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Index Reduction of Second Order, Discrete Time Descriptor Systems ⋆

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Abstract This paper is devoted to the analysis of linear, second order *discrete time descriptor systems* (or singular difference equations (SiDEs) with control). Following the algebraic approach proposed in [10, 11], first we present a theoretical framework to analyze the corresponding initial value problem for SiDEs, which is followed by the analysis of descriptor systems. We also describe numerical methods to analyze structural properties related to the solvability analysis of these systems. This work extends and completes the researches in [2, 14, 18].
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1 Introduction and Preliminaries

In this paper we study second order, discrete time descriptor systems of the form

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) + D_n u(n) = f(n), \quad \text{for all } n \geqslant n_0. \tag{1.1}$$

We will also discuss the initial value problem of the associated singular difference equation (SiDE)

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) = f(n), \quad \text{for all } n \geqslant n_0, \tag{1.2}$$

together with some given initial conditions

$$x(n_0+1) = x_1, \; x(n_0) = x_0. \tag{1.3}$$

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Here the solution/state $x = \{x(n)\}_{n \geqslant n_0}$, the inhomogeneity $f = \{f(n)\}_{n \geqslant n_0}$, the input function $u = \{u(n)\}_{n \geqslant n_0}$, where $x(n) \in \mathbb{R}^d$, $f(n) \in \mathbb{R}^m$ and $u(n) \in \mathbb{R}^p$ for each $n \geqslant n_0$. The coefficients contain three matrix sequences $\{A_n\}_{n \geqslant n_0}$, $\{B_n\}_{n \geqslant n_0}$, $\{C_n\}_{n \geqslant n_0}$ which always take values in $\mathbb{R}^{m,d}$, and $\{D_n\}_{n \geqslant n_0}$ which take values in $\mathbb{R}^{m,p}$. We notice, that all the results in this paper also carry over to the complex case, and they can also be easily extended to systems of higher than second order, but for ease of notation and because this is the most important case in practice, we restrict ourselves to the real, second order case.

The SiDE (1.2), on one side, can be consider as the resulting equation, obtained by finite difference or discretization of some continuous-time DAEs or constrained PDEs. One the other side, there are also many models/applications in real-life, which lead to SiDEs, for example Leotief economic models, backward Leslie model in biology, etc, see e.g. [1, 5, 9, 15].

While both DAEs and SiDEs of first order have been well-studied from both theoretical and numerical sides, the same maturity has not been reached for higher order systems. In classical literature [1, 5, 9], usually new variables are introduced to present some chosen derivatives of the state variable x such that a high order system can be reformulated as a first order one. This method, however, is not only non-unique but also has presented some substantial disadvantages. As have been fully discussed in [14, 18] for continuous time systems, these disadvantages include: (1st) increase the index of the system, and therefore the complexity of a numerical method to solve it; (2nd) increase the computational effort, due to the bigger size of a new system; (3rd) affect the controllability/observability of the corresponding descriptor system, since there exist situations where a new system is uncontrollable while the original one is. Therefore, the *algebraic approach*, which treats the system directly without reformulating it, has been presented in [14, 18, 22, 23] in order to overcome the disadvantages mentioned above. Nevertheless, even for second order SiDEs, this method has not yet been considered.

Another motivation of this work comes from recent researches on the stability analysis of high order, discrete time systems with time-dependent coefficients [13, 19]. There, considered systems are in either strangeness-free form or linear state-space form. Nevertheless, it is not always the case in applications, and hence, a reformulation procedure is necessary.

Therefore, the main aim of this article is to set up a comparable framework for second order SiDEs/descriptor systems. It is worth marking that the algebraic method proposed in [14, 18] is applicable theoretically but not numerically, due to two reasons: (1) The condensed form of the matrix coefficients are really big and complicated. (2) The system's transformations are not orthogonal, and hence, not numerically stable. In this work, we will modify this method to make it more concise and also be computable in a stable way.

The outline of this paper is as follows. After recalling some preliminary concepts and some auxiliary lemmata, in Sections 2 and 3 we consecutively introduce *index reduction procedures* for SiDEs and for descriptor systems. Resulting systems from these procedures allow us to determine structural properties such as existence and uniqueness of a solution, consistency and hidden constraints, etc. For the numerical solution of these systems, in Section 4 we study the *difference array approach* in order to bring the original system to its strangeness-free form. Finally, we finish with some conclusion.

In the following example we demonstrate some difficulties that may arise in the analysis of second order SiDEs.

Example 1 Consider the following second order SiDE, motivated from Example 2, [18].

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \end{bmatrix}, \ n \geqslant n_0. \tag{1.4}$$

Clearly, from the second equation $\begin{bmatrix} 1 & 0 \end{bmatrix} x(n) = f_2(n)$, we can shift forward the time n to obtain

$$\begin{bmatrix} 1 & 0 \end{bmatrix} x(n+1) = f_2(n+1) \quad \text{and} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} x(n+2) = f_2(n+2).$$

Inserting these into the first equation of (1.4), we find out the hidden constraint $f_2(n+2)+f_2(n+1)+\begin{bmatrix} 0 & 1 \end{bmatrix} x(n) = f_1(n)$. Consequently, we obtain the following system, which possess a unique solution

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) - f_2(n+2) - f_2(n+1) \\ f_2(n) \end{bmatrix}, \ n \geqslant n_0.$$

Let $n = n_0$ in this new system, we obtain a constraint that $x(n_0)$ must obey. This example showed us some important facts. Firstly, one can use some shift operators and row-manipulation (Gaussian eliminations) to derive hidden constraints. Secondly, a solution only exists if an initial condition fulfills some consistency conditions.

For matrices $Q \in \mathbb{R}^{q,d}$, $P \in \mathbb{R}^{p,d}$, the pair (Q, P) is said to *have no hidden redundancy* if

$$\operatorname{rank}\left(\begin{bmatrix} Q \\ P \end{bmatrix}\right) = \operatorname{rank(Q)} + \operatorname{rank(P)}.$$

Otherwise, (Q, P) is said to *have hidden redundancy.* The geometrical meaning of this concept is that the intersection space $\operatorname{span}(P^T) \cap \operatorname{span}(Q^T)$ contains only the zero-vector $\mathbf{0}$. Here by $\operatorname{span}(P^T)$ (resp., $\operatorname{span}(Q^T)$) we denote the real vector space spanned by the rows of P (resp., rows of Q). We further notice that, if $\begin{bmatrix} Q \\ P \end{bmatrix}$ is of full row rank then obviously, the pair (Q, P) has no hidden redundancy. However, the converse is not true as is obvious for $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Lemma 1 *([7]) Suppose that for $Q \in \mathbb{R}^{q,d}$, $P \in \mathbb{R}^{p,d}$, the pair (Q, P) has no hidden redundancy. Then, for any matrix $U \in C^{q,q}$ and any $V \in C^{p,p}$, the pair (UQ, VP) has no hidden redundancy.*

Lemma 2 *([7]) Consider $k+1$ full row rank matrices $R_0 \in \mathbb{R}^{r_0,d}, \ldots, R_k \in \mathbb{R}^{r_k,d}$, and assume that for $j = k, \ldots, 1$ none of the matrix pairs $\left(R_j, \begin{bmatrix} R_{j-1} \\ \vdots \\ R_0 \end{bmatrix}\right)$ has a hidden redundancy. Then, $\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}$ has full row rank.*

Lemma 3 below will be very useful later for our analysis, in order to remove hidden redundancy in the coefficients of (1.2).

Lemma 3 *Consider two matrix sequences* $\{P_n\}_{n\geqslant n_0}$, $\{Q_n\}_{n\geqslant n_0}$ *which take values in* $\mathbb{R}^{m,d}$, *and assume that they satisfy the constant rank assumptions*

$$\operatorname{rank}(Q_n) = r_Q, \quad \textit{and} \ \operatorname{rank}\left(\begin{bmatrix} P_n \\ Q_n \end{bmatrix}\right) = r_{[P;Q]}, \ \textit{for all } n \geqslant n_0 \ .$$

Then, there exists a matrix sequence $\left\{ \begin{bmatrix} S_n & 0 \\ Z_n^{(1)} & Z_n^{(2)} \end{bmatrix} \right\}_{n\geqslant n_0}$ *in* $\mathbb{R}^{p,p+q}$ *such that the following conditions hold.*

i) $S_n \in \mathbb{R}^{r_{[P;Q]}-r_Q,\ p}$, $Z_n^{(1)} \in \mathbb{R}^{p-r_{[P;Q]}+r_Q,\ p}$, $Z_n^{(2)} \in \mathbb{R}^{p-r_{[P;Q]}+r_Q,\ q}$,

ii) $\begin{bmatrix} S_n \\ Z_n^{(1)} \end{bmatrix} \in \mathbb{R}^{p,p}$ *is orthogonal, and* $Z_n^{(1)} P_n + Z_n^{(2)} Q_n = 0$,

iii) the matrix $S_n P_n$ *has full row rank, and the pair* $(S_n P_n, Q_n)$ *has no hidden redundancy.*

Proof. First using SVD we factorize Q_n and then partition P_n conformably to get

$$U_1^T Q_n V_1 = \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } P_n V_1 = \begin{bmatrix} P_{n,1} & P_{n,2} \end{bmatrix}, \tag{1.5}$$

where the matrices $U_1 = \begin{bmatrix} U_{11} & U_{12} \end{bmatrix} \in \mathbb{R}^{q,q}$, $V_1 = \begin{bmatrix} V_{11} & V_{12} \end{bmatrix} \in \mathbb{R}^{d,d}$ are orthogonal and $\Sigma_n \in \mathbb{R}^{r_Q, r_Q}$ is diagonal. Now we use a second SVD to factorize $P_{n,2}$ and to find an orthogonal matrix $U_2^T = \begin{bmatrix} S \\ Z_n^{(1)} \end{bmatrix} \in \mathbb{R}^{p,p}$ such that $U_2^T P_{n,2} = \begin{bmatrix} P_{n,12} \\ 0 \end{bmatrix}$, where $P_{n,12}$ has full row rank. Thus, we obtain

$$\left[\begin{array}{cc} S_n & 0 \\ Z_n^{(1)} & 0 \\ \hline 0 & U_{11}^T \\ 0 & U_{12}^T \end{array}\right] \left[\begin{array}{c} P_n \\ \hline Q_n \end{array}\right] \begin{bmatrix} V_{11} & V_{12} \end{bmatrix} = \left[\begin{array}{cc} P_{n,11} & P_{n,12} \\ P_{n,21} & 0 \\ \hline \Sigma_n & 0 \\ 0 & 0 \end{array}\right] \begin{array}{c} r_{[P;Q]} - r_Q \\ p - r_{[P;Q]} + r_Q \\ r_Q \\ q - r_Q \end{array} .$$

Since $P_{n,12}$ has full row rank, $S_n P_n = \begin{bmatrix} P_{n,11} & P_{n,12} \end{bmatrix} V_1^{-1}$ also has full row rank. Moreover, one sees that

$$\operatorname{rank}\left(\begin{bmatrix} S_n P_n \\ Q_n \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} 0 & P_{n,12} \end{bmatrix}\right) + \operatorname{rank}\left(\begin{bmatrix} \Sigma_n & 0 \end{bmatrix}\right) = \operatorname{rank}(S_n P_n) + \operatorname{rank}(Q_n),$$

which follows that the pair $(S_n P_n, Q_n)$ has no hidden redundancy.
Finally, setting $Z_n^{(2)} := -P_{n,21} \Sigma_n^{-1} U_{11}^T$, we obtain

$$Z_n^{(1)} P_n + Z_n^{(2)} Q_n = \left([P_{n,21} \quad 0] - P_{n,21} \Sigma_n^{-1} [\Sigma_n \quad 0] \right) V_1^{-1} = 0,$$

which completes the proof. □

Remark 1 i) In the special case, where P_n has full row rank and the pair (P_n, Q_n) has no hidden redundancy, we will adapt the notation of an empty matrix and take $S_n = I_p$, $Z_n^{(1)} = [\ ]^{0,p}$, $Z_n^{(2)} = [\ ]^{0,q}$.
ii) Furthermore, we notice that the matrices U_1, U_2, V_1 in the proof of Lemma 3 are orthogonal. Therefore, in case that the singular values of Q_n are neither too small nor too big, then Σ_n^{-1} is well-conditioned, and hence we can stably compute the matrix $Z_n^{(2)}$. Both matrices $Z_n^{(1)}$ and $Z_n^{(2)}$ will play the key role in our *index reduction procedure* presented in the next section.

For any given matrix M, by M^T we denote its transpose. By $T_0(M)$ we denote an orthogonal matrix whose columns span the left null space of M. By $T_\perp(M)$ we denote an orthogonal matrix whose columns span the vector space range(M). From basic linear algebra, we have the following three lemmata.

Lemma 4 *The matrix* $\begin{bmatrix} T_\perp^T(M) \\ T_0^T(M) \end{bmatrix}$ *is nonsingular, the matrix* $T_\perp^T(M)\ M$ *has full row rank, and the following identity holds*

$$\begin{bmatrix} T_\perp^T(M) \\ T_0^T(M) \end{bmatrix} M = \begin{bmatrix} T_\perp^T(M)\ M \\ 0 \end{bmatrix}.$$

Proof. A simple proof can be found, for example, in [6]. □

Lemma 5 *Given four matrices* $\breve{A}$, $\breve{B}$, $\breve{C}$ *in* $\mathbb{R}^{m,d}$ *and* $\breve{D}$ *in* $\mathbb{R}^{m,p}$. *Let us consider the following matrices whose columns span orthogonal bases of the associated vector spaces*

$$\begin{array}{llll}
T_1 \ \textit{basis of}\ \mathrm{kernel}(\breve{A}^T), & \textit{and} & T_{1,\perp} & \textit{basis of}\ \mathrm{range}(\breve{A}), \\
W_1 \ \textit{basis of}\ \mathrm{kernel}(T_1^T \breve{D})^T, & \textit{and} & W_{1,\perp} & \textit{basis of}\ \mathrm{range}(T_1^T \breve{D}), \\
 & & J_{\mathrm{D}} & := W_{1,\perp}^T T_1^T \breve{D}, \\
J_{\mathrm{B1}} := W_1^T T_1^T \breve{B}, & \textit{and} & J_{\mathrm{B2}} & := W_{1,\perp}^T T_{1,\perp}^T \breve{B}, \\
J_{\mathrm{C1}} := W_1^T T_1^T \breve{C}, & \textit{and} & J_{\mathrm{C2}} & := W_{1,\perp}^T T_1^T \breve{C}, \\
T_2 \ \textit{basis of}\ \mathrm{kernel}(J_{\mathrm{B1}}^T), & \textit{and} & T_{2,\perp} & \textit{basis of}\ \mathrm{range}(J_{\mathrm{B1}}), \\
T_3 \ \textit{basis of}\ \mathrm{kernel}(J_{\mathrm{B2}}^T), & \textit{and} & T_{3,\perp} & \textit{basis of}\ \mathrm{range}(J_{\mathrm{B2}}), \\
T_4 \ \textit{basis of}\ \mathrm{kernel}(T_2^T J_{\mathrm{C1}})^T, & \textit{and} & T_{4,\perp} & \textit{basis of}\ \mathrm{range}(T_2^T J_{\mathrm{C1}}).
\end{array}$$

Then, the following assertions hold true.

i) The matrices $\begin{bmatrix} T_{i,\perp} \\ T_i \end{bmatrix}$, $i = 1, ..., 4$, $\begin{bmatrix} W_{1,\perp} \\ W_1 \end{bmatrix}$ *are orthogonal.*

ii) The matrices $T_{1,\perp}^T \breve{A}$, $T_{2,\perp}^T J_{\mathrm{B1}}$, $T_{3,\perp}^T J_{\mathrm{B2}}$, $T_{4,\perp}^T T_2^T J_{\mathrm{C1}}$, *and* J_{D} *have full row rank.*

iii) Moreover, there exists a nonsingular matrix $\breve{U}$ *such that*

$$\breve{U} \left[\breve{A}\ \breve{B}\ \breve{C} \,\middle|\, \breve{D} \right] = \left[\begin{array}{ccc|c} \breve{A}_1 & \breve{B}_1 & \breve{C}_1 & \breve{D}_1 \\ 0 & \breve{B}_2 & \breve{C}_2 & 0 \\ 0 & 0 & \breve{C}_3 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \breve{B}_4 & \breve{C}_4 & \breve{D}_4 \\ 0 & 0 & \breve{C}_5 & \breve{D}_5 \end{array} \right], \tag{1.6}$$

where the matrices $\breve{A}_1$, $\breve{B}_2$, $\breve{B}_4$, $\breve{C}_3$, $\begin{bmatrix}\breve{D}_4\\ \breve{D}_5\end{bmatrix}$ *have full row rank.*

Proof. The first two claims followed directly from Lemma 4. To prove the third claim, we construct the desired matrix $\breve{U}$ as follows

$$\breve{U} := \left[\begin{array}{c|cc|c} I & & & \\ \hline & I & & \\ & & T_{4,\perp}^T & \\ & & T_4^T & \\ \hline & & & I \end{array}\right] \cdot \left[\begin{array}{c|c|c} I & & \\ \hline & T_{2,\perp}^T & \\ & T_2^T & \\ \hline & & T_{3,\perp}^T \\ & & T_3^T \end{array}\right] \cdot \left[\begin{array}{c|c} I & \\ \hline & W_1^T \\ & W_{1,\perp}^T \end{array}\right] \cdot \begin{bmatrix} T_{1,\perp}^T \\ T_1^T \end{bmatrix} .$$

Thus, we have that

$$\breve{U}\left[\breve{A}\ \breve{B}\ \breve{C}\,\middle|\,\breve{D}\right] = \left[\begin{array}{cccc} T_{1,\perp}^T\breve{A} & T_{1,\perp}^T\breve{B} & T_{1,\perp}^T\breve{C} & T_{1,\perp}^T\breve{D} \\ 0 & T_{2,\perp}^T J_{\mathrm{B1}} & T_{2,\perp}^T J_{\mathrm{C1}} & 0 \\ 0 & 0 & T_{4,\perp}^T T_2^T J_{\mathrm{C1}} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & T_{3,\perp}^T J_{\mathrm{B2}} & T_{3,\perp}^T J_{\mathrm{C2}} & T_{3,\perp}^T J_{\mathrm{D}} \\ 0 & 0 & T_3^T J_{\mathrm{C2}} & T_3^T J_{\mathrm{D}} \end{array}\right] .$$

Due to the parts i) and ii), we see that this is exactly the desired form (1.6). □

Lemma 6 *Let* $P \in \mathbb{R}^{p,d}$, $Q \in \mathbb{R}^{q,d}$ *be two full row rank matrices and* $p + q \leqslant d$. *Then, the following assertions hold true.*

i) There exists a matrix $F \in \mathbb{R}^{d,d}$ *such that* $H := \begin{bmatrix} P \\ QF \end{bmatrix}$ *has full row rank.*

ii) For any $G \in \mathbb{R}^{q,d}$, *there exists a matrix* $F \in \mathbb{R}^{d,d}$ *such that* $\begin{bmatrix} P \\ G + QF \end{bmatrix}$ *has full row rank.*

Proof. i) First we consider the SVDs of P and G that reads

$$U_P P V_P = \begin{bmatrix} \Sigma_P & 0_{p,d-p} \end{bmatrix}, \quad U_Q Q V_Q = \begin{bmatrix} \Sigma_Q & 0_{q,d-q} \end{bmatrix},$$

where Σ_P, Σ_Q are nonsingular, diagonal matrices, and $0_{p,d-p}$ (resp. $0_{q,d-q}$) are the zero matrix of size p by $d-p$ (resp. q by $d-q$).
By choosing $F := V_Q \begin{bmatrix} 0 & I_q \\ I_{d-q} & 0 \end{bmatrix} V_P^{-1}$ we see that

$$\begin{bmatrix} U_P & 0 \\ 0 & U_Q \end{bmatrix} \begin{bmatrix} P \\ QF \end{bmatrix} V_P = \begin{bmatrix} U_P P V_P \\ U_Q Q F V_P \end{bmatrix} = \begin{bmatrix} \Sigma_P & 0_{p,d-p-q} & 0_{p,q} \\ 0_{q,p} & 0_{p,d-p-q} & \Sigma_Q \end{bmatrix},$$

and hence, the claim i) is proven.
ii) Clearly, in case that the matrix F is very big, then G is only a small perturbation, and hence for sufficiently large η, by choosing

$$F := \eta V_Q \begin{bmatrix} 0 & I_q \\ I_{d-q} & 0 \end{bmatrix} V_P^{-1} ,$$

we obtain the full row rank property of $\begin{bmatrix} P \\ G + QF \end{bmatrix}$. □

Remark 2 It should be noted that, the proof of Lemmata 5 and 6 are constructive, and all the matrices $T_{i,\perp}$, T_i, $i = 1, ..., 4$, $W_{1,\perp}$, W_1 and F can be stably computed.

2 Strangeness-index of second-order SiDEs

In this section, we study the solvability analysis of the second-order SiDE (1.2) and of its corresponding IVP (1.2)–(1.3). Many regularization procedures and their associated index concepts have been proposed for first order systems, see the survey [17] and the references therein. Nevertheless, for second order systems, only the strangeness-index has been proposed for only continuous but not discrete time systems in [18, 23]. Thus, it is our purpose to construct a comparable regularization and index concept for system (1.2).

Let

$$M_n := \begin{bmatrix} A_n \; B_n \; C_n \end{bmatrix}, \; X(n) := \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix},$$

we call $\{M_n\}_{n \geqslant n_0}$ the *behavior matrix sequence* of system (1.2). Thus, (1.2) can be rewritten as

$$M_n X(n) = f(n), \text{ for all } n \geqslant n_0. \tag{2.1}$$

Clearly, by scaling (1.2) with a pointwise nonsingular matrix sequence $\{P_n\}_{n \geqslant n_0}$ in $\mathbb{R}^{d,d}$, we obtain a new system

$$\begin{bmatrix} P_n A_n \; P_n B_n \; P_n C_n \end{bmatrix} X(n) = P_n f(n), \text{ for all } n \geqslant n_0, \tag{2.2}$$

without changing the solution space. This motivates the following definition.

Definition 1 Two behavior matrix sequences $\{M_n = \begin{bmatrix} A_n \; B_n \; C_n \end{bmatrix}\}_{n \geqslant n_0}$ and $\{\tilde{M}_n = \begin{bmatrix} \tilde{A}_n \; \tilde{B}_n \; \tilde{C}_n \end{bmatrix}\}_{n \geqslant n_0}$ are called *(strongly) left equivalent* if there exists a pointwise nonsingular matrix sequence $\{P_n\}_{n \geqslant n_0}$ such that $\tilde{M}_n = P_n M_n$ for all $n \geqslant n_0$. We denote this equivalence by $\{M_n\}_{n \geqslant n_0} \overset{\ell}{\sim} \{\tilde{M}_n\}_{n \geqslant n_0}$. If this is the case, we also say that two SiDEs (1.2), (2.2) are left equivalent.

Lemma 7 *Consider the behavior matrix sequence $\{M_n\}_{n \geqslant n_0}$ of system* (1.2). *Then, for all $n \geqslant n_0$, we have that*

$$\{M_n\}_{n \geqslant n_0} \overset{\ell}{\sim} \left\{ \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{bmatrix} \right\}_{n \geqslant n_0}, \quad \begin{matrix} r_{2,n} \\ r_{1,n} \\ r_{0,n} \\ v_n \end{matrix} \tag{2.3}$$

where the matrices $A_{n,1}$, $B_{n,2}$, $C_{n,3}$ on the main diagonal have full row rank. Here the numbers $r_{2,n}$, $r_{1,n}$, $r_{0,n}$, v_n are row-sizes of the block rows of M_n. Furthermore, these numbers are invariant under left equivalent transformations. Thus, we can call them the local characteristic invariants of the SiDE (1.2).

Proof. The block diagonal form (2.3) is obtained directly by consecutively compressing the block columns A_n, B_n, C_n of M_n via Lemma 4. In details, we have that

$$\begin{aligned}
&\text{rows of } A_{n,1} \text{ form the basis of the space } \operatorname{range}(A_n^T),\\
&\text{rows of } B_{n,2} \text{ form the basis of the space } \operatorname{range}(T_0^T(A_n)\ B_n)^T,\\
&\text{rows of } C_{n,3} \text{ form the basis of the space } \operatorname{range}\left(T_0^T\left(\begin{bmatrix} A_n \\ B_n \end{bmatrix}\right)\ C_n\right)^T.
\end{aligned}$$

Moreover, from (2.3), we obtain the following identities

$$\begin{aligned}
r_{2,n} &= \operatorname{rank}(A_n),\\
r_{1,n} &= \operatorname{rank}(\begin{bmatrix} A_n & B_n \end{bmatrix}) - \operatorname{rank}(A_n),\\
r_{0,n} &= \operatorname{rank}(\begin{bmatrix} A_n & B_n & C_n \end{bmatrix}) - \operatorname{rank}(\begin{bmatrix} A_n & B_n \end{bmatrix}),
\end{aligned}$$

which proves the second claim. □

Analogous to the continuous-time case, we will apply an *algebraic approach* (see [2, 18]), which aims to reformulate (1.2) into a so-called *strangeness-free* form, as stated in the following definition.

Definition 2 ([13]) System (1.2) is called *strangeness-free* if there exists a pointwise nonsingular matrix sequence $\{P_n\}_{n \geqslant n_0}$ such that by scaling the SiDE (1.2) at each point n with P_n, we obtain a new system of the form

$$\begin{matrix} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hat{v} \end{matrix} \quad \begin{bmatrix} \hat{A}_{n,1} \\ 0 \\ 0 \\ 0 \end{bmatrix} x(n+2) + \begin{bmatrix} \hat{B}_{n,1} \\ \hat{B}_{n,2} \\ 0 \\ 0 \end{bmatrix} x(n+1) + \begin{bmatrix} \hat{C}_{n,1} \\ \hat{C}_{n,2} \\ \hat{C}_{n,3} \\ 0 \end{bmatrix} x(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hat{f}_4(n) \end{bmatrix}, \text{ for all } n \geqslant n_0, \tag{2.4}$$

where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ has full row rank for all $n \geqslant n_0$.

Remark 3 We notice that, if the SiDE (1.2) is of the strangeness-free form (2.4), then the existence and uniqueness of the solution $\{x(n)\}_{n \geqslant n_0}$ can be achieved if and only if $\hat{r}_2 + \hat{r}_1 + \hat{r}_0 = d$. Furthermore, either the last block row equation of (2.4) do not appear , i.e. $\hat{v} = 0$, or $\hat{f}_4(n) = 0$ for all $n \geqslant n_0$.

In order to perform an algebraic approach, an additional assumption below is usually needed.

Assumption 1. *Assume that the local characteristic invariants $r_{2,n}$, $r_{1,n}$, $r_{0,n}$ become global, i.e., they are constant for all $n \geqslant n_0$. Furthermore, assume that two matrix sequences $\left\{ \begin{bmatrix} A_{n,1} \\ B_{n,2} \\ C_{n,3} \end{bmatrix} \right\}_{n \geqslant n_0}$ and $\left\{ \begin{bmatrix} B_{n,2} \\ C_{n,3} \end{bmatrix} \right\}_{n \geqslant n_0}$ have constant rank for all $n \geqslant n_0$.*

Remark 4 Following directly from the proof of Lemma 7, we see that Assumption 1 is satisfied if and only if five following constant rank conditions are satisfied

$$\begin{aligned}
&\operatorname{rank}(A_n) \equiv \text{const.},\ \operatorname{rank}(\begin{bmatrix} A_n & B_n \end{bmatrix}) \equiv \text{const.},\ \operatorname{rank}(\begin{bmatrix} A_n & B_n & C_n \end{bmatrix}) \equiv \text{const.},\\
&\operatorname{rank}(T_0^T(A_n)\ B_n) \equiv \text{const.},\ \operatorname{rank}\left(T_0^T\left(\begin{bmatrix} A_n \\ B_n \end{bmatrix}\right)\ C_n\right) \equiv \text{const.}
\end{aligned} \tag{2.5}$$

Remark 5 In the context of continuous-time systems, the quantities r_2, r_1, and r_0 are the dimensions of the second order derivative part, the first order derivative part, and the algebraic part, respectively. Furthermore, $r_2 + r_1$ is exactly the degree of freedoms of the considered system.

Let us call the number

$$r_u := 3r_2 + 2r_1 + r_0$$

the *upper rank* of system (1.2). Clearly, r_u is invariant under left equivalence transformations. Rewrite (2.1) block row-wise, we obtain the following system for all $n \geqslant n_0$.

$$\begin{aligned} A_{n,1}x(n+2) + B_{n,1}x(n+1) + C_{n,1}x(n) &= f_1(n), \quad r_2 \text{ equations}, &\quad (2.6a)\\ B_{n,2}x(n+1) + C_{n,2}x(n) &= f_2(n), \quad r_1 \text{ equations}, &\quad (2.6b)\\ C_{n,3}x(n) &= f_3(n), \quad r_0 \text{ equations}, &\quad (2.6c)\\ 0 &= f_4(n), \quad v \text{ equations}. &\quad (2.6d) \end{aligned}$$

Since the matrices $A_{n,1}$, $B_{n,2}$, $C_{n,3}$ have full row rank, the number of scalar difference equations of order 2 (resp. 1, and 0) in (1.2) is exactly r_2 (resp. r_1 and r_0), while v is the number of redundant equations. Now we are able to define the shift-forward operator Δ, which acts on some or whole equations of system (2.6). This operator maps each equation of system (2.6) at the time instant n to the equation itself at the time $n+1$, for example

$$\Delta : C_{n,3}x(n) = f_3(n) \mapsto C_{n+1,3}x(n+1) = f_3(n+1). \tag{2.7}$$

Clearly, under Assumption 1, this shift operator can be applied to equations of system (2.6). In order to reveal all hidden constraints of (2.6) we propose the idea, that for each $j = 1, 2$, we use equations of order less than j to reduce the number of scalar equations of order j. This task will be performed in Lemmata 9 and 10 below. In details, if the matrix pair $(B_{n,2}, C_{n+1,3})$ has hidden redundancy then we will make use of the shifted equation (2.7). Analogously, if the pair $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}\right)$ has hidden redundancy then we will make use of the shifted equation

$$B_{n+1,2}x(n+2) + C_{n+1,2}x(n+1) = f_2(n+1), \tag{2.8}$$

and may be also the double shifted equation

$$C_{n+2,3}x(n+2) = f_3(n+2). \tag{2.9}$$

Lemma 8 *Consider the SiDE* (1.2) *and the equivalent system* (2.6)*. Then,* (1.2) *has an identical solution set as the extended system*

$$\begin{array}{c} r_2 \\ r_1 \\ r_0 \\ v \\ \hline r_0 \\ r_1 \\ r_0 \end{array} \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \\ \hline 0 & C_{n+1,3} & 0 \\ B_{n+1,2} & C_{n+1,2} & 0 \\ C_{n+2,3} & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \\ f_4(n) \\ \hline f_3(n+1) \\ f_2(n+1) \\ f_3(n+2) \end{bmatrix}, \tag{2.10}$$

for all $n \geqslant n_0$.

Proof. Since all equations in the lower part of (2.10) at any time point n is the consequence of the upper part (which is exactly (2.6)) at the time instants $n+1$ and $n+2$, the proof is directly followed. $\square$

Lemma 9 *Consider the behavior matrix sequence* $\{M_n\}_{n \geqslant n_0}$ *in* (2.3). *Assume that Assumption 1 is satisfied. Then, there exist matrix sequences* $\{S_n^{(i)}\}_{n \geqslant n_0}$, $i = 1, 2$, *and* $\{Z_n^{(j)}\}_{n \geqslant n_0}$, $j = 1, ..., 5$, *of appropriate sizes such that for all* $n \geqslant n_0$, *the following conditions hold true.*

i) For $i = 1, 2$, *the matrices* $\begin{bmatrix} S_n^{(i)} \\ Z_n^{(i)} \end{bmatrix} \in \mathbb{R}^{r_i, r_i}$ *are orthogonal.*

ii) The following identities hold true.

$$Z_n^{(1)} B_{n,2} + Z_n^{(3)} C_{n+1,3} = 0, \tag{2.11a}$$

$$Z_n^{(2)} A_{n,1} + Z_n^{(4)} B_{n+1,2} + Z_n^{(5)} C_{n+2,3} = 0. \tag{2.11b}$$

iii) Both matrix pairs $\left(S_n^{(2)} A_n, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix} \right)$, $\left(S_n^{(1)} B_{n,2}, C_{n+1,3} \right)$ *have no hidden redundancy.*

Proof. The proof can be directly obtained by applying Lemma 3 to two matrix pairs $(B_{n,2}, C_{n+1,3})$ and $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix} \right)$. $\square$

Lemma 10 *Under the condition of Lemma 9, the SiDE* (1.2) *has exactly the same solution set as the transformed system*

$$\begin{matrix} d_2 \\ s_2 \\ d_1 \\ s_1 \\ r_0 \\ v \end{matrix} \left[\begin{array}{ccc} S_n^{(2)} A_{n,1} & S_n^{(2)} B_{n,1} & S_n^{(2)} C_{n,1} \\ 0 & Z_n^{(2)} B_{n,1} + Z_n^{(4)} C_{n+1,2} & Z_n^{(2)} C_{n,1} \\ \hline 0 & S_n^{(1)} B_{n,2} & S_n^{(1)} C_{n,2} \\ 0 & 0 & Z_n^{(1)} C_{n,2} \\ \hline 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} =$$

$$= \left[\begin{array}{c} S_n^{(2)} f_1(n) \\ Z_n^{(2)} f_1(n) + Z_n^{(4)} f_2(n+1) + Z_n^{(5)} f_3(n+2) \\ \hline S_n^{(1)} f_2(n) \\ Z_n^{(1)} f_2(n) + Z_n^{(3)} f_3(n+1) \\ \hline f_3(n) \\ f_4(n) \end{array} \right], \quad \text{for all } n \geqslant n_0. \tag{2.12}$$

Furthermore, both matrix pairs $\left(S_n^{(2)} A_n, \begin{bmatrix} S_n^{(1)} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix} \right)$, $\left(S_n^{(1)} B_{n,2}, C_{n+1,3} \right)$ *have no hidden redundancy.*

Proof. The proof is simple but quite long and technical, so we leave it to Appendix A. $\square$

Consider system (2.12), we see that the upper rank of the behavior matrix is

$$\begin{aligned} r_u^{new} &\leqslant 3d_2 + 2(s_2 + d_1) + (s_1 + r_0) \\ &= 3(r_2 - s_2) + 2(s_2 + r_1 - s_1) + (s_1 + r_0) \\ &= r - (s_2 + s_1) \leqslant r. \end{aligned}$$

In conclusion, after performing a so-called *index reduction step*, which passes from (2.6) to (2.12), we have reduced the upper rank r_u at least by $s_2 + s_1$. Continue in this fashion until $s_1 = s_2 = 0$, we obtain the following algorithm.

Algorithm 1 Index reduction steps for SiDEs at the time point n

1: **Input:** The SiDE (1.2) and its behavior form (2.1). Set $i = 0$, $\mu = 0$.
2: **Return:** A strangeness-free SiDE of the form (2.4).
3: Transform the behavior matrix $\begin{bmatrix} A_n & B_n & C_n \end{bmatrix}$ to the block upper triangular form

$$\tilde{M}_n := \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{bmatrix},$$

where all the matrices $A_{n,1}$, $B_{n,2}$, $C_{n,3}$ on the main diagonal have full row rank.
4: **if** both matrix pairs $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix} \right)$ and $(B_{n,2}, C_{n+1,3})$ have no hidden redundancy **then** STOP.
5: **else** set $i := i + 1$ and go to 6
6: Find the matrices $S_n^{(j)}$, $j = 1, 2$, and $Z_n^{(j)}$, $j = 1, ..., 5$ as in Lemma 9.
7: **if** $Z_n^{(5)} \neq [\ ]$ **then** set $\mu := \mu + 2$.
8: **else** set $\mu := \mu + 1$
9: **end if**
10: **end if**
11: Go back to 3.

After each index reduction step the upper rank r_u^i has been decreased at least by $s_2^i + s_1^i$, so Algorithm 1 terminates after a finite number μ of iterations, which will be called the *strangeness-index* of the SiDE (1.2).

Theorem 2 *Consider the SiDE* (2.1) *and assume that Assumption 1 is satisfied for any n and any i considered within the loop, such that the strangeness-index μ is well-defined by Algorithm 1. Then, the SiDE* (1.2) *has the same solution set as the strangeness-free SiDE*

$$\begin{array}{l} r_2^\mu \\ r_1^\mu \\ r_0^\mu \\ v^\mu \end{array} \begin{bmatrix} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} \hat{g}_1(n) \\ \hat{g}_2(n) \\ \hat{g}_3(n) \\ \hat{g}_4(n) \end{bmatrix}, \text{ for all } n \geqslant n_0, \qquad (2.13)$$

where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ *has full row rank for all $n \geqslant n_0$. Here $\hat{g}_2$ and $\hat{g}_3$ are functions of $f(n+1), \ldots, f(n+\mu)$.*

Proof. The proof is a direct consequence of Algorithm 1, where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ has full row rank due to Lemma 2. □

To illustrate Algorithm 1, we consider the following example.

Example 2 Given a parameter $\alpha \in \mathbb{R}$, we consider the second order SiDE

$$\begin{bmatrix} 1 & n{+}1 & n{+}4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n{+}2) + \begin{bmatrix} 0 & \alpha & 2n{+}3 \\ 1 & n & 1 \\ 0 & 0 & 0 \end{bmatrix} x(n{+}1) + \begin{bmatrix} 0 & n{+}1 & 0 \\ 0 & 0 & n \\ 0 & 0 & n{+}1 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \end{bmatrix}, \tag{2.14}$$

for all $n \geqslant 0$. Fortunately, the behavior matrix

$$M = \left[\begin{array}{ccc|ccc|ccc} 1 & \text{n+1} & \text{n+4} & 0 & \alpha & \text{2n+3} & 0 & \text{n+1} & 0 \\ \hline 0 & 0 & 0 & 1 & \text{n} & 1 & 0 & 0 & \text{n} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{n+1} \end{array}\right] = \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \end{bmatrix}$$

is already in the block diagonal form, so we do not need to perform Step 3 in Algorithm 1. Furthermore, all constant rank conditions required in Assumption 1 are satisfied. We observe that

$$B_{n+1,2} = \begin{bmatrix} 1 & n+1 & 1 \end{bmatrix}, \quad C_{n+1,2} = \begin{bmatrix} 0 & 0 & n+1 \end{bmatrix},$$
$$C_{n+1,3} = \begin{bmatrix} 0 & 0 & n+2 \end{bmatrix}, \quad C_{n+2,3} = \begin{bmatrix} 0 & 0 & n+3 \end{bmatrix}.$$

By directly verifying, we see that the matrix pair $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}\right)$ has hidden redundancy, while the pair $(B_{n,2}, C_{n+1,3})$ does not. Due to Lemma 9 we choose $S_n^{(2)} = [\;]$, $Z_n^{(2)} = 1$, $Z_n^{(4)} = -1$, $Z_n^{(5)} = -1$. Notice that the fact $Z_n^{(5)}$ is non-empty leads to the appearance of $f_3(n+2)$. Furthermore, the resulting system (2.12) reads

$$\begin{bmatrix} 0 & \alpha & n{+}2 \\ 1 & n & 1 \\ 0 & 0 & 0 \end{bmatrix} x(n{+}1) + \begin{bmatrix} 0 & n{+}1 & 0 \\ 0 & 0 & n \\ 0 & 0 & n{+}1 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) - f_2(n{+}1) - f_3(n{+}2) \\ f_2(n) \\ f_3(n) \end{bmatrix}. \tag{2.15}$$

Here the matrix coefficient associated with $x(n+2)$ becomes zero, so for notational convenience we do not write this term. Go back to Step 3, we see that two following cases may happen.

i) If $\alpha \neq 0$, then Algorithm 1 terminates here, and the strangeness-index is $\mu = 2$, which is exactly the number of time-shift appear in the inhomogeneity f in the strangeness-free formulation (2.15).

ii) If $\alpha = 0$, then the matrix pair $\left(\begin{bmatrix} 0 & \alpha & n{+}2 \\ 1 & n & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & n+2 \end{bmatrix}\right)$ have hidden redundancy. Due to Lemma 9 we choose $S_n^{(1)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Z_n^{(1)} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $Z_n^{(2)} = -\begin{bmatrix} 0 & 1 \end{bmatrix}$.

The resulting system (2.12) now reads

$$\begin{bmatrix} 1 & n & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & 0 & n \\ \hline 0 & n+1 & 0 \\ 0 & 0 & n+1 \end{bmatrix} x(n)$$

$$= \begin{bmatrix} f_2(n) \\ \hline f_1(n) - f_2(n+1) - f_3(n+2) - f_3(n+1) \\ f_3(n) \end{bmatrix}. \tag{2.16}$$

Algorithm 1 terminates here, and the strangeness-index is $\mu = 3$, which is bigger than the number of time-shift appear in the inhomogeneity f in the strangeness-free formulation (2.16).

A direct consequence of Theorem 2 is, that we can deduce the theoretical solvability for (1.2) as follows.

Corollary 1 *Under the assumption of Theorem 2, the following statements hold true.*

i) The corresponding IVP for the SiDE (1.2) *is solvable if and only if either $v^{\mu} = 0$ or $\hat{g}_4(n) = 0$ for all $n \geqslant n_0$. Furthermore, it is uniquely solvable if, in addition, we have $r_2^{\mu} + r_1^{\mu} + r_0^{\mu} = d$.*

ii) The initial condition (1.3) *is consistent if and only if the following equalities hold.*

$$\begin{aligned} \hat{B}_{n_0,2}x_1 + \hat{C}_{n_0,2}x_0 &= \hat{g}_2(n_0), \\ \hat{C}_{n_0,3}x_0 &= \hat{g}_3(n_0). \end{aligned}$$

Another direct consequence of Theorem 2 is, that we can obtain an underlying difference equation as follows.

Corollary 2 *Assume that the IVP* (1.2)-(1.3) *is uniquely solvable for any consistent initial condition. Under the assumption of Theorem 2, the solution x to this IVP is also a solution to the* (implicit) underlying difference equation

$$\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix} x(n+2) + \begin{bmatrix} \hat{B}_{n,1} \\ \hat{C}_{n+1,2} \\ 0 \end{bmatrix} x(n+1) + \begin{bmatrix} \hat{C}_{n,1} \\ 0 \\ 0 \end{bmatrix} x(n) = \begin{bmatrix} \hat{g}_1(n) \\ \hat{g}_2(n+1) \\ \hat{g}_3(n+2) \end{bmatrix}, \tag{2.17}$$

where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ *is invertible for all $n \geqslant n_0$.*

Remark 6 Unlike in [14, 18], we do not change the variable x. This trick permits us to simplify significantly the condensed forms in [2, 18]. We emphasize that as in (2.5), we only require five constant rank conditions within one step of index reduction, instead of seven as in [18]. Therefore, this trick will enlarge the domain of application for SiDEs (and also for DAEs, in the continuous time case). This trick is also useful for the control analysis of the descriptor system (1.1), as will be seen later.

Remark 7 i) Within one loop of Algorithm 1, for each n, we have used 4 SVDs to remove the hidden redundancies in two matrix pairs. The total cost depends on the problems itself, i. e., depending on sizes of the matrix pairs which applied SVDs. Nevertheless, it would not exceed $\mathcal{O}(m^2 d^2)$.
ii) Unfortunately, since $Z_n^{(3)}$, $Z_n^{(4)}$,$Z_n^{(5)}$ are not orthogonal, in general Algorithm 1 could not be stably implemented. For the numerical solution to the IVP (1.2)-(1.3), we will consider a suitable numerical scheme in Section 4.
iii) Furthermore, similar to the case of continuous time systems, the strangeness index μ constructed here only gives an upper bound for the number of shift-forward operator that have been used, in order to achieve the strangeness-free form (2.13). For further details, see Remark 17, [18]. To overcome this obstacle, another approach will be presented in Section 4.

3 Strangeness-index of second order descriptor systems

Based on the index reduction procedure for SiDEs in Section 2, in this section we construct the strangeness-index concept for the descriptor system (1.1). The solvability analysis for first order descriptor systems with variable coefficients have been carefully discussed in [3, 12, 20]. Nevertheless, for second order descriptor systems, this problem has been rarely considered. We refer the interested readers to [14, 23] for continuous time systems.

It is well known, that in regularization procedures of continuous time systems, one should avoid differentiating equations that involve an input function, due to the fact that it may not be differentiable. Here, we will also keep this spirit, and hence, will not shift any equation that involve an input function, since it may destroy the causality of the considered system. In the following lemma, we give the condensed form for system (1.1).

Lemma 11 *Consider the descriptor system* (1.1). *Then, there exist two pointwise nonsingular matrix sequences $\{U_n\}_{n\geqslant n_0}$, $\{V_n\}_{n\geqslant n_0}$ such that the following identities hold.*

$$
\begin{array}{l}
\left(U_n \begin{bmatrix} A_n & B_n & C_n \end{bmatrix},\ U_n D_n V_n\right) \\
= \left(\left[\begin{array}{ccc} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ \hline 0 & B_{n,4} & C_{n,4} \\ 0 & 0 & C_{n,5} \\ 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc} D_{n,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \\ 0 & 0 & 0 \end{array}\right] \right), \quad \begin{array}{c} r_{2,n} \\ r_{1,n} \\ r_{0,n} \\ \hline \varphi_{1,n} \\ \varphi_{0,n} \\ v_n \end{array} \quad \textit{for all } n \geqslant n_0. \qquad (3.1)
\end{array}
$$

Here sizes of the block rows are $r_{2,n}$, $r_{1,n}$, $r_{0,n}$, $\varphi_{1,n}$, $\varphi_{0,n}$, v_n, the matrices $A_{n,1}$, $B_{n,2}$, $B_{n,4}$, $C_{n,3}$ are of full row rank and the matrices $\Sigma_{\varphi,1}$, $\Sigma_{\varphi,0}$ are nonsingular and diagonal.

Proof. First we apply Lemma 5 to four matrices A_n, B_n, C_n and D_n to obtain the matrix U_n that satisfies (1.6). Decompose the matrix $\begin{bmatrix} \check{D}_4 \\ \check{D}_5 \end{bmatrix}$ via one SVD, we then obtain the block $\begin{bmatrix} 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \end{bmatrix}$. Finally, we use Gaussian elimination to cancel out all matrices on the two columns of $\check{D}$ that contain $\Sigma_{\varphi,1}$ and $\Sigma_{\varphi,0}$, and hence, we obtain the desired form (3.1). □

In order to build an index reduction procedure for (1.1), we also need the following assumption.

Assumption 3. *Assume that the local characteristic invariants $r_{2,n}$, $r_{1,n}$, $r_{0,n}$, $\varphi_{1,n}$, $\varphi_{0,n}$, v_n, become global, i.e., they are constant for all $n \geqslant n_0$.*

Make use of Lemma 11, we can transform the descriptor system (1.1) to the following system

$$\begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \left[\begin{array}{ccc} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ \hline 0 & B_{n,4} & C_{n,4} \\ 0 & 0 & C_{n,5} \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \left[\begin{array}{ccc} D_{n,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix} = \tilde{f}(n), \quad (3.2)$$

where $u(n) = V_n v(n)$, $v(n) := \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix}$, $\tilde{f}(n) := U_n f(n)$, for all $n \geqslant n_0$.

Moreover, we notice that the third and fourth block rows, whose sizes are φ_1 and φ_0, are related to the feedback regularization of (1.1), as shown in the following proposition.

Proposition 1 *i) Assume that for each $n \geqslant n_0$, the matrix $\begin{bmatrix} A_{n,1} \\ B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}$ is of full row rank. Then, there exist two matrices sequences $\{F_{n,1}\}_{n \geqslant n_0}$, $\{F_{n,0}\}_{n \geqslant n_0}$ which take values $\mathbb{R}^{m,d}$ such that the following matrix has full row rank*

$$\left[\begin{array}{c} A_{n,1} \\ B_{n+1,2} \\ C_{n+2,3} \\ \hline B_{n+1,4} + \begin{bmatrix} 0 & \Sigma_{\varphi,1} & 0 \end{bmatrix} F_{n+1,1} \\ C_{n+2,5} + \begin{bmatrix} 0 & 0 & \Sigma_{\varphi,0} \end{bmatrix} F_{n+2,0} \end{array}\right].$$

ii) Consequently, if the upper part of (3.2) *is strangeness-free then there exists a first order feedback of the form*

$$v(n) = F_{n,1} x(n+1) + F_{n,0} x(n), \ \text{for all } n \geqslant n_0, \quad (3.3)$$

such that the closed loop system

$$A_n x(n+2) + (B_n + D_n F_{n,1})\, x(n+1) + (C_n + D_n F_{n,0})\, x(n) = f(n),$$

is strangeness-free.

Proof. Since the part ii) is a direct consequence of part i), we only need to prove i). The part i) is directly followed by applying Lemma 6 for $P = \begin{bmatrix} A_{n,1} \\ B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}$, $Q = \begin{bmatrix} 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \end{bmatrix}$ and $G = \begin{bmatrix} B_{n+1,4} \\ C_{n+2,5} \end{bmatrix}$. □

From Proposition 1, we see that we only need to remove the hidden redundancies in the upper part of (3.2) as follows. By performing one index reduction step for the upper part of (3.2), as in Section 2, we obtain the following lemma.

Lemma 12 *Assume that the upper part of the descriptor system* (3.2) *is not strangeness-free. Then, for each input sequence* $\{v(n)\}_{n\geqslant n_0}$, *it has exactly the same solution set as the following system*

$$\begin{array}{c} \tilde{r}_2 \\ \tilde{r}_1 \\ \tilde{r}_0 \\ \hline \varphi_1 \\ \varphi_0 \\ \tilde{v} \end{array} \begin{bmatrix} \tilde{A}_{n,1} & \tilde{B}_{n,1} & \tilde{C}_{n,1} \\ 0 & \tilde{B}_{n,2} & \tilde{C}_{n,2} \\ 0 & 0 & \tilde{C}_{n,3} \\ \hline 0 & B_{n,4} & C_{n,4} \\ 0 & 0 & C_{n,5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \tilde{D}_{n,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix} = \tilde{f}(n), \quad (3.4)$$

where $\tilde{r}_2 = r_2 - s_2$, $\tilde{r}_0 = r_0 + s_0$, $\tilde{v} \geqslant v$, *for some* $s_2 > 0$, $s_1 > 0$.

Proof. System (3.4) is directly obtained by applying Lemma 10 to the upper part of (3.2). To keep the brevity of this paper, we will omit the details here. □

Similar to the observation made in Section 2, here we also see, that an index reduction step, which passes system (3.2) to the new form (3.4) has reduced the upper rank r^u by at least $s_2 + s_1$. Continue in this way, finally we obtain the strangeness-free descriptor system in the next theorem.

Theorem 4 *Consider the descriptor system* (1.1). *Furthermore, assume that Assumption 3 is fulfilled whenever needed. Then, for each fixed input sequence* $\{u(n)\}_{n\geqslant n_0}$, *system* (1.1) *has the same solution set as the so-called* strangeness-free descriptor system

$$\begin{array}{c} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hline \hat{\varphi}_1 \\ \hat{\varphi}_0 \\ \hat{v} \end{array} \begin{bmatrix} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ \hline 0 & \hat{B}_{n,5} & \hat{C}_{n,5} \\ 0 & 0 & \hat{C}_{n,6} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \\ \hline \hat{D}_{n,4} \\ \hat{D}_{n,5} \\ 0 \end{bmatrix} u(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hline \hat{f}_4(n) \\ \hat{f}_5(n) \\ \hat{f}_6(n) \end{bmatrix}, \quad \textit{for all } n \geqslant n_0, \quad (3.5)$$

where the matrices $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$, $\begin{bmatrix} \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix}$ *have full row rank for all* $n \geqslant n_0$.

Proof. By repeating index reduction steps until the upper rank r^u stop decreasing, we obtain the system

$$\begin{array}{c} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hline \hat{\varphi}_1 \\ \hat{\varphi}_0 \\ \hat{v} \end{array} \begin{bmatrix} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ \hline 0 & \hat{B}_{n,5} & \hat{C}_{n,5} \\ 0 & 0 & \hat{C}_{n,6} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\hat{\varphi}_1} & 0 \\ 0 & 0 & \Sigma_{\hat{\varphi}_0} \\ 0 & 0 & 0 \end{bmatrix} v(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hline \hat{f}_4(n) \\ \hat{f}_5(n) \\ \hat{f}_6(n) \end{bmatrix},$$

for all $n \geqslant n_0$, where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ has full row rank for all $n \geqslant n_0$. Here the new input sequence $\{v(n)\}_{n \geqslant n_0}$ satisfies $u(n) = V_n v(n)$, V_n is nonsingular for all $n \geqslant n_0$. Transform back $v(n) = V_n^{-1} u(n)$, and set

$$\begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \\ \hline \hat{D}_{n,4} \\ \hat{D}_{n,5} \\ 0 \end{bmatrix} := \begin{bmatrix} \hat{D}_{n,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\hat{\varphi}_1} & 0 \\ 0 & 0 & \Sigma_{\hat{\varphi}_0} \\ 0 & 0 & 0 \end{bmatrix} V^{-1},$$

we obtain exactly the strangeness-free descriptor system (3.5). □

As a direct corollary of Theorem 4, we obtain the existence and uniqueness of a solution to the closed-loop system via feedback as follows.

Corollary 3 *Under the conditions of Theorem 4, the following statements hold true.*
i) There exists a first order feedback of the form (3.3) *such that the closed-loop system is solvable if and only if either* $\hat{v} = 0$ *or* $\hat{f}_6(n) = 0$ *for all* $n \geqslant n_0$.
ii) Furthermore, the solution to the corresponding IVP (of the closed-loop system) is unique if and only if in addition, $d = \sum_{i=0}^{2} \hat{r}_i + \sum_{i=0}^{1} \hat{\varphi}_i$.

Remark 8 It should be noted that, in analogous to SiDEs, each index reduction step of the descriptor system (1.1) also makes use of Lemma 10, where the matrices $Z_n^{(i)}$, $i = 3, 4, 5$, may not be orthogonal. Furthermore, in Lemma 11, two matrices U_n, V_n are only nonsingular but not orthogonal. Therefore, in general, the strangeness-free formulation (3.5) could not be stably computed. For the numerical treatment of (continuous time) second order DAEs, in [23] a different approach was developed. We will modify it for SiDEs/descriptor systems in the next section.

Remark 9 Another interesting method while considering descriptor systems is the *behavior approach*, where we combine both the state x and input u in one behavior vector. Then, (1.1) will become a SiDE of this behavior variable, and hence, we can apply the results in Section 2 for this system. However, to keep the brevity of this research, we will not present the details here. For the interested readers, we refer to [12, 20, 21] for the case of first order DAEs, and [23] for the case of second order DAEs.

4 Difference arrays of second-order SiDEs/descriptor systems

As have shown in two previous sections, to analyze the theoretical solvability of the SiDE (1.2) or of the descriptor system (1.1), first one needs to bring it to a strangeness-free formulation. Nevertheless, this task is not always doable, for example when Assumptions 1, 3 are violated at some index reduction steps. These difficulties have also been observed for continuous time systems of both first and higher orders, and they have been addressed in [12, 23]. The basic idea, thanks to Campbell [4], while considering DAEs, is to differentiate a given system a number of times and put every one of them, including the original one, into a so-called *inflated system*. Then, the strangeness-free formulation will be determined by appropriate selection of equations inside this inflated system. In this section we will examine this approach to the descriptor system (1.1). The analysis for SiDEs of the form (1.2) can be obtained by simply setting D_n to be $0^{m,p}$ for all n. We further assume the following condition.

Assumption 5. *Consider the descriptor system* (1.1)*. Assume that there exists a first order feedback of the form* (3.3) *such that the corresponding IVP of the closed-loop system is uniquely solvable.*

Notice that, in case of the SiDE (1.2), Assumption 5 means that the IVP (1.2)-(1.3) is uniquely solvable. Now let us introduce the *difference-inflated system of level* $\ell \in \mathbb{N}$ as follows.

$$\begin{aligned}
A_n x(n+2) + B_n x(n+1) + C_n x(n) + D_n u(n) &= f(n), \\
A_{n+1} x(n+3) + B_{n+1} x(n+2) + C_{n+1} x(n+1) + D_{n+1} u(n+1) &= f(n+1), \\
&\cdots \\
A_{n+\ell} x(n+\ell+2) + B_{n+\ell} x(n+\ell+1) + C_{n+\ell} x(n+\ell) + D_{n+\ell} u(n+\ell) &= f(n+\ell)\ .
\end{aligned}$$

We rewrite this system as

$$\underbrace{\begin{bmatrix} C_n & B_n & A_n & & & \\ & C_{n+1} & B_{n+1} & A_{n+1} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & C_{n+\ell} & B_{n+\ell} & A_{n+\ell} \end{bmatrix}}_{=:\mathcal{M}} \underbrace{\begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \\ \vdots \\ x(n+\ell) \end{bmatrix}}_{=:\mathcal{X}} +$$

$$+ \underbrace{\begin{bmatrix} D_n & & & \\ & D_{n+1} & & \\ & & \ddots & \\ & & & D_{n+\ell} \end{bmatrix}}_{=:\mathcal{N}} \underbrace{\begin{bmatrix} u(n) \\ u(n+1) \\ \vdots \\ u(n+\ell) \end{bmatrix}}_{=:\mathcal{U}} = \underbrace{\begin{bmatrix} f(n) \\ f(n+1) \\ \vdots \\ f(n+\ell) \end{bmatrix}}_{=:\mathcal{G}}, \quad \text{for all } n \geqslant n_0. \tag{4.1}$$

Definition 3 Suppose that the descriptor system (1.1) satisfies Assumption 5. The minimum number ℓ such that by using elementary matrix's row operations, a strangeness-free descriptor system of the form (3.5) can be extracted from (4.1) is called the *shift-index* of (1.1), and be denoted by ν.

We give the relation between this shift-index ν and the strangeness-index μ in the following proposition.

Proposition 2 *Suppose that the descriptor system* (1.1) *satisfies Assumption 5. If the strangeness-index μ is well-defined, then so is the shift-index ν. Furthermore, we have that $\nu \leqslant \mu$.*

Proof. The first claim is straight forward, since every reformulation step performed in Algorithm 1 is a consequence of an inflated system (4.1) with $\ell = \mu$. □

Remark 10 As will be seen later in Example 3, for second order SiDEs, the shift index can be strictly smaller than the strangeness index.

Assume that ν is already known, now we construct an algorithm to select the strangeness-free descriptor system (3.5) from the inflated system (4.1). For notational convenience, we will follow the Matlab language, [16]. Consider the following spaces and matrices

$$\begin{aligned} &\mathcal{W} := \left[\mathcal{M}(:, 3n+1 : end) \quad \mathcal{N}(:, n+1 : end)\right], \\ &U_1 \text{ basis of } \operatorname{kernel}(\mathcal{W}^T), \text{ and } U_{1,\perp} \text{ basis of } \operatorname{range}(\mathcal{W}), \end{aligned} \tag{4.2}$$

due to Lemma 4 we have that $U_1^T \mathcal{W} = 0$ and $U_{1,\perp}^T \mathcal{W}$ has full row rank. Furthermore, the matrix $\begin{bmatrix} U_1^T \\ U_{1,\perp}^T \end{bmatrix}$ is nonsingular, and hence system (4.1) is equivalent to the system below.

$$U_1^T \mathcal{M}(:, 1:3n) \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \end{bmatrix} + U_1^T \mathcal{N}(:, 1:n) u(n) = U_1^T \mathcal{G}, \tag{4.3}$$

$$U_{1,\perp}^T \mathcal{W} \begin{bmatrix} x(n+3) \\ \vdots \\ x(n+\nu) \\ \hline u(n+1) \\ \vdots \\ u(n+\nu) \end{bmatrix} + U_{1,\perp}^T \left[\mathcal{M}(:, 1:3n) \;\; \mathcal{N}(:, 1:n)\right] \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \\ \hline u(n) \end{bmatrix} = U_{1,\perp}^T \mathcal{G}. \tag{4.4}$$

Notice that due to the full row rank property of $U_{1,\perp}^T \mathcal{W}$, (4.4) plays no role in the determination of the strangeness-free descriptor system (3.5). Thus, (3.5) is a consequence of (4.3). In the following proposition we show that system (4.3) is not affected by left equivalence transformation.

Proposition 3 *Consider two left equivalent systems. Then, at the same level ℓ, their difference-inflated systems of the form* (4.1) *are also left equivalent. Consequently, system* (4.3) *is not affected by left equivalence transformation.*

Proof. Let us assume that (1.1) is left equivalent to the SiDE

$$\tilde{A}_n x(n+2) + \tilde{B}_n x(n+1) + \tilde{C}_n x(n) + \tilde{D}_n u(n) = \tilde{f}(n), \text{ for all } n \geqslant n_0. \tag{4.5}$$

Thus, there exists a pointwise nonsingular matrix sequence $\{P_n\}_{n \geqslant n_0}$ such that

$$\left[\tilde{A}_n \; \tilde{B}_n \; \tilde{C}_n \; \tilde{D}_n\right] = P_n \left[A_n \; B_n \; C_n \; D_n\right] \text{ and } \tilde{f}(n) = P_n f(n), \text{ for all } n \geqslant n_0.$$

Therefore, the difference-inflated system of level ℓ for system (4.5) takes the form

$$\tilde{\mathcal{M}} \mathcal{X} + \tilde{\mathcal{N}} \mathcal{U} = \tilde{\mathcal{G}}, \tag{4.6}$$

where the matrix coefficients are

$$\tilde{\mathcal{M}} = \operatorname{diag}(P_n, ..., P_{n+\ell}) \; \mathcal{M}, \; \tilde{\mathcal{N}} = \operatorname{diag}(P_n, ..., P_{n+\ell}) \; \mathcal{N}, \; \tilde{\mathcal{G}} = \operatorname{diag}(P_n, ..., P_{n+\ell}) \mathcal{G}.$$

This follows that two systems (4.1) and (4.6) are left equivalent, which finishes the proof. □

For notational convenience, let us rewrite system (4.3) as

$$\left[\check{A}\;\;\check{B}\;\;\check{C}\,\middle|\,\check{D}\right]\begin{bmatrix} x(n+2)\\ x(n+1)\\ x(n)\\ \hline u(n)\end{bmatrix} = \check{G}.$$

Scale this system with the matrix $\check{U}$ obtained in Lemma 5, we have

$$\left[\begin{array}{ccc|c} \check{A}_1 & \check{B}_1 & \check{C}_1 & \check{D}_1\\ 0 & \check{B}_2 & \check{C}_2 & 0\\ 0 & 0 & \check{C}_3 & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & \check{B}_4 & \check{C}_4 & \check{D}_4\\ 0 & 0 & \check{C}_5 & \check{D}_5 \end{array}\right]\begin{bmatrix} x(n+2)\\ x(n+1)\\ x(n)\\ \hline u(n)\end{bmatrix} = \begin{bmatrix} \check{G}_1\\ \check{G}_2\\ \check{G}_3\\ 0\\ \hline \check{G}_4\\ \check{G}_5\end{bmatrix}. \tag{4.7}$$

Here the matrices $\check{A}_1$, $\check{B}_2$, $\check{B}_4$, $\check{C}_3$, and $\begin{bmatrix}\check{D}_4\\ \check{D}_5\end{bmatrix}$ have full row rank. Notice that the presence of the 0 block on the right hand side vector is due to Assumption 5. In the following theorem we answer the question how to derive the strangeness-free formulation (3.5) from (4.7).

Theorem 6 *Assume that the shift index ν of the descriptor system* (1.1) *is well-defined. Furthermore, suppose that* (1.1) *satisfies Assumption 5. Then, any solution to the descriptor system* (1.1) *is also a solution to the following system*

$$\begin{array}{c} \hat{r}_2\\ \hat{r}_1\\ \hat{r}_0\\ \hline \hat{\varphi}_1\\ \hat{\varphi}_0 \end{array}\left[\begin{array}{ccc} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1}\\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2}\\ 0 & 0 & \hat{C}_{n,3}\\ \hline 0 & \hat{B}_{n,5} & \hat{C}_{n,5}\\ 0 & 0 & \hat{C}_{n,6} \end{array}\right]\begin{bmatrix} x(n+2)\\ x(n+1)\\ x(n)\end{bmatrix} + \begin{bmatrix} \hat{D}_{n,1}\\ 0\\ 0\\ \hline \hat{D}_{n,4}\\ \hat{D}_{n,5}\end{bmatrix} u(n) = \begin{bmatrix} \hat{G}_{n,1}\\ \hat{G}_{n,2}\\ \hat{G}_{n,3}\\ \hline \hat{G}_{n,4}\\ \hat{G}_{n,5}\end{bmatrix}, \text{ for all } n \geqslant n_0, \tag{4.8}$$

where the matrices $\begin{bmatrix}\hat{A}_{n,1}\\ \hat{B}_{n+1,2}\\ \hat{C}_{n+2,3}\end{bmatrix}$, $\begin{bmatrix}\hat{D}_{n,4}\\ \hat{D}_{n,5}\end{bmatrix}$ *have full row rank for all $n \geqslant n_0$. Furthermore, $\sum_{i=0}^{2} \hat{r}_i + \sum_{i=0}^{1} \hat{\varphi}_i = d$, or equivalently,*

$$\operatorname{rank}\left(\begin{bmatrix}\hat{A}_{n,1}\\ \hat{B}_{n+1,2}\\ \hat{C}_{n+2,3}\end{bmatrix}\right) + \operatorname{rank}\left(\begin{bmatrix}\hat{D}_{n,4}\\ \hat{D}_{n,5}\end{bmatrix}\right) = d\;. \tag{4.9}$$

Proof. First we will extract the first two block row equations of system (4.8) from (4.7), by suitably removing the existence hidden redundancy. Applying Lemma 4 consecutively for two following matrix pairs $\left(\check{B}_2, \check{C}_3\right)$, $\left(\check{A}_1, \begin{bmatrix}\check{B}_2\\ \check{C}_3\end{bmatrix}\right)$, we obtain two orthogonal matrices $\begin{bmatrix} S_n^{(i)}\\ Z_n^{(i)}\end{bmatrix} \in \mathbb{R}^{r_i, r_i}$, $i = 1, 2$ such that both pairs

$\left(S_n^{(1)}\breve{B}_2, \breve{C}_3\right)$, $\left(S_n^{(2)}\breve{A}_1, \begin{bmatrix}\breve{B}_2\\ \breve{C}_3\end{bmatrix}\right)$ have no hidden redundancy. Scale the first and second block row equations of (4.7) with $S_n^{(2)}$ and $S_n^{(1)}$ respectively, we obtain

$$\left[\begin{array}{ccc|c} S_n^{(2)}\breve{A}_1 & S_n^{(2)}\breve{B}_1 & S_n^{(2)}\breve{C}_1 & S_n^{(2)}\breve{D}_1 \\ 0 & S_n^{(1)}\breve{B}_2 & S_n^{(1)}\breve{C}_2 & 0 \end{array}\right] \left[\begin{array}{c} x(n+2) \\ x(n+1) \\ x(n) \\ \hline u(n) \end{array}\right] = \begin{bmatrix} S_n^{(2)}\breve{G}_1 \\ S_n^{(1)}\breve{G}_2 \end{bmatrix}.$$

Combining these equations with the third, fifth and sixth block equations of (4.7), we obtain the system

$$\left[\begin{array}{ccc|c} S_n^{(2)}\breve{A}_1 & S_n^{(2)}\breve{B}_1 & S_n^{(2)}\breve{C}_1 & S_n^{(2)}\breve{D}_1 \\ 0 & S_n^{(1)}\breve{B}_2 & S_n^{(1)}\breve{C}_2 & 0 \\ 0 & 0 & \breve{C}_3 & 0 \\ \hline 0 & \breve{B}_4 & \breve{C}_4 & \breve{D}_4 \\ 0 & 0 & \breve{C}_5 & \breve{D}_5 \end{array}\right] \left[\begin{array}{c} x(n+2) \\ x(n+1) \\ x(n) \\ \hline u(n) \end{array}\right] = \left[\begin{array}{c} S_n^{(2)}\breve{G}_1 \\ S_n^{(1)}\breve{G}_2 \\ \breve{G}_3 \\ \hline \breve{G}_4 \\ \breve{G}_5 \end{array}\right]. \tag{4.10}$$

which is exactly our desired system (4.8). Moreover, due to Lemma 2, the matrix $\begin{bmatrix} S_n^{(2)}\breve{A}_1 \\ S_n^{(1)}\breve{B}_2 \\ \breve{C}_3 \end{bmatrix}$ has full row rank. Finally, the identity (4.9) holds true due to Assumption 5. □

We summarize our result in the following algorithm.

Algorithm 2 Strangeness-free formulation for SiDEs using difference arrays

1: **Input:** The SiDE (1.1).
2: **Return:** The strangeness-free descriptor system (4.8).
3: Set $\ell := 0$.
4: Construct the difference-inflated system of level ℓ, and rewrite it in the form (4.1).
5: Find U_1 as in (4.2) and construct system (4.3).
6: Find $\breve{U}$ as in Lemma 5 and construct system (4.7).
7: Find the matrices $S_n^{(1)}$, $S_n^{(2)}$ in the process used to remove the hidden redundancies in two matrix pairs $\left(\breve{B}_2,\ \breve{C}_3\right)$, $\left(\breve{A}_1, \begin{bmatrix}\breve{B}_2\\ \breve{C}_3\end{bmatrix}\right)$, respectively.
8: Construct the system (4.10).
9: **if** rank $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ + rank $\begin{bmatrix} \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix} = d$ **then** STOP.
10: **else** set $\ell := \ell + 1$ and go to 4
11: **end if**

In order to illustrate Algorithm 2, we consider two following examples.

Example 3 Let us revisit system (2.14) for the case $\alpha = 0$. In this system, $D_n = 0$ for all $n \geqslant 0$. For $\ell = 2$, the inflated system (4.1) reads

$$\left[\begin{array}{ccc|cc} C_n & B_n & A_n & 0 & 0 \\ 0 & C_{n+1} & B_{n+1} & A_{n+1} & 0 \\ 0 & 0 & C_{n+2} & B_{n+2} & A_{n+2} \end{array}\right] \left[\begin{array}{c} x(n) \\ x(n+1) \\ x(n+2) \\ \hline x(n+3) \\ x(n+4) \end{array}\right] = \begin{bmatrix} f(n) \\ f(n+1) \\ f(n+2) \end{bmatrix} \tag{4.11}$$

Let U_1 be the basis of kernel$(\mathcal{W}^T)$, where $\mathcal{W} = \begin{bmatrix} 0 & 0 \\ A_{n+1} & 0 \\ B_{n+2} & A_{n+2} \end{bmatrix}$. We then compute system (4.3) by scaling (4.11) with U_1^T. The resulting system reads

$$U_1^T \begin{bmatrix} C_n & B_n & A_n \\ 0 & C_{n+1} & B_{n+1} \\ 0 & 0 & C_{n+2} \end{bmatrix} \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \end{bmatrix} = U_1^T \begin{bmatrix} f(n) \\ f(n+1) \\ f(n+2) \end{bmatrix} . \tag{4.12}$$

Finally, by performing Steps 6 to 10 we can extract the strangeness-free form (2.16) from (4.12). Thus, we conclude that the shift index is $\nu = 2$.

Example 4 Our consider system, which describes a three link robot arm [8], is of the form

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} K_0 & H_0^T \\ H_0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(t).$$

Here M_0 represents the nonsingular mass matrix, G_0 the coefficient matrix associated with damping, centrifugal, gravity, and Coriolis forces, K_0 the stiffness matrix, and H_0 the constraint. A simple discretized version of this system takes the form

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \frac{x(n+2) - 2x(n+1) + x(n)}{h^2} + \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix} \frac{x(n+2) - x(n+1)}{h}$$
$$+ \begin{bmatrix} K_0 & H_0^T \\ H_0 & 0 \end{bmatrix} x(n) = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(n).$$

where h is the discretized stepsize.

As a simple example, let us take $M_0 = G_0 = K_0 = H_0 = B_0 = 1$, $h = 0.01$. Then, Algorithm 2 terminates after two steps and hence, the shift index is $\nu = 2$ for all $n \geqslant n_0$. Furthermore, we notice that no matter forward or backward approximations has been chosen for discretizing the derivative $\dot{x}(t)$, the shift index remains unchanged $\nu = 2$. Nevertheless, the resulting strangeness-free descriptor systems are different.

5 Conclusion

By using the algebraic approach, we have analyzed the solvability analysis of second order SiDEs/descriptor systems, based on derived condensed forms constructed under certain constant rank assumptions. In comparison to well-known results [18, 22], we have reduce the number of constant rank conditions in every index reduction step from seven to five. This would enlarge the domain of application for SiDEs (and also for DAEs). However, requiring constant rank assumptions in the discrete-time case seems less nature than in the continuous-time case. To overcome this limitation, we also consider the difference-array method, which is numerically applicable. We also notice that the backward time case ($n \leqslant n_0$) can be directly extended from the forward time case, as it has been done in [2]. The analysis of two way case, which happens while considering boundary value problems for DAEs, have presented many difficulties, is under our research.

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A Proof of Lemma 10

First we prove that any solution to (2.10) is also a solution to (2.12). Notice that, due to Lemma 8, two systems (2.6) and (2.10) have identical solution set. Thus, we only need to prove that (2.10) and (2.12) are equivalent.
Necessity: The main idea here is to apply elementary row transformations to system (2.10) to obtain (2.12). Notice that we use only two elementary block row operations:

i) scaling a block row equation with a nonsingular matrix,

ii) add to one row a linear combinations of another rows.

Firstly, by scaling the first (resp., second) block row equation of (2.10) with a unitary matrix $\begin{bmatrix} S_n^{(2)} \\ Z_n^{(2)} \end{bmatrix}$ (resp., $\begin{bmatrix} S_n^{(1)} \\ Z_n^{(1)} \end{bmatrix}$), we obtain an equivalent system to (2.6), as follows

$$\left[\begin{array}{ccc} S_n^{(2)}A_{n,1} & S_n^{(2)}B_{n,1} & S_n^{(2)}C_{n,1} \\ Z_n^{(2)}A_{n,1} & Z_n^{(2)}B_{n,1} & Z_n^{(2)}C_{n,1} \\ \hline 0 & S_n^{(1)}B_{n,2} & S_n^{(1)}C_{n,2} \\ 0 & Z_n^{(1)}B_{n,2} & Z_n^{(1)}C_{n,2} \\ \hline 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \\ \hline 0 & C_{n+1,3} & 0 \\ B_{n+1,2} & C_{n+1,2} & 0 \\ C_{n+2,3} & 0 & 0 \end{array}\right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \left[\begin{array}{c} S_n^{(2)}f_1(n) \\ Z_n^{(2)}f_1(n) \\ \hline S_n^{(1)}f_2(n) \\ Z_n^{(1)}f_2(n) \\ \hline f_3(n) \\ f_4(n) \\ \hline f_3(n+1) \\ f_2(n+1) \\ f_3(n+2) \end{array}\right], \quad \begin{array}{c} d_2 \\ s_2 \\ \hline d_1 \\ s_1 \\ \hline r_0 \\ v \\ \hline r_0 \\ r_1 \\ r_0 \end{array}. \tag{A.1}$$

By adding the seventh row scaled with $Z_n^{(3)}$ to the fourth row of (A.1) and making use of (2.11a) we obtain the first hidden constraint

$$Z_n^{(1)}C_{n,2}x(n) = Z_n^{(1)}f_2(n) + Z_n^{(3)}f_3(n+1),$$

which is exactly the fourth row of (2.12).

We continue by adding the seventh row scaled with $Z_n^{(4)}$ and the eighth row scaled with $Z_n^{(5)}$ to the second row of (A.1) and making use of (2.11b) to obtain

$$\begin{aligned} &\left(Z_n^{(2)}B_{n,1} + Z_n^{(4)}C_{n+1,2}\right)x(n+1) + Z_n^{(2)}C_{n,1}x(n) \\ &\quad = Z_n^{(2)}f_1(n) + Z_n^{(4)}f_2(n+1) + Z_n^{(5)}f_3(n+2). \end{aligned}$$

This is exactly the second row of (2.12). Therefore, any solution to (2.6) is also a solution to (2.12).

Sufficiency: Let x be an arbitrary solution to (2.12). Thus, x is also a solution to the shifted system

$$\begin{array}{c} d_2 \\ s_2 \\ \hline d_1 \\ s_1 \\ \hline r_0 \\ v \\ \hline r_0 \\ r_0 \end{array} \left[\begin{array}{ccc} S_n^{(2)}A_{n,1} & S_n^{(2)}B_{n,1} & S_n^{(2)}C_{n,1} \\ 0 & Z_n^{(2)}B_{n,1} + Z_n^{(4)}C_{n+1,2} & Z_n^{(2)}C_{n,1} \\ \hline 0 & S_n^{(1)}B_{n,2} & S_n^{(1)}C_{n,2} \\ 0 & 0 & Z_n^{(1)}C_{n,2} \\ \hline 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \\ \hline 0 & C_{n+1,3} & 0 \\ C_{n+2,3} & 0 & 0 \end{array}\right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix}$$

$$= \left[\begin{array}{c} S_n^{(2)}f_1(n) \\ Z_n^{(2)}f_1(n) + Z_n^{(4)}f_2(n+1) + Z_n^{(5)}f_3(n+2) \\ \hline S_n^{(1)}f_2(n) \\ Z_n^{(1)}f_2(n) + Z_n^{(3)}f_3(n+1) \\ \hline f_3(n) \\ f_4(n) \\ \hline f_3(n+1) \\ f_3(n+2) \end{array}\right], \quad \text{for all } n \geqslant n_0. \tag{A.2}$$

Since elementary matrix row operations are reversible, we can reverse the transformations performed in the necessity part. Consequently, we see that any solution to (A.2) is also a solution to (A.1), and hence, this completes the proof.