, Hartree-Fock theory, physics of laser beams and so on, which we refer to [13, 15, 23].

Problems of Choquard type have been studied extensively by several authors in the last decades. A good introduction to mathematical treatment of Choquard equations is a survey paper by Moroz-Schaftingen [22]. Some existence and multiplicity results for Choquard equations involving the fractional Laplacian and p-Laplacian can be found in recent papers [1, 2, 19, 26] and references therein. Choquard equations with weights were also investigated by some authors, see [11] for instance.

It is not easy to investigate the qualitative properties of solutions to (1) directly due to the presence of the convolution term in the right hand side. To overcome this difficulty, one possible approach when $p = 2$ is to set $v = \frac{1}{|x|^{n-\beta}} * \frac{u^q}{|x|^\alpha}$ and transform (1) into an equivalent integral system of the form

$$\begin{cases} u(x) = C \displaystyle\int_{\mathbb{R}^n} \frac{u^{q-1}(y)v(y)}{|x-y|^{n-2s}|y|^\alpha} dy & \text{ in } \mathbb{R}^n, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\beta}|y|^\alpha} dy & \text{ in } \mathbb{R}^n. \end{cases}$$

Then integral equation methods, such as the method of moving planes in integral forms, may be used to establish symmetry, nonexistence or classification of solutions, see [17, 21, 25] for more details about this approach. However, such transformation does not exist for the case $p > 2$. Nevertheless, when $\alpha = 0$ and $0 < \beta < 2$, equation (1) is equivalent to the fractional system

$$\begin{cases} (-\Delta)^s_p u = v u^{q-1} & \text{in } \mathbb{R}^n, \\ (-\Delta)^{\frac{\beta}{2}} v = u^q & \text{in } \mathbb{R}^n, \end{cases} \tag{2}$$

up to a suitable scaling. Using this approach, Ma-Zhang [18] proved the symmetry and nonexistence of positive solutions of (2) in the case $0 < \beta < 2$ and $p = 2$. Very recently, Ma-Zhang [20] partially extended this result to the case $p > 2$. They proved that if $u \in L_{sp} \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $v \in L_\beta \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is a positive solution pair of (2) with $q > p - 1$ and u, v satisfy some asymptotic properties at infinity, then u must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^n$. However, they still assume $0 < \beta < 2$ as this assumption is unavoidable in their approach.

Motivated by the above works, in this paper, we study the symmetry and monotonicity of positive solutions of (1) without the restriction $\beta < 2$ or $q > p - 1$. Moreover, in our result, u is allowed to blow up at zero. In stead of using the equivalent fractional system as in [18, 20], we exploit a direct method of moving planes for the weighted Choquard nonlinearity. This approach allows us to cover the full range $0 < \beta < n$ and $\alpha > 0$ in our results.

Our first result is the following.

Theorem 1.1. *Assume $0 < s < 1$, $0 < \beta < n$, $p > 1$, $q > 0$ and $\alpha > 0$.*

(i) If equation (1) has a positive solution $u \in L_{sp} \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\liminf_{y \to 0} u(y) > 0,$$

then $\alpha < n$.

(ii) If equation (1) has a positive solution $u \in L_{sp} \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ such that for some $\gamma \in \mathbb{R}$ and $c > 0$,

$$u(x) \geq \frac{c}{|x|^\gamma} \quad \text{as} \quad |x| \to \infty,$$

then $\gamma q > \beta - \alpha$.

In this paper, we deal with solutions having properties mentioned in Theorem 1.1. Therefore, it is necessary to assume $\alpha < n$ and $\gamma q > \beta - \alpha$. For two positive functions f and g defined in $\mathbb{R}^n \setminus \{x^0\}$, we denote $f \sim g$ as $x \to x^0$ if $0 < \liminf_{x \to x^0} \frac{f(x)}{g(x)} \leq \limsup_{x \to x^0} \frac{f(x)}{g(x)} < \infty$. Our main result is the following theorem.

Theorem 1.2. *Assume $0 < s < 1$, $0 < \alpha, \beta < n$, $p > 2$, $q \geq 1$ and $u \in L_{sp} \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ is a positive solution of (1) such that*

$$u(x) \sim \frac{1}{|x|^\gamma} \quad \text{as} \quad |x| \to \infty \tag{3}$$

for some $\gamma > \max\{0, \frac{\beta - \alpha}{q}\}$ satisfying

$$\min\{\gamma(q - p) + n + \alpha, \gamma(2q - p) + 2\alpha\} > sp + \beta. \tag{4}$$

Assume in addition that

$$u(x) \leq \frac{C}{|x|^{\delta}} \quad \textit{as} \quad x \to 0 \tag{5}$$

for some $C > 0$, $\delta \in (0, \frac{n-\alpha}{q})$ *and*

$$\liminf_{y \to 0} u(y) > u(x) \quad \textit{for all } x \in \mathbb{R}^n. \tag{6}$$

Then u must be radially symmetric and monotone decreasing about the origin.

Remark 1. Because the Kelvin transform is not valid for the fractional p-Laplacian, we need to impose the additional assumptions on the behavior of u at infinity. Furthermore, since u is not defined at zero, we need some control around this singular point to make the moving plane method work. Obviously, if (6) does not hold, then u cannot be monotone decreasing about the origin.

As a consequence of Theorem 1.2, we have the following symmetry result for positive solutions which blow up at rate $\delta < \frac{n-\alpha}{q}$ near the origin.

Corollary 1. *Assume* $0 < s < 1$, $0 < \alpha, \beta < n$, $p > 2$, $q \geq 1$ *and* $u \in L_{sp} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ *is a positive solution of* (1) *such that*

$$u(x) \sim \frac{1}{|x|^{\gamma}} \quad \textit{as} \quad |x| \to \infty$$

for some $\gamma > \max\{0, \frac{\beta-\alpha}{q}\}$ *satisfying* $\min\{\gamma(q-p)+n+\alpha, \gamma(2q-p)+2\alpha\} > sp+\beta$ *and*

$$u(x) \sim \frac{1}{|x|^{\delta}} \quad \textit{as} \quad x \to 0$$

for some $\delta \in (0, \frac{n-\alpha}{q})$. *Then u must be radially symmetric and monotone decreasing about the origin.*

The remainder of this paper is organized as follows. In Section 2, we recall some basic notations and lemmas used in the direct method of moving planes for the fractional p-Laplacian. Then the proofs of Theorem 1.1 and 1.2 are given in the last section.

2. Preliminaries.

In order to apply the method of moving planes, we first introduce some basic notations and lemmas. For $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}$$

be the moving plane,

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

be the region to the left of the plane and

$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$$

be the reflection of the point $x = (x_1, x_2, \ldots, x_n)$ about the plane T_λ.

Moreover, to compare the values of $u(x)$ with $u(x^\lambda)$, we denote

$$w_\lambda(x) = u_\lambda(x) - u(x), \quad \text{where} \quad u_\lambda(x) = u(x^\lambda).$$

For the convenience, we also denote by C, C' the generic positive constants whose concrete values may change from line to line or even in the same line.

The following *maximum principle* and *boundary estimate* lemmas were established in [6]. They plays the essential roles in the direct method for the fractional p-Laplacian.

Lemma 2.1 (A maximum principle for anti-symmetric functions [6])**.** *Let Ω be a bounded domain in Σ_λ. Assume that $u \in L_{sp} \cap C^{1,1}_{\mathrm{loc}}(\Omega)$. If*

$$\begin{cases} (-\Delta)^s_p u_\lambda - (-\Delta)^s_p u \geq 0 & \text{in } \Omega, \\ w_\lambda \geq 0 & \text{in } \Sigma_\lambda \setminus \Omega, \end{cases}$$

then

$$w_\lambda \geq 0 \quad \text{in } \Omega.$$

Furthermore, if $w_\lambda = 0$ at some point in Ω, then $w_\lambda = 0$ almost everywhere in $\mathbb{R}^n$.

These conclusions hold for unbounded region Ω if we further assume that

$$\liminf_{|x|\to\infty} w_\lambda(x) \geq 0.$$

Lemma 2.2 (A boundary estimate [6])**.** *Assume that $w_{\lambda_0} > 0$ in Σ_{λ_0}. Suppose $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k}$ such that*

$$w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} \leq 0 \quad \text{and} \quad x^k \to x^0 \in T_{\lambda_0}.$$

Let

$$\delta_k = dist(x^k, T_{\lambda_k}) = \lambda_k - x^k_1.$$

Then

$$\limsup_{\delta_k \to 0} \frac{1}{\delta_k} \left((-\Delta)^s_p u_{\lambda_k}(x^k) - (-\Delta)^s_p u(x^k) \right) < 0.$$

The following elementary inequality is also useful in the next section.

Lemma 2.3. *For any $0 < t_1 \leq t_2$ and $q \geq 0$, we have*

$$t_1^q - t_2^q \geq \max\{q, 1\} t_2^{q-1}(t_1 - t_2).$$

Proof. If $q \geq 1$, then there exists $\xi \in (t_1, t_2)$ such that

$$t_1^q - t_2^q = q\xi^{q-1}(t_1 - t_2) \geq q t_2^{q-1}(t_1 - t_2),$$

while if $0 \leq q < 1$, then

$$t_1^q - t_2^q \geq t_1 t_2^{q-1} - t_2^q = t_2^{q-1}(t_1 - t_2).$$

□

3. Proof of the main results.

3.1. Necessary conditions for the existence of positive solutions.

Proof of Theorem 1.1. Assume that (1) has a positive solution $u \in L_{sp} \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. Taking any $x^0 \neq 0$, we have

$$\begin{aligned} \infty > \left(\frac{1}{|x|^{n-\beta}} * \frac{u^q}{|x|^\alpha} \right)(x^0) &= \int_{\mathbb{R}^n} \frac{u^q(y)}{|x^0 - y|^{n-\beta}|y|^\alpha} dy \\ &= \int_{B_{|x^0|}(0)} \frac{u^q(y)}{|x^0 - y|^{n-\beta}|y|^\alpha} dy + \int_{\mathbb{R}^n \setminus B_{|x^0|}(0)} \frac{u^q(y)}{|x^0 - y|^{n-\beta}|y|^\alpha} dy \\ &\geq \frac{1}{(2|x^0|)^{n-\beta}} \int_{B_{|x^0|}(0)} \frac{u^q(y)}{|y|^\alpha} dy + \frac{1}{2^{n-\beta}} \int_{\mathbb{R}^n \setminus B_{|x^0|}(0)} \frac{u^q(y)}{|y|^{n+\alpha-\beta}} dy. \end{aligned} \tag{7}$$

If $\liminf_{y\to 0} u(y) > 0$, then $\inf_{B_{|x^0|}(0)} u > 0$ and (7) yields

$$\infty > \int_{B_{|x^0|}(0)} \frac{u^q(y)}{|y|^\alpha} dy \geq \left(\inf_{B_{|x^0|}(0)} u \right)^q \int_{B_{|x^0|}(0)} \frac{dy}{|y|^\alpha},$$

which indicates $\alpha < n$.

Now we assume $u(x) \geq \frac{c}{|x|^\gamma}$ for all large $|x|$, then (7) leads to

$$\infty > \int_{\mathbb{R}^n \setminus B_{|x^0|}(0)} \frac{u^q(y)}{|y|^{n+\alpha-\beta}} dy \geq C \int_{\mathbb{R}^n \setminus B_{|x^0|}(0)} \frac{dy}{|y|^{n+\alpha-\beta+\gamma q}},$$

which implies $\gamma q > \beta - \alpha$. $\square$

3.2. **Symmetry of positive solutions.** In this subsection, we apply the method of moving planes to prove Theorem 1.2. Beside using notations mentioned in Section 2, we also denote for $\lambda \leq 0$,

$$\Sigma_\lambda^* = \Sigma_\lambda \setminus \{0^\lambda\}$$

and

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda^* : w_\lambda(x) < 0\}.$$

We define for all $x \in \Sigma_\lambda$,

$$\begin{aligned}
F(x) &= \left(\frac{1}{|x|^{n-\beta}} * \frac{u^q}{|x|^\alpha} \right)(x) = \int_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\beta}|y|^\alpha} dy, \\
H(x) &= \left(\frac{1}{|x|^{n-\beta}} * \frac{u^{q-1}}{|x|^\alpha} \right)(x) = \int_{\mathbb{R}^n} \frac{u^{q-1}(y)}{|x-y|^{n-\beta}|y|^\alpha} dy, \\
P_\lambda(x) &= \int_{\Sigma_\lambda^-} \frac{u^{q-1}(y) w_\lambda(y)}{|x-y|^{n-\beta}|y|^\alpha} dy.
\end{aligned}$$

Lemma 3.1. *If $\lambda \leq 0$ and $x \in \Sigma_\lambda^-$, then*

$$(-\Delta)_p^s u_\lambda(x) - (-\Delta)_p^s u(x) \geq \max\{q-1, 1\} F(x) \frac{u^{q-2}(x)}{|x|^\alpha} w_\lambda(x) + q P_\lambda(x) \frac{u^{q-1}(x)}{|x|^\alpha}.$$

Proof. Using (1) and the fact $\lambda \leq 0 < \alpha$, we have

$$\begin{aligned}
(-\Delta)_p^s u_\lambda(x) - (-\Delta)_p^s u(x) &= F(x^\lambda) \frac{u_\lambda^{q-1}(x)}{|x^\lambda|^\alpha} - F(x) \frac{u^{q-1}(x)}{|x|^\alpha} \\
&\geq F(x^\lambda) \frac{u_\lambda^{q-1}(x)}{|x|^\alpha} - F(x) \frac{u^{q-1}(x)}{|x|^\alpha} \\
&= F(x) \frac{u_\lambda^{q-1}(x) - u^{q-1}(x)}{|x|^\alpha} + \left(F(x^\lambda) - F(x) \right) \frac{u_\lambda^{q-1}(x)}{|x|^\alpha}. \qquad (8)
\end{aligned}$$

From Lemma 2.3 and the fact $w_\lambda(x) < 0$, we have

$$\begin{aligned}
\frac{u_\lambda^{q-1}(x) - u^{q-1}(x)}{|x|^\alpha} &\geq \frac{\max\{q-1, 1\} u^{q-2}(x)[u_\lambda(x) - u(x)]}{|x|^\alpha} \\
&= \max\{q-1, 1\} \frac{u^{q-2}(x)}{|x|^\alpha} w_\lambda(x) \qquad (9)
\end{aligned}$$

and

$$
\begin{aligned}
&\left(F(x^\lambda) - F(x)\right) \frac{u_\lambda^{q-1}(x)}{|x|^\alpha} \\
&= \left(\int_{\mathbb{R}^n} \frac{u^q(y)}{|x^\lambda - y|^{n-\beta}|y|^\alpha} dy - \int_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\beta}|y|^\alpha} dy \right) \frac{u_\lambda^{q-1}(x)}{|x|^\alpha} \\
&= \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^\lambda - y|^{n-\beta}} \right) \left(\frac{u_\lambda^q(y)}{|y^\lambda|^\alpha} - \frac{u^q(y)}{|y|^\alpha} \right) dy \cdot \frac{u_\lambda^{q-1}(x)}{|x|^\alpha} \qquad (10) \\
&\geq \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^\lambda - y|^{n-\beta}} \right) \frac{u_\lambda^q(y) - u^q(y)}{|y|^\alpha} dy \cdot \frac{u_\lambda^{q-1}(x)}{|x|^\alpha} \\
&\geq \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\beta}} \frac{u_\lambda^q(y) - u^q(y)}{|y|^\alpha} dy \cdot \frac{u_\lambda^{q-1}(x)}{|x|^\alpha} \\
&\geq \int_{\Sigma_\lambda^-} \frac{q u^{q-1}(y)[u_\lambda(y) - u(y)]}{|x-y|^{n-\beta}|y|^\alpha} dy \cdot \frac{u_\lambda^{q-1}(x)}{|x|^\alpha} \\
&\geq q P_\lambda(x) \frac{u^{q-1}(x)}{|x|^\alpha}. \qquad (11)
\end{aligned}
$$

The conclusion follows immediately from (8), (9) and (11). □

Lemma 3.2 (Decay at infinity)**.** *Assume $\lambda \leq \overline{\lambda} < 0$ and for some $x^* \in \Sigma_\lambda^*$ we have*

$$w_\lambda(x^*) = \min_{\Sigma_\lambda} w_\lambda < 0.$$

Then there exist $R_0 > 0$ (depending on $\overline{\lambda}$ but independent of λ) such that $|x^*| < R_0$.

Proof. From Lemma 3.1, we have

$$(-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \geq c(x^*) w_\lambda(x^*), \tag{12}$$

where

$$c(x) = \max\{q-1, 1\} F(x) \frac{u^{q-2}(x)}{|x|^\alpha} + q H(x) \frac{u^{q-1}(x)}{|x|^\alpha}, \quad x \in \Sigma_\lambda. \tag{13}$$

We claim that

$$0 < c(x) < \frac{C + C' \ln|x|}{|x|^{n+\alpha-\beta+\gamma(q-2)}} + \frac{C}{|x|^{2\alpha-\beta+2\gamma(q-1)}} \quad \text{for all } x \in \Sigma_\lambda, \tag{14}$$

where $C, C' > 0$ are independent of x and λ.

To prove (14), we first deduce from (3) that

$$\frac{c}{|y|^\gamma} < u(y) < \frac{C}{|y|^\gamma} \quad \text{for all } y \in \mathbb{R}^n \setminus B_{\frac{|\overline{\lambda}|}{2}}(0), \tag{15}$$

where $C, c > 0$ depend on $\overline{\lambda}$ but are independent of y. Moreover, from (5), we have for all $x \in \Sigma_\lambda$,

$$\begin{aligned}
\int\limits_{|y|<\frac{|x|}{2}} \frac{u^q(y)}{|y|^\alpha} dy &= \int\limits_{|y|<\frac{|\overline{\lambda}|}{2}} \frac{u^q(y)}{|y|^\alpha} dy + \int\limits_{\frac{|\overline{\lambda}|}{2} \le |y| < \frac{|x|}{2}} \frac{u^q(y)}{|y|^\alpha} dy \\
&\le C \int\limits_{|y|<\frac{|\overline{\lambda}|}{2}} \frac{dy}{|y|^{\alpha+\delta q}} + C \int\limits_{\frac{|\overline{\lambda}|}{2} \le |y| < \frac{|x|}{2}} \frac{dy}{|y|^{\alpha+\gamma q}} \\
&\le \begin{cases} C & \text{if } \gamma q > n-\alpha, \\ C + C|x|^{n-\alpha-\gamma q} & \text{if } \gamma q < n-\alpha, \\ C + C' \ln|x| & \text{if } \gamma q = n-\alpha. \end{cases} \qquad (16)
\end{aligned}$$

Similarly,

$$\int\limits_{|y|<\frac{|x|}{2}} \frac{u^{q-1}(y)}{|y|^\alpha} dy \le \begin{cases} C & \text{if } \gamma(q-1) > n-\alpha, \\ C + C'|x|^{n-\alpha-\gamma(q-1)} & \text{if } \gamma(q-1) < n-\alpha, \\ C + C' \ln|x| & \text{if } \gamma(q-1) = n-\alpha. \end{cases} \qquad (17)$$

We then estimate $F(x)$ for $x \in \Sigma_\lambda$ as follows

$$\begin{aligned}
F(x) &= \int_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\beta}|y|^\alpha} dy \\
&= \left\{ \int\limits_{|y|<\frac{|x|}{2}} + \int\limits_{\substack{|x-y|\ge\frac{|x|}{2} \\ |y|\ge\frac{|x|}{2}}} + \int\limits_{|x-y|<\frac{|x|}{2}} \right\} \frac{u^q(y)}{|x-y|^{n-\beta}|y|^\alpha} dy \\
&\le \frac{2^{n-\beta}}{|x|^{n-\beta}} \int\limits_{|y|<\frac{|x|}{2}} \frac{u^q(y)}{|y|^\alpha} dy + \int\limits_{\substack{|x-y|\ge\frac{|x|}{2} \\ |y|\ge\frac{|x|}{2}}} \frac{C}{|x-y|^{n-\beta}|y|^{\alpha+\gamma q}} dy \\
&\qquad + \int\limits_{|x-y|<\frac{|x|}{2}} \frac{C}{|x-y|^{n-\beta}|y|^{\alpha+\gamma q}} dy \\
&\le \frac{2^{n-\beta}}{|x|^{n-\beta}} \int\limits_{|y|<\frac{|x|}{2}} \frac{u^q(y)}{|y|^\alpha} dy + 3^{n-\beta} \int\limits_{|y|\ge\frac{|x|}{2}} \frac{C}{|y|^{n+\alpha-\beta+\gamma q}} dy \\
&\qquad + \frac{2^{\alpha+\gamma q}}{|x|^{\alpha+\gamma q}} \int\limits_{|x-y|<\frac{|x|}{2}} \frac{C}{|x-y|^{n-\beta}} dy.
\end{aligned}$$

The following elementary facts have been used in the above estimation

$$\begin{aligned}
|x-y| \ge |x| - |y| > \frac{|x|}{2} \quad & \text{if } |y| < \frac{|x|}{2}, \\
|x-y| \ge \frac{|x| + |x-y|}{3} \ge \frac{|y|}{3} \quad & \text{if } |x-y| \ge \frac{|x|}{2}. \\
|y| \ge |x| - |x-y| > \frac{|x|}{2} \quad & \text{if } |x-y| < \frac{|x|}{2}.
\end{aligned}$$

Therefore, we may use (16) to obtain

$$F(x) \leq \begin{cases} \frac{C}{|x|^{n-\beta}} + \frac{C}{|x|^{\alpha-\beta+\gamma q}} & \text{if } \gamma q \neq n-\alpha, \\ \frac{C+C' \ln|x|}{|x|^{n-\beta}} & \text{if } \gamma q = n-\alpha. \end{cases} \tag{18}$$

Similarly, we may use (17) to get

$$H(x) \leq \begin{cases} \frac{C}{|x|^{n-\beta}} + \frac{C}{|x|^{\alpha-\beta+\gamma(q-1)}} & \text{if } \gamma(q-1) \neq n-\alpha, \\ \frac{C+C' \ln|x|}{|x|^{n-\beta}} & \text{if } \gamma(q-1) = n-\alpha. \end{cases} \tag{19}$$

By collecting (13), (15), (18) and (19), we deduce

$$0 < c(x) < \begin{cases} \frac{C}{|x|^{n+\alpha-\beta+\gamma(q-2)}} + \frac{C}{|x|^{2\alpha-\beta+2\gamma(q-1)}} & \text{if } \gamma q \neq n-\alpha, \\ \frac{C+C' \ln|x|}{|x|^{n+\alpha-\beta+\gamma(q-2)}} + \frac{C}{|x|^{2\alpha-\beta+2\gamma(q-1)}} & \text{if } \gamma q = n-\alpha, \end{cases}$$

which clearly implies (14).

For simplicity, we denote $G(t) = |t|^{p-2}t$. Then G is strictly increasing in $\mathbb{R}$ and for all $t_1 \leq t_2$, we have

$$G(t_1) - G(t_2) \leq C|t_2|^{p-2}(t_1 - t_2) \tag{20}$$

(see [6, Lemma 5.1]).

Using the definition of the fractional p-Laplacian and the monotonicity of G, we can compute

$$\begin{aligned}
&(-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \\
&= C_{n,sp} PV \int_{\mathbb{R}^n} \frac{G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y))}{|x^* - y|^{n+sp}} dy \\
&= C_{n,sp} PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y))}{|x^* - y|^{n+sp}} dy \\
&\quad + C_{n,sp} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u(y)) - G(u(x^*) - u_\lambda(y))}{|x^* - y^\lambda|^{n+sp}} dy \\
&\leq C_{n,sp} PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y))}{|x^* - y^\lambda|^{n+sp}} dy \\
&\quad + C_{n,sp} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u(y)) - G(u(x^*) - u_\lambda(y))}{|x^* - y^\lambda|^{n+sp}} dy \\
&\leq C_{n,sp} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u(y)) - G(u(x^*) - u(y))}{|x^* - y^\lambda|^{n+sp}} dy,
\end{aligned}$$

where we have used

$$\frac{1}{|x^* - y|^{n+sp}} \geq \frac{1}{|x^* - y^\lambda|^{n+sp}},$$

$$G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y)) = G'(\zeta_1)(w_\lambda(x^*) - w_\lambda(y)) \leq 0$$

and

$$G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u_\lambda(y)) = G'(\zeta_2) w_\lambda(x^*) \leq 0$$

for some ζ_1, ζ_2.

Now we apply the inequality (20) to get

$$(-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \leq C w_\lambda(x^*) \int_{\Sigma_\lambda} \frac{|u(x^*) - u(y)|^{p-2}}{|x^* - y^\lambda|^{n+sp}} dy. \tag{21}$$

Let us denote $M = \left(\frac{2C}{c}\right)^{\frac{1}{\gamma}}$, where C, c are the same as in (15). Then for any x, y such that $|y| > M|x|$ and $|x| \geq \frac{|\bar{\lambda}|}{2}$, we deduce from (15) that

$$u(y) < \frac{C}{|y|^\gamma} < \frac{c}{2|x|^\gamma} < \frac{u(x)}{2}.$$

We choose some point $z \in \Sigma_\lambda$ such that $|z| = (M+1)|x^*|$ and $B_{|x^*|}(z) \subset \Sigma_\lambda$, then $u(y) < \frac{1}{2}u(x^*)$ for all $y \in B_{|x^*|}(z)$. Therefore,

$$\begin{aligned}
\int_{\Sigma_\lambda} \frac{|u(x^*) - u(y)|^{p-2}}{|x^* - y^\lambda|^{n+sp}} dy &\geq \frac{u^{p-2}(x^*)}{2^{p-2}} \int_{B_{|x^*|}(z)} \frac{dy}{|x^* - y^\lambda|^{n+sp}} \\
&= \frac{u^{p-2}(x^*)}{2^{p-2}} \int_{B_{|x^*|}(z^\lambda)} \frac{dy}{|x^* - y|^{n+sp}} \\
&\geq \frac{u^{p-2}(x^*)}{2^{p-2}} \int_{B_{|x^*|}(z^\lambda)} \frac{dy}{[(M+3)|x^*|]^{n+sp}} \\
&\geq C \frac{u^{p-2}(x^*)}{|x^*|^{sp}},
\end{aligned} \tag{22}$$

where we have used the fact

$$\begin{aligned}
|x^* - y| &\leq |x^*| + |-z^\lambda| + |z^\lambda - y| \\
&\leq |x^*| + |z| + |z^\lambda - y| \leq (M+3)|x^*| \quad \text{ for all } y \in B_{|x^*|}(z^\lambda).
\end{aligned}$$

Substituting (22) into (21) and using (3), we deduce

$$(-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \leq C w_\lambda(x^*) \frac{u^{p-2}(x^*)}{|x^*|^{sp}} \leq \frac{C w_\lambda(x^*)}{|x^*|^{\gamma(p-2)+sp}}. \tag{23}$$

The inequalities (12), (14) and (23) imply

$$\frac{C + C' \ln|x^*|}{|x^*|^{n+\alpha-\beta+\gamma(q-2)}} + \frac{C}{|x^*|^{2\alpha-\beta+2\gamma(q-1)}} \geq \frac{1}{|x^*|^{\gamma(p-2)+sp}},$$

or equivalently,

$$\frac{C + C' \ln|x^*|}{|x^*|^{n+\alpha-\beta+\gamma(q-p)-sp}} + \frac{C}{|x^*|^{2\alpha-\beta+\gamma(2q-p)-sp}} \geq 1.$$

This fact and (4) would imply $|x^*| < R_0$ for some $R_0 > 0$ independent of λ. $\square$

We are ready to prove our main result.

Proof of Theorem 1.2. We will show the symmetry of u about T_0 by moving plane T_λ along x_1 direction from $-\infty$ to the right.

For $\lambda < 0$, since

$$\liminf_{x \to 0^\lambda} w_\lambda(x) = \liminf_{x \to 0^\lambda} (u_\lambda(x) - u(x)) = \liminf_{x \to 0} u(x) - u(0^\lambda) > 0,$$

w_λ is strictly positive near its singular point 0^λ.

This fact indicates that if w_λ is negative somewhere in Σ_λ, then the negative minima of w_λ are attained in the interior of Σ_λ^*. We carry on the method of moving planes in two steps.

Step 1. (Move the plane along x_1 direction from near $-\infty$)

In this step, we show that for λ sufficiently negative,

$$w_\lambda \geq 0 \quad \text{ in } \Sigma_\lambda^*. \tag{24}$$

Indeed, we may choose $\lambda < -R_0$, where R_0 is defined in Lemma 3.2 with $\overline{\lambda} = -1$. If (24) does not hold for such λ, then there exists $x^* \in \Sigma_\lambda^*$ such that $w_\lambda(x^*) = \min\limits_{\Sigma_\lambda} w_\lambda < 0$. Hence $|x^*| < R_0$ by Lemma 3.2. This contradicts $x^* \in \Sigma_\lambda^*$ and $\lambda < -R_0$. Hence, (24) is proved.

Step 2. (Move the plane to the limiting position)

Step 1 provides a starting point, from which we can now move the plane T_λ to the right as long as (24) holds to its limiting position. Define

$$\lambda_0 = \sup\{\lambda \leq 0 \mid w_\mu \geq 0 \text{ in } \Sigma_\mu^* \text{ for all } \mu \leq \lambda\}.$$

In this step, we show that

$$\lambda_0 = 0. \tag{25}$$

Suppose (25) is false, i.e., $\lambda_0 < 0$. Since w_λ depends on λ continuously, we have

$$w_{\lambda_0} \geq 0 \quad \text{ in } \Sigma_{\lambda_0}^*.$$

Then we may use (8) and (10) in the proof of Lemma 3.1 to get

$$\begin{aligned}
&(-\Delta)_p^s u_{\lambda_0}(x) - (-\Delta)_p^s u(x) \geq \big(F(x^{\lambda_0}) - F(x)\big) \frac{u_{\lambda_0}^{q-1}(x)}{|x|^\alpha} \\
&= \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^{\lambda_0}-y|^{n-\beta}} \right) \left(\frac{u_{\lambda_0}^q(y)}{|y^{\lambda_0}|^\alpha} - \frac{u^q(y)}{|y|^\alpha} \right) dy \cdot \frac{u_{\lambda_0}^{q-1}(x)}{|x|^\alpha} \\
&\geq \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x^{\lambda_0}-y|^{n-\beta}} \right) \left(\frac{1}{|y^{\lambda_0}|^\alpha} - \frac{1}{|y|^\alpha} \right) u^q(y) dy \cdot \frac{u_{\lambda_0}^{q-1}(x)}{|x|^\alpha} \\
&> 0
\end{aligned}$$

for all $x \in \Sigma_{\lambda_0}^*$. This indicates $w_{\lambda_0} \not\equiv 0$.

Then by the strong maximum principle (see Lemma 2.1), we have

$$w_{\lambda_0} > 0 \quad \text{ in } \Sigma_{\lambda_0}^*.$$

On the other hand, by the definition of λ_0, there exists a sequence $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k}^*$ such that

$$\lambda_k < \frac{\lambda_0}{2}, \quad w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} < 0 \quad \text{ and } \quad \nabla w_{\lambda_k}(x^k) = 0. \tag{26}$$

By Lemma 3.2, we have $|x^k| < R_0$, where R_0 is defined in Lemma 3.2 with $\overline{\lambda} = \frac{\lambda_0}{2}$. Therefore, we may assume that $x^k \to x^0$. Now from (26), we have

$$w_{\lambda_0}(x^0) \leq 0, \text{ hence } x^0 \in T_{\lambda_0}; \quad \text{ and } \quad \nabla w_{\lambda_0}(x^0) = 0.$$

It follows that

$$\frac{w_{\lambda_k}(x^k)}{\delta_k} \to 0 \quad \text{ as } k \to \infty.$$

Consequently, we may use (12) and (14) to get

$$\begin{aligned}
\frac{1}{\delta_k} \big((-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k)\big) &\geq c(x^k) \frac{w_{\lambda_k}(x^k)}{\delta_k} \\
&\geq \left(\sup_{x \in \Sigma_{\frac{\lambda_0}{2}} \cap B_{R_0}(0)} c(x) \right) \frac{w_{\lambda_k}(x^k)}{\delta_k} \to 0.
\end{aligned}$$

This contradicts Lemma 2.2 and hence (25) is proved. Consequently, $w_0 \geq 0$ in Σ_0. By moving the plane along x_1 direction from near $+\infty$, we also have $w_0 \leq 0$ in Σ_0. Therefore, u is symmetric and monotone decreasing about plane T_0.

We may repeat the above argument for any direction to conclude that u is radially symmetric and monotone decreasing about the origin. This completes the proof of Theorem 1.2. □

REFERENCES

[1] P. d'Avenia, G. Siciliano and M. Squassina, On fractional Choquard equations, *Math. Models Methods Appl. Sci.*, **25** (2015), 1447–1476.

[2] P. Belchior, H. Bueno, O.H. Miyagaki and G.A. Pereira, Remarks about a fractional Choquard equation: Ground state, regularity and polynomial decay, *Nonlinear Anal.*, **164** (2017), 38–53.

[3] C. Bjorland, L. Caffarelli and A. Figalli, Non-local gradient dependent operators, *Adv. Math.*, **230** (2012), 1859–1894.

[4] C. Bjorland, L. Caffarelli and A. Figalli, Nonlocal tug-of-war and the infinity fractional Laplacian, *Comm. Pure Appl. Math.*, **65** (2012), 337–380.

[5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations*, **32** (2007), 1245–1260.

[6] W. Chen and C. Li, Maximum principles for the fractional p-Laplacian and symmetry of solutions, *Adv. Math.*, **335** (2018), 735–758.

[7] W. Chen, C. Li and Y. Li, A direct method of moving planes for the fractional Laplacian, *Adv. Math.*, **308** (2017), 404–437.

[8] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.*, **59** (2006), 330–343.

[9] W. Dai, Y. Fang and G. Qin, Classification of positive solutions to fractional order Hartree equations via a direct method of moving planes, *J. Differential Equations*, **265** (2018), 2044–2063.

[10] J. Dou and H. Zhou, Liouville theorem for fractional Hénon equation and system on $\mathbb{R}^n$, *Comm. Pure Appl. Anal.*, **14** (2015), 1915–1927.

[11] L. Du, F. Gao and M. Yang, Existence and qualitative analysis for nonlinear weighted Choquard equations, preprint, `arXiv:1810.11759`.

[12] A.T. Duong and P. Le, Symmetry and nonexistence results for a fractional Hénon-Hardy system on a half-space, *Rocky Mountain J. Math.*, (2019), to appear. Available from: `https://projecteuclid.org/euclid.rmjm/1552186836`.

[13] E.P. Gross, *Physics of Many-Particle Systems*, Vol.1, Gordon Breach, New York, 1966.

[14] P. Le, Liouville theorem and classification of positive solutions for a fractional Choquard type equation, *Nonlinear Anal.*, **185** (2019), 123-141.

[15] E.H. Lieb and B. Simon, The Hartree-Fock theory for Coulomb systems, *Comm. Math. Phys.*, **53** (1977), 185–194.

[16] B. Liu and L. Ma, Radial symmetry results for fractional Laplacian systems, *Nonlinear Anal.*, **146** (2016), 120–135.

[17] S. Liu, Regularity, symmetry, and uniqueness of some integral type quasilinear equations, *Nonlinear Anal.*, **71** (2009), 1796–1806.

[18] P. Ma and J. Zhang, Symmetry and Nonexistence of Positive Solutions for Fractional Choquard Equations, preprint, `arXiv:1704.02190`.

[19] P. Ma and J. Zhang, Existence and multiplicity of solutions for fractional Choquard equations, *Nonlinear Anal.*, **164** (2017), 100–117.

[20] L. Ma and Z. Zhang, Symmetry of positive solutions for Choquard equations with fractional p-Laplacian, *Nonlinear Anal.*, **182** (2019), 248–262.

[21] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.*, **195** (2010), 455–467.

[22] V. Moroz and J.V. Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.*, **19** (2017), 773–813.

[23] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie-Verlag, Berlin, 1954.

[24] L. Wu and P. Niu, Symmetry and nonexistence of positive solutions to fractional p-Laplacian equations, *Discrete Contin. Dyn. Syst.*, **39** (2018), 1573–1583.
[25] D. Xu and Y. Lei, Classification of positive solutions for a static Schrodinger-Maxwell equation with fractional Laplacian, *Applied Math. Letters*, **43** (2015), 85–89.
[26] W. Zhang and X. Wu, Nodal solutions for a fractional Choquard equation, *J. Math. Anal. Appl.*, **464** (2018), 1167–1183.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: lephuong@tdtu.edu.vn