Random projection method for stochastic split feasibility problems

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Abstract: We focus on the multiple-sets split feasibility problem of two arbitrary (possibly infinite) collections of closed convex sets. Under some conditions, it can be reformulated as a stochastic optimization problem. We propose a class of random projection algorithms and prove the almost sure convergence of these algorithms. We also provided convergence rates and some numerical experiments to illustrate the behavior of the algorithms.

1 Introduction

Consider the classical multiple-sets split feasibility problem (MSF)

$$\text{Find } x \in \cap_{i=1}^{t} C_i \text{ such that } Ax \in \cap_{j=1}^{r} Q_j, \qquad (MSF)$$

where A is a given real $m \times n$ matrix, $C_1, \dots, C_t$, $Q_1, \dots, Q_r$ are closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The MSFP was firstly defined by Censor et al in [6] for modeling many pratical applications especially intensity- modulated radiation therapy. It also generalizes both the convex feasibility problem and the split feasibility problem. When $Q_j \equiv \mathbb{R}^m$ for all j, the (MSF) problem becomes the convex feasibility problem ([1, 7])

$$\text{Find } x \in \cap_{i=1}^{t} C_i,$$

and when $t = r = 1$, it becomes the split feasibility problem ([3, 5, 9])

$$\text{Find } x \in C \text{ such that } Ax \in Q.$$

Optimization problems involving a large number of constraints appear more and more in the pratical applications such as inverse problems, computer science, machine learning and statistics (see [11] and the references therein). The convex feasibility problem of a (possibly infinite) collections of closed convex sets also called stochastic feasibility problem was firstly considered in [2] and then formulated as a stochastic optimization problem in [10]. Also in [10], the authour proposed a random projection algorithm and studied its convergence rate for stochastic feasibility problem . In [12], the authours proposed several stochastic reformulations and develop a general projection algorithm for the stochastic convex feasibility problem that can be paralleled. Recently, the stochastic fixed point problem has been investigated in [8]. Motivated by these works, we are interested in the stochastic split feasiblity problem (SSF)

$$\text{Find } x \in \cap_{i\in\mathcal{I}} C_i \text{ such that } Ax \in \cap_{j\in\mathcal{J}} Q_j, \qquad (SSF)$$

where A is a given real $m \times n$ matrix and $\{C_i\}_{i\in\mathcal{I}}, \{Q_j\}_{j\in\mathcal{J}}$ are arbitrary collections of closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Comparing to the clasical (MSF) problem, in (SSF), the sets I and J may be infinite. In the next section, we reformulate the (SSF) problem as a stochastic optimization problem and study the equivalence of these problems. In Section 3, we propose a random projection algorithm for solving the (SSF) and study its convergence analysis. Numerical experimental results are provided in Section 4.

2 Problem formulation

Let C be a closed convex set in $\mathbb{R}^n$. We denote by P_C the projection on C. In the following lemma, we recall some important properties of P_C that will be useful for the next part of the paper.

Lemma 2.1. *(see for example [4, 6])*

(i) P_C is firmly non-expansive,i.e. for all $x, y \in \mathbb{R}^n$

$$\|P_C(x) - P_C(y)\|^2 \le \langle x - y, P_C(x) - P_C(y)\rangle.$$

(ii) $I - P_C$ is firmly non-expansive,i.e. for all $x, y \in \mathbb{R}^n$

$$\|(I - P_C)(x) - (I - P_C)(y)\|^2 \le \langle x - y, (I - P_C)(x) - (I - P_C)(y)\rangle.$$

Now, we consider the stochastic split feasibility problem:

$$\text{Find } x \in \cap_{i\in\mathcal{I}} C_i \text{ such that } Ax \in \cap_{j\in\mathcal{J}} Q_j, \tag{SSF}$$

where A is a given real $m \times n$ matrix and $\{C_i\}_{i\in\mathcal{I}}, \{Q_j\}_{j\in\mathcal{J}}$ are finite or infinite collections of closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Problem (SSF) can be reformulated as the following stochastic optimization problem:

$$\min_{x\in\mathbb{R}^n} \mathcal{F}(x) = \frac{1}{2}\mathbf{E}\left[\|x - P_{C_\omega}(x)\|^2 + \beta\|Ax - P_{Q_\theta}(Ax)\|^2\right], \tag{SOP}$$

where β is an arbitrary positive number, $\omega \sim \mathcal{P}$, $\theta \sim \mathcal{Q}$, $\mathcal{P}$ is a probability distribution over $\mathcal{I}$, $\mathcal{Q}$ is a probability distribution over $\mathcal{J}$ and the expectation is taken with respect to ω, θ.

We denote the solution set of Problem (SSF) by S, and the solution set of Problem (SOP) by S_1. It is clear that a solution of (SSF) is also a solution of (SOP), *i.e.* $S \subset S_1$, but the inverse inclusion is not always true, for example, when the random variable ω takes only one value in the set $\mathcal{I}$ or the random variable θ takes only one value in the set $\mathcal{J}$.

Lemma 2.2. *Assume that $S \neq \emptyset$, then (SSF) and (SOP) are equivalent, i.e. the solution set of (SSF) equals the solution set of (SOP) if one of the following conditions holds:*

(i) $\mathcal{P}\{\omega = i\} > 0$ for any $i \in \mathcal{I}$ and $\mathcal{Q}\{\theta = j\} > 0$ for any $j \in \mathcal{J}$.

(ii) Linear regularity condition: There exists $\kappa < \infty$ such that

$$dist_S^2(x) \leq \kappa\mathcal{F}(x) \quad \forall x \in \mathbb{R}^n. \tag{1}$$

Proof. (i) Let $x \in S$, then $x \in C_i$ and $Ax \in Q_j$ for any $i \in \mathcal{I}, j \in \mathcal{J}$. Since ω and θ are random variables taking values in the set $\mathcal{I}, \mathcal{J}$, respectively, we have $x = P_{C_\omega}(x), Ax = P_{Q_\theta}(Ax)$ or $\mathcal{F}(x) = 0$. Therefore, $S \subset S_1$.

Now, let $x \in S_1$, we have $\mathcal{F}(x) = 0$. For any $i \in I$,

$$0 = \mathcal{F}(x) \geq \|x - P_{C_i}(x)\|^2\mathcal{P}\{\omega = i\}.$$

But $\mathcal{P}\{\omega = i\} > 0$, then $x = P_{C_i}(x)$ or $x \in C_i$. Similarly, we have $Ax \in Q_j$ for any $j \in \mathcal{J}$. It means that $x \in S$.

(ii) As proved in (i), $S \subset S_1$. If $x \in S_1$, we have

$$dist_S^2(x) \leq \kappa\mathcal{F}(x) = 0.$$

So, $x \in S$.

□

Remark 2.3. *Condition (i) is similar to the condition used in [10] and Condition (ii) (Linear regularity condition) was used in several works ([8, 10, 12]). Note that the linear regularity condition is quite conservative and does not hold for any collection of closed convex sets (see Example 1, [12]).*

Let

$$F(x,\omega,\theta) = \frac{1}{2}\left[\|x - P_{C_\omega}(x)\|^2 + \beta\|Ax - P_{Q_\theta}(Ax)\|^2\right], \tag{2}$$

then

$$\mathcal{F}(x) = \mathbf{E}\left[F(x,\omega,\theta)\right], \tag{3}$$

and

$$\begin{aligned} \nabla\mathcal{F}(x) &= \mathbf{E}(\nabla_x F(x,\omega,\theta)) & (4)\\ &= x - \mathbf{E}\left[P_{C_\omega}(x)\right] + \beta A^T Ax - \beta A^T\mathbf{E}\left[P_{Q_\theta}(Ax)\right]. & (5) \end{aligned}$$

Lemma 2.4. *(i) For each $\omega \in I, \theta \in J$, $F(x,\omega,\theta)$ has Lipschitz gradient with constant $L = 1+\beta\lambda(A^TA)$, where $\lambda(A^TA)$ is the largest eigenvalue of A^TA.*

(ii) The function $\mathcal{F}(x)$ also has Lipschitz gradient with constant L.

Proof. (i) It is easy to see that

$$\nabla_x F(x,\omega,\theta) = x - P_{C_\omega}(x) + \beta A^T(Ax - P_{Q_\theta}(Ax)).$$

For fixed $\omega \in \mathcal{E}, \theta \in \mathcal{F}$,

$$\begin{aligned} & \|\nabla_x F(x,\omega,\theta) - \nabla_x F(y,\omega,\theta)\| \\ \le\ & \|x - P_{C_\omega}(x) - y + P_{C_\omega}(y)\| \\ & +\beta\left\|A^T\left[Ax - P_{Q_\theta}(Ax) - Ay + P_{Q_\theta}(Ay)\right]\right\| \\ \le\ & (1+\beta\lambda(A^TA))\|x-y\|. \end{aligned}$$

The last inequality follows from the firmly non-expansive property of $I - P_{C_\omega}$ and $I - P_{Q_\theta}$.

(ii) Since $\nabla\mathcal{F}(x) = \mathbf{E}\left[\nabla_x F(x,\omega,\theta)\right]$,

$$\begin{aligned} & \|\nabla\mathcal{F}(x) - \nabla\mathcal{F}(y)\| \\ =\ & \|\mathbf{E}\left[\nabla_x F(x,\omega,\theta)\right] - \mathbf{E}\left[\nabla_x F(y,\omega,\theta)\right]\| \\ \le\ & \mathbf{E}\left[\|\nabla_x F(x,\omega,\theta) - \nabla_x F(y,\omega,\theta)\|\right] \\ \le\ & L\|x-y\|. \end{aligned}$$

□

Lemma 2.5.

$$\boldsymbol{E}\left[\|\nabla_x F(x,\omega,\theta)\|^2\right] \le 2L\mathcal{F}(x). \tag{6}$$

Proof. We have

$$\begin{aligned} & \|\nabla_x F(x,\omega,\theta)\|^2 \\ = & \|x - P_{C_\omega}(x) + \beta A^T(Ax - P_{Q_\theta}(Ax))\|^2 \\ = & \|x - P_{C_\omega}(x)\|^2 + \beta^2\|A^T(Ax - P_{Q_\theta}(Ax))\|^2 \\ & +2\beta\langle x - P_{C_\omega}(x), A^T(Ax - P_{Q_\theta}(Ax))\rangle. \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} & \langle x - P_{C_\omega}(x), A^T(Ax - P_{Q_\theta}(Ax))\rangle \\ \le & \lambda(A^TA)\|x - P_{C_\omega}(x)\|^2 + \frac{1}{\lambda(A^TA)}\|A^T(Ax - P_{Q_\theta}(Ax))\|^2. \end{aligned}$$

Therefore,

$$\|\nabla_x F(x,\omega,\theta)\|^2 \le 2LF(x,\omega,\theta).$$

By taking expectation with respect to ω and θ, we obtain

$$\mathbf{E}\left[\|\nabla_x F(x,\omega,\theta)\|^2\right] \le 2L\mathcal{F}(x).$$

□

Lemma 2.6. *(Supermartingale convergence lemma[10, 13])*
Let $\{v_k\}, \{u_k\}, \{a_k\}$ and $\{b_k\}$ be sequences of nonnegative random variables such that

$$\boldsymbol{E}[v_{k+1}|\mathcal{V}_k] \le (1+a_k)v_k - u_k + b_k \quad a.s. \text{ for all } k \ge 0,$$

$$\sum_{k=0}^{\infty} a_k < \infty \quad a.s., \quad \sum_{k=0}^{\infty} b_k < \infty \quad a.s.,$$

where $\mathcal{V}_k$ denotes the σ-algebraic generated by random variables $v_0, \dots, v_k$, $u_0, \dots, u_k$, $a_0, \dots, a_k$, $b_0, \dots, b_k$. Then, we have $\lim_{k\to\infty} v_k = v$ for a random variable $v \ge 0 \quad a.s.$, and $\sum_{k=0}^{\infty} u_k < \infty \quad a.s.$

3 Algorithm and it convergence analysis

Algorithm:

Take a mini-batch size $N \ge 1$, and a positive sequence $\{\alpha_k\}_{k\ge1}$

Iter 0**:** Let x_0 be arbitrary.

Iter k: Draw $2N$ independent samples $\omega_k^1, \omega_k^2, \ldots, \omega_k^N \sim \mathcal{P}$, $\theta_k^1, \theta_k^2, \ldots, \theta_k^N \sim \mathcal{Q}$.
Compute

$$x_k \quad = \quad x^{k-1} - \frac{\alpha_k}{N} \sum_{i=1}^{N} \nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i).$$

We denote by $\mathcal{X}_k$ the history of the method up to time $k \geq 1$

$$\mathcal{X}_k = \left\{ x_0, (\omega_t^i, 1 \leq i \leq N, 1 \leq t \leq k), (\theta_t^i, 1 \leq i \leq N, 1 \leq t \leq k) \right\}.$$

Proposition 3.1. *If the sequence $\{\alpha_k\}$ satisfy the following condition*

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty, \sum_{k=1}^{\infty} \alpha_k = \infty,$$

then there exists a nonnegative random variable c such that

$$\lim_{k \to \infty} \mathcal{F}(x^k) = c \quad a.s., \tag{7}$$

and

$$\liminf_{k \to \infty} \nabla \mathcal{F}(x^k) = 0 \quad a.s. \tag{8}$$

Proof. As proved in Lemma 2.4, the function $\mathcal{F}(x)$ has Lipschitz gradient with constant $L = 1 + \lambda(A^T A)$. Therefore,

$$\begin{aligned} \mathcal{F}(x^k) &\leq \mathcal{F}(x^{k-1}) + \nabla \mathcal{F}(x^{k-1})^T (x^k - x^{k-1}) + \frac{L}{2} \|x^k - x^{k-1}\|^2 \\ &= \mathcal{F}(\S^{\|-\infty}) - \frac{\alpha_k}{N} \sum_{i=1}^{N} \nabla \mathcal{F}(x^{k-1})^T \nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i) \\ &\quad + \frac{L\alpha_k^2}{2N^2} \| \sum_{i=1}^{N} \nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i) \|^2. \end{aligned}$$

Take the expectation on $\mathcal{X}_{k-1}$ and note that ω_k^i and θ_k^i are independent of the past $\mathcal{X}_{k-1}$ when x^{k-1} is determined by $\mathcal{X}_{k-1}$, we have

$$\begin{aligned} \mathbf{E}\left[\mathcal{F}(x^k) | \mathcal{X}_{k-1}\right] &\leq F(x^{k-1}) - \frac{\alpha_k}{N} \sum_{i=1}^{N} \nabla \mathcal{F}(x^{k-1})^T \mathbf{E}\left[\nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i)\right] \\ &\quad + \frac{L\alpha_k^2}{2N^2} \mathbf{E}\left[\| \sum_{i=1}^{N} \nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i) \|^2 \right] \quad a.s. \end{aligned} \tag{9}$$

By definitions of F and $\mathcal{F}$, it is easy to see that

$$\mathbf{E}\left[\nabla_x F(x^{k-1},\omega_k^i,\theta_k^i)\right] = \nabla\mathcal{F}(x^{k-1}). \tag{10}$$

In addition, thanks to Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathbf{E}\left[\|\sum_{i=1}^{N}\nabla_x F(x^{k-1},\omega_k^i,\theta_k^i)\|^2\right] &\leq \mathbf{E}\left[N\sum_{i=1}^{N}\|\nabla_x F(x^{k-1},\omega_k^i,\theta_k^i)\|^2\right] \\ &= N\sum_{i=1}^{N}\mathbf{E}\left[\|\nabla_x F(x^{k-1},\omega_k^i,\theta_k^i\|^2\right].\end{aligned}$$

By using Lemma 2.5, we obtain

$$\mathbf{E}\left[\|\sum_{i=1}^{N}\nabla_x F(x^{k-1},\omega_k^i,\theta_k^i)\|^2\right] \leq 2N^2L\mathcal{F}(x^{k-1}). \tag{11}$$

Combinning (9), (10), (11), we have

$$\mathbf{E}\left[\mathcal{F}(x)|\mathcal{X}_{k-1}\right] \leq (1+L^2\alpha_k^2)\mathcal{F}(x^{k-1}) - \alpha_k\|\nabla\mathcal{F}(x^{k-1})\|^2 \quad a.s. \tag{12}$$

Now, thanks to the supermartingale convergence Lemma 2.6, we can conclude that

$$\mathcal{F}(x^k) \to c \quad a.s.,$$

for some nonegative random variable c and

$$\sum_{k=1}^{\infty}\alpha_k\|\nabla\mathcal{F}(x^{k-1})\|^2 < \infty \quad a.s.$$

But by assumption, $\sum_{k=1}^{\infty}\alpha_k = \infty$. It implies that

$$\liminf_{k\to\infty}\|\nabla\mathcal{F}(x^{k-1})\|^2 = 0 \quad a.s.$$

Hence,

$$\liminf_{k\to\infty}\nabla\mathcal{F}(x^k) = 0 \quad a.s.$$

□

Theorem 3.2. *Assume that the solution set S_1 of (SOP) is nonempty. Then if $\sum_{k=1}^{\infty}\alpha_k^2 < \infty$ and $\sum_{i=1}^{\infty}\alpha_k = \infty$ the sequence $\{x^k\}$ generated by the Algorithm converges almost surely to a random point in the solution set S_1.*

Proof. Let z belong to S_1 and $\mathcal{F}^* = \mathcal{F}(z)$ be the optimal value of (SOP). We have

$$\begin{aligned} \|x_{k+1} - z\|^2 &= \|x^{k-1} - z\|^2 + 2\langle x^{k-1} - z, x^k - x^{k-1}\rangle + \|x^k - x^{k-1}\|^2. \\ &= \|x^{k-1} - z\|^2 - 2\frac{\alpha_k}{N}\sum_{i=1}^{N}\langle x^{k-1} - z, \nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i)\rangle \\ &\quad + \frac{\alpha_k^2}{N^2}\|\sum_{i=1}^{N}\nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i)\|^2. \end{aligned}$$

Taking the conditional expectation on $\mathcal{X}_{k-1}$ and using

$$\mathbf{E}\left[\nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i)\right] = \nabla \mathcal{F}(x^{k-1}),$$

$$\mathbf{E}\left[\|\sum_{i=1}^{N}\nabla_x F(x^{k-1}, \omega_k^i, \theta_k^i)\|^2\right] \leq 2N^2 L\mathcal{F}(x^{k-1}),$$

we obtain

$$\begin{aligned} &\mathbf{E}\left[\|x^k - z\|^2 | \mathcal{X}_{k-1}\right] \\ \leq\ & \|x^{k-1} - z\|^2 - 2\alpha_k \nabla\mathcal{F}(x^{k-1})^T(x^{k-1} - z) + 2\alpha_k^2 L\mathcal{F}(x^{k-1}). \end{aligned}$$

Since $\mathcal{F}$ is convex, we have

$$\nabla\mathcal{F}(x^{k-1})^T(x^{k-1} - z) \geq \mathcal{F}(x^{k-1}) - \mathcal{F}(z) = \mathcal{F}(x^{k-1}) - \mathcal{F}^*.$$

So,

$$\begin{aligned} \mathbf{E}\left[\|x^k - z\|^2 \| \mathcal{X}_{k-1}\right] &\leq \|x^{k-1} - z\|^2 - 2\alpha_k(\mathcal{F}(x^{k-1}) - \mathcal{F}^*) \\ &\quad + 2\alpha_k^2 L\mathcal{F}(x^{k-1}). \end{aligned} \tag{13}$$

By Proposition 3.1, the sequence $\mathcal{F}(x^{k-1})$ converge almost surely, hence it is bounded almost surely. Combining this with the condition $\sum_{k=1}^{\infty}\alpha_k^2 < \infty$, we imply that

$$\sum_{k=1}^{\infty}\alpha_k^2\mathcal{F}(x^{k-1}) < \infty \quad a.s.$$

Clearly, $\mathcal{F}(x^{k-1}) \geq \mathcal{F}^* \geq 0$. Thanks to the supermartigale convergence lemma, we have the sequence $\left\{\|x^k - z\|\right\}$ is convergent almost surely for z arbitrary in S_1. Moreover,

$$\sum_{k=1}^{\infty}\alpha_k(\mathcal{F}(x^{k-1} - \mathcal{F}^*) < \infty \quad a.s.$$

Therefore, $\liminf_{k\to\infty}(\mathcal{F}(x^{k-1} - \mathcal{F}^*) = 0 \quad a.s.$ or

$$\liminf_{k\to\infty} \mathcal{F}(x^{k-1}) = \mathcal{F}^* \quad a.s. \tag{14}$$

On the other hand, the sequence $\{\|x^k - z\|\}$ is convergent almost surely. Therefore, almost surely, $\{x^k\}$ is bounded and has limit points. By using (14) and the continuity of $\mathcal{F}$, we can conclude that $\{x^k\}$ converges and its limit point is in S_1 almost surely. □

Theorem 3.3. *Assume that the solution set S of (MSFP) is nonempty. Then if there exist positive numbers $\underline{\alpha}$ and overlineα such that*

$$0 < \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < \frac{1}{L},$$

then the sequence $\{x^k\}$ generated by the Algorithm converges almost surely to a random point in the solution set S.

Proof. Let z be a solution of (SSF) then $z \in S_1$ and the optimal value of (SOP) is $\mathcal{F}^* = \mathcal{F}(z) = 0$. So (13) becomes

$$\mathbf{E}\left[\|x^k - z\|^2 | \mathcal{X}_{k-1}\right] \leq \|x^{k-1} - z\|^2 - 2\alpha_k(1 - \alpha_k L)\mathcal{F}(x^{k-1}). \tag{15}$$

Note that $\mathcal{F}(x^{k-1}) \geq 0$ and $\alpha_k(1 - \alpha_k L) > 0$. By using the supermartingle convergence lemma, we obtain that the sequence $\{\|x^k - z\|\}$ is convergent almost surely for any $z \in S$. In addition,

$$\sum_{k=1}^{\infty} \alpha_k(1 - \alpha_k L)\mathcal{F}(x^{k-1}) < \infty \quad a.s.$$

Since $\alpha_k(1 - \alpha_k L) \geq \underline{\alpha}(1 - \overline{\alpha} L) > 0$, it implies that

$$\liminf_{k\to\infty} \mathcal{F}(x^{k-1}) = 0.$$

By the same argument as in the proof of Theorem 3.2, we can conclude that $\{x^k\}$ converges and its limit point is in S almost surely. □

The following proposition provides the convergence rate of our algorithm.

Proposition 3.4. *Assume that S is nonempty and $0 < \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < \frac{1}{L}$ for every k.*

(i) Let $\hat{x}^k$ be the average point, i.e.

$$\hat{x}^k = \frac{1}{\sum_{i=0}^k \alpha_{i+1}} \sum_{i=0}^k \alpha_{i+1} x^i.$$

Then, we have

$$\boldsymbol{E}[\mathcal{F}(\hat{x}^k)] \leq \frac{dist_S^2(x_0)}{2(1-\overline{\alpha}L)\sum_{i=0}^k \alpha_{i+1}}.$$

(ii) If the linear regularity condition (1) holds, then we have

$$\boldsymbol{E}[dist_S(x^k)] \leq \left(1 - \frac{2\underline{\alpha}(1-\overline{\alpha}L)}{\kappa}\right) \boldsymbol{E}[dist_S(x^{k-1})],$$

and

$$\boldsymbol{E}[\mathcal{F}(x^k)] \leq \left(1 - \frac{2\underline{\alpha}(1-\overline{\alpha}L)}{\kappa}\right)^k \frac{dist_S(x^0)}{2}.$$

Proof. (i) Taking expectation on (15), we obtain, for any $z \in S$,

$$\mathbf{E}[\|x^k - z\|^2] \leq \mathbf{E}[\|x^{k-1} - z\|^2] - 2\alpha_k(1-\alpha_k L)\mathbf{E}[\mathcal{F}(x^{k-1})]. \quad (16)$$

For any α_k satisfied $0 < \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < \frac{1}{L}$, we have

$$\alpha_k(1-\alpha_k L) \geq \alpha_k(1-\overline{\alpha}L).$$

It implies that

$$2\alpha_k(1-\overline{\alpha}L)\mathbf{E}[\mathcal{F}(x^{k-1})] \leq \mathbf{E}[\|x^{k-1} - z\|^2] - \mathbf{E}[\|x^k - z\|^2]. \quad (17)$$

By taking the sum of (17) from 1 to $k+1$, we have

$$2(1-\overline{\alpha}L)\mathbf{E}[\sum_{i=0}^k \alpha_{i+1}\mathcal{F}(x^i)] \leq \|x_0 - z\|^2.$$

Thanks to the convexity of $\mathcal{F}$ and by taking $z = P_S(x_0)$, we can conclude that

$$\mathbf{E}[\mathcal{F}(\hat{x}^k)] \leq \frac{dist_S^2(x_0)}{2(1-\overline{\alpha}L)\sum_{i=0}^k \alpha_{i+1}}.$$

(ii) If the linear regularity condition holds then there exists $\kappa < \infty$ such that

$$dist_S^2(x) \leq \kappa \mathcal{F}(x) \quad \forall x \in \mathbb{R}^n.$$

From (15), we have

$$\mathbf{E}\left[\|x^k - z\|^2 | \mathcal{X}_{k-1}\right] \leq \|x^{k-1} - z\|^2 - \frac{2\alpha_k(1-\alpha_k L)}{\kappa} dist_S^2(x^{k-1}).$$

Taking expectation, we get

$$\mathbf{E}[\|x^k - z\|^2] \leq \mathbf{E}[\|x^{k-1} - z\|^2] - \frac{2\alpha_k(1-\alpha_k L)}{\kappa}\mathbf{E}[dist_S^2(x^{k-1})]. \quad (18)$$

We can choose $z = P_S(x^{k-1})$ and note that

$$\|x^k - P_S(x^{k-1})\|^2 \geq dist_S^2(x^k).$$

From (18), it implies that

$$\begin{aligned} \mathbf{E}[dist_S^2(x^k)] &\leq \left(1 - \frac{2\alpha_k(1-\alpha_k L)}{\kappa}\right)\mathbf{E}[dist_S^2(x^{k-1})] \\ &\leq \left(1 - \frac{2\underline{\alpha}(1-\overline{\alpha} L)}{\kappa}\right)\mathbf{E}[dist_S^2(x^{k-1})]. \end{aligned}$$

□

4 Numerical experiments

In this section, we report several numerical experimental results to illustrate the behavior of our algorithm. We implement the algorithm in Matlab on a Corei5 computer with 512Mb RAM.

Example 4.1. *In this example, we suppose A is an $m \times n$ matrix and*

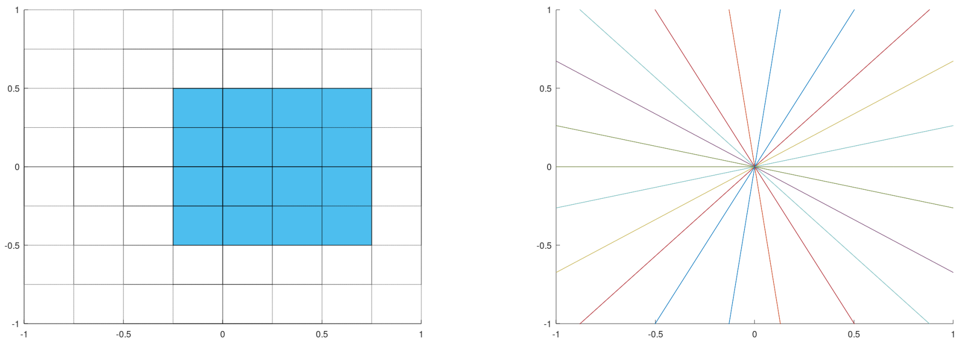
- *The random vectors $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ are the uniformly random vectors in $[0,1]^n$ and $[0,1]^m$, respectively;*
- *C_ω is a box in $\mathbb{R}^n$ defined by*

$$C_\omega = \{x \in \mathbb{R}^n | \quad -1 + \omega_i \leq x_i \leq \omega_i \quad \forall i\}.$$

- *Q_θ is a half-space in $\mathbb{R}^m$ defined by*

$$Q_\theta = \left\{y \in \mathbb{R}^m | \quad c_\theta^T y \leq 0\right\},$$

with $c_\theta = (-1, -1, \ldots, -1) + 2\theta$.



(a) The boxes C_ω (b) The half-spaces Q_θ

Figure 1: Boxes and half-spaces

Figure 1 illustrates the sets C_ω *and* Q_θ*. It is clear that*

$$\cap_\omega C_\omega = \{0\},$$

and

$$\cap_\theta Q_\theta = \{0\},$$

hence the (SSF) problem has unique solution that is the origin $(0, 0, \ldots, 0)$ *of* $\mathbb{R}^n$*. To test our algorithm, we take*

$$\beta = 1; \quad \alpha_k \equiv \frac{1}{1.5(\beta + \lambda(A^T A))} \quad \forall k,$$

and each entry of the matrix A *is uniformly generated in* $[0, 1]$*. For each size* (m, n) *of problem, we test the algorithm on* 100 *samples of* A *and report the average time and error* $\|x^k\|$ *corresponding to different values of mini-batch size* N *in Table 1 and 2. We stop the algorithm if* $\|x^k\| \leq 10^{-2}$ *or the number of iterations exceeds* 500*.*

Example 4.2. *Suppose that* A *is a* 2×2 *matrix and*

- ω *is uniformly distribution on the unit circle* $C(0, 1) \subset \mathbb{R}^2$ *and* C_ω *is the disc with center* ω *and radius* 2 *in* $\mathbb{R}^2$*.*
- θ *is uniformly distribution on the circle* $C(0, 2) \subset \mathbb{R}^2$ *and* Q_θ *is the disc with center* θ *and radius* 4 *in* $\mathbb{R}^2$*.*

Table 1: Average CPUs time corresponding to different problem and mini-batch sizes

(n, m)	$N = 1$	$N = 5$	$N = 10$
(5,5)	3.2953	15.631	31.286
(10,5)	3.8206	18.314	36.382
(20,5)	4.1570	20.735	41.819
(50,5)	4.5560	22.590	45.254
(100.5)	6.8337	33.558	67.650

Table 2: Average error corresponding to different problem and mini-batch sizes

(n, m)	$N = 1$	$N = 5$	$N = 10$
(5,5)	0.073978	0.073496	0.072076
(10,5)	0.23751	0.23911	0.23908
(20,5)	0.59016	0.58459	0.58973
(50,5)	1.5513	1.5517	1.5513
(100,5)	3.0151	3.0033	3.0245

We can easily check that

$$\cap_\omega C_\omega = \{x \in \mathbb{R}^2 | \|x\| \leq 1\} = B(0, 1),$$

and

$$\cap_\theta Q_\theta = \{y \in \mathbb{R}^2 | \|y\| \leq 2\} = B(0, 2).$$

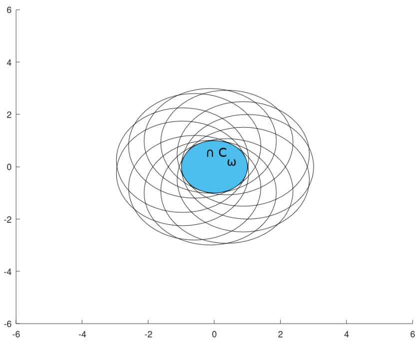
Figure 2 helps us understand the discs C_ω and Q_θ. The (SSF) problem has infinitely many solutions and x^* is a solution of (SSF) if and only if $\|x\| \leq 1$ and $\|Ax\| \leq 2$. Therefore, we set

$$er := \max(0, \|x\| - 1) + \max(0, \|Ax\| - 2),$$

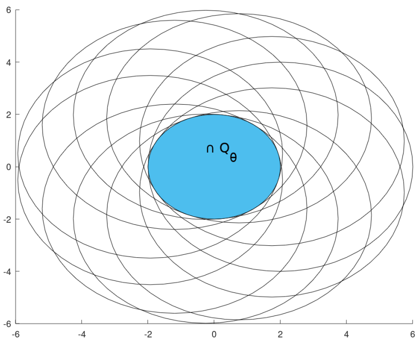
as the error for this example.

We take $\beta = 1$ and test the algorithm with different choices of mini-batch sizes $N = 1, 5, 10$. The stopping rules are $er < 10^{-4}$ and the number of iterations exceeds 500. The average errors over 100 problems corresponding to 100 randomly generated matrix A are plotted in Figure 3.

We also test the algorithm with different choices of α_k. The average times and errors are reported in Table 3. We observe that the choice of the parameter α_k plays an important role for the efficiency of the algorithm. The error goes to 0 quite quick when α_k close to $\frac{1}{L}$.



(a) The discs C_ω



(b) The discs Q_θ

Figure 2: Circles

Table 3: Average CPUs times and errors corresponding to different choices of α_k

	$\alpha_k = \frac{1}{1.1L}$	$\alpha_k = \frac{1}{1.5L}$	$\alpha_k = \frac{1}{k+1}$
Average CPUs time	0.33749	0.35064	2.8761
Average error	0.00031	0.00050	0.01419

5 Conclusion

We proposed a stochastic reformulation of the stochastic split feasibility problem and studied the equivalence between these problems. Then, we introduced a mini-batch random projection algorithm and proved the convergence in consistent and non-consistent cases. We also derived the linear convergence rates for this algorithm under linear regularity condition.

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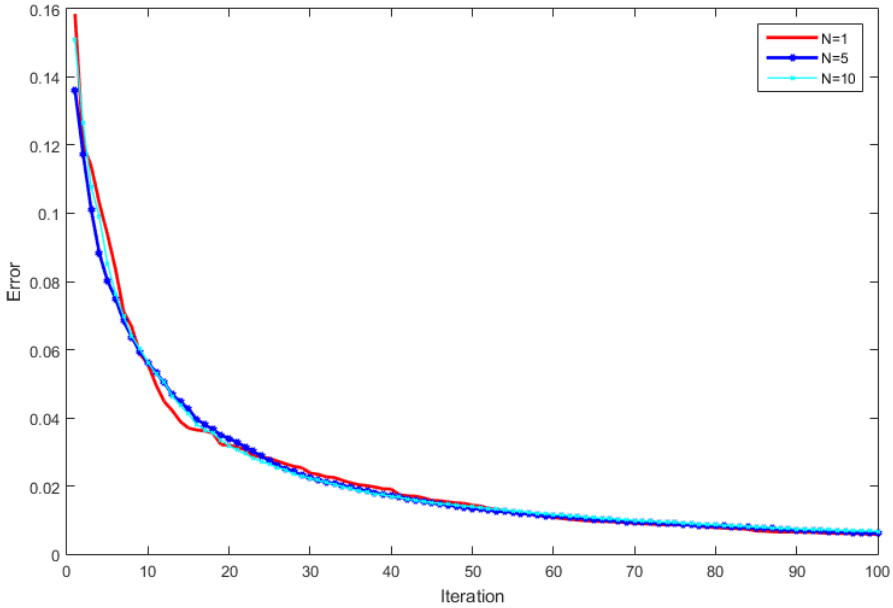

Figure 3: Convergence behavior corresponding to different choice of mini-batch size.

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