THE KERNEL OF KAMEKO'S HOMOMORPHISM AND THE PETERSON HIT PROBLEM

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Abstract. Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$ with the degree of each generator x_i being 1, where $\mathbb{F}_2$ denote the prime field of two elements.

The *hit problem* of Frank Peterson asks for a minimal generating set for the polynomial algebra P_k as a module over the mod-2 Steenrod algebra $\mathcal{A}$. Equivalently, we want to find a vector space basis for $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ in each degree.

In this paper, we study a generating set for the kernel of Kameko's homomorphism $\widetilde{Sq}^0_* : \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \longrightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ in a so-called 'generic degree'. By using these results, we explicitly compute the hit problem for $k = 5$ in respective generic degree.

1. Introduction

Denote by $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ the polynomial algebra over the field of two elements, $\mathbb{F}_2$, in k generators $x_1, x_2, \ldots, x_k$, each of degree 1. This algebra arises as the cohomology with coefficients in $\mathbb{F}_2$ of a classifying space of an elementary abelian 2-group of rank k. Therefore, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula $Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f)Sq^{n-i}(g)$, for $f, g \in P_k$ (see Steenrod and Epstein [30]).

We study the *Peterson hit problem* of determining a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. Equivalently, we want to find a vector space basis for the quotient

$$QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k,$$

where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$.

The hit problem was first studied by Peterson [18], Wood [39], Singer [28] and Priddy [23], who showed its relation to several classical problems in the homotopy theory. Then, this problem was investigated by Boardman [1], Bruner, Hà and Hưng [2], Carlisle and Wood [3], Crabb and Hubbuck [4], Hưng and Nam [6, 7], Janfada and Wood [8, 9], Kameko [10, 11, 12], Mothebe [13, 14], Nam [16, 17], Repka and Selick [24], Silverman [25], Silverman and Singer [27], Singer [29], Walker and Wood [34, 35, 36, 37], Wood [40], the present author [31, 33] and others.

The vector space QP_k was explicitly calculated by Peterson [18] for $k = 1, 2$, by Kameko [10] for $k = 3$ and by us [33] and Kameko [12] for $k = 4$. Recently, the hit

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problem and it's applications to representations of general linear groups have been presented in the monographs of Walker and Wood [37, 38].

The μ-function is one of the numerical functions that have much been used in the context of the hit problem. For a nonnegative integer n, $\mu(n)$ is the minimum number of terms of the form $2^d-1, d>0$, with repetitions allowed, whose sum is n. A routine computation shows that $\mu(n)=s$ if and only if there exists a unique sequence of integers $d_1>d_2>\ldots>d_{s-1}\geqslant d_s>0$ such that

$$n=2^{d_1}+2^{d_2}+\ldots+2^{d_{s-1}}+2^{d_s}-s=\sum_{1\leqslant i\leqslant s}(2^{d_i}-1). \tag{1.1}$$

Denote by $(P_k)_n$ the subspace of P_k consisting of the homogeneous polynomials of degree n in P_k and $(QP_k)_n$ the subspace of QP_k consisting of all the classes represented by the elements in $(P_k)_n$.

The following is Peterson's conjecture, which was established by Wood.

Theorem 1.1 (Wood [39]). *If $\mu(n)>k$, then $(QP_k)_n=0$.*

One of the main tools in the study of the hit problem is Kameko's homomorphism $\widetilde{Sq}^0_*: QP_k\to QP_k$. This homomorphism is induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_*: P_k\to P_k$, given by

$$\widetilde{Sq}^0_*(x)=\begin{cases} y, & \text{if } x=x_1x_2\ldots x_ky^2,\\ 0, & \text{otherwise,}\end{cases}$$

for any monomial $x\in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_*Sq^{2t}=Sq^t\widetilde{Sq}^0_*$, and $\widetilde{Sq}^0_*Sq^{2t+1}=0$ for any non-negative integer t.

Theorem 1.2 (Kameko [10]). *Let n be a positive integer. If $\mu(n)=k$, then $(\widetilde{Sq}^0_*)_{(k,n)}:=\widetilde{Sq}^0_*:(QP_k)_n\to(QP_k)_{\frac{n-k}{2}}$ is an isomorphism of the $\mathbb{F}_2$-vector spaces.*

Based on Theorems 1.1 and 1.2, the hit problem is reduced to the case of degree n with $\mu(n)=s<k$.

The hit problem in the case of degree n of the form (1.1) with $s=k-1$, was studied by Crabb and Hubbuck [4], Nam [16], Repka and Selick [24], Walker and Wood [36] and the present author [31, 33].

For $s=k-2$, we studied the kernel of Kameko's homomorphism $(\widetilde{Sq}^0_*)_{(k,n)}$. We give in [31] a prediction for the dimension of $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(k,n)}$ in this case.

Conjecture 1.3 (See Sum [31]). *Let $n=\sum_{i=1}^{k-2}(2^{d_i}-1)$ with d_i positive integers. If $d_{i-2}-d_{i-1}>i$ for $3\leqslant i\leqslant k-1$ and $d_{k-2}>k\geqslant 3$, then*

$$\dim\mathrm{Ker}(\widetilde{Sq}^0_*)_{(k,n)}=\prod_{3\leqslant i\leqslant k}(2^i-1).$$

This conjecture is true for $k\leqslant 4$. Recently, Walker and Wood [38] give a lower bound for the dimension of $(QP_k)_n$.

Theorem 1.4 (Walker and Wood [38]). *Let $k\geqslant 3$ and $n=\sum_{i=1}^{k-2}(2^{d_i}-1)$ with d_i positive integers. If $d_i-d_{i+1}\geqslant 4$ for $1\leqslant i\leqslant k-3$ and $d_{k-2}\geqslant 5$, then*

$$\dim(QP_k)_n\geqslant(k-1)\prod_{3\leqslant i\leqslant k}(2^i-1). \tag{1.2}$$

From our result in [31, Theorem 1.6], we can see that if Conjecture 1.3 is true, then the inequality (1.2) is an equality for n as given in this conjecture.

In this paper, we study some properties of a generating set for the subspace $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(k,n)}$ of $(QP_k)_n$ and explicitly compute the space $(QP_k)_n$ for $k = 5$ and $d_3 > 5$. One of our main results (Theorem 3.3.3) implies an upper bound for the dimension of $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(k,n)}$.

Theorem 1.5. *Let $n = \sum_{i=1}^{k-2}(2^{d_i} - 1)$ with d_i positive integers. If $d_{i-2} - d_{i-1} > i$ for $3 \leqslant i \leqslant k-1$ and $d_{k-2} > \binom{k}{2}$, then*

$$\dim \mathrm{Ker}(\widetilde{Sq}^0_*)_{(k,n)} < 2^{(k-1)^2} \prod_{1\leqslant i\leqslant k}(2^i - 1).$$

Based on the construction of generators for $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(k,n)}$ as given in the proof of Theorem 1.5, we prove the following.

Theorem 1.6. *Let $n = 2^{d+s+t} + 2^{d+s} + 2^d - 3$ with d, s, t non-negative integers. If $d \geqslant 6$, t, $s \geqslant 4$, then*

$$\dim \mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,n)} = (2^3 - 1)(2^4 - 1)(2^5 - 1) = 3255. \tag{1.3}$$

Thus, we obtain the following.

Corollary 1.7. *For $k = 5$, Conjecture 1.3 is true and the inequality* (1.2) *is an equality.*

Based on the proof of Theorem 1.6 and our results in [33, Theoren 1.4], we explicitly compute $(QP_5)_n$.

Theorem 1.8. *Let $n = 2^{d+s+t} + 2^{d+s} + 2^d - 3$, with d, s, t integers such that $d \geqslant 6$, $s \geqslant 0$ and $t > 0$. The dimension of the $\mathbb{F}_2$-vector space $(QP_5)_n$ is given by the following table:*

n	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t \geqslant 6$
$s = 0$	1116	2790	3813	4960	5735	6045
$s = 1$	3410	6231	7285	7719	7595	7595
$s = 2$	5766	9207	10726	11160	11160	11160
$s = 3$	7254	10695	12090	12555	12555	12555
$s \geqslant 4$	7595	11160	12555	13020	13020	13020

Note that the case $s = 0$, $t = 1$ of this theorem have been proved by Phúc [20].

This paper is organized as follows. In Section 2, we recall some needed information on the admissible monomials in P_k and criterions of Singer [29] and Silverman [26] on hit monomials. In Section 3, we present the results for generators of the kernel of Kameko's homomorphism. As an application of the results of Section 3, in Section 4, we explicitly compute the space QP_5 in the degree $n = 2^{d+s+t} + 2^{d+s} + 2^d - 3$ for $d > 5$, $s \geqslant 0$ and $t > 0$.

2. Preliminaries

In this section, we recall some results from Kameko [10], Singer [29] and Silverman [26] which will be used in the next sections.

Notation 2.1. From now on, we use the following notations.

$$\mathbb{N}_k = \{1, 2, \ldots, k\},$$
$$X_{\mathbb{J}} = X_{j_1,j_2,\ldots,j_s} = \prod_{j\in\mathbb{N}_k\setminus\mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k,$$

In particular, we have $X_{\mathbb{N}_k} = 1$, $X_\emptyset = x_1x_2\ldots x_k$, $X_k = x_1x_2\ldots x_{k-1} \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 and $i \geqslant 0$. Denote by $\alpha(a)$ the number of 1's in dyadic expansion of a.

Let $x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \in P_k$. Denote by $\nu_j(x) = a_j, 1 \leqslant j \leqslant k$. Set

$$\mathbb{J}_i(x) = \{j \in \mathbb{N}_k : \alpha_i(\nu_j(x)) = 0\},$$

for $i \geqslant 0$. Then we have

$$x = \prod_{i\geqslant 0} X_{\mathbb{J}_i(x)}^{2^i}.$$

For a polynomial f in P_k, we denote by $[f]$ the class in QP_k represented by f. For a subset $S \subset P_k$, we denote

$$[S] = \{[f] : f \in S\} \subset QP_k.$$

Definition 2.2. A weight vector ω is a sequence of non-negative integers $(\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ such that $\omega_i = 0$ for $i \gg 0$. For a monomial x in P_k, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots),$$
$$\sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)},\ i \geqslant 1$. The sequences $\omega(x)$ and $\sigma(x)$ are respectively called the weight vector and the exponent vector of x.

The sets of the weight vectors and the exponent vectors are given the left lexicographical order.

For weight vectors $\omega = (\omega_1, \omega_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots)$, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$, the length $\ell(\omega) = \max\{i : \omega_i > 0\}$, the concatenation of weight vectors $\omega|\eta = (\omega_1, \ldots, \omega_r, \eta_1, \eta_2, \ldots)$ if $\ell(\omega) = r$ and $(a)|^b = (a)|(a)|\ldots|(a)$, ($b$ times of (a)'s), where a, b are positive integers. Denote by $P_k(\omega)$ the subspace of P_k spanned by monomials y such that $\deg y = \deg\omega$ and $\omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of $P_k(\omega)$ spanned by monomials y such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$, then f is said to be *hit*.

ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+P_k + P_k^-(\omega)$.

iii) $f \simeq_{(s,\omega)} g$ if and only if $f - g \in A_s^+P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$, $\equiv_\omega$ and $\simeq_{(s,\omega)}$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have

$$(QP_k)_n \cong \bigoplus_{\deg\omega=n} QP_k(\omega). \tag{2.1}$$

We recall some elementary properties on the action of the Steenrod squares on P_k.

Proposition 2.4. *Let f be a polynomial in P_k.*
i) *If $i > \deg f$, then $Sq^i(f) = 0$. If $i = \deg f$, then $Sq^i(f) = f^2$.*
ii) *If i is not divisible by 2^s, then $Sq^i(f^{2^s}) = 0$ while $Sq^{r2^s}(f^{2^s}) = (Sq^r(f))^{2^s}$.*

Proposition 2.5 (See [33, Proposition 2.5])**.** *Let x, y be monomials and let f, g be polynomials in P_k such that $\deg x = \deg f$, $\deg y = \deg g$.*
i) *If $\omega_i(x) \leqslant 1$ for $i > s$ and $x \simeq_t f$ with $t \leqslant s$, then $xy^{2^s} \simeq_t fy^{2^s}$.*
ii) *If $\omega_i(x) = 0$ for $i > s$, $x \simeq_s f$ and $y \simeq_r g$, then $xy^{2^s} \simeq_{s+r} fg^{2^s}$.*

Definition 2.6. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds:
i) $\omega(x) < \omega(y)$;
ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.7. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$ for $j = 1, 2, \ldots, t$ and $x - \sum_{j=1}^{t} y_j \in \mathcal{A}^+P_k$.
A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

Definition 2.8. A monomial x is said to be strictly inadmissible if and only if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$, for $j = 1, 2, \ldots, t$ and $x - \sum_{j=1}^{t} y_j \in \mathcal{A}_s^+P_k$ with $s = \max\{i \ : \ \omega_i(x) > 0\}$.

It is easy to see that if x is strictly inadmissible, then it is inadmissible. The following theorem is a modification of a result in [10].

Theorem 2.9 (Kameko [10], Sum [31])**.** *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*
i) *If w is inadmissible, then xw^{2^r} is also inadmissible.*
ii) *If w is strictly inadmissible, then $xw^{2^r}y^{2^{r+s}}$ is also strictly inadmissible.*

Proposition 2.10 (See [31])**.** *Let x be an admissible monomial in P_k. Then we have*
i) *If there is an index i_0 such that $\omega_{i_0}(x) = 0$, then $\omega_i(x) = 0$ for all $i > i_0$.*
ii) *If there is an index i_0 such that $\omega_{i_0}(x) < k$, then $\omega_i(x) < k$ for all $i > i_0$.*

For $1 \leqslant i \leqslant k$, define a homomorphism $f_i : P_{k-1} \to P_k$ of $\mathcal{A}$-algebras by substituting $f_i(x_u) = x_u$ for $1 \leqslant u < i$ and $f_i(x_u) = x_{u+1}$ for $i \leqslant u < k$. For $1 \leqslant i < j \leqslant k$, denote $f_{(i,j)} = f_i f_{j-1} : P_{k-2} \xrightarrow{f_{j-1}} P_{k-1} \xrightarrow{f_i} P_k$.

Proposition 2.11 (See Mothebe and Uys [15])**.** *Let i, d be positive integers such that $1 \leqslant i \leqslant k$. If x is an admissible monomial in P_{k-1} then $x_i^{2^d-1}f_i(x)$ is also an admissible monomial in P_k.*

Now, we recall Singer's criterion on the hit monomials in P_k.

Definition 2.12. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{s_j} - 1$ for s_j a non-negative integer and $j = 1, 2, \ldots, k$. If z is a spike with $s_1 > s_2 > \ldots > s_{r-1} \geqslant s_r > 0$ and $s_j = 0$ for $j > r$, then it is called a minimal spike.

The following is a criterion for the hit monomials in P_k.

Theorem 2.13 (See Singer [29])**.** *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

From this theorem, we see that if z is a minimal spike, then $P_k^-(\omega(z)) \subset \mathcal{A}^+P_k$.

We need Silverman's criterion for the hit polynomials in P_k.

Theorem 2.14 (See Silverman [26, Theorem 1.2]). *Let p be a polynomial of the form fg^{2^m} for some homogeneous polynomials f and g. If $\deg f < (2^m-1)\mu(\deg g)$, then p is hit.*

This result leads to a criterion in terms of minimal spike which is a stronger version of Theorem 2.13.

Theorem 2.15 (See Walker-Wood [37, Theorem 14.1.3]). *Let $x \in P_k$ be a monomial of degree n, where $\mu(n) \leqslant k$ and let z be the minimal spike of degree n. If there is an index h such that $\sum_{i=1}^h 2^{i-1}\omega_i(x) < \sum_{i=1}^h 2^{i-1}\omega_i(z)$, then x is hit.*

For $1 \leqslant r \leqslant k$, we set

$$P_r^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_r^{a_r} \ : \ a_1a_2\ldots a_r = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_r^{a_r} \ : \ a_1a_2\ldots a_r > 0\}\rangle.$$

It is easy to see that P_r^0 and P_r^+ are the $\mathcal{A}$-submodules of P_k.

For $J = (j_1, j_2, \ldots, j_r) : 1 \leqslant j_1 < \ldots < j_r \leqslant k$, we define a monomorphism $\theta_J : P_r \to P_k$ of $\mathcal{A}$-algebras by substituting $\theta_J(x_t) = x_{j_t}$ for $1 \leqslant t \leqslant r$. It is easy to see that, for any weight vector ω of degree n,

$$Q\theta_J(P_r^+)(\omega) \cong QP_r^+(\omega) \text{ and } (Q\theta_J(P_r^+))_n \cong (QP_r^+)_n$$

for $1 \leqslant r \leqslant k$. So, by a simple computation using Theorem 1.1 and (2.1), we get the following.

Proposition 2.16 (See Walker-Wood [37]). *For a weight vector ω of degree n, we have direct summand decompositions of the $\mathbb{F}_2$-vector spaces*

$$QP_k(\omega) = \bigoplus_{\mu(n)\leqslant r\leqslant k}\ \bigoplus_{\ell(J)=r} Q\theta_J(P_r^+)(\omega),$$

where $\ell(J)$ is the length of J. Consequently

$$\dim QP_k(\omega) = \sum_{\mu(n)\leqslant r\leqslant k}\binom{k}{r}\dim QP_r^+(\omega),$$
$$\dim (QP_k)_n = \sum_{\mu(n)\leqslant r\leqslant k}\binom{k}{r}\dim (QP_r^+)_n.$$

We recall a result in our work [21] which is used in Section 4.

Definition 2.17. For any $(i;I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i,\\ \sum_{s\in I} x_{s-1}, & \text{if } j = i,\\ x_{j-1}, & \text{if } i < j \leqslant k.\end{cases}$$

Then, $p_{(i;I)}$ is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = \emptyset$, $p_{(i;\emptyset)}(x_i) = 0$ and $p_{(i;I)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

Lemma 2.18 (See Phúc and Sum [21]). *If x is a monomial in P_k, then $p_{(i;I)}(x) \in P_{k-1}(\omega(x))$. So, $p_{(i;I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$ for any weight vector ω.*

From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k, $B_k^0(n) = B_k(n) \cap P_k^0$, $B_k^+(n) = B_k(n) \cap P_k^+$. For a weight vector ω of degree n, we set $B_k(\omega) = B_k(n) \cap P_k(\omega)$, $B_k^+(\omega) = B_k^+(n) \cap P_k(\omega)$. Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the basses of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

3. On the kernel of Kameko's homomorphism

3.1. A construction for the generators of P_k.

Notation 3.1.1. We denote

$$\mathcal{N}_k = \{(i;I)\colon\ I = (i_1, i_2, \ldots, i_r), 1 \leqslant i < i_1 < \ldots < i_r \leqslant k,\ 0 \leqslant r < k\}.$$

For $(i;I) \in \mathcal{N}_k$, denote by $r = \ell(I)$ the length of I.

Definition 3.1.2 (See Sum [33]). Let $(i;I) \in \mathcal{N}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1}+\ldots+2^{r-u}} \prod_{u<t\leqslant r} x_{i_t}^{2^{r-t}}$ for $r = \ell(I) > 0$, For any monomial x in P_{k-1}, we define the monomial $\phi_{(i;I)}(x)$ in P_k by setting

$$\phi_{(i;I)}(x) = \begin{cases} f_i(x), & \text{if } r = \ell(I) = 0,\\ (x_i^{2^r-1} f_i(x))/x_{(I,u)}, & \text{if there exists } 1 \leqslant u \leqslant r \text{ such that}\\ & \quad \nu_{i_1-1}(x) = \ldots = \nu_{i_{(u-1)}-1}(x) = 2^r - 1,\\ & \quad \nu_{i_u-1}(x) > 2^r - 1,\\ & \quad \alpha_{r-t}(\nu_{i_u-1}(x)) = 1,\ \forall t,\ 1 \leqslant t \leqslant u,\\ & \quad \alpha_{r-t}(\nu_{i_t-1}(x)) = 1,\ \forall t,\ u < t \leqslant r,\\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.1.3 (See Sum [33]). *Let d be a positive integer and let $j_0, j_1, \ldots, j_{d-1} \in \mathbb{N}_k$. We set $i = \min\{j_0, \ldots, j_{d-1}\}$, $I = (i_1, \ldots, i_r)$ with $\{i_1, \ldots, i_r\} = \{j_0, \ldots, j_{d-1}\} \setminus \{i\}$. Then, we have*

$$\prod_{0\leqslant t<d} X_{j_t}^{2^t} \simeq_{(d-1} \phi_{(i;I)}(X_k^{2^d-1}).$$

Proposition 3.1.4. *For any positive integer d, $\{\phi_{(i;I)}(X_k^{2^d-1}) :\ (i;I) \in \mathcal{N}_k\}$ is the set of all admissible monomials in P_k such that their weight vectors are $(k-1)|^d$.*

Proof. Let x be an admissible monomial in P_k such that $\omega(x) = (k-1)|^d$. Then, there are integers $j_1, j_2, \ldots, j_d$ such that $x = \prod_{1\leqslant t\leqslant d} X_{j_t}^{2^{d-t}}$.

If there is an index t_0 such that $j_{t_0} > j_{t_0+1}$, then

$$x = \prod_{1\leqslant t<t_0} X_{j_t}^{2^{d-t}} (X_{j_{t_0}}^2 X_{j_{t_0+1}})^{2^{d-t_0-1}} \prod_{t_0+2\leqslant t<t_0} X_{j_t}^{2^{d-t}}.$$

We have

$$\begin{aligned} X_{j_{t_0}}^2 X_{j_{t_0+1}} = x_{j_{t_0+1}}^2 x_{j_{t_0}} X_{t_0,j_{t_0+1}}^3 &= x_{j_{t_0+1}} x_{j_{t_0}}^2 X_{t_0,j_{t_0+1}}^3\\ &+ Sq^1(x_{j_{t_0+1}} x_{j_{t_0}} X_{t_0,j_{t_0+1}}^3) \ \mathrm{mod}(P_k^-((k-1)|^2). \end{aligned}$$

This equality shows that the monomial $X_{j_{t_0}}^2 X_{j_{t_0+1}}$ is strictly inadmissible, hence by Theorem 2.9, x is strictly inadmissible. Since x is admissible, we get $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_d$. Break the sequence $J = (j_1, j_2, \ldots, j_d)$ into sections $I_0, I_1, \ldots, I_r$ of lengths $a_u = \ell(I_u)$, $0 \leqslant u \leqslant r \leqslant k-1$, so that j_{t-1} and j_t are in the same section if and only if $j_{t-1} = j_t$. If there is an index $u > 0$ such that $a_u > 1$, then there is t_1 such that $j_{t_1} < j_{t_1+1} = j_{t_1+2}$, ($j_{t_1} \in I_{u-1}$ and $j_{t_1+1}, j_{t_1+2} \in I_u$). Then, we have

$$x = \prod_{1 \leqslant t < t_1} X_{j_t}^{2^{d-t}} (X_{j_{t_1}}^4 X_{j_{t_0+1}}^3)^{2^{d-t_1-2}} \prod_{t_1+3 \leqslant t < t_0} X_{j_t}^{2^{d-t}}.$$

By a direct computation, we have

$$\begin{aligned} X_{j_{t_1}}^4 X_{j_{t_1+1}}^3 &= x_{j_{t_1}}^3 x_{j_{t_1+1}}^4 X_{j_{t_1}, j_{t_1+1}}^3 = x_{j_{t_1}}^2 x_{j_{t_1+1}}^5 X_{j_{t_1}, j_{t_1+1}}^7 + Sq^1(x_{j_{t_1}}^3 x_{j_{t_0}}^3 X_{j_{t_1}, j_{t_1+1}}^7) \\ &+ Sq^2(x_{j_{t_1}}^2 x_{j_{t_0}}^3 X_{j_{t_1}, j_{t_1+1}}^7) \bmod(P_k^-((k-1)|^3). \end{aligned}$$

This equality implies $X_{j_{t_1}}^4 X_{j_{t_1+1}}^3$ is strictly inadmissible. By Theorem 2.9, x is strictly inadmissible. Since x is admissible, we obtain $a_u = \ell(I_u) = 1,\ 1 \leqslant u \leqslant r$ and $x = \phi_{(i;I)}(X_k^{2^d-1})$, where $i = j_1 \in I_0$, $I = (i_1, i_2, \ldots, i_r)$ with $I_u = \{i_u\}$ for $1 \leqslant u \leqslant r$. In [22, Proposition 3.7], we have proved that the set

$$[\{\phi_{(i;I)}(X_k^{2^d-1}) :\ (i;I) \in \mathcal{N}_k\}]_{(k-1)|^d}$$

is linearly independent in $QP_k((k-1)|^d)$. So, the proposition is proved. □

Theorem 3.1.5 (See Sum [33, Proposition 3.3])**.** *Let $n = \sum_{i=1}^{k-1}(2^{d_i}-1)$ with d_i positive integers such that $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1} := d \geqslant k-1 \geqslant 3$, and let $m = \sum_{i=1}^{k-2}(2^{d_i - d_{k-1}} - 1)$. Then*

$$\bigcup_{(i;I) \in \mathcal{N}_k} \big\{\phi_{(i;I)}(X_k^{2^d-1} z^{2^d}) :\ z \in B_{k-1}(m)\big\}.$$

is a minimal set of generators for $\mathcal{A}$-module P_k in degree n. Consequently

$$\dim(QP_k)_n = (2^k - 1)\dim(QP_{k-1})_m.$$

3.2. **Some properties of admissible monomials in $P_k((k-2)|^d)$.**

Lemma 3.2.1. *Let $n = \sum_{i=1}^{k-2}(2^{d_i}-1)$ with d_i positive integers such that $d_1 > d_2 > \ldots d_{k-3} \geqslant d_{k-2}$. If x is an admissible monomial of degree n in P_k such that $[x] \in \operatorname{Ker}((\widetilde{Sq}_*^0)_{(k,n)})$, then $\omega_i(x) = k-2$ for $1 \leqslant i \leqslant d_{k-2}$.*

Proof. Note that $z = \prod_{t=1}^{k-2} x_t^{2^{d_t}-1}$ is the minimal spike of degree n and $\omega_i(z) = k-2$ for $1 \leqslant i \leqslant d_{k-2}$. Since x is admissible, $[x] \neq 0$. By Theorem 2.13, either $\omega_1(x) = k-2$ or $\omega_1(x) = k$. If $\omega_1(x) = k$, then $x = X_\emptyset y^2$ with y a monomial in P_k. Since x is admissible, by Theorem 2.9, y is also admissible. So, $(\widetilde{Sq}_*^0)_{(k,n)}([x]) = [y] \neq 0$. This contradict the hypothesis that $x \in \operatorname{Ker}((\widetilde{Sq}_*^0)_{(k,n)})$, hence $\omega_1(x) = k-2$. Now the lemma follows from Proposition 2.10. □

Lemma 3.2.2. *If x is a monomial of degree n in P_k such that $[x] \in \operatorname{Ker}((\widetilde{Sq}_*^0)_{(k,n)})$, then $x \equiv \sum \bar{x}$ with $\bar{x}$ monomials in P_k such that $\omega_i(\bar{x}) = k-2$, for $1 \leqslant i \leqslant d = d_{k-2}$.*

Proof. If $\omega_1(x) < k-2$, then by Theorem 2.13, x is hit, hence the lemma holds. Suppose $\omega_1(x) = k-2$ and let $s > 1$ be the smallest index such that $\omega_s(x) \neq k-2$. If $\omega_s(x) < k-2$, then by Theorem 2.13, x is hit, hence the lemma holds.

Assume that $\omega_s(x) = k$. Then we have $x = \prod_{t=0}^{s-3} X_{\mathbb{J}_t(x)}^{2^t} y^{2^{s-2}} \prod_{t\geqslant s} X_{\mathbb{J}_t(x)}^{2^t}$, where $y = X_{\mathbb{J}_{s-2}(x)} X_{\mathbb{J}_{s-1}(x)}^2 = X_{u,v}^3 x_u^2 x_v^2$ with $1 \leqslant u < v \leqslant k$. Then we have

$$y = \sum_{i\neq u,v} X_{i,u,v}^3 x_u x_v^2 x_i^4 + Sq^1(X_{u,v}^3 x_u x_v^2).$$

By using this equality, the Cartan formula and Theorem 2.13, we obtain $x \equiv \sum_{i\neq u,v} x_{(i,u,v)}$, where

$$x_{(i,u,v)} = \prod_{t=0}^{s-3} X_{\mathbb{J}_t(x)}^{2^t} (X_{i,u,v}^3 x_u x_v^2 x_i^4)^{2^{s-2}} \prod_{t\geqslant s} X_{\mathbb{J}_t(x)}^{2^t}.$$

A simple computation shows that $\omega_i(x_{(i,u,v)}) = k-2$ for $1 \leqslant i \leqslant s$. By repeating this argument we see that the lemma is true.

If $\omega_1(x) = k$, then $x = X_\emptyset y^2$ with y a monomial in P_k. Then, we have $(\widetilde{Sq}^0_*)_{(k,n)}([x]) = [y] = 0$. Hence, $y = \sum_{r>0} Sq^r(g_r)$ with suitable polynomial g_r in P_k. Then, using Proposition 2.4, Theorem 2.13 and the Cartan formula we get

$$\begin{aligned} x = X_\emptyset y^2 &= \sum_{r>0} X_\emptyset Sq^{2r}(g_r^2) \\ &= \sum_{r>0} Sq^{2r}(X_\emptyset g_r^2) + \sum_{r>0}\sum_{t=1}^{r} Sq^{2t}(X_\emptyset)(Sq^{r-t}(g_r))^2 \equiv \sum x', \end{aligned}$$

where x' are monomials in $(P_k)_n$ such that $\omega_1(x') = k-2$. The lemma is proved. □

Lemma 3.2.3. *Let x be a monomial of degree $(k-2)(2^d-1)$. If $\omega_1(x) < k$ and there is $r > d$ such that $\omega_r(x) > 0$, then $x \in P_k^-((k-2)|^d) + \mathcal{A}^+P_k$.*

Proof. If $\omega_1(x) < k-2$, then $x \in P_k^-((k-2)|^d)$, hence the lemma holds. If $\omega_1(x) \geqslant k-2$, then $\omega_1(x) = k-2$. Let s be the smallest index such that $\omega_s(x) > k-2$, then $\omega_s(x) = k$. If $s \geqslant d$, then $(k-2)(2^d-1) = \deg x \geqslant (k-2)(2^d-1) + 2^{t-1}\omega_t(x) > (k-2)(2^d-1)$. This is a contradiction, so $s < d$. If there is $1 < r < s$ such that $\omega_r(x) < k-2$, then $x \in P_k^-((k-2)|^d)$, so the lemma holds. Suppose $\omega_r = k-2$ for $1 \leqslant r < s$. We have

$$y := X_{\mathbb{J}_{s-1}(x)}^2 X_{\mathbb{J}_{s-2}(x)} = X_{u,v}^3 x_u^2 x_v^2 = \sum_{i\neq u,v} X_{1,u,v}^3 x_i^4 x_u x_v^2 + Sq^1(X_{u,v}^3 x_u x_v^2).$$

for $1 \leqslant u < v \leqslant k$. Using Proposition 2.4 and the Cartan formula we get $x \equiv_{(k-2)|^d} \sum_{i\neq u,v} x_{(i,u,v)}$, where

$$x_{(i,u,v)} = \prod_{t=0}^{s-3} X_{\mathbb{J}_t(x)}^{2^t} (X_{i,u,v}^3 x_u x_v^2 x_i^4)^{2^{s-2}} \prod_{t\geqslant s} X_{\mathbb{J}_t(x)}^{2^t}.$$

It is easy to see that $\omega_i(x_{(i,u,v)}) = k-2$ for $1 \leqslant i \leqslant s$ and $\omega_r(x_{(i,u,v)}) > 0$ for suitable $r > d$. By repeating this argument we obtain $x \equiv_{(k-2)|^d} \sum \bar{x}$ with $\bar{x}$ monomials such that $\omega_i \leqslant k-2$ for $1 \leqslant i \leqslant d$ and $\omega_r(\bar{x}) > 0$ for suitable $r > d$. Then we have $\sum_{i=1}^d 2^{i-1}\omega_i(\bar{x}) < \deg \bar{x} = (k-2)(2^d-1)$. Hence, there is an index i such that $\omega_i(\bar{x}) < k-2$ and $\bar{x} \in P_k^-((k-2)|^d)$. The lemma is proved. □

Lemma 3.2.4. *Let $i_1, i_2, j_1, j_2 \in \mathcal{N}_k$ such that $i_1 < j_1$, $i_2 < j_2$.*

i) *If either $i_1 > i_2$ or $i_1 = i_2$ and $j_1 > j_2$, then $X_{i_1,j_1}^2 X_{i_2,j_2}$ is strictly inadmissible.*

ii) *If $j_1 > j_2$ and $i, j \in \mathcal{N}_k$, $i < j$, then the monomial $X_{i_1,j_1}^4 X_{i_2,j_2}^2 X_{i,j}$ is strictly inadmissible.*

iii) *If either $i_1 < i_2 \leqslant j_1$ or $i_1 = i_2, j_1 \neq j_2$, then $X_{i_1,j_1}^4 X_{i_2,j_2}^3$ is strictly inadmissible.*

iv) *If either $i_1 < i_2$ and $j_1 \leqslant j_2$, then $X_{i_1,j_1}^8 X_{i_2,j_2}^7$ is strictly inadmissible.*

Proof. We prove i). If $i_1 > i_2$ and $j_1 = j_2 = j$, then $x = x_{i_1} x_{i_2}^2 X_{i_1,i_2,j}^3$. We have

$$x = x_{i_2} x_{i_1}^2 X_{i_1,i_2,j}^3 + \sum_{t \neq i_1,i_2,j} x_{i_1} x_{i_2} x_t^4 X_{i_1,i_2,j,t}^3 + Sq^1(x_{i_1} x_{i_2} X_{i_1,i_2,j}^3).$$

This equality shows that x is strictly inadmissible.

If $j_1 \neq j_2$, then $x = x_{i_1} x_{i_2}^2 x_{j_1} x_{j_2}^2 X_{i_1,i_2,j_1,j_2}^3$. Then we have

$$\begin{aligned} x &= x_{i_2} x_{i_1}^2 x_{j_1} x_{j_2}^2 X_{i_1,i_2,j_1,j_2}^3 + x_{i_2} x_{i_1} x_{j_1}^2 x_{j_2}^2 X_{i_1,i_2,j_1,j_2}^3 \\ &\quad + \sum_{t \neq i_1,i_2,j_1,j_2} x_{i_1} x_{i_2} x_{j_1} x_{j_2}^2 x_t^4 X_{i_1,i_2,j_1,j_2,t}^3 \\ &\quad + Sq^1(x_{i_1} x_{i_2} x_{j_1} x_{j_2}^2 X_{i_1,i_2,j_1,j_2}^3). \end{aligned}$$

Since $i_2 < i_1 < j_1$, the above equality implies x is strictly inadmissible.

We now prove ii). If $i_1 = i_2 = i$, then $x = x_{j_1} x_{j_2}^2 X_{i,j_1,j_2}^3$. We have

$$x = x_{j_1}^2 x_{j_2} X_{i,j_1,j_2}^3 + \sum_{t \neq i,j_1,j_2} x_{j_1} x_{j_2} x_t^4 X_{i,j_1,j_2,j}^3 + Sq^1(x_{j_1} x_{j_2} X_{i,j_1,j_2,j}^3).$$

This equality shows that x is strictly inadmissible. By Theorem 2.9, $X_{i,j} x^2$ is also strictly inadmissible.

Suppose $i_1 < i_2$. Then $x = x_{i_1} x_{i_2}^2 x_{j_2}^2 x_{j_1} X_{i_1,i_2,j_1,j_2}^3$. We have

$$\begin{aligned} x_{i_1} x_{i_2}^2 x_{j_2}^2 x_{j_1} &= x_{i_1} x_{i_2} x_{j_2}^2 x_{j_1}^2 + x_{i_1} x_{i_2}^2 x_{j_2} x_{j_1}^2 \\ &\quad + Sq^1(x_{i_1}^2 x_{i_2} x_{j_2} x_{j_1}) + Sq^2(x_{i_1} x_{i_2} x_{j_2} x_{j_1}) \end{aligned}$$

So, using the Cartan formula, we get

$$x = x_{i_1} x_{i_2} x_{j_2}^2 x_{j_1}^2 X_{i_1,i_2,j_1,j_2}^3 + x_{i_1} x_{i_2}^2 x_{j_2} x_{j_1}^2 X_{i_1,i_2,j_1,j_2}^3 + A + B + C$$

where

$$\begin{aligned} A &= x_{i_1}^2 x_{i_2} x_{j_1} x_{j_2} Sq^1(X_{i_1,i_2,j_1,j_2}^3) + Sq^1(x_{i_1} x_{i_2} x_{j_1} x_{j_2}) Sq^1(X_{i_1,i_2,j_1,j_2}^3), \\ B &= x_{i_1} x_{i_2} x_{j_1} x_{j_2} Sq^2(X_{i_1,i_2,j_1,j_2}^3), \\ C &= Sq^1(x_{i_1}^2 x_{i_2} x_{j_1} x_{j_2} X_{i_1,i_2,j_1,j_2}^3) + Sq^2(x_{i_1} x_{i_2} x_{j_1} x_{j_2} X_{i_1,i_2,j_1,j_2}^3). \end{aligned}$$

A direct computation shows that $X_{i,j} A^2 \in P_k^-((k-2)|^3)$, $X_{i,j} B^2 \in P_k^-((k-2)|^3) + \mathcal{A}_1^+ P_5$ and $X_{i,j} C^2 \in P_k^-((k-2)|^3) + \mathcal{A}_3^+ P_5$. Hence, the monomial $X_{i,j} x^2$ is strictly inadmissible.

We prove Part iii). If $i_2 = j_1 = i$, then $x := X_{i_1,j_1}^4 X_{i_2,j_2}^3 = x_{i_1}^3 x_{j_2}^4 X_{i_1,i_2,j_2}^7$. Then we have

$$\begin{aligned} x &= x_{i_1}^2 x_{j_2}^5 X_{i_1,i_2,j_2}^7 + Sq^1(x_{i_1}^3 x_{j_2}^3 X_{i_1,i_2,j_2}^7) \\ &\quad + Sq^2(x_{i_1}^2 x_{j_2}^3 X_{i_1,i_2,j_2}^7) \bmod(P_k^-((k-2)|^3)). \end{aligned}$$

Hence, x is strictly inadmissible. If $i_2 < j_1$, then $x = x_{i_1}^3 x x_{i_2}^4 x_{j_1}^3 x_{j_2}^4 Y^7$ with $Y := X_{i_1,i_2,j_1,j_2}^7$. Then we have

$$\begin{aligned} x &= x_{i_1}^3 x x_{i_2}^3 x_{j_1}^4 x_{j_2}^4 Y^7 + x_{i_1}^2 x x_{i_2}^5 x_{j_1}^3 x_{j_2}^4 Y^7 + x_{i_1}^3 x x_{i_2}^3 x_{j_1}^5 x_{j_2}^4 Y^7 \\ &\quad + Sq^1(x_{i_1}^3 x x_{i_2}^3 x_{j_1}^3 x_{j_2}^4 Y^7) + Sq^2(x_{i_1}^2 x x_{i_2}^3 x_{j_1}^3 x_{j_2}^4 Y^7) \bmod(P_k^-((k-2)|^3)). \end{aligned}$$

The above equality shows that the monomial $X_{1,j_{t_0}}^4 X_{1,j_{t_0+1}}^3$ is strictly inadmissible.

We now prove Part iv). If $j_1 = j_2$, then $x = x_{i_1}^7 x_{i_2}^8 X_{i_1,i_2,j_1}^{15}$. We have

$$x = x_{i_1}^6 x_{i_2}^9 X_{i_1,i_2,j_1}^{15} + Sq^1(x_{i_1}^7 x_{i_2}^7 X_{i_1,i_2,j_1}^{15}) + Sq^2(x_{i_1}^6 x_{i_2}^7 X_{i_1,i_2,j_1}^{15}) \bmod(P_k^-((k-2)|^4)).$$

Hence, x is strictly inadmissible. If $j_1 = i_2$, then $x = x_{i_1}^7 x_{j_2}^8 X_{i_1,i_2,j_1}^{15}$. By a similar computation, we see that x is strictly inadmissible.

Suppose $j_1 \neq i_1, j_2$, then we have $x = x_{i_1}^7 x_{i_2}^8 x_{j_1}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15}$. If $i_2 < j_1$, then

$$\begin{aligned} x &= x_{i_1}^6 x_{i_2}^9 x_{j_1}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} + x_{i_1}^6 x_{i_2}^7 x_{j_1}^9 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} \\ &\quad + x_{i_1}^7 x_{i_2}^7 x_{j_1}^8 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} + Sq^1(x_{i_1}^7 x_{i_2}^7 x_{j_1}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15}) \\ &\quad + Sq^2(x_{i_1}^6 x_{i_2}^7 x_{j_1}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15}) \bmod(P_k^-((k-2)|^4)). \end{aligned}$$

Hence, x is strictly inadmissible. If $j_1 < i_2$, then

$$\begin{aligned} x &= x_{i_1}^4 x_{j_1}^7 x_{i_2}^{11} x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} + x_{i_1}^5 x_{j_1}^6 x_{i_2}^{11} x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} + x_{i_1}^7 x_{j_1}^6 x_{i_2}^9 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} \\ &\quad + Sq^1(x_{i_1}^7 x_{j_1}^7 x_{i_2}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} + x_{i_1}^3 x_{j_1}^{11} x_{i_2}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15}) \\ &\quad + Sq^2((x_{i_1}^7 x_{j_1}^6 x_{i_2}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} + x_{i_1}^3 x_{j_1}^{10} x_{i_2}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15}) \\ &\quad + Sq^4(x_{i_1}^4 x_{j_1}^7 x_{i_2}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15} + x_{i_1}^5 x_{j_1}^6 x_{i_2}^7 x_{j_2}^8 X_{i_1,i_2,j_1,j_2}^{15}) \bmod(P_k^-((k-2)|^4)). \end{aligned}$$

Hence, x is strictly inadmissible.

The lemma is proved. ☐

We see that there is a monomial x in P_k such that x is inadmissible but it is not strictly inadmissible. We defined the notion of strongly inadmissible monomial in $(P_k)_{(k-2)(2^d-1)}$.

For a positive integer a, denote by $\alpha(a)$ the number of ones in dyadic expansion of a and by $\zeta(a)$ the greatest integer u such that a is divisible by 2^u. That means $a = 2^{\zeta(a)}b$ with b an odd integer. We set $d(a) = a - \alpha(a) - \zeta(a)$.

For any positive integer d such that $d > d(k-2)$, denote by $\mathcal{P}_{(k,d)}$ the subspace of P_k spanned by all monomials x of degree $(k-2)(2^d-1)$ such that

$$\sum_{j=1}^{h} \omega_j(x) < (k-2)(2^h-1),$$

for some h, $1 \leqslant h \leqslant d - d(k-2)$.

Definition 3.1. A monomial x of of weight vector $(k-2)|^d$ in P_k is said to be strongly inadmissible if there exist monomials $y_1, y_2, \ldots, y_t$ of the same weight vector $(k-2)|^d$ such that $y_u < x$, $1 \leqslant u \leqslant t$ and

$$x \simeq_d y_1 + y_2 + \ldots y_t \mod(\mathcal{P}_{(k,d)}).$$

Obviously, if x is strictly inadmissible, then x is strongly inadmissible.

By a direct computation we can show that the monomial $x = x_1 x_2^3 x_3^6 x_4^6 x_5^5$ of weight vector $(3)|^3$ in P_5 is strongly inadmissible but it is not strictly inadmissible.

Proposition 3.2. *Let x be a monomial of weight vector $(k-2)|^d$ in P_k. If x is strongly inadmissible, then x is inadmissible.*

Proof. Set $s = \alpha(k-2)$. Then

$$k-2 = 2^{t_1} + 2^{t_2} + \ldots + 2^{t_{s-1}} + 2^{t_s},$$

where $t_1 > t_2 > \ldots > t_{s-1} > t_s = \zeta(k-2) \geqslant 0$. Then, we have

$$\begin{aligned}(k-2)(2^d-1) &= 2^{d+t_1} + 2^{d+t_2} + \ldots + 2^{d+t_{s-1}} + 2^{d+t_s} - k + 2\\ &= \sum_{1\leqslant i\leqslant k-2}(2^{d_i}-1),\end{aligned}$$

where

$$d_i = \begin{cases} d+t_i, & 1 \leqslant i < s,\\ d+t_s - i + s - 1, & s \leqslant i \leqslant k-3,\\ d_{k-3} = d + t_s - k + s + 1 = d - d_(k-2), & i = k-2.\end{cases}$$

It is easy to see that $d_1 > d_2 > \ldots > d_{k-3} = d_{k-2} = d - d(k-2)$. Hence $z = \prod_{i=1}^{k-2} x_i^{2^{d_i-1}}$ is a minimal spike of degree $(k-2)(2^d-1)$ and $\omega_j(z) = k-2$ for $1 \leqslant j \leqslant k-2$. By Theorem 2.14, we have $\mathcal{P}_{(k,d)} \subset \mathcal{A}^+P_k$. Hence x is inadmissible. $\square$

Proposition 3.2.5. *Let c, d, e be positive integers and let u, x, $y \in P_k$ be monomials such that $\omega(u) = (k-2)|^c$, $\omega(x) = (k-2)|^d$ and $\omega(y) = (k-2)|^e$. If x is strongly inadmissible, then $ux^{2^c}y^{2^{c+d}}$ is also strongly inadmissible.*

Proof. Since x is strongly inadmissible, there exist monomials $y_1, y_2, \ldots, y_t$ of the same weight vector $(k-2)|^d$, $g_1 \in P_k^-((k-2)|^d)$ and $g_2 \in \mathcal{P}_{(k,d)}$ such that $y_i < x$ for $i = 1, 2, \ldots, t$ and

$$x = y_1 + y_2 + \ldots + y_t + g_1 + g_2 + \sum_{1\leqslant j<2^d} Sq^j(h_j),$$

where h_j are suitable polynomials in P_k. Since $\omega(u) = (k-2)|^c$ and $\omega(x) = (k-2)|^d$, using Proposition 2.4 and the Cartan formula, we get

$$u(Sq^j(h_j))^{2^c}y^{2^{c+d}} = Sq^{j2^c}\left(uh_j^{2^c}y^{2^{c+d}}\right) \mod(P_k^-((k-2)|^{c+d+e})),$$

for $1 \leqslant j < 2^d$. Combining the above equalities gives

$$\begin{aligned}ux^{2^c}y^{2^{c+d}} &= \sum_{1\leqslant i\leqslant t} uy_i^{2^c}y^{2^{c+d}} + ug_1^{2^c}y^{2^{c+d}} + ug_2^{2^c}y^{2^{c+d}}\\ &\quad + \sum_{1\leqslant j<2^d} Sq^{j2^c}\left(uh_j^{2^c}y^{2^{c+d}}\right) \mod(P_k^-((k-2)|^{c+d+e})).\end{aligned}$$

Since $\omega(u) = (k-2)|^c$, we can easily check that $uy_i^{2^c}y^{2^{c+d}} < ux^{2^c}y^{2^{c+d}}$ for $1 \leqslant i \leqslant t$, $ug_1^{2^c}y^{2^{c+d}} \in P_k^-((k-2)|^{c+d+e})$ and $ug_2^{2^c}y^{2^{c+d}} \in \mathcal{P}_{(k,c+d+e)}$. Hence, the last equality implies that $ux^{2^c}y^{2^{c+d}}$ is strongly inadmissible. $\square$

Proposition 3.2.6. *Let $d \geqslant d(k-2)+2$ and let $x = \prod_{t=1}^{d} X_{i_t,j_t}^{2^{d-t}}$, where $1 \leqslant i_t < j_t \leqslant k$ for $1 \leqslant t \leqslant d$. If the monomial x is admissible, then $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_d$ and $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_d$.*

Proof. Suppose x is admissible. By using Lemma 3.2.4(i) and Theorem 2.9, one gets $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_d$. Combining Lemma 3.2.4(ii) and Theorem 2.9 gives $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_{d-1}$. So, we need only to prove $j_{d-1} \leqslant j_d$.

Suppose the contrary that $j_d < j_{d-1}$. We proved the proposition for the case $i_{d-2} < i_{d-1} < i_d$ and $j_{d-2} \notin \{i_{d-2}, i_{d-1}, i_d, j_{d-2}, j_{d-1}\}$. The other cases can be proved by similar computations. Then, we have $x = yY^8$ with

$$y = X^4_{i_{d-2},j_{d-2}} X^2_{i_{d-1},j_{d-1}} X_{i_d,j_d} = x^3_{i_{d-2}} x^5_{i_{d-1}} x^6_{i_d} x^3_{j_{d-2}} x^6_{j_{d-1}} x^5_{i_d} Z^7,$$

and $Y = \prod_{t=1}^{d-3} X^{2^{d-3-t}}_{i_t,j_t}$, $\quad Z = X_{i_{d-2},\, i_{d-1},\, i_d,\, j_{d-2}, j_{d-1},\, j_d}$. By a direct computation, we get

$$\begin{aligned}x &= x^3_{i_{d-2}} x^5_{i_{d-1}} x^5_{i_d} x^3_{j_{d-2}} x^6_{j_{d-1}} x^6_{i_d} Z^7 Y^8 + x^3_{i_{d-2}} x^5_{i_{d-1}} x^6_{i_d} x^3_{j_{d-2}} x^5_{j_{d-1}} x^6_{i_d} Z^7 Y^8 \\ &\quad + g + Sq^1(x^3_{i_{d-2}} x^6_{i_{d-1}} x^5_{i_d} x^3_{j_{d-2}} x^5_{j_{d-1}} x^5_{i_d} Z^7 Y^8) \\ &\quad + Sq^2(x^3_{i_{d-2}} x^5_{i_{d-1}} x^5_{i_d} x^3_{j_{d-2}} x^5_{j_{d-1}} x^5_{i_d} Z^7 Y^8) \mod(P_k^-((k-2)|^d)),\end{aligned}$$

with

$$\begin{aligned}g &= x^5_{i_{d-2}} x^5_{i_{d-1}} x^5_{i_d} x^3_{j_{d-2}} x^5_{j_{d-1}} x^5_{i_d} Z^7 + x^3_{i_{d-2}} x^5_{i_{d-1}} x^5_{i_d} x^5_{j_{d-2}} x^5_{j_{d-1}} x^5_{i_d} Z^7 \\ &\quad + \sum_u x^3_{i_{d-2}} x^5_{i_{d-1}} x^5_{i_d} x^3_{j_{d-2}} x^5_{j_{d-1}} x^5_{i_d} x^9_u Z^7_u,\end{aligned}$$

where the last sum runs over all $u \in \mathbb{N}_k \setminus \{i_{d-2}, i_{d-1}, i_d, j_{d-2}, j_{d-1}, j_d\}$ and $Z_u = Z/x_u$. It is easy to see that if a monomial v appears as a term of g, then $\omega_1(v) = k$ and $\omega_2(v) = k-5$. This implies $\omega_1(v) + 2\omega_2(v) = 3k - 10 < (k-2)(2^2-1)$. Since $d \geqslant d(k-2)+2$, one gets $g \in \mathcal{P}_{(k,d)}$. This shows that x is strongly inadmissible. By Proposition 3.2, x is inadmissible. This is a contradiction, so $j_{d-1} \leqslant j_d$. □

Notation 3.2.7. Let $S = (s_1, s_2, \ldots, s_d)$ be a sequences of integers.

Break S into sections of lengths $c_0, c_1, \ldots, c_r$, so that s_{t-1} and s_t are in the same section if and only if $s_{t-1} = s_t$. We denote $\mathrm{rl}(S) = c_1 + c_2 + \ldots + c_r$, the reduced length of S. For example, for $S = (2,2,3,1,1,1)$, we have $\mathrm{rl}(S) = 4$.

We denote by PSeq^d_k the set of all pairs $(\mathcal{I}, \mathcal{J})$ of sequences $\mathcal{I} = (i_1, i_2, \ldots, i_d)$, $\mathcal{J} = (j_1, j_2, \ldots, j_d)$, where i_t, j_t are integers such that $1 \leqslant i_t < j_t \leqslant k$, for $1 \leqslant t \leqslant d$, and by PInc^d_k the set of all $(\mathcal{I}, \mathcal{J}) \in \mathsf{PSeq}^d_k$ such that $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_d$ and $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_d$. For $(\mathcal{I}, \mathcal{J}) \in \mathsf{PSeq}^d_k$, we denote

$$X_{(\mathcal{I},\mathcal{J})} = \prod_{1 \leqslant t \leqslant d} X^{2^{d-t}}_{i_t,j_t} \in P_k.$$

Proposition 3.2.8. *Let $(\mathcal{I}, \mathcal{J}) \in \mathsf{PInc}^d_k$. If the monomial $X_{(\mathcal{I},\mathcal{J})}$ is admissible, then $\mathrm{rl}(\mathcal{I}) < \binom{k}{2}$ and $\mathrm{rl}(\mathcal{J}) < \binom{k}{2}$.*

Proof. We prove $\mathrm{rl}(\mathcal{I}) < \binom{k}{2}$. Break $\mathcal{I}$ into sections of lengths $b_0, b_1, \ldots, b_r$ so that i_{t-1} and i_t are in the same section if and only if $i_{t-1} = i_t$. We prove $b_u \leqslant k - u$ for $1 \leqslant u \leqslant r$. Suppose the contrary that there is an index u, $1 \leqslant u \leqslant r$, such that $b_u \geqslant k - u + 1$. Then, there is t_0 such that $i_{(t_0+t)} = i_{(t_0+1)}$ for $1 \leqslant t \leqslant b_u$ and $i_{t_0} < i_{(t_0+1)}$. Note that $i_{(t_0+1)} \geqslant u+1$. We have

$$X_{(\mathcal{I},\mathcal{J})} = \prod_{t=1}^{t_0} X^{2^{d-t}}_{i_t,j_t} \Big(\prod_{t=1}^{b_u} X^{2^{b_u-t}}_{i_{(t_0+t)},j_{(t_0+t)}} \Big)^{2^{d-t_0-b_u}} \prod_{t=t_0+b_u+1}^{d} X^{2^{d-t}}_{i_t,j_t}.$$

Since $i_{t_0+t} = i_{t_0+1}$ for $1 \leqslant t \leqslant b_u$, we have

$$y := \prod_{t=1}^{b_u} X^{2^{b_u-t}}_{i_{(t_0+t)},j_{(t_0+t)}} = f_i\Big(\prod_{t=1}^{b_u} Y^{2^{b_u-t}}_{j_{(t_0+t-1)}}\Big),$$

where $i = i_{t_0+1}$ and $Y_h = x_1 \ldots \hat{x}_h \ldots x_{k-1} \in P_{k-1}$ for $1 \leqslant h \leqslant k-1$. Since f_i is a homomorphism of $\mathcal{A}$-algebras, by Proposition 3.1.3, if $y \neq \phi_{(h;H)}(Y^{2^{b_u-1}})$ for all $(h;H) \in \mathcal{N}_{k-1}$ and $Y = x_1x_2\ldots x_{k-1} \in P_{k-1}$, then y is strictly inadmissible. Hence, by Theorem 2.9, $X_{(\mathcal{I},\mathcal{J})}$ is also strictly inadmissible. Suppose $y = \phi_{(h;H)}(Y^{2^{b_u-1}})$ for suitable $(h;H) \in \mathcal{N}_{k-1}$. Since $b_u \geqslant k-u$ and $u+1 \leqslant i_{t_0+1} < i_{t_0+2} < j_{t_0+2} = h+1$, we have $h \geqslant u+1$, $\ell(H) \leqslant k-u-2$ and $b_u - \ell(H) \geqslant 3$. Hence, we get $j_{t_0+1} = j_{t_0+2} = j_{t_0+3} = h+1$. Then, we obtain

$$X_{(\mathcal{I},\mathcal{J})} = \prod_{t=1}^{t_0-1} X^{2^{d-t}}_{i_t,j_t}\Big(X^8_{i_{t_0},j_{t_0}}X^7_{i_{t_0+1},j_{t_0+1}}\Big)^{2^{d-t_0-3}}\prod_{t=t_0+4}^{d} X^{2^{d-t}}_{i_t,j_t}.$$

By Lemma 3.2.4(iv), the monomial $X^8_{i_{t_0},j_{t_0}}X^7_{i_{t_0+1},j_{t_0+1}}$ is strictly inadmissible. Hence, by Theorem 2.9, $X_{(\mathcal{I},\mathcal{J})}$ is strictly inadmissible. This contradicts the hypothesis that x is admissible. Since $b_u \leqslant k-u$ for $1 \leqslant u \leqslant r \leqslant k-2$, we get

$$\mathrm{rl}(\mathcal{I}) = \sum_{u=1}^{r} b_u \leqslant \sum_{u=1}^{r}(k-u) < \sum_{u=1}^{k-1}(k-u) = \binom{k}{2}.$$

We now prove $\mathrm{rl}(\mathcal{J}) < \binom{k}{2}$. Break $\mathcal{J}$ into sections of lengths $c_0, c_1, \ldots, c_s$ so that j_{t-1} and j_t are in the same section if and only if $j_{t-1} = j_t$. We prove $c_v < k+v-s$ for all v, $1 \leqslant v \leqslant s$. Suppose there is an index v, $1 \leqslant v \leqslant s$ such that $c_v \geqslant k+v-s$, then there is t_0 such that $j_{t_0+t} = j_{t_0+1} > j_{t_0}$ for $1 \leqslant t \leqslant c_v$. It is easy to see that $j_{t_0+1} \leqslant k+v-s$. Then we have

$$X_{(\mathcal{I},\mathcal{J})} = \prod_{t=1}^{t_0} X^{2^{d-t}}_{i_t,j_t}\Big(\prod_{t=1}^{c_v} X^{2^{c_v-t}}_{i_{(t_0+t)},j_{(t_0+t)}}\Big)^{2^{d-t_0-c_v}}\prod_{t=t_0+c_v+1}^{d} X^{2^{d-t}}_{i_t,j_t}.$$

Since $j_{t_0+t} = j_{t_0+1}$ for $1 \leqslant t \leqslant c_v$, we have

$$z := \prod_{t=1}^{c_v} X^{2^{c_v-t}}_{i_{(t_0+t)},j_{(t_0+t)}} = f_j\Big(\prod_{t=1}^{c_v} Y^{2^{c_v-t}}_{i_{(t_0+t-1)}}\Big),$$

where $j = j_{t_0+1}$. Since f_j is a homomorphism of $\mathcal{A}$-algebras, by Proposition 3.1.3, if $z \neq \phi_{(h;H)}(Y^{2^{c_v-1}})$ for all $(h;H) \in \mathcal{N}_{k-1}$, then z is strictly inadmissible. Hence, by Theorem 2.9, $X_{(\mathcal{I},\mathcal{J})}$ is also strictly inadmissible. Suppose $z = \phi_{(h;H)}(Y^{2^{c_v-1}})$ for suitable $(h;H) \in \mathcal{N}_{k-1}$. Since $c_v \geqslant k+v-s$ and $i_{t_0+t} < j_{t_0+1} \leqslant k+v-s$, for $1 \leqslant t \leqslant c_v$, we get $\ell(H) \leqslant k+v-s-2$. If $\ell(H) < k+v-s-2$, then $c_v - \ell(H) \geqslant 3$. Hence, we get $i_{t_0+1} = i_{t_0+2} = i_{t_0+3} = h+1$. Then, we obtain

$$X_{(\mathcal{I},\mathcal{J})} = \prod_{t=1}^{t_0-1} X^{2^{d-t}}_{i_t,j_t}\Big(X^8_{i_{t_0},j_{t_0}}X^7_{i_{t_0+1},j_{t_0+1}}\Big)^{2^{d-t_0-3}}\prod_{t=t_0+4}^{d} X^{2^{d-t}}_{i_t,j_t}.$$

By Lemma 3.2.4(iv), the monomial $X^8_{i_{t_0},j_{t_0}}X^7_{i_{t_0+1},j_{t_0+1}}$ is strictly inadmissible. Hence, by Theorem 2.9, $X_{(\mathcal{I},\mathcal{J})}$ is strictly inadmissible. If $\ell(H) = k+v-s-2$, then $i_t = 1$

for $1 \leqslant t \leqslant t_0+1$ and $c_v - \ell(H) \geqslant 2$. This implies $i_{t_0+2} = i_{t_0+1} = 1$. Then, we get

$$X_{(\mathcal{I},\mathcal{J})} = \prod_{t=1}^{t_0-1} X_{1,j_t}^{2^{d-t}} \left(X_{1,j_{t_0}}^4 X_{1,j_{t_0+1}}^3\right)^{2^{d-t_0-2}} \prod_{t=t_0+3}^{d} X_{i_t,j_t}^{2^{d-t}}.$$

By Lemma 3.2.4(iii), the monomial $X_{1,j_{t_0}}^4 X_{1,j_{t_0+1}}^7$ is strictly inadmissible. Hence, by Theorem 2.9, $X_{(\mathcal{I},\mathcal{J})}$ is strictly inadmissible. This contradicts the fact that x is admissible.

Since $c_v \leqslant k+v-s-1$ for $1 \leqslant v \leqslant s \leqslant k-2$, one gets

$$\mathrm{rl}(\mathcal{J}) = \sum_{v=1}^{s} c_v \leqslant \sum_{v=1}^{s}(k+v-s-1) = \sum_{u=1}^{s}(k-u) < \binom{k}{2}.$$

The proposition is completely proved. □

From the proofs of Lemmas 3.2.4, Proposition 3.2.8 and Theorem 2.9, we easily obtain the following.

Proposition 3.2.9. *Let $d \geqslant \binom{k}{2}$. For any $(\mathcal{H},\mathcal{K}) \in \mathsf{PSeq}_k^d$, we have*

$$X_{(\mathcal{H},\mathcal{K})} \simeq_d \sum_{u=\min\mathcal{H}+1}^{\min\mathcal{K}} \sum_{(\mathcal{I},\mathcal{J})\in\mathcal{B}_u} X_{(\mathcal{I},\mathcal{J})} \ \mathrm{mod}(\mathcal{P}_{(k,d)}),$$

where $\mathcal{B}_u$ is a set of some pairs $(\mathcal{I},\mathcal{J}) \in \mathsf{PInc}_k^d$ such that $rl(\mathcal{I}) < \binom{k}{2}$, $rl(\mathcal{J}) < \binom{k}{2}$, $\min\mathcal{I} = \min\mathcal{H}$ and $\min\mathcal{J} = u$.

3.3. **Proof of Theorem 1.5.**

Let $n = \sum_{i=1}^{k-2}(2^{d_i}-1)$ with d_i positive integers such that $d_1 > d_2 > \ldots > d_{k-3} \geqslant d_{k-2} := d$, and $m = \sum_{i=1}^{k-3}(2^{d_i-d}-1)$.

Lemma 3.3.1. *Let $d \geqslant d(k-2)$ and let $f, g \in (P_k)_{(k-2)(2^d-1)}$ be homogeneous polynomials and let $y \in (P_k)_m$ be a monomial. If $f \simeq_{(d,(k-2)|^d)} g \ \mathrm{mod}(\mathcal{P}_{(k,d)})$, then $fy^{2^d} \equiv gy^{2^d}$.*

Proof. Note that $z = \prod_{i=1}^{k-2} x^{2^{d_i}-1}$ is the minimal spike of degree n and $\omega_t(z) = k-2$ for $1 \leqslant t \leqslant d$. Suppose $f = g+g_1+\sum_{1\leqslant j<2^d} Sq^j(h_j)$, where $g_1 \in \mathcal{P}_{(k,d)}$ and suitable polynomials $h_j \in P_k$. By Proposition 2.4, $Sq^j(h_j)y^{2^d} = Sq^j(h_jy^{2^d})$ for $1 \leqslant j < 2^d$. If a monomial w appears as a term of the polynomial g_1, then there is an integer h, $1 \leqslant h \leqslant d - d(k-2)$, such that

$$\sum_{i=1}^{h} 2^{i-1}\omega_i(w) < (k-2)(2^h-1) = \sum_{i=1}^{h} 2^{i-1}\omega_i(z).$$

By Theorem 2.15, wy^{2^d} is hit, hence the polynomial $g_1y^{2^d}$ is hit. This implies $fy^{2^d} \equiv gy^{2^d}$. The lemma is proved. □

Definition 3.3.2. Suppose $d_0 > d(k-2)$, and $\mathcal{B}_{(k,d_0)}$ is a subset of $\mathsf{PInc}_k^{d_0}$. The set $\mathcal{B}_{(k,d_0)}$ is said to be compatible with $(k-2)|^{d_0}$ if all of the following hold:

i) For any $(\mathcal{I},\mathcal{J}) \in \mathcal{B}_{(k,d_0)}$, $\mathrm{rl}(\mathcal{I}) \leqslant d_0-2$ and $\mathrm{rl}(\mathcal{J}) \leqslant d_0-2$,

ii) For any $(\mathcal{H},\mathcal{K}) \in \mathsf{PSeq}_k^{d_0}$, we have

$$X_{(\mathcal{H},\mathcal{K})} \simeq_{d_0} \sum_{u=\min\mathcal{H}+1}^{\min\mathcal{K}} \sum_{(\mathcal{I},\mathcal{J})\in\mathcal{B}_u} X_{(\mathcal{I},\mathcal{J})} \bmod(\mathcal{P}_{(k,d_0)}), \tag{3.1}$$

where $\mathcal{B}_u$ is a set of some pairs $(\mathcal{I},\mathcal{J}) \in \mathcal{B}_{(k,d_0)}$ such that $\min\mathcal{I} = \min\mathcal{H}$ and $\min\mathcal{J} = u$.

Obviously, Proposition 3.2.9 shows that for any $d_0 > \binom{k}{2}$, the set

$$\bar{\mathcal{B}}_{(k,d_0)} = \{((i)|^{d_0-\binom{k}{2}}|\mathcal{I},(j)|^{d_0-\binom{k}{2}}|\mathcal{J}) : (\mathcal{I},\mathcal{J}) \in \mathsf{PInc}_k^{\binom{k}{2}},\ i=\min\mathcal{I},\ j=\min\mathcal{J}\}$$

is compatible with $(k-2)|^{d_0}$. By a simple computation, we get

$$|\bar{\mathcal{B}}_{(k,d_0)}| = |\mathsf{PInc}_k^{\binom{k}{2}}| < 2^{(k-1)^2}(2^k-1)(2^{k-1}-1).$$

Theorem 1.5 follows from this inequality, Proposition 3.2.9, Theorem 3.1.5 and the following.

Theorem 3.3.3. *Let $k \geqslant 4$, $n = \sum_{i=1}^{k-2}(2^{d_i}-1)$ with d_i positive integers such that $d_1 > d_2 > \ldots > d_{k-3} \geqslant d_{k-2} = d \geqslant d_0 \geqslant d(k-2)$, and $m = \sum_{i=1}^{k-3}(2^{d_i-d}-1)$. Suppose the set $\mathcal{B}_{(k,d_0)} \subset \mathsf{PInc}_k^{d_0}$ is compatible with $(k-2)|^{d_0}$. Then,*

$$\bigcup_{(\mathcal{I},\mathcal{J})\in\mathcal{B}_{(k,d_0)}} \Big\{X_{(\mathcal{I},\mathcal{J})}(X_{i,j})^{2^d-2^{d_0}}(f_{(i,j)}(z))^{2^d} : z \in B_{k-2}(m),\ i=\min\mathcal{I},\ j=\min\mathcal{J}\Big\}$$

is a set of generators for $\operatorname{Ker}(\widetilde{Sq}^0)_{(k,n)}$. Consequently

$$\dim\operatorname{Ker}(\widetilde{Sq}^0)_{(k,n)} \leqslant |\mathcal{B}_{(k,d_0)}|\dim(QP_{k-2})_m.$$

We need the following lemma for the proof of the theorem.

Lemma 3.3.4. *Let n, m and $\mathcal{B}_{(k,d_0)}$ be as in Theorem 3.3.3. Let y_0 be a monomial in $(P_k)_{m_0-1}$, $y_u = y_0x_u$ for $1 \leqslant u \leqslant k$, and $(\mathcal{I},\mathcal{J}) \in \mathcal{B}_{(k,d_0)}$, $i=\min\mathcal{I}$, $j=\min\mathcal{J}$. Then we have*

$$X_{(\mathcal{I},\mathcal{J})}y_i^{2^{d_0}} \equiv \sum_{\substack{1\leqslant u\leqslant k\\ u\neq i,j}} \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_u} X_{(\mathcal{U},\mathcal{V})}y_u^{2^{d_0}}, \tag{3.2}$$

$$X_{(\mathcal{I},\mathcal{J})}y_j^{2^{d_0}} \equiv \sum_{\substack{1\leqslant v\leqslant k\\ v\neq i,j}} \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{C}_v} X_{(\mathcal{U},\mathcal{V})}y_v^{2^{d_0}}, \tag{3.3}$$

where $\mathcal{B}_u$ is a set of some $(\mathcal{U},\mathcal{V}) \in B_{(k,d_0)}$ such that $\min\mathcal{U} = u$ for $u < i$ and $\min\mathcal{U} = i$ for $u > i$; $\mathcal{C}_v$ is a set of some $(\mathcal{U},\mathcal{V}) \in B_{(k,d_0)}$ such that $\min\mathcal{V} = v$ for $v < j$ and $\min\mathcal{V} = j$ for $v > j$.

Proof. Applying the Cartan formula, we have

$$X_{i,j}y_i^2 = \sum_{\substack{1\leqslant u<j\\ u\neq i}} X_{u,j}y_u^2 + \sum_{j<u\leqslant k} X_{j,u}y_u^2 + Sq^1(X_jy_0^2).$$

We have $X_{(\mathcal{I},\mathcal{J})} = X_{(\mathcal{I}\setminus i,\mathcal{J}\setminus j)}(X_{i,j})^{2^{d_0-1}}$ and

$$X_{(\mathcal{I}\setminus i,\mathcal{J}\setminus j)}(X_{u,j})^{2^{d_0-1}} = \begin{cases} X_{(u|(\mathcal{I}\setminus i),\mathcal{J})}, & \text{if } u<j,\\ X_{(j|(\mathcal{I}\setminus i),u|(\mathcal{J}\setminus j))}, & \text{if } u>j.\end{cases}$$

Using the Cartan formula, Proposition 2.4 and Theorem 2.13 we can see that the polynomial $X_{(\mathcal{I}\setminus i,\mathcal{J}\setminus j)}(Sq^1(X_j y_0^2))^{2^{d_0-1}}$ is hit. So, we get

$$X_{(\mathcal{I},\mathcal{J})}y_i^{2^{d_0}} \equiv \sum_{\substack{1\leqslant u<j\\ u\neq i}} X_{(u|(\mathcal{I}\setminus i),\mathcal{J})}y_u^{2^{d_0}} + \sum_{j<u\leqslant k} X_{(j|(\mathcal{I}\setminus i),u|(\mathcal{J}\setminus j))}y_u^{2^{d_0}}.$$

Since $\mathrm{rl}(\mathcal{I}) < d_0 - 1$ and $\mathrm{rl}(\mathcal{J}) < d_0 - 1$, we have $\min(u|(\mathcal{I}\setminus i)) = u$ for $u < i$, $\min(j|(\mathcal{I}\setminus i)) = \min(u|(\mathcal{I}\setminus i)) = i$ for $i < u < j$ and $\min(u|(\mathcal{J}\setminus j)) = j$ for $u > j$. Hence the relation (3.2) follows from Lemma 3.3.1 and the condition of $\mathcal{B}_{(k,d_0)}$ in Assumption 3.3.2.

The relation (3.3) is proved by a similar computation. □

Lemma 3.3.5. *Let n, m_0 be as in Lemma 3.3.4 and let $\mathcal{P}_k^1(n)$ denote the subspace of $(P_k)_n$ spanned by all monomials of the form $X_{(\mathcal{J},\mathcal{J})}(f_i(y))^{2^{d_0}}$ with $(\mathcal{I},\mathcal{J}) \in \mathcal{B}_{(k,d_0)}$, $i = \min\mathcal{I}$ and $y \in (P_{k-1})_{m_0}$. Then $\mathrm{Ker}(\widetilde{Sq}^0)_{(k,n)} \subset [\mathcal{P}_k^1(n)]$.*

Proof. Let x be a monomial of degree n such that $[x] \in \mathrm{Ker}(\widetilde{Sq}^0)_{(k,n)}$. Using Lemmas 3.2.1 and 3.2.2, we can assume that $\omega_i(x) = k-2$, for $1 \leqslant i \leqslant d$. Since $d \geqslant d_0$, there are sequences of integers $\mathcal{H} = (\alpha_1,\alpha_2,\ldots,\alpha_{d_0})$, $\mathcal{K} = (\beta_1,\beta_2,\ldots,\beta_{d_0})$ such that $1 \leqslant \alpha_t < \beta_t \leqslant k$, for $1 \leqslant t \leqslant d_0$, and $x = X_{(\mathcal{H},\mathcal{K})}\bar{y}^{2^{d_0}}$, where $\bar{y}$ is a monomial of degree m_0 in P_k. By the condition of $\mathcal{B}_{(k,d_0)}$ in Definition 3.3.2, the monomial $X_{(\mathcal{H},\mathcal{K})}$ is of the form (3.1). Hence, using Lemma 3.3.1, one gets

$$x = X_{(\mathcal{H},\mathcal{K})}\bar{y}^{2^{d_0}} \equiv \sum_{u=\min\mathcal{H}+1}^{\min\mathcal{K}} \sum_{(\mathcal{I},\mathcal{J})\in\mathcal{B}_u} X_{(\mathcal{I},\mathcal{J})}\bar{y}^{2^{d_0}},$$

where $\mathcal{B}_u$ is a set of some $(\mathcal{I},\mathcal{J}) \in B_{(k,d_0)}$ such that $\min\mathcal{J} = \min\mathcal{H}$ and $\min\mathcal{J} = u$. For $i = \min\mathcal{H}$, we have $\bar{y} = x_i^a f_i(y)$ with a a non-negative integer and $y \in (P_{k-1})_{m_0-a}$. We prove the lemma by proving $[X_{(\mathcal{I},\mathcal{J})}(x_i^a f_i(y))^{2^{d_0}}] \in [\mathcal{P}_k^1(n)]$ for all $(\mathcal{I},\mathcal{J}) \in \mathcal{B}_u$, $i < u \leqslant \min\mathcal{K}$. We prove this claim by double induction on (a,i).

If $a = 0$, then the claim is true for all $1 \leqslant i < k$. Suppose $a > 0$ and the claim is true for $(a-1,i)$ with $1 \leqslant i < \min\mathcal{J}$.

For $i = 1$, using Lemma 3.3.4 with $y_0 = x_1^{a-1}f_1(y)$, we get

$$X_{(\mathcal{I},\mathcal{J})}(x_1^a f_1(y))^{2^{d_0}} \equiv \sum_{\substack{2\leqslant t\leqslant k\\ t\neq\min\mathcal{J}}} \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_{(t,1)}} X_{(\mathcal{U},\mathcal{V})}(x_1^{a-1}f_1(x_{t-1}y))^{2^{d_0}}, \tag{3.4}$$

where $\mathcal{B}_{(t,1)}$ is a set of some $(\mathcal{U},\mathcal{V}) \in \mathcal{B}_{(k,d_0)}$ such that $\min\mathcal{U} = 1$. By the inductive hypothesis, $[X_{(\mathcal{U},\mathcal{V})}(x_1^{a-1}f_1(x_{t-1}y))^{2^{d_0}}] \in [\mathcal{P}_k^1(n)]$ for all $(\mathcal{U},\mathcal{V}) \in \mathcal{B}_{(t,1)}$ with $1 < t \neq \min\mathcal{J}$. Hence, the claim is true for $(a,1)$.

Suppose $i > 1$ and the claim is true for all (a',t), $1 \leqslant t < i$, and for $(a-1,i)$. Applying Lemma 3.3.4 for $y_0 = x_i^{a-1}f_i(y)$, we have

$$\begin{aligned} X_{(\mathcal{I},\mathcal{J})}(x_i^a f_i(y))^{2^{d_0}} \equiv &\sum_{1\leqslant t<i} \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_{(t,i)}} X_{(\mathcal{U},\mathcal{V})}(x_t x_i^{a-1}f_i(y))^{2^{d_0}}\\ &+ \sum_{\substack{i<t\leqslant k\\ t\neq\min\mathcal{J}}} \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_{(t,i)}} X_{(\mathcal{U},\mathcal{V})}(x_i^{a-1}f_i(x_{t-1}y))^{2^{d_0}}, \end{aligned} \tag{3.5}$$

where $\mathcal{B}_{(t,i)}$ is a set of some $(\mathcal{U},\mathcal{V}) \in \mathcal{B}_{(k,d_0)}$ such that $\min\mathcal{U} = t$ for $t < i$ and $\min\mathcal{U} = i$ for $t > i$. From the relation (3.5) and the inductive hypothesis, we see that our claim is true for (a, i). This completes the proof. □

We now prove Theorem 3.3.3.

Proof of Theorem 3.3.3. Denote by $\mathcal{P}_k(n)$ the subspace of $(P_k)_n$ spanned by the set $B_k(n)$. We prove that $\mathrm{Ker}(\widetilde{Sq}^0)_{(k,n)} \subset [P_k(n)]$. By using Lemma 3.3.5, we prove that $[X_{(\mathcal{I},\mathcal{J})}(f_i(y^*))^{2^d}] \in \mathcal{P}_k(n)$ for all $(\mathcal{I},\mathcal{J}) \in \mathcal{B}_{(k,d_0)}$ with $\min\mathcal{I} = i$ and $y^* \in (P_{k-1})_{m_0}$, where $m_0 = \sum_{t=1}^{k-2}(2^{d_t-d_0}-1)$.

Set $j = \min\mathcal{J}$, we have $f_i(y^*) = x_j^b f_{(i,j)}(y)$ with b a nonnegative integer and $y \in (P_{k-2})_{m_0-b}$. We prove $[X_{(\mathcal{I},\mathcal{J})}(x_j^b f_{(i,j)}(y))^{2^d}] \in [\mathcal{P}_k(n)]$ by double induction on (b, j).

If $b = 0$, then $y \in (P_{k-2})_{m_0}$. Since $\omega_u(y) = k-2$ for $1 \leqslant u \leqslant d-d_0$, we get $y = Y^{2^{d-d_0}-1}(\tilde{y})^{2^{d-d_0}}$, with $\tilde{y} \in (P_{k-2})_m$ and $Y = x_1x_2\ldots x_{k-2}$. Note that $f_{(i,j)}(Y) = X_{i,j}$, hence $f_{(i,j)}(y) = X_{i,j}^{2^{d-d_0}-1}(f_{(i,j)}(\tilde{y}))^{2^{d-d_0}}$. Since $B_{k-2}(m)$ is a set of generators for $(P_{k-2})_m$, there are $z_1, z_2, \ldots, z_r \in B_{k-2}(m)$ such that

$$\tilde{y} \equiv z_1 + z_2 + \ldots + z_r + \sum_{t>0} Sq^t(h_t),$$

where h_t are suitable polynomials in P_{k-2}. By using the Cartan formula and Theorem 2.13, we see that the polynomial $X_{(\mathcal{I},\mathcal{J})}(X_{i,j})^{2^d-2^{d_0}}(Sq^t(f_{(i,j)}(h_t)))^{2^d}$ is hit. Since $f_{(i,j)} : P_{k-2} \to P_k$ is a homomorphism of $\mathcal{A}$-algebras, we get

$$X_{(\mathcal{I},\mathcal{J})}(f_{(i,j)}(y))^{2^d} \equiv \sum_{1\leqslant u\leqslant r} X_{(\mathcal{I},\mathcal{J})}(X_{i,j})^{2^d-2^{d_0}}(f_{(i,j)}(z_u))^{2^d} \in \mathcal{P}_k(n).$$

Hence, our claim is true for $(0, j)$, $i < j \leqslant k$. We assume $b > 0$ and our claim holds for all $(b-1, j)$ with $i < j \leqslant k$.

For $j = 2$, we have $i = 1$. Applying Lemma 3.3.4 for $y_0 = x_2^{b-1}f_{(1,2)}(y)$ we obtain

$$X_{(\mathcal{I},\mathcal{J})}(x_2^b f_{(1,2)}(y))^{2^{d_0}} \equiv \sum_{3\leqslant t\leqslant k}\ \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_t} X_{(\mathcal{U},\mathcal{V})}(x_2^{b-1}f_{(1,2)}(x_{t-2}y))^{2^{d_0}},$$

where $\mathcal{B}_t$ is a set of some $(\mathcal{U},\mathcal{V}) \in \mathcal{B}_{(k,d_0)}$ such that $\min\mathcal{V} = 2$. The last equality and the inductive hypothesis imply our claim for $(b, 2)$.

Suppose $j > 2$ and the claim holds for all (b', t) with $1 \leqslant i < t < j$ and for $(b-1, j)$. Using Lemma 3.3.4 with $y_0 = x_j^{b-1}f_{(i,j)}(y)$, we have

$$\begin{aligned}
X_{(\mathcal{I},\mathcal{J})}(x_j^b f_{(i,j)}(y))^{2^{d_0}} \equiv &\sum_{1\leqslant t<i}\ \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_t} X_{(\mathcal{U},\mathcal{V})}(f_i(x_t x_{j-1}^{b-1}y))^{2^{d_0}}\\
&+ \sum_{i<t<j}\ \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_t} X_{(\mathcal{U},\mathcal{V})}(f_i(x_{t-1}x_{j-1}^{b-1}y))^{2^{d_0}}\\
&+ \sum_{j<t\leqslant k}\ \sum_{(\mathcal{U},\mathcal{V})\in\mathcal{B}_t} X_{(\mathcal{U},\mathcal{V})}(x_j^{b-1}f_{(i,j)}(x_{t-2}y))^{2^{d_0}},
\end{aligned}$$

where $\mathcal{B}_t$ is a set of some $(\mathcal{U},\mathcal{V}) \in \mathcal{B}_{(k,d_0)}$ such that $\min\mathcal{V} = i$ for $t < i$, $\min\mathcal{V} = t$ for $i < t < j$ and $\min\mathcal{V} = j$ for $t > j$. From the last equality and the inductive hypothesis, our claim is true for (b, j). The theorem is proved □

4. Applications to the case $k = 5$

In this section, we explicitly determine the spaces $QP_5((3)|^d)$ for $d \geqslant 5$. By using this results and Theorem 3.3.3 and prove Theorems 1.6 and 1.8.

4.1. The admissible monomials of the weight vector $(3)|^d$ in P_5.

From the results of Kameko [10] and our work [33] we see that if $d \geqslant 4$, then $QP_3^+((3)|^d) = \langle [(x_1x_2x_3)^{2^d-1}]_{(3)|^d}\rangle$ and $QP_4^+((3)|^d) = \langle \{[w_{d,u}]_{(3)|^d} : 1 \leqslant u \leqslant 11\}\rangle$, where

$$\begin{array}{ll}
w_{d,1} = x_1x_2^{2^d-2}x_3^{2^d-1}x_4^{2^d-1} & w_{d,2} = x_1x_2^{2^d-1}x_3^{2^d-2}x_4^{2^d-1}\\
w_{d,3} = x_1x_2^{2^d-1}x_3^{2^d-1}x_4^{2^d-2} & w_{d,4} = x_1^{2^d-1}x_2x_3^{2^d-2}x_4^{2^d-1}\\
w_{d,5} = x_1^{2^d-1}x_2x_3^{2^d-1}x_4^{2^d-2} & w_{d,6} = x_1^{2^d-1}x_2^{2^d-1}x_3x_4^{2^d-2}\\
w_{d,7} = x_1^{3}x_2^{2^d-3}x_3^{2^d-2}x_4^{2^d-1} & w_{d,8} = x_1^{3}x_2^{2^d-3}x_3^{2^d-1}x_4^{2^d-2}\\
w_{d,9} = x_1^{3}x_2^{2^d-1}x_3^{2^d-3}x_4^{2^d-2} & w_{d,10} = x_1^{2^d-1}x_2^{3}x_3^{2^d-3}x_4^{2^d-2}\\
w_{d,11} = x_1^{7}x_2^{2^d-5}x_3^{2^d-3}x_4^{2^d-2} &
\end{array}$$

Applying Proposition 2.16, we get $\dim QP_5^0((3)|^d) = \binom{5}{3} + 11\binom{5}{4} = 65$. So we need only to determine $QP_5^+((3)|^d)$. The main result of the subsection is the following.

Theorem 4.1.1. *Let d be a positive integer. If $d \geqslant 5$, then $QP_5^+((3)|^d)$ is an $\mathbb{F}_2$-vector space of dimension 90 with a basis consisting the classes represented by the mononials $a_{d,t}$, $1 \leqslant t \leqslant 90$, which are determined as follows:*

1. $x_1x_2x_3^{2^d-2}x_4^{2^d-2}x_5^{2^d-1}$ 2. $x_1x_2x_3^{2^d-2}x_4^{2^d-1}x_5^{2^d-2}$ 3. $x_1x_2x_3^{2^d-1}x_4^{2^d-2}x_5^{2^d-2}$
4. $x_1x_2^{2^d-2}x_3x_4^{2^d-2}x_5^{2^d-1}$ 5. $x_1x_2^{2^d-2}x_3x_4^{2^d-1}x_5^{2^d-2}$ 6. $x_1x_2^{2^d-2}x_3^{2^d-1}x_4x_5^{2^d-2}$
7. $x_1x_2^{2^d-1}x_3x_4^{2^d-2}x_5^{2^d-2}$ 8. $x_1x_2^{2^d-1}x_3^{2^d-2}x_4x_5^{2^d-2}$ 9. $x_1^{2^d-1}x_2x_3x_4^{2^d-2}x_5^{2^d-2}$
10. $x_1^{2^d-1}x_2x_3^{2^d-2}x_4x_5^{2^d-2}$ 11. $x_1x_2^2x_3^{2^d-4}x_4^{2^d-1}x_5^{2^d-1}$ 12. $x_1x_2^2x_3^{2^d-1}x_4^{2^d-4}x_5^{2^d-1}$
13. $x_1x_2^2x_3^{2^d-1}x_4^{2^d-1}x_5^{2^d-4}$ 14. $x_1x_2^{2^d-1}x_3^2x_4^{2^d-4}x_5^{2^d-1}$ 15. $x_1x_2^{2^d-1}x_3^2x_4^{2^d-1}x_5^{2^d-4}$
16. $x_1x_2^{2^d-1}x_3^{2^d-1}x_4^2x_5^{2^d-4}$ 17. $x_1^{2^d-1}x_2x_3^2x_4^{2^d-4}x_5^{2^d-1}$ 18. $x_1^{2^d-1}x_2x_3^2x_4^{2^d-1}x_5^{2^d-4}$
19. $x_1^{2^d-1}x_2x_3^{2^d-1}x_4^2x_5^{2^d-4}$ 20. $x_1^{2^d-1}x_2^{2^d-1}x_3x_4^2x_5^{2^d-4}$ 21. $x_1x_2^2x_3^{2^d-3}x_4^{2^d-2}x_5^{2^d-1}$
22. $x_1x_2^2x_3^{2^d-3}x_4^{2^d-1}x_5^{2^d-2}$ 23. $x_1x_2^2x_3^{2^d-1}x_4^{2^d-3}x_5^{2^d-2}$ 24. $x_1x_2^{2^d-1}x_3^2x_4^{2^d-3}x_5^{2^d-2}$
25. $x_1^{2^d-1}x_2x_3^2x_4^{2^d-3}x_5^{2^d-2}$ 26. $x_1x_2^3x_3^{2^d-4}x_4^{2^d-2}x_5^{2^d-1}$ 27. $x_1x_2^3x_3^{2^d-4}x_4^{2^d-1}x_5^{2^d-2}$
28. $x_1x_2^3x_3^{2^d-2}x_4^{2^d-4}x_5^{2^d-1}$ 29. $x_1x_2^3x_3^{2^d-2}x_4^{2^d-1}x_5^{2^d-4}$ 30. $x_1x_2^3x_3^{2^d-1}x_4^{2^d-4}x_5^{2^d-2}$
31. $x_1x_2^3x_3^{2^d-1}x_4^{2^d-2}x_5^{2^d-4}$ 32. $x_1x_2^{2^d-1}x_3^3x_4^{2^d-4}x_5^{2^d-2}$ 33. $x_1x_2^{2^d-1}x_3^3x_4^{2^d-2}x_5^{2^d-4}$
34. $x_1^3x_2x_3^{2^d-4}x_4^{2^d-2}x_5^{2^d-1}$ 35. $x_1^3x_2x_3^{2^d-4}x_4^{2^d-1}x_5^{2^d-2}$ 36. $x_1^3x_2x_3^{2^d-2}x_4^{2^d-4}x_5^{2^d-1}$
37. $x_1^3x_2x_3^{2^d-2}x_4^{2^d-1}x_5^{2^d-4}$ 38. $x_1^3x_2x_3^{2^d-1}x_4^{2^d-4}x_5^{2^d-2}$ 39. $x_1^3x_2x_3^{2^d-1}x_4^{2^d-2}x_5^{2^d-4}$
40. $x_1^3x_2^{2^d-1}x_3x_4^{2^d-4}x_5^{2^d-2}$ 41. $x_1^3x_2^{2^d-1}x_3x_4^{2^d-2}x_5^{2^d-4}$ 42. $x_1^{2^d-1}x_2x_3^3x_4^{2^d-4}x_5^{2^d-2}$
43. $x_1^{2^d-1}x_2x_3^3x_4^{2^d-2}x_5^{2^d-4}$ 44. $x_1^{2^d-1}x_2^3x_3x_4^{2^d-4}x_5^{2^d-2}$ 45. $x_1^{2^d-1}x_2^3x_3x_4^{2^d-2}x_5^{2^d-4}$
46. $x_1^3x_2^{2^d-3}x_3^2x_4^{2^d-4}x_5^{2^d-1}$ 47. $x_1^3x_2^{2^d-3}x_3^2x_4^{2^d-1}x_5^{2^d-4}$ 48. $x_1^3x_2^{2^d-3}x_3^{2^d-1}x_4^2x_5^{2^d-4}$
49. $x_1^3x_2^{2^d-1}x_3^{2^d-3}x_4^2x_5^{2^d-4}$ 50. $x_1^{2^d-1}x_2^3x_3^{2^d-3}x_4^2x_5^{2^d-4}$ 51. $x_1^3x_2^5x_3^{2^d-6}x_4^{2^d-4}x_5^{2^d-1}$
52. $x_1^3x_2^5x_3^{2^d-6}x_4^{2^d-1}x_5^{2^d-4}$ 53. $x_1^3x_2^5x_3^{2^d-1}x_4^{2^d-6}x_5^{2^d-4}$ 54. $x_1^3x_2^{2^d-1}x_3^5x_4^{2^d-6}x_5^{2^d-4}$
55. $x_1^{2^d-1}x_2^3x_3^5x_4^{2^d-6}x_5^{2^d-4}$ 56. $x_1x_2^3x_3^{2^d-3}x_4^{2^d-2}x_5^{2^d-2}$ 57. $x_1x_2^3x_3^{2^d-2}x_4^{2^d-3}x_5^{2^d-2}$
58. $x_1x_2^{2^d-2}x_3^3x_4^{2^d-3}x_5^{2^d-2}$ 59. $x_1^3x_2x_3^{2^d-3}x_4^{2^d-2}x_5^{2^d-2}$ 60. $x_1^3x_2x_3^{2^d-2}x_4^{2^d-3}x_5^{2^d-2}$
61. $x_1^3x_2^{2^d-3}x_3x_4^{2^d-2}x_5^{2^d-2}$ 62. $x_1^3x_2^{2^d-3}x_3^{2^d-2}x_4x_5^{2^d-2}$ 63. $x_1^3x_2^{2^d-3}x_3^2x_4^{2^d-3}x_5^{2^d-2}$
64. $x_1x_2^6x_3^{2^d-5}x_4^{2^d-3}x_5^{2^d-2}$ 65. $x_1x_2^7x_3^{2^d-6}x_4^{2^d-3}x_5^{2^d-2}$ 66. $x_1^7x_2x_3^{2^d-6}x_4^{2^d-3}x_5^{2^d-2}$

67. $x_1x_2^7x_3^{2^d-5}x_4^{2^d-4}x_5^{2^d-2}$	68. $x_1x_2^7x_3^{2^d-5}x_4^{2^d-2}x_5^{2^d-4}$	69. $x_1^7x_2x_3^{2^d-5}x_4^{2^d-4}x_5^{2^d-2}$
70. $x_1^7x_2x_3^{2^d-5}x_4^{2^d-2}x_5^{2^d-4}$	71. $x_1^7x_2^{2^d-5}x_3x_4^{2^d-4}x_5^{2^d-2}$	72. $x_1^7x_2^{2^d-5}x_3x_4^{2^d-2}x_5^{2^d-4}$
73. $x_1^7x_2^{2^d-5}x_3^{2^d-3}x_4^2x_5^{2^d-4}$	74. $x_1^3x_2^3x_3^{2^d-4}x_4^{2^d-3}x_5^{2^d-2}$	75. $x_1^3x_2^3x_3^{2^d-3}x_4^{2^d-4}x_5^{2^d-2}$
76. $x_1^3x_2^3x_3^{2^d-3}x_4^{2^d-2}x_5^{2^d-4}$	77. $x_1^3x_2^{2^d-3}x_3^3x_4^{2^d-4}x_5^{2^d-2}$	78. $x_1^3x_2^{2^d-3}x_3^3x_4^{2^d-2}x_5^{2^d-4}$
79. $x_1^3x_2^4x_3^{2^d-5}x_4^{2^d-3}x_5^{2^d-2}$	80. $x_1^3x_2^5x_3^{2^d-6}x_4^{2^d-3}x_5^{2^d-2}$	81. $x_1^3x_2^5x_3^{2^d-5}x_4^{2^d-4}x_5^{2^d-2}$
82. $x_1^3x_2^5x_3^{2^d-5}x_4^{2^d-2}x_5^{2^d-4}$	83. $x_1^3x_2^7x_3^{2^d-7}x_4^{2^d-4}x_5^{2^d-2}$	84. $x_1^3x_2^7x_3^{2^d-7}x_4^{2^d-2}x_5^{2^d-4}$
85. $x_1^7x_2^3x_3^{2^d-7}x_4^{2^d-4}x_5^{2^d-2}$	86. $x_1^7x_2^3x_3^{2^d-7}x_4^{2^d-2}x_5^{2^d-4}$	87. $x_1^3x_2^7x_3^{2^d-3}x_4^{2^d-6}x_5^{2^d-4}$
88. $x_1^7x_2^3x_3^{2^d-3}x_4^{2^d-6}x_5^{2^d-4}$	89. $x_1^7x_2^{2^d-5}x_3^5x_4^{2^d-6}x_5^{2^d-4}$	90. $x_1^7x_2^{11}x_3^{2^d-11}x_4^{2^d-6}x_5^{2^d-4}$

4.1.1. *Generating sets for $QP_5((3)|^d)$ with $d \leqslant 4$.*

Proposition 4.1.2. *We have*

i) $B_5((3)|^1) = \{X_{\alpha,\beta} : 1 \leqslant \alpha < \beta \leqslant 5\}$.

ii) $B_5^+((3)|^2)$ *is the set of the monomials* $a_{2,t}$, $1 \leqslant t \leqslant 15$, *which are determine as follows:*

1. $x_1x_2x_3^2x_4^2x_5^3$	2. $x_1x_2x_3^2x_4^3x_5^2$	3. $x_1x_2x_3^3x_4^2x_5^2$	4. $x_1x_2^2x_3x_4^2x_5^3$
5. $x_1x_2^2x_3x_4^3x_5^2$	6. $x_1x_2^2x_3^2x_4x_5^3$	7. $x_1x_2^2x_3^2x_4^3x_5$	8. $x_1x_2^2x_3^3x_4x_5^2$
9. $x_1x_2^2x_3^3x_4^2x_5$	10. $x_1x_2^3x_3x_4^2x_5^2$	11. $x_1x_2^3x_3^2x_4x_5^2$	12. $x_1x_2^3x_3^2x_4^2x_5$
13. $x_1^3x_2x_3x_4^2x_5^2$	14. $x_1^3x_2x_3^2x_4x_5^2$	15. $x_1^3x_2x_3^2x_4^2x_5$.	

Proof. For $d = 1$, if $x \in P_5((3)|^1)$, then $\omega(x) = (3)|^1$ if and only if $x = X_{\alpha,\beta}$ with $1 \leqslant \alpha < \beta \leqslant 5$. Since $X_{\alpha,\beta}$ is admissible, we see that the first of Proposition 4.1.2 is true.

From the results in Kameko [10] and our work [33], we have $B_3^+((3)|^2) = \{x_1^3x_2^3x_3^3\}$ and $B_4^+((3)|^2) = \{w_{2,1}, w_{2,2}, \ldots, w_{2,6}\}$, where

$$w_{2,1} = x_1x_2^2x_3^3x_4^3,\ w_{2,2} = x_1x_2^3x_3^2x_4^3,\ w_{2,3} = x_1x_2^3x_3^3x_4^2,$$
$$w_{2,4} = x_1^3x_2x_3^2x_4^3,\ w_{2,5} = x_1^3x_2x_3^3x_4^2,\ w_{2,6} = x_1^3x_2^3x_3x_4^2.$$

Hence, applying Proposition 2.16, we get $\dim QP_5^0((3)|^2) = \binom{5}{3} + 6\binom{5}{4} = 40$. So, we need only to determine $QP_5^+((3)|^2)$. By a direct computation we see that if $x \in P_5^+((3)|^2)$ and $x \neq a_{2,t}$ for all t, $1 \leqslant t \leqslant 15$, then $x = X_{i_1,j_1}^2X_{i_2,j_2}$ with $i_1 > i_2$. By Lemma 3.2.4, x is inadmissible.

We observe that for $1 \leqslant t \leqslant 15$, $a_{2,t} = x_if_i(b_{2,t})$ with $b_{2,t}$ an admissible monomial of degree 8 in P_4 and $1 \leqslant i \leqslant 5$. By Proposition 2.11, $a_{2,t}$ is admissible. The proposition is proved. □

Consider the case $d = 3$. From the results in Kameko [10] and our work [33], we have $B_3^+((3)|^3) = \{(x_1x_2x_3)^7\}$ and $B_4^+((3)|^3) = \{w_{3,i} : 1 \leqslant i \leqslant 10\}$, where $w_{3,i}$, $1 \leqslant i \leqslant 10$, are determined as in the beginning of this subsection. Namely,

$w_{3,1} = x_1x_2^6x_3^7x_4^7$	$w_{3,2} = x_1x_2^7x_3^6x_4^7$	$w_{3,3} = x_1x_2^7x_3^7x_4^6$	$w_{3,4} = x_1^7x_2x_3^6x_4^7$
$w_{3,5} = x_1^7x_2x_3^7x_4^6$	$w_{3,6} = x_1^7x_2^7x_3x_4^6$	$w_{3,7} = x_1^3x_2^5x_3^6x_4^7$	$w_{3,8} = x_1^3x_2^5x_3^7x_4^6$
$w_{3,9} = x_1^3x_2^7x_3^5x_4^6$	$w_{3,10} = x_1^7x_2^3x_3^5x_4^6$		

So, using Proposition 2.16, we get $\dim QP_5^0((3)|^3) = \binom{5}{3} + 10\binom{5}{4} = 60$. We need to compute $QP_5^+((3)|^3)$.

We denote by $A(3)$ the set of the monomilas $a_{3,t}$, $1 \leqslant t \leqslant 50$, which are given in Theorem 4.1.1 for $d = 3$ and five monomials:

$a_{3,51} = x_1^3x_2^3x_3^4x_4^4x_5^7$	$a_{3,52} = x_1^3x_2^3x_3^4x_4^7x_5^4$	$a_{3,53} = x_1^3x_2^3x_3^7x_4^4x_5^4$
$a_{3,54} = x_1^3x_2^7x_3^3x_4^4x_5^4$	$a_{3,55} = x_1^7x_2^3x_3^3x_4^4x_5^4$	

Proposition 4.1.3. *$B_5^+((3)|^3) = A(3) \cup C(3)$, where $C(3)$ is the set of the monomials $a_{3,t}$, $56 \leqslant t \leqslant 70$ which are determined as follows:*

56. $x_1x_2^3x_3^5x_4^6x_5^6$	57. $x_1x_2^3x_3^6x_4^5x_5^6$	58. $x_1x_2^6x_3^3x_4^5x_5^6$	59. $x_1^3x_2x_3^5x_4^6x_5^6$
60. $x_1^3x_2x_3^6x_4^5x_5^6$	61. $x_1^3x_2^5x_3x_4^6x_5^6$	62. $x_1^3x_2^5x_3^6x_4x_5^6$	63. $x_1^3x_2^5x_3^2x_4^5x_5^6$
64. $x_1^3x_2^3x_3^4x_4^5x_5^6$	65. $x_1^3x_2^3x_3^5x_4^4x_5^6$	66. $x_1^3x_2^3x_3^5x_4^6x_5^4$	67. $x_1^3x_2^4x_3^3x_4^5x_5^6$
68. $x_1^3x_2^5x_3^3x_4^4x_5^6$	69. $x_1^3x_2^5x_3^3x_4^6x_5^4$	70. $x_1^3x_2^5x_3^6x_4^3x_5^4$	

We prepare some lemmas for the proof of this proposition.

Lemma 4.1.4. *Let x is one of the monomials: $x_1x_2^6x_3^6x_4$, $x_1x_2^2x_3^6x_4^5$, $x_1x_2^6x_3^2x_4^5$, $x_1x_2^6x_3^3x_4^4$, $x_1^3x_2^4x_3x_4^6$, $x_1^3x_2^4x_3^5x_4^2$, $x_1^3x_2^5x_3^4x_4^2$. Then, the monomial $x_i^7f_i(x)$, $1 \leqslant i \leqslant 5$, is strictly inadmissible.*

Proof. We prove the lemma for some monomials of the form $f_5(x)$. The others can be proved by similar computations. We have

$$\begin{aligned}
x_1x_2^6x_3^6x_4x_5^7 &= x_1x_2^5x_3^6x_4^2x_5^7 + x_1x_2^6x_3^5x_4^2x_5^7 + Sq^1(x_1^2x_2^5x_3^5x_4x_5^7)\\
&\quad + Sq^2(x_1x_2^5x_3^5x_4x_5^7 + x_1x_2^3x_3^5x_4x_5^9) \ \mathrm{mod}(P_5^-((3)|^3)),\\
x_1x_2^2x_3^6x_4^5x_5^7 &= x_1x_2x_3^6x_4^6x_5^7 + x_1x_2^2x_3^5x_4^6x_5^7 + Sq^1(x_1^2x_2x_3^5x_4^5x_5^7)\\
&\quad + Sq^2(x_1x_2x_3^5x_4^5x_5^7 + x_1x_2x_3^3x_4^5x_5^9) \ \mathrm{mod}(P_5^-((3)|^3)),\\
x_1x_2^6x_3^3x_4^4x_5^7 &= x_1x_2^3x_3^4x_4^6x_5^7 + x_1x_2^3x_3^6x_4^4x_5^7 + x_1x_2^4x_3^3x_4^6x_5^7 + x_1x_2^4x_3^6x_4^3x_5^7\\
&\quad + x_1x_2^6x_3^2x_4^5x_5^7 + Sq^1(x_1^2x_2^3x_3^3x_4^5x_5^7 + x_1^2x_2^3x_3^5x_4^3x_5^7 + x_1^2x_2^5x_3^3x_4^3x_5^7)\\
&\quad + Sq^2(x_1x_2^3x_3^3x_4^3x_5^9 + x_1x_2^3x_3^3x_4^5x_5^7 + x_1x_2^3x_3^5x_4^3x_5^7\\
&\qquad + x_1x_2^5x_3^3x_4^3x_5^7 + x_1x_2^6x_3^2x_4^3x_5^7) \ \mathrm{mod}(P_5^-((3)|^3)),\\
x_1^3x_2^5x_3^4x_4^2x_5^7 &= x_1^3x_2^3x_3^4x_4^4x_5^7 + x_1^3x_2^5x_3^2x_4^4x_5^7 + Sq^1(x_1^3x_2^3x_3x_4^2x_5^{11})\\
&\quad + Sq^2(x_1^5x_2^3x_3^2x_4^2x_5^7) + Sq^4(x_1^3x_2^3x_3^2x_4^2x_5^7) \ \mathrm{mod}(P_5^-((3)|^3)).
\end{aligned}$$

Hence, the above monomials are strictly inadmissible. ☐

Lemma 4.1.5. *The following monomials are strictly inadmissible:*

$x_1^3x_2^5x_3^5x_4^2x_5^6$	$x_1^3x_2^5x_3^5x_4^6x_5^2$	$x_1^3x_2^5x_3^6x_4^5x_5^2$	$x_1^3x_2^4x_3^5x_4^3x_5^6$
$x_1^3x_2^4x_3^5x_4^6x_5^3$	$x_1^3x_2^5x_3^4x_4^3x_5^6$	$x_1^3x_2^5x_3^4x_4^6x_5^3$	$x_1^3x_2^5x_3^6x_4^4x_5^3$

Proof. We prove the lemma for the monomial $x = x_1x_2^6x_3^6x_4x_5^7$. The others can be proved by a similar computation. We have

$$\begin{aligned}
x &= x_1^3x_2^3x_3^6x_4^4x_5^5 + x_1^3x_2^5x_3^6x_4^2x_5^5 + Sq^1(x_1^3x_2^3x_3^9x_4^2x_5^3)\\
&\quad + Sq^2(x_1^5x_2^3x_3^6x_4^2x_5^3) + Sq^4(x_1^3x_2^3x_3^6x_4^2x_5^3) \ \mathrm{mod}(P_5^-((3)|^3)).
\end{aligned}$$

These equality implies x is strictly inadmissible. ☐

Proof of Proposition 4.1.3. By a direct computation we see that if $x \in P_5^+((3)|^3)$ and $x \neq a_{3,t}$ for all t, $1 \leqslant t \leqslant 70$, then either x is one of the monomials as given in Lemmas 4.1.4, 4.1.5, or x is of the form $X_{i,j}^4X_{i_1,j_1}^2X_{i_2,j_2}$ with $i_1 > i_2$. Hence, by Lemma 3.2.4(i) and Theorem 2.9, x is inadmissible.

We now prove that the set $\{[a_{3,t}] : 1 \leqslant t \leqslant 70\}$ is linearly independent in $QP_5^+((3)|^3)$.

Consider $\langle [A(3)]_{(3)|^3} \rangle \subset QP_5((3)|^3)$ and $\langle [C(3)]_{(3)|^3} \rangle \subset QP_5((3)|^3)$. We see that for $1 \leqslant t \leqslant 55$, $a_{3,t} = x_i^7f_i(b_{3,t})$ with $b_{3,t}$ an admissible monomial of degree 14 in

P_4. By Proposition 2.11, $a_{3,t}$ is admissible. This implies $\dim\langle[A(3)]_{(3)|^3}\rangle = 55$. On the other hand, we have $\langle[A(3)]_{(3)|^3}\rangle \cap \langle[C(3)]_{(3)|^3}\rangle = \{0\}$. Hence, we need only to prove the set $[C(3)]_{(3)|^3}$ is linearly independent. Suppose there is a linear relation

$$S := \sum_{56\leqslant t\leqslant 70} \gamma_t a_{3,t} \equiv_{(3)|^3} 0, \tag{4.1}$$

where $\gamma_t \in \mathbb{F}_2$.

Applying the homomorphism $p_{(i;j)} : P_5 \to P_4$ to (4.1), we obtain

$$\begin{aligned}
&p_{(1;2)}(S) \equiv_{(3)|^3} (\gamma_{58} + \gamma_{67})w_{3,10} \equiv_{(3)|^3} 0,\\
&p_{(1;3)}(S) \equiv_{(3)|^3} (\gamma_{57} + \gamma_{64})w_{3,9} \equiv_{(3)|^3} 0,\\
&p_{(1;4)}(S) \equiv_{(3)|^3} (\gamma_{56} + \gamma_{65} + \gamma_{68})w_{3,8} \equiv_{(3)|^3} 0,\\
&p_{(1;5)}(S) \equiv_{(3)|^3} (\gamma_{56} + \gamma_{57} + \gamma_{58} + \gamma_{66} + \gamma_{69} + \gamma_{70})w_{3,7} \equiv_{(3)|^3} 0,\\
&p_{(2;3)}(S) \equiv_{(3)|^3} (\gamma_{60} + \gamma_{63} + \gamma_{64} + \gamma_{67})w_{3,9} \equiv_{(3)|^3} 0,\\
&p_{(2;4)}(S) \equiv_{(3)|^3} (\gamma_{59} + \gamma_{65})w_{3,8} \equiv_{(3)|^3} 0,\\
&p_{(2;5)}(S) \equiv_{(3)|^3} (\gamma_{59} + \gamma_{60} + \gamma_{66})w_{3,7} \equiv_{(3)|^3} 0,\\
&p_{(3;4)}(S) \equiv_{(3)|^3} (\gamma_{61} + \gamma_{62} + \gamma_{63} + \gamma_{68})w_{3,8} \equiv_{(3)|^3} 0,\\
&p_{(3;5)}(S) \equiv_{(3)|^3} (\gamma_{61} + \gamma_{69})w_{3,7} \equiv_{(3)|^3} 0,\\
&p_{(4;5)}(S) \equiv_{(3)|^3} (\gamma_{62} + \gamma_{70})w_{3,9} \equiv_{(3)|^3} 0.
\end{aligned}$$

From these equalities, we get $\gamma_{67} = \gamma_{58}$, $\gamma_{64} = \gamma_{57}$, $\gamma_{65} = \gamma_{59}$, $\gamma_{69} = \gamma_{61}$, $\gamma_{70} = \gamma_{62}$. Then, applying the homomorphism $p_{(1;(i,j))} : P_5 \to P_4$ to (4.1), we get

$$\begin{aligned}
&p_{(1;(2,3))}(S) \equiv_{(3)|^3} \gamma_{60}w_{3,9} + \gamma_{63}w_{3,10} \equiv_{(3)|^3} 0,\\
&p_{(1;(2,4))}(S) \equiv_{(3)|^3} (\gamma_{56} + \gamma_{58} + \gamma_{68})w_{3,9} + \gamma_{68}w_{3,10} \equiv_{(3)|^3} 0,\\
&p_{(1;(3,4))}(S) \equiv_{(3)|^3} (\gamma_{59} + \gamma_{60} + \gamma_{66})w_{3,3} + (\gamma_{59} + \gamma_{62})w_{3,9}\\
&\qquad + (\gamma_{56} + \gamma_{57} + \gamma_{59} + \gamma_{61} + \gamma_{63} + \gamma_{68})w_{3,8} \equiv_{(3)|^3} 0.
\end{aligned}$$

From these above equalities we obtain $\gamma_{60} = \gamma_{63} = \gamma_{68} = 0$ and $\gamma_t = \gamma_{56}$ for $56 \leqslant t \leqslant 70$ and $t \neq 60, 63, 68$. Hence, the relation (4.1) becomes $S = \gamma_{56}\theta \equiv_{(3)|^3} 0$, where $\theta = \sum_{t\neq 60,63,68} a_{3,t}$. By a direct computation, we can show that $a_{3,70} = x_1^3x_2^5x_3^6x_4^3x_5^4$ is admissible. So, we get $\gamma_{56} = 0$. The proposition follows. □

We consider the case $d = 4$. From the results in Kameko [10], we get $B_3^+((3)|^4) = \{(x_1x_2x_3)^7\}$ and $B_4^+((3)|^4) = \{w_{4,t} : 1 \leqslant t \leqslant 11\}$, where $w_{3,t}$ are determined as in the beginning of this subsection. By Proposition 2.16, $\dim QP_5((3)|^4) = \binom{5}{3} + 11\binom{5}{4} = 65$. We need to determined the set $B_5^+((3)|^4)$.

For $d \geqslant 4$, we denote $A(d)$ the set of monomials $a_{d,t}$, $1 \leqslant t \leqslant 55$, which are determined as in Theorem 4.1.1.

Proposition 4.1.6. *$B_5^+((3)|^4) = A(4) \cup C(4)$, where $C(4)$ is the set of the monomials $a_{4,t}$, $56 \leqslant t \leqslant 89$ which are determined as in Theorem 4.1.1 for $d = 4$ and the following monomials:*

$$\begin{array}{lll}
a_{4,90} = x_1^3x_2^7x_3^8x_4^{13}x_5^{14} & a_{4,91} = x_1^7x_2^3x_3^8x_4^{13}x_5^{14} & a_{4,92} = x_1^7x_2^7x_3^8x_4^9x_5^{14}\\
a_{4,93} = x_1^7x_2^7x_3^9x_4^8x_5^{14} & a_{4,94} = x_1^7x_2^7x_3^9x_4^{10}x_5^{12}. &
\end{array}$$

We need the following lemma for the proof of this proposition.

Lemma 4.1.7. *If x is one of the monomials: $x_1x_2^7x_3^{10}x_4^{12}$, $x_1^7x_2x_3^{10}x_4^{12}$, $x_1^3x_2^3x_3^{12}x_4^{12}$, $x_1^3x_2^5x_3^8x_4^{14}$, $x_1^3x_2^5x_3^{14}x_4^8$, then the monomial $x_i^{15}f_i(x)$, $1 \leqslant i \leqslant 5$, is strictly inadmissible.*

Proof. We prove the lemma for monomials of the form $f_5(x)$. The others can be proved by similar computations. We have

$$\begin{aligned}
x_1x_2^7x_3^{10}x_4^{12}x_5^{15} &= x_1x_2^4x_3^{11}x_4^{14}x_5^{15} + x_1x_2^6x_3^{11}x_4^{12}x_5^{15} + x_1x_2^7x_3^8x_4^{13}x_5^{14}\\
&+ Sq^1(x_1x_2^5x_3^{11}x_4^{12}x_5^{15} + x_1x_2^7x_3^8x_4^{13}x_5^{15}\\
&+ x_1x_2^7x_3^9x_4^{12}x_5^{15} + x_1x_2^8x_3^7x_4^{13}x_5^{15} + x_1^4x_2^7x_3^7x_4^{11}x_5^{15})\\
&+ Sq^2(x_1^2x_2^7x_3^7x_4^{12}x_5^{15} + x_1^2x_2^7x_3^8x_4^{11}x_5^{15} + x_1^2x_2^8x_3^7x_4^{11}x_5^{15})\\
&+ Sq^4(x_1x_2^4x_3^7x_4^{14}x_5^{15} + x_1^2x_2^5x_3^7x_4^{12}x_5^{15}) \bmod(P_5^-((3)|^4))\\
x_1^3x_2^3x_3^{12}x_4^{12}x_5^{15} &= x_1^2x_2^3x_3^{13}x_4^{12}x_5^{15} + x_1^2x_2^4x_3^{11}x_4^{13}x_5^{15} + x_1^2x_2^5x_3^{11}x_4^{12}x_5^{15}\\
&+ Sq^1(x_1^3x_2^3x_3^{11}x_4^{12}x_5^{15} + x_1^3x_2^8x_3^7x_4^{11}x_5^{15})\\
&+ Sq^2(x_1^2x_2^3x_3^{11}x_4^{12}x_5^{15} + x_1^2x_2^8x_3^7x_4^{11}x_5^{15})\\
&+ Sq^4(x_1^3x_2^4x_3^7x_4^{12}x_5^{15} + x_1^2x_2^4x_3^7x_4^{13}x_5^{15}) \bmod(P_5^-((3)|^4))\\
x_1^3x_2^5x_3^8x_4^{14}x_5^{15} &= x_1^2x_2^5x_3^9x_4^{14}x_5^{15} + x_1^3x_2^4x_3^9x_4^{14}x_5^{15}\\
&+ Sq^1(x_1^3x_2^3x_3^9x_4^{14}x_5^{15}) + Sq^2(x_1^2x_2^3x_3^9x_4^{14}x_5^{15} + x_1^5x_2^3x_3^6x_4^{14}x_5^{15})\\
&+ Sq^4(x_1^3x_2^3x_3^6x_4^{14}x_5^{15}) \bmod(P_5^-((3)|^4)).
\end{aligned}$$

Hence, the above monomials are strictly inadmissible. □

Lemma 4.1.8. i) *The following monomials are strictly inadmissible.*

$$\begin{array}{llll}
x_1^3x_2^{12}x_3^3x_4^{13}x_5^{14} & x_1^3x_2^{13}x_3^{14}x_4^3x_5^{12} & x_1^3x_2^5x_3^9x_4^{14}x_5^{14} & x_1^3x_2^5x_3^{14}x_4^9x_5^{14}\\
x_1^3x_2^5x_3^{14}x_4^{11}x_5^{12} & x_1^3x_2^7x_3^{13}x_4^8x_5^{14} & x_1^3x_2^7x_3^{13}x_4^{14}x_5^8 & x_1^7x_2^3x_3^{13}x_4^8x_5^{14}\\
x_1^7x_2^3x_3^{13}x_4^{14}x_5^8 & x_1^3x_2^7x_3^{11}x_4^{12}x_5^{12} & x_1^7x_2^3x_3^{11}x_4^{12}x_5^{12} & x_1^7x_2^{11}x_3^3x_4^{12}x_5^{12}\\
x_1^7x_2^{11}x_3^5x_4^8x_5^{14} & x_1^7x_2^{11}x_3^5x_4^{14}x_5^8 & x_1^7x_2^{11}x_3^{13}x_4^6x_5^8 & x_1^7x_2^9x_3^7x_4^{10}x_5^{12}.
\end{array}$$

ii) *The following monomials are strongly inadmissible.*

$$x_1^3x_2^{13}x_3^6x_4^{11}x_5^{12} \quad x_1^3x_2^{13}x_3^7x_4^{10}x_5^{12} \quad x_1^7x_2^7x_3^9x_4^{14}x_5^8.$$

Proof. We prove Part i) for $x = x_1^3x_2^{12}x_3^3x_4^{13}x_5^{14}$, $y = x_1^7x_2^3x_3^{13}x_4^8x_5^{14}$ and $z = x_1^7x_2^{11}x_3^5x_4^8x_5^{14}$. We have

$$\begin{aligned}
x &= x_1^2x_2^{11}x_3^5x_4^{13}x_5^{14} + x_1^2x_2^{13}x_3^3x_4^{13}x_5^{14} + x_1^3x_2^9x_3^5x_4^{14}x_5^{14} + x_1^3x_2^{11}x_3^4x_4^{13}x_5^{14}\\
&+ Sq^1(x_1^3x_2^7x_3^3x_4^{13}x_5^{18} + x_1^3x_2^7x_3^3x_4^{17}x_5^{14} + x_1^3x_2^{11}x_3^3x_4^{13}x_5^{14}) + Sq^2(x_1^2x_2^{11}x_3^3x_4^{13}x_5^{14}\\
&+ x_1^5x_2^7x_3^3x_4^{14}x_5^{14}) + Sq^4(x_1^3x_2^7x_3^3x_4^{14}x_5^{14}) \bmod(P_5^-((3)|^4)),\\
y &= x_1^5x_2^3x_3^{11}x_4^{12}x_5^{14} + x_1^5x_2^3x_3^{13}x_4^{10}x_5^{14} + x_1^7x_2^2x_3^{13}x_4^9x_5^{14} + x_1^7x_2^3x_3^9x_4^{12}x_5^{14}\\
&+ x_1^7x_2^3x_3^{12}x_4^9x_5^{14} + Sq^1(x_1^7x_2^3x_3^{11}x_4^9x_5^{14}) + Sq^2(x_1^7x_2^2x_3^{11}x_4^9x_5^{14} + x_1^7x_2^5x_3^7x_4^{10}x_5^{14}\\
&+ x_1^7x_2^5x_3^{11}x_4^6x_5^{14}) + Sq^4(x_1^5x_2^3x_3^7x_4^{12}x_5^{14} + x_1^5x_2^3x_3^{13}x_4^6x_5^{14} + x_1^{11}x_2^3x_3^7x_4^6x_5^{14})\\
&+ Sq^8(x_1^7x_2^3x_3^7x_4^6x_5^{14}) \bmod(P_5^-((3)|^4)),\\
z &= x_1^5x_2^{11}x_3^3x_4^{12}x_5^{14} + x_1^5x_2^{11}x_3^5x_4^{10}x_5^{14} + x_1^5x_2^{11}x_3^9x_4^6x_5^{14} + x_1^5x_2^{13}x_3^3x_4^{10}x_5^{14}\\
&+ x_1^7x_2^9x_3^5x_4^{10}x_5^{14} + x_1^7x_2^{10}x_3^5x_4^9x_5^{14} + x_1^7x_2^{11}x_3^4x_4^9x_5^{14} + Sq^1(x_1^7x_2^7x_3^3x_4^5x_5^{22}\\
&+ x_1^7x_2^7x_3^3x_4^9x_5^{18} + x_1^7x_2^{11}x_3^3x_4^5x_5^{18} + x_1^7x_2^{11}x_3^3x_4^9x_5^{14}) + Sq^2(x_1^7x_2^7x_3^2x_4^5x_5^{22}
\end{aligned}$$

$$+ x_1^7x_2^7x_3^3x_4^{12}x_5^{14} + x_1^7x_2^{10}x_3^3x_4^9x_5^{14} + x_1^7x_2^{13}x_3^3x_4^6x_5^{14} + x_1^9x_2^7x_3^3x_4^{10}x_5^{14})$$
$$+ Sq^4(x_1^5x_2^7x_3^3x_4^{12}x_5^{14} + x_1^5x_2^{11}x_3^5x_4^6x_5^{14} + x_1^5x_2^{13}x_3^3x_4^6x_5^{14} + x_1^{11}x_2^7x_3^3x_4^6x_5^{14})$$
$$+ Sq^8(x_1^7x_2^7x_3^3x_4^6x_5^{14}) \bmod(P_5^-((3)|^4)).$$

The above equalities show that x, y, z are strictly inadmissible.

We prove Part ii) for $w = x_1^3x_2^{13}x_3^6x_4^{11}x_5^{12}$. We have

$$\begin{aligned}w &= x_1^2x_2^{13}x_3^3x_4^{14}x_5^{13} + x_1^2x_2^{13}x_3^5x_4^{14}x_5^{11} + x_1^3x_2^{11}x_3^5x_4^{14}x_5^{12} + x_1^3x_2^{13}x_3^3x_4^{14}x_5^{12}\\
&+ x_1^3x_2^{13}x_3^4x_4^{14}x_5^{11} + Sq^1(x_1^3x_2^{11}x_3^3x_4^{17}x_5^{10} + x_1^3x_2^{13}x_3^3x_4^{13}x_5^{12} + x_1^3x_2^{13}x_3^3x_4^{14}x_5^{11}\\
&+ x_1^3x_2^{13}x_3^3x_4^{18}x_5^7 + x_1^3x_2^{13}x_3^5x_4^{11}x_5^{12} + x_1^3x_2^{13}x_3^6x_4^{13}x_5^9 + x_1^3x_2^{17}x_3^3x_4^{11}x_5^{10}\\
&+ x_1^3x_2^{17}x_3^3x_4^{14}x_5^7) + Sq^2(x_1^2x_2^{13}x_3^3x_4^{14}x_5^{11} + x_1^3x_2^{13}x_3^5x_4^{13}x_5^9 + x_1^5x_2^{11}x_3^3x_4^{14}x_5^{10}\\
&+ x_1^5x_2^{14}x_3^3x_4^{11}x_5^{10} + x_1^5x_2^{14}x_3^3x_4^{14}x_5^7) + Sq^4(x_1^3x_2^{11}x_3^3x_4^{14}x_5^{10} + x_1^3x_2^{14}x_3^3x_4^{11}x_5^{10}\\
&+ x_1^3x_2^{14}x_3^3x_4^{14}x_5^7) + x_1^5x_2^{13}x_3^5x_4^{13}x_5^9 \bmod(P_5^-((3)|^4)).\end{aligned}$$

Since $x_1^5x_2^{13}x_3^5x_4^{13}x_5^9 \in \mathcal{P}_{(5,4)}$, this equality shows that w is strongly inadmissible. The lemma follows. □

Proof of Proposition 4.1.6. Let $x \in P_5^+((3)|^4)$ be an admissible monomial, then $x = X_{i,j}y^2$ with $1 \leqslant i < j \leqslant k$. Since x is admissible, by Theorem 2.9, y is admissible. By a direct computation we see that if $z \in B_5((3)|^3)$ such that $X_{i,j}z^2 \in P_5^+((3)|^4)$ and $X_{i,j}z^2 \neq a_{4,t}$ for all t, $1 \leqslant t \leqslant 94$, then either $X_{i,j}z^2$ is one of the monomials as given in Lemmas 3.2.4(iv), 4.1.7, 4.1.8, or $X_{i,j}z^2$ is of the form uv^{2^r}, where u is a monomial as given in one of Lemmas 3.2.4, 4.1.4, 4.1.5 and r is a suitable integer. Hence, by Theorem 2.9, $X_{i,j}z^2$ is inadmissible. Since $x = X_{i,j}y^2$ is admissible, we have $x = a_{4,t}$ for some t, $1 \leqslant t \leqslant 94$.

We now prove that the set $\{[a_{4,t}] : 1 \leqslant t \leqslant 94\}$ is linearly independent in $QP_5^+((3)|^4)$.

Consider $\langle[A(4)]_{(3)|^4}\rangle \subset QP_5((3)|^4)$ and $\langle[C(4)]_{(3)|^4}\rangle \subset QP_5((3)|^4)$. We see that for $1 \leqslant t \leqslant 55$, $a_{4,t} = x_i^{15}f_i(b_{4,t})$ with $b_{4,t}$ an admissible monomial of degree 30 in P_4. By Proposition 2.11, $a_{4,t}$ is admissible. This implies $\dim\langle[A(4)]_{(3)|^4}\rangle = 55$. On the other hand, we have $\langle[A(4)]_{(3)|^4}\rangle \cap \langle[C(4)]_{(3)|^4}\rangle = \{0\}$. Hence, we need only to prove the set $[C(4)]_{(3)|^4}$ is linearly independent. Suppose there is a linear relation

$$S := \sum_{56\leqslant t\leqslant 94} \gamma_t a_{4,t} \equiv_{(3)|^4} 0, \tag{4.2}$$

where $\gamma_t \in \mathbb{F}_2$. We denote $\gamma_{\mathbb{J}} = \sum_{t\in\mathbb{J}}\gamma_t$ for $\mathbb{J} \subset \mathbb{N}$.

Applying the homomorphism $p_{(i;j)} : P_5 \to P_4$ to (4.2), we obtain

$$p_{(1;2)}(S) \equiv_{(3)|^4} \gamma_{58}w_{4,10} + \gamma_{\{64,79\}}w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(1;3)}(S) \equiv_{(3)|^4} \gamma_{\{57,74,91,92\}}w_{4,9} + \gamma_{\{65,90\}}w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(1;4)}(S) \equiv_{(3)|^4} \gamma_{\{56,68,75,77,81,83,93\}}w_{4,8} + \gamma_{67}w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(1;5)}(S) \equiv_{(3)|^4} \gamma_{\{56,57,58,64,65,67,76,78,82,84,87\}}w_{4,7} + \gamma_{68}w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(2;3)}(S) \equiv_{(3)|^4} \gamma_{\{60,63,74,79,80,90,92\}}w_{4,9} + \gamma_{\{66,91\}}w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(2;4)}(S) \equiv_{(3)|^4} \gamma_{\{59,70,75,85,93\}}w_{4,8} + \gamma_{69}w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(2;5)}(S) \equiv_{(3)|^4} \gamma_{\{59,60,66,69,76,86,88\}}w_{4,7} + \gamma_{70}w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(3;4)}(S) \equiv_{(3)|^4} \gamma_{\{61,62,63,72,73,77,89\}}w_{4,8} + \gamma_{71}w_{4,11} \equiv_{(3)|^4} 0,$$

$$p_{(3;5)}(S) \equiv_{(3)|^4} \gamma_{\{61,71,78\}} w_{4,7} + \gamma_{72} w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(4;5)}(S) \equiv_{(3)|^4} \gamma_{62} w_{4,7} + \gamma_{73} w_{4,11} \equiv_{(3)|^4} 0.$$

From these equalities, we get $\gamma_{58} = \gamma_{62} = \gamma_{67} = \gamma_{68} = \gamma_{69} = \gamma_{70} = \gamma_{71} = \gamma_{72} = \gamma_{73} = 0$, $\gamma_{79} = \gamma_{64}$, $\gamma_{90} = \gamma_{65}$, $\gamma_{91} = \gamma_{66}$, $\gamma_{78} = \gamma_{61}$. Then, applying the homomorphism $p_{(1;(i,j))} : P_5 \to P_4$ to (4.2), we get

$$p_{(1;(2,3))}(S) \equiv_{(3)|^4} \gamma_{\{57,60,74,92\}} w_{4,9} + \gamma_{63} w_{4,10} + \gamma_{80} w_{4,11} \equiv_{(3)|^4} 0,$$
$$p_{(1;(2,4))}(S) \equiv_{(3)|^4} \gamma_{\{56,59,64,75,77,81,82,83,85,93\}} w_{4,8} + \gamma_{77} w_{4,10} + \gamma_{81} w_{4,11} \equiv_{(3)|^4} 0,$$
$$\begin{aligned} p_{(1;(3,4))}(S) \equiv_{(3)|^4}\ & \gamma_{\{56,61,63,65,66,74,75,77,80,81,83,84,85,89,92,93,94\}} w_{4,8} + \gamma_{83} w_{4,11} \\ & + \gamma_{\{59,60,66,76,86,88\}} w_{4,3} + \gamma_{\{57,66,74,75,85,87,92,93,94\}} w_{4,9} \equiv_{(3)|^4} 0. \end{aligned}$$

From these above equalities we obtain $\gamma_{63} = \gamma_{77} = \gamma_{80} = \gamma_{81} = \gamma_{83} = 0$ and $\gamma_{89} = \gamma_{61}$. Then we have

$$\begin{aligned} p_{(1;(2,5))}(S) \equiv_{(3)|^4}\ & \gamma_{\{56,57,59,60,61,65,66,76,82,84,86,87,88\}} w_{4,7} \\ & + \gamma_{61} w_{4,10} + \gamma_{82} w_{4,11} \equiv_{(3)|^4} 0, \\ p_{(1;(3,5))}(S) \equiv_{(3)|^4}\ & \gamma_{\{59,75,85,93\}} w_{4,2} + \gamma_{\{56,57,64,66,74,76,82,84,86,87,92,94\}} w_{4,7} \\ & + \gamma_{\{57,66,74,76,86,87,92,94\}} w_{4,9} + \gamma_{84} w_{4,11} \equiv_{(3)|^4} 0, \\ p_{(1;(4,5))}(S) \equiv_{(3)|^4}\ & \gamma_{\{60,64,65,74,92\}} w_{4,1} + \gamma_{\{56,57,64,65,75,76,82,84,87,88,93,94\}} w_{4,7} \\ & + \gamma_{\{56,75,76,82,84,88,93,94\}} w_{4,8} + \gamma_{87} w_{4,11} \equiv_{(3)|^4} 0. \end{aligned}$$

Computing from these above equalities we get $\gamma_t = 0$ for $t \notin \{56, 57, 59, 60, 64, 65, 66, 74, 75, 76, 79, 90, 91, 92, 93, 94\}$, and the relation (4.2) becomes

$$S = \delta_1\theta_1 + \delta_2\theta_2 + \delta_3\theta_3 + \delta_4\theta_4 + \delta_5\theta_5 \equiv_{(3)|^4} 0, \tag{4.3}$$

where $\delta_1 = \gamma_{56} + \gamma_{57}$, $\delta_2 = \gamma_{60}$, $\delta_3 = \gamma_{57} + \gamma_{60} + \gamma_{74}$, $\delta_4 = \gamma_{56} + \gamma_{75}$, $\delta_5 = \gamma_{56}$ and θ_u, $1 \leqslant u \leqslant 5$, is determined as follows:

$$\begin{aligned} \theta_1 &= x_1x_2^3x_3^{14}x_4^{13}x_5^{14} + x_1x_2^7x_3^{10}x_4^{13}x_5^{14} + x_1^3x_2^3x_3^{12}x_4^{13}x_5^{14} + x_1^3x_2^7x_3^8x_4^{13}x_5^{14}, \\ \theta_2 &= x_1^3x_2x_3^{14}x_4^{13}x_5^{14} + x_1^3x_2^3x_3^{12}x_4^{13}x_5^{14} + x_1^7x_2x_3^{10}x_4^{13}x_5^{14} + x_1^7x_2^3x_3^8x_4^{13}x_5^{14}, \\ \theta_3 &= x_1^3x_2^3x_3^{12}x_4^{13}x_5^{14} + x_1^7x_2^7x_3^8x_4^9x_5^{14}, \\ \theta_4 &= x_1^3x_2^3x_3^{13}x_4^{12}x_5^{14} + x_1^7x_2^7x_3^9x_4^8x_5^{14}, \\ \theta_5 &= x_1x_2^3x_3^{13}x_4^{14}x_5^{14} + x_1x_2^3x_3^{14}x_4^{13}x_5^{14} + x_1x_2^6x_3^{11}x_4^{13}x_5^{14} + x_1^3x_2x_3^{13}x_4^{14}x_5^{14} \\ &\quad + x_1^3x_2^3x_3^{12}x_4^{13}x_5^{14} + x_1^3x_2^3x_3^{13}x_4^{12}x_5^{14} + x_1^3x_2^3x_3^{13}x_4^{14}x_5^{12} + x_1^3x_2^4x_3^{11}x_4^{13}x_5^{14} \\ &\quad + x_1^7x_2^7x_3^9x_4^{10}x_5^{12}. \end{aligned}$$

By a direct computation we can show that the monomial $a_{4,94} = x_1^7x_2^7x_3^9x_4^{10}x_5^{12}$ is admissible, hence from (4.3) one gets $\delta_5 = 0$. By applying the homomorphism $\rho_4 : P_5 \to P_5$ to (4.3), we get

$$\rho_4(S) \equiv_{(3)|^4} \delta_1\theta_1 + \delta_2\theta_2 + (\delta_3 + \delta_4)\theta_3 + \delta_4\theta_4 + \delta_4\theta_5 \equiv_{(3)|^4} 0.$$

This implies $\delta_4 = 0$. Applying the homomorphism $\rho_3 : P_5 \to P_5$ to (4.3) gives

$$\rho_3(S) \equiv_{(3)|^4} \delta_1\theta_1 + \delta_2\theta_2 + \delta_3\theta_4 \equiv_{(3)|^4} 0.$$

Hence, $\delta_3 = 0$. Now by applying the homomorphism $\rho_2 : P_5 \to P_5$ to (4.3) we obtain

$$\rho_2(S) \equiv_{(3)|^4} \delta_1\theta_1 + \delta_2\theta_3 \equiv_{(3)|^4} 0.$$

This implies $\delta_2 = 0$. Finally, applying the homomorphism $\rho_1 : P_5 \to P_5$ to (4.3) we obtain $\rho_1(S) \equiv_{(3)|^4} \delta_1\theta_2 \equiv_{(3)|^4} 0$, hence $\delta_1 = 0$. From the above equalities we get $\gamma_t = 0$ for all t, $56 \leqslant r \leqslant 94$. The proposition is proved. $\square$

4.1.2. *Proof of Theorem 4.1.1.*

Lemma 4.1.9.

i) *The following monomials are strictly inadmissible.*

$$x_1^3x_2^7x_3^{24}x_4^{29}x_5^{30} \quad x_1^7x_2^3x_3^{24}x_4^{29}x_5^{30} \quad x_1^7x_2^7x_3^{25}x_4^{26}x_5^{28} \quad x_1^{15}x_2^{15}x_3^{17}x_4^{18}x_5^{28}.$$

ii) *The following monomials are strongly inadmissible.*

$$x_1^3x_2^{15}x_3^{21}x_4^{26}x_5^{28} \quad x_1^{15}x_2^3x_3^{21}x_4^{26}x_5^{28}.$$

Proof. By a direct computation, we have

$$\begin{aligned}
x_1^3x_2^7x_3^{24}x_4^{29}x_5^{30} &\simeq_5 x_1x_2^3x_3^{30}x_4^{29}x_5^{30} + x_1x_2^7x_3^{26}x_4^{29}x_5^{30} + x_1^3x_2^3x_3^{28}x_4^{29}x_5^{30},\\
x_1^7x_2^7x_3^{25}x_4^{26}x_5^{28} &\simeq_5 x_1x_2^3x_3^{29}x_4^{30}x_5^{30} + x_1x_2^3x_3^{30}x_4^{29}x_5^{30} + x_1x_2^6x_3^{27}x_4^{29}x_5^{30}\\
&\quad + x_1^3x_2x_3^{29}x_4^{30}x_5^{30} + x_1^3x_2^3x_3^{28}x_4^{29}x_5^{30} + x_1^3x_2^3x_3^{29}x_4^{28}x_5^{30}\\
&\quad + x_1^3x_2^3x_3^{29}x_4^{30}x_5^{28} + x_1^3x_2^4x_3^{27}x_4^{29}x_5^{30},\\
x_1^{15}x_2^{15}x_3^{17}x_4^{18}x_5^{28} &\simeq_5 x_1x_2^3x_3^{29}x_4^{30}x_5^{30} + x_1x_2^3x_3^{30}x_4^{29}x_5^{30} + x_1x_2^6x_3^{27}x_4^{29}x_5^{30}\\
&\quad + x_1^3x_2x_3^{29}x_4^{30}x_5^{30} + x_1^3x_2^3x_3^{28}x_4^{29}x_5^{30} + x_1^3x_2^3x_3^{29}x_4^{28}x_5^{30}\\
&\quad + x_1^3x_2^3x_3^{29}x_4^{30}x_5^{28} + x_1^3x_2^4x_3^{27}x_4^{29}x_5^{30},\\
x_1^3x_2^{15}x_3^{21}x_4^{26}x_5^{28} &\simeq_5 x_1x_2^3x_3^{30}x_4^{29}x_5^{30} + x_1x_2^6x_3^{27}x_4^{29}x_5^{30} + x_1x_2^7x_3^{27}x_4^{28}x_5^{30}\\
&\quad + x_1^3x_2x_3^{29}x_4^{30}x_5^{30} + x_1^3x_2^3x_3^{28}x_4^{29}x_5^{30} + x_1^3x_2^3x_3^{29}x_4^{28}x_5^{30}\\
&\quad + x_1^3x_2^3x_3^{29}x_4^{30}x_5^{28} + x_1^3x_2^4x_3^{27}x_4^{29}x_5^{30} + x_1^3x_2^7x_3^{25}x_4^{28}x_5^{30}\\
&\quad + x_1^3x_2^7x_3^{25}x_4^{30}x_5^{28} + x_1^3x_2^7x_3^{29}x_4^{26}x_5^{28} \ \mathrm{mod}(\mathcal{P}_{(5,5)}).
\end{aligned}$$

The lemma follows. $\square$

Proof of Theorem 4.1.1. Denote $A(d) = \{a_{d,t} : 1 \leqslant t \leqslant 55\}$ and $C(d) = \{a_{d,t} : 56 \leqslant t \leqslant 90\}$. We prove that $B_5^+((3)|^d) \subset A(d) \cup C(d)$ by induction on $d \geqslant 5$.

Let $x \in P_5^+((3)|^d)$ is an admissible monomial. Then, $\omega(x) = (3)|^d$ and $x = X_{i,j}y^2$ with y a monomial in $P_5((3)|^{d-1})$ and $1 \leqslant i < j \leqslant 5$. Since x is admissible, by Theorem 2.9, y is also admissible.

Let $d = 5$ and $z \in A(4) \cup C(4) \cup B_5^0((3)|^4)$. By a direct computation we see that if $X_{i,j}z^2 \in P_5^+((3)|^5)$ and $X_{i,j}z^2 \neq a_{5,t}$ for all t, $1 \leqslant t \leqslant 90$, then either $X_{i,j}z^2$ is one of the monomials as given in Lemma 4.1.9, or $X_{i,j}z^2$ is of the form uv^{2^r}, where u is a monomial as given in one of Lemmas 3.2.4, 4.1.4, 4.1.5, 4.1.7, 4.1.8 and r is a suitable integer. Hence, by Theorem 2.9, $X_{i,j}z^2$ is inadmissible. Since $x = X_{i,j}y^2$ is admissible and $y \in B_5((3)|^4) \subset A(4) \cup C(4) \cup B_5^0((3)|^4)$, we have $x = a_{5,t}$ for some t, $1 \leqslant t \leqslant 90$. Hence, $B_5^+((3)|^5 \subset A(5) \cup C(5)$.

Suppose $d > 5$ and $B_5^+((3)|^{d-1}) \subset A(d-1) \cup C(d-1)$. Let $z \in A(d-1) \cup C(d-1) \cup B_5^0((3)|^{d-1})$. By a direct computation we can check that if $X_{i,j}z^2 \in P_5^+((3)|^d)$ and $X_{i,j}z^2 \neq a_{d,t}$ for all t, $1 \leqslant t \leqslant 90$, then $X_{i,j}z^2$ is is of the form uv^{2^r}, where u is a monomial as given in one of Lemmas 3.2.4, 4.1.4, 4.1.5, 4.1.7, 4.1.8, 4.1.9 and r is a suitable integer. By Theorem 2.9, $X_{i,j}z^2$ is inadmissible. Since $x = X_{i,j}y^2$ is admissible and $y \in B_5((3)|^{d-1}) \subset A(d-1) \cup C(d-1) \cup B_5^0((3)|^{d-1})$, we have $x = a_{d,t}$ for some t, $1 \leqslant t \leqslant 90$. That means $B_5^+((3)|^d) \subset A(d) \cup C(d)$.

Now we prove that the set $[A(d)\cup C(d)]_{(3)|^d}$ is linearly independent in $QP_5((3)|^d)$. Consider $\langle [A(d)]_{(3)|^d}\rangle \subset QP_5((3)|^4)$ and $\langle [C(d)]_{(3)|^d}\rangle \subset QP_5((3)|^d)$. By a simple computation, we can see that for $1 \leqslant t \leqslant 55$, $a_{d,t} = x_i^{2^d-1}f_i(b_{d,t})$ with $b_{d,t}$ an admissible monomial of degree $2(2^d-1)$ in P_4 and $1 \leqslant i \leqslant 5$. By Proposition 2.11, $a_{d,t}$ is admissible. This implies $\dim\langle [A(d)]_{(3)|^d}\rangle = 55$. On the other hand, we have $\langle [A(d)]_{(3)|^d}\rangle \cap \langle [C(d)]_{(3)|^d}\rangle = \{0\}$. Hence, we need only to prove the set $[C(d)]_{(3)|^d}$ is linearly independent in $QP_5((3)|^d)$. Suppose there is a linear relation

$$S := \sum_{56\leqslant t\leqslant 90}\gamma_t a_{d,t} \equiv_{(3)|^d} 0, \tag{4.4}$$

where $\gamma_t \in \mathbb{F}_2$. Applying the homomorphism $p_{(i;j)} : P_5 \to P_4$, $1 \leqslant i < j \leqslant 5$, to (4.4), we obtain

$$\begin{aligned}
&p_{(1;2)}(S) \equiv_{(3)|^d} \gamma_{58}w_{d,10} + \gamma_{\{64,79\}}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(1;3)}(S) \equiv_{(3)|^d} \gamma_{\{57,74\}}w_{d,9} + \gamma_{65}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(1;4)}(S) \equiv_{(3)|^d} \gamma_{\{56,68,75,77,81,83\}}w_{d,8} + \gamma_{67}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(1;5)}(S) \equiv_{(3)|^d} \gamma_{\{56,57,58,64,65,67,76,78,82,84,87\}}w_{d,7} + \gamma_{68}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(2;3)}(S) \equiv_{(3)|^d} \gamma_{\{60,63,74,79,80\}}w_{d,9} + \gamma_{66}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(2;4)}(S) \equiv_{(3)|^d} \gamma_{\{59,70,75,85\}}w_{d,8} + \gamma_{69}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(2;5)}(S) \equiv_{(3)|^d} \gamma_{\{59,60,66,69,76,86,88\}}w_{d,7} + \gamma_{70}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(3;4)}(S) \equiv_{(3)|^d} \gamma_{\{61,62,63,72,73,77,89\}}w_{d,8} + \gamma_{71}w_{d,11} + \equiv_{(3)|^d} 0,\\
&p_{(3;5)}(S) \equiv_{(3)|^4} \gamma_{\{61,71,78\}}w_{d,7} + \gamma_{72}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(4;5)}(S) \equiv_{(3)|^d} \gamma_{62}w_{d,7} + \gamma_{73}w_{d,11} \equiv_{(3)|^d} 0.
\end{aligned}$$

From these equalities, we get $\gamma_{58} = \gamma_{62} = \gamma_{65} = \gamma_{66} = \gamma_{67} = \gamma_{68} = \gamma_{69} = \gamma_{70} = \gamma_{71} = \gamma_{72} = \gamma_{73} = 0$, $\gamma_{74} = \gamma_{57}$, $\gamma_{78} = \gamma_{61}$, $\gamma_{79} = \gamma_{64}$. Then, applying the homomorphism $p_{(1;(i,j))} : P_5 \to P_4$ to (4.4), we get

$$\begin{aligned}
&p_{(1;(2,3))}(S) \equiv_{(3)|^d} \gamma_{60}w_{d,9} + \gamma_{63}w_{d,10} + \gamma_{80}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(1;(2,4))}(S) \equiv_{(3)|^d} \gamma_{\{56,59,64,75,77,81,82,83,85\}}w_{d,8} + \gamma_{77}w_{d,10} + \gamma_{81}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(1;(3,4))}(S) \equiv_{(3)|^d} \gamma_{\{59,60,76,86,88\}}w_{d,3} + \gamma_{\{56,57,61,63,75,77,80,81,83,84,85,89,90\}}w_{d,8}\\
&\qquad\qquad + \gamma_{\{75,85,87\}}w_{d,9} + \gamma_{83}w_{d,11} \equiv_{(3)|^d} 0.
\end{aligned}$$

Computing from these above equalities gives $\gamma_{60} = \gamma_{63} = \gamma_{77} = \gamma_{80} = \gamma_{81} = \gamma_{83} = 0$ and $\gamma_{64} = \gamma_{57}$, $\gamma_{75} = \gamma_{65}$, $\gamma_{89} = \gamma_{61}$. Then we have

$$\begin{aligned}
&p_{(1;(2,5))}(S) \equiv_{(3)|^d} \gamma_{\{56,57,59,61,76,82,84,86,87,88,90\}}w_{d,7} + \gamma_{61}w_{d,10} + \gamma_{82}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(1;(3,5))}(S) \equiv_{(3)|^d} \gamma_{\{56,59,85\}}w_{d,2} + \gamma_{\{56,57,76,82,84,86,87\}}w_{d,7}\\
&\qquad\qquad + \gamma_{\{76,86,87\}}w_{d,9} + \gamma_{84}w_{d,11} \equiv_{(3)|^d} 0,\\
&p_{(1;(4,5))}(S) \equiv_{(3)|^d} \gamma_{\{76,82,84,87,88,90\}}w_{d,7} + \gamma_{\{76,82,84,88,90\}}w_{d,8} + \gamma_{87}w_{d,11} \equiv_{(3)|^d} 0.
\end{aligned}$$

By a direct computation from the above equalities we get $\gamma_t = 0$ for all t, $56 \leqslant t \leqslant 90$. The theorem is proved. □

Proof of Theorem 1.6. Let $n = 2^{d+s+t} + 2^{d+s} + 2^d - 3$ and $m = 2^{s+t} + 2^s - 2$. We have $\frac{n-5}{2} = 2^{d-1+s+t} + 2^{d-1+s} + 2^{d-2} + 2^{d-2} - 4$. By Theorem 3.1.5 and Theorem 1.4 in [33], if $d \geqslant 6$, $s \geqslant 4$ and $t \geqslant 4$, then

$$\dim(QP_5)_{\frac{n-5}{2}} = (2^5-1)\dim(QP_4)_{2^{s+t+1}+2^{s+1}-2} = 3(2^3-1)(2^4-1)(2^5-1).$$

Kameko's homomorphism $(\widetilde{Sq}^0)_{(5,n)} : (QP_5)_n \longrightarrow (QP_5)_{\frac{n-5}{2}}$ is an epimorphism, hence using Theorem 2.15, we get

$$\begin{aligned}4(2^3-1)(2^4-1)(2^5-1) \leqslant \dim(QP_5)_n &= \dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} + \dim(QP_5)_{\frac{n-5}{2}}\\ &= \dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} + 3(2^3-1)(2^4-1)(2^5-1).\end{aligned}$$

This implies $\dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} \geqslant (2^3-1)(2^4-1)(2^5-1)$.

We set $\mathcal{B}_{(5,d)} = \{(\mathcal{I},\mathcal{J}) \in \mathsf{PInc}_5^d : X_{(\mathcal{I},\mathcal{J})} \in B_5((3)|^d)\}$. By a direct computation using Lemma 3.2.4, Lemmas in Subsection 4.1 on inadmissible monomials and Proposition 3.2.5 we can check that the set $\mathcal{B}_{(5,d)}$ is compatible with $((3)|^d)$ for any $d \geqslant 6$. By applying Theorem 3.3.3 we obtain

$$\dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} \leqslant |\mathcal{B}_{((5,d)}| \dim(QP_3)_m = 155\dim(QP_3)_m.$$

By Kameko [10], we have $\dim(QP_3)_m = 21$ for any $s,\ t \geqslant 2$. Hence, we get

$$\begin{aligned}\dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} &\leqslant |B_5((3)|^6)| \dim(QP_3)_m\\ &= 155 \times 21 = (2^3-1)(2^4-1)(2^5-1).\end{aligned}$$

Thus, $\dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} = (2^3-1)(2^4-1)(2^5-1)$, for any $d \geqslant 6,\ s,\ t \geqslant 4$. The theorem is proved ☐

4.2. **Proof of Theorem 1.8.**

First, we prove the following.

Theorem 4.2.1. *Let $n = 2^{d+s+t} + 2^{d+s} + 2^d - 3$, with d, s, t integers such that $d \geqslant 6, s \geqslant 0$ and $t > 0$. Then,* $\dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} = 155\dim(QP_3)_m$.

We recall the following.

Theorem 4.2.2 (See Kameko [10])**.** *Let $m = 2^{s+t} + 2^s - 2$ with $s,\ t$ integers such that $s \geqslant 0,\ t \geqslant 1$. The dimension of the $\mathbb{F}_2$-vector space $(QP_3)_m$ is given by the following table:*

m	$t=1$	$t=2$	$t=3$	$t=4$	$t\geqslant 5$
$s=0$	3	7	10	13	14
$s=1$	8	15	14	14	14
$s\geqslant 2$	14	21	21	21	21

Sketch proof of Theorem 4.2.1. From the proof of Theorem 1.6, we see that the theorem holds for $s,\ t \geqslant 4$ and $\dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} \leqslant 155\dim(QP_3)_m$ for any $s \geqslant 0,\ t \geqslant 1$. We prove $\dim\mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} = 155\dim(QP_3)_m$ by proving that the set

$$\{[X_{(\mathcal{I},\mathcal{J})}(f_{(i,j)(z)})^{2^d}] : (\mathcal{I},\mathcal{J}) \in \mathcal{B}_{(5,d)},\ z \in B_3(m),\ i = \min\mathcal{I},\ j = \min\mathcal{J}\}$$

is linearly independent in $(QP_5)_n$. Suppose that there is a linear relation

$$S := \sum_{(\mathcal{I},\mathcal{J})\in\mathcal{B}_{(5,d)}, z\in B_3(m)} \gamma_{(\mathcal{I},\mathcal{J}),z} X_{(\mathcal{I},\mathcal{J})}(f_{(i,j)(z)})^{2^d} \equiv 0,$$

with $\gamma_{(\mathcal{I},\mathcal{J}),z} \in \mathbb{F}_2$. By a direct computation from the relations $p_{(i;j)}(S) \equiv 0$, $1 \leqslant i < j \leqslant 5$ and $p_{(1;(u,v))}(S) \equiv 0$, $2 \leqslant u < v \leqslant 5$, we get $\gamma_{(\mathcal{I},\mathcal{J}),z} = 0$ for all $(\mathcal{I},\mathcal{J}) \in \mathcal{B}_{(5,d)}, z \in B_3(m)$.

Now the theorem follows from the relation $\dim \mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} = 155 \dim(QP_3)_m$ and Theorem 4.2.2. □

Proposition 4.2.3 (See Sum [33, Theorem 1.4])**.** *Let $n = 2^{d+s+t} + 2^{d+s} + 2^d - 3$, with d, s, t integers such that $d \geqslant 6$, $s \geqslant 0$ and $t > 0$. The dimension of the $\mathbb{F}_2$-vector space $(QP_4)_{\frac{n-5}{2}}$ is given by the following table:*

n	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t \geqslant 6$
$s=0$	21	55	73	95	115	125
$s=1$	70	126	165	179	175	175
$s=2$	116	192	241	255	255	255
$s=3$	164	240	285	300	300	300
$s \geqslant 4$	175	255	300	315	315	315

Proof of Theorem 1.8. By Theorem 3.1.5, we have

$$\dim(QP_5)_{\frac{n-5}{2}}) = (2^5 - 1) \dim(QP_4)_{\frac{n-5}{2}}$$

for any $d \geqslant 6$. Since $(\widetilde{Sq}^0)_{(5,n)} : (QP_5)_n \to (QP_5)_{\frac{n-5}{2}})$ is an epimorphism, we get

$$\dim(QP_5)_n = \dim \mathrm{Ker}(\widetilde{Sq}^0)_{(5,n)} + \dim(QP_4)_{\frac{n-5}{2}}.$$

Now the theorem follows from the last equality, Theorem 4.2.1 and Proposition 4.2.3. □

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