BERNSTEIN-MARKOV PROPERTIES ASSOCIATED TO COMPACT SETS IN $\mathbb{R}^d$

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ABSTRACT. Given a body convex P and a sequence $\{K_j\}$ of Borel subsets of a non-pluripolar Borel set $K \subset \mathbb{C}^d$. We prove some properties about the convergence of the sequence of the P-extremal functions $\{V^*_{P,K_j}\}$. This is used to give a sufficient condition guaranteeing that the triple (P, K, μ) where μ is a finite positive Borel measure with compact support K satisfy a Bernstein-Markov inequality. Our work expands results in [3] for P-plurpotential theory.

1. INTRODUCTION

Let K be a compact subset of $\mathbb{C}^d$ and μ be a positive Borel measure on $K \subset \mathbb{C}^d$. Obviously the $L^2(\mu)-$ norm on K of a polynomial p is majorized by its sup-norm. It is a natural problem to see whether the above estimate can be reversed. For this purpose, we say that the pair (K, μ) has the *Bernstein-Markov property* if for each $\varepsilon > 0$ there exists a positive constant $C = C_\varepsilon > 0$ such that

$$\|p\|_K := \sup_{z\in K} |p(z)| \le Ce^{\varepsilon \deg p}\|p\|_{L^2(\mu)},\ \forall p \in \mathbb{C}[z_1, \cdots, z_d]. \tag{1.1}$$

The Bernstein-Markov property is a classical concept and was studied thoroughly in [2], [3], [7],... One use of this property is to approximate the global extremal function V_K by functions of the form $\frac{1}{\deg p}\log|p|$ where p are polynomials that form an orthognormal system for $L^2(K, \mu)$. In [3], T. Bloom and N. Levenberg proved the following interesting result about sufficient conditions such that (K, μ) has the Bernstein-Markov property.

Theorem 1.1. *Let K be a compact regular subset of the unit ball in $\mathbb{C}^d$ and μ be a finite positive Borel measure on K. Set*

$$E_r = \{z \in K : \mu(K \cap B(z, r)) \ge r^T\},\ \forall r > 0.$$

Suppose that there exists a positive constant T such that one of the following (equivalent) conditions holds true:

(i) $\lim_{r\to 0^+} C(E_r, B) = C(K, B)$, where $C(E, B)$ is the relative capacity of E in B;

(ii) $V^*_{E_r} \to V^*_E$ pointwise as $r \to 0$ on $\mathbb{C}^d$, where V_{E_j} and V_E are the global extremal function of E_j and E respectively;

(iii) $u^*_{E_j,B} \to u^*_{E,B}$ pointwise as $r \to 0$ on B, where $u_{E,B}$ and $u_{E_j,B}$ are the relative extremal functions of E and E_j respectively.

Then (K, μ) has the Bernstein-Markov property (1.1).

The aim of this note is to expand some of mains results about sufficient conditions for Bernstein-Markov property of measures living on K, but for $P-$ polynomials on $\mathbb{C}^d$, where P is a compact subset of $(\mathbb{R}^+)^d$ satisfying $P \cap (\mathbb{Z}^+)^d \ne \emptyset$. Let us now recall the notion of

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$P-$polynomials associated to such a compact set P. Following [1], for each $n \geq 1$ we consider the finite-dimensional polynomial space

$$\text{Poly}(nP) := \{p \in \mathbb{C}[z_1, \cdots, z_d] : p(z) = \sum_{J \in nP \cap (Z^+)^d} a_J z^J\}.$$

Here we use the multi-dimensional notation $z^J = z_1^{j_1}...z_d^{j_d}$ for $J = (j_1, ..., j_d)$.

In the case $P = \Sigma := \{(x_1, ..., x_d) \in (\mathbb{R}^+)^d : x_1 + ... + x_d \leq 1\}$, the standard unit simplex in $\mathbb{R}^d$ we have $\text{Poly}(n\Sigma) = \mathscr{P}_n$ the usual space of holomorphic polynomials of degree at most n in $\mathbb{C}^d$. On the other hand, since there exists $A \in \mathbb{Z}^+$ such that $P \subset A\Sigma$ we get

$$\text{Poly}(nP) \subset \text{Poly}(nA\Sigma) = \mathscr{P}_{nA}, \forall n \geq 1.$$

Sometimes we also assume further that P is a *convex body*, i.e, P is a compact, convex set in $(\mathbb{R}^+)^d$ with non-empty interior. Moreover, we require that P is *admissible* in the sense that

$$\Sigma \subset kP, \quad \text{for some } k \in \mathbb{Z}^+. \tag{1.2}$$

These last restrictions were emphasized in [1] to exploit the approximability of the $P-$global extremal functions by (normalized) logarithms of $P-$polynomials.

2. PRELIMINARIES

Throughout this paper, unless otherwise specify, we always denote by K a compact subset of $\mathbb{C}^d, \mu$ a positive finite measure whose support equals to K and for P a compact subset of $(\mathbb{R}^+)^d$ satisfying $P \cap (\mathbb{Z}^+)^d \neq \emptyset$.

We first recall some elements about global $P-$extremal functions associated to P. Most of the material that follows is taken from [9] (in the case $P = \sigma$) and [1], [5] (in the case P is a convex body). The first function to be defined is the logarithmic indicator function of P

$$H_P(z) := \sup_{J=(j_1,...,j_d) \in P} \log(|z_1|^{j_1} ....|z_d|^{j_d}) = \sup_{J=(j_1,...,j_d)} (j_1 \log|z_1| + ... + j_d \log|z_d|), \ z \neq 0$$

and $H_P(0) = 0$. Since H_P is the maximum of finite plurisubharmonic functions we conclude that $H_P \in \text{PSH}(\mathbb{C}^d)$. In the standard case $P = \Sigma$, an easy reasoning yields

$$H_\Sigma(z) = \max_{1 \leq j \leq d} \log^+ |z_j|, \ \forall z \in \mathbb{C}^d.$$

In general, since (1.2), $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$ we have

$$H_P(z) \geq \frac{1}{k} \max_{1 \leq j \leq d} \log^+ |z_j|. \tag{2.1}$$

We will now use $H_P(z)$ to provide a generalization of the standard Lelong class

$$\mathscr{L}_P := \mathscr{L}_P(\mathbb{C}^d) = \{u \in \text{PSH}(\mathbb{C}^d) : u(z) \leq c_u + H_P(z), z \in \mathbb{C}^d\},$$

where c_u is a constant depending only on u. If $P = \Sigma$ then $\mathscr{L}_P = \mathscr{L}(\mathbb{C}^d)$ the usual Lelong class in $\mathbb{C}^d$.

For a bounded subset $E \subset \mathbb{C}^d$, the $P-$global extremal function of E is defined by

$$V_{P,E}(z) := \sup\{u(z) : u \in \mathscr{L}_P(\mathbb{C}^d), u \leq 0 \text{ on } E\}.$$

We also let $V_E^*(z) := \limsup_{\xi \to z} V_E(\xi)$ be the upper semicontinous regularization of $V_{P,E}$. For $P = \Sigma$ we have $V_{\Sigma,E} = V_E$, the standard Siciak global extremal function.

It is well-known that $V_E^* \equiv +\infty \Longleftrightarrow E$ is pluripolar, i.e there exists a plurisubharmonic function u on $\mathbb{C}^d$ such that $E \subset \{z \in \mathbb{C}^d : u(z) = -\infty\}$. According to a result of Siciak we can even choose $u \in \mathscr{L}(\mathbb{C}^d)$. One use of these extremal functions is to define certain concepts of regularity.

Definition 2.1. *A compact set $K \subset \mathbb{C}^d$ is said to be L-regular (resp. PL$-$ regular) if V_K (resp. $V_{P,K}$) is continuous on $\mathbb{C}^d$.*

We can show, under some restrictions on P that the two notions $L-$ regularity and $PL-$ regularity is actually equivalent.

3. Convergence of P- extremal functions

Let E be a subset of $\mathbb{C}^d$. The P- extremal function of E given by

$$V_{P,E}(z) = \sup\{u(z) : u \in \mathscr{L}_P, u \leq 0 \text{ on } E\}.$$

and $V^*_{P,E}(z) := \limsup_{\xi \to z} V_{P,E}(\xi)$ is the upper semicontinous regularization of $V_{P,E}$. For $P = \Sigma$ we have

$$V_{\Sigma,E} = V_E = \sup\{u(z) : u \in \mathscr{L}(\mathbb{C}^d), u \leq 0 \text{ on } E\}$$

is the usual global extremal function of E. Note that since $\frac{1}{n}\log|p| \in \mathscr{L}_P$ for any $p \in \text{Poly}(nP)$, we have the following (generalized) Bernstein- Walsh inequality

Proposition 3.1. *Le E be non-pluripolar. Then for any $p \in Poly(nP)$,*

$$|p(z)| \leq \|p\|_E e^{nV_{P,E}(z)}, \; z \in \mathbb{C}^d.$$

In the special but important case where P is *convex* we have (see [5])

$$p \in \text{Poly}(nP), q \in \text{Poly}(nP) \Longrightarrow pq \in \text{Poly}\big((n+m)P\big).$$

Using this fact and some standard technique on solving $\bar{\partial}-$ equation with L^2- estimates, Bayraktar [1] (see also Proposition 2.1 in [5]) proved in the theorem below that $V_{P,K}$ can be defined by means of polynomials. In case $P = \Sigma$, this result of course reduces to the famous Siciak-Zakharyuta approximation theorem.

Theorem 3.2. *Let P be an admissible convex body and K be a non-pluripolar compact subset in $\mathbb{C}^d$. Then*

$$V_{P,K} = \lim_{n \to \infty} \frac{1}{n} \log \Phi_n(z), z \in \mathbb{C}^d,$$

where

$$\Phi_n(z) = \sup\{|p_n(z)| : p_n \in Poly(nP), \|p_n\|_K \leq 1\}.$$

Furthermore, if $V_{P,K}$ is continuous then the convergence is locally uniform on $\mathbb{C}^d$.

Using the above theorem we can compare the two notions of regularity introduced in the last section. The simple lemma below is needed for this task.

Lemma 3.3. *Let P be an admissible convex body in $(\mathbb{R}^+)^d$. Then there exist constants $a, A > 0$ such that for every bounded non-pluripolar subset E of $\mathbb{C}^d$ and any compact set K of $\mathbb{C}^d$ we have*

$$aV_E \leq V_{P,E}, V_{P,K} \leq AV_K \text{ on } \mathbb{C}^d.$$

So in case P is an admissible convex body, K is $L-$regular if and only if K is $PL-$regular.

Proof. Since $P \subset A\Sigma$, using Theorem 3.2 we conclude easily that $V_{P,K} \leq AV_K$ on $\mathbb{C}^d$. On the other hand, in view of (2.1) we infer that $aV_K \leq V_{P,K}$ for $a := 1/k$. In particular, if P is an admissible convex body then we have $V^*_K = 0$ if and only if $V^*_{P,K} = 0$. The proof is thereby completed. □

We have the following simple facts which will be useful in the sequel.

Proposition 3.4. *(i) Let $P(a,r)$ be the open polydisc with center $a=(a_1,...,a_d)$, radius r. Then*
$$V_{P,\overline{P}(a,r)} = H_P(\frac{z-a}{r}) = \sup_{J\in P}\log^+|\frac{z-a}{r}|^J, z\in\mathbb{C}^d.$$
(ii) If $u\in\mathscr{L}_P$ then
$$u(z)\le \max_{\overline{P}(a,r)} u + H_P(\frac{z-a}{r}), \forall z\in\mathbb{C}^d.$$
(iii) If $\{u_\alpha\}_{\alpha\in I}\subset\mathscr{L}_P$ and $u=\sup_{\alpha\in I} u_\alpha$ then either $u^*\equiv+\infty$ or $u^*\in\mathscr{L}_P$.

Proof. (i) For simplicity of notation, we may assume that $a=0$ and $r=1$. It is then enough to show
$$V_{P,\overline{P}(0,1)}(z) = H_P(z) = \sup_{J\in P}\log^+|z|^J,\ z\in\mathbb{C}^d.$$
Since $H_P\in \mathrm{PSH}(\mathbb{C}^d), H_P=0$ on $\overline{P}(0,1)$, it is clear that $H_P\le V_{P,\overline{P}(0,1)}$ on $\mathbb{C}^d$. For the reverse inequality, we take $z\in\mathbb{C}^d$. If $|z|:=\max(|z_1|,...,|z_d|)\le 1$ then the inequality is obvious. Consider the case $|z|>1$. Then for every $u\in\mathscr{L}_P, u\le 0$ on $\overline{P}(0,1))$ the function
$$\varphi(\lambda)=u(\lambda z)-H_P(\lambda z)$$
is bounded, subharmonic on $\{\lambda\in\mathbb{C}:|\lambda|>\frac{1}{|z|}\}$ and $\varphi(\lambda)\le 0$ as $|\lambda|=\frac{1}{|z|}$. By the maximum principle we get $\varphi(\lambda)\le 0$ for all $|\lambda|\ge\frac{1}{|z|}$. In particular with $\lambda=1$ we obtain the required inequality.

(ii) Set $v(z)=u(z)-\max_{\overline{P}(a,r)} u,\ z\in\mathbb{C}^d$. Then $v\in\mathscr{L}_P, v\le 0$ on $\overline{P}(a,r)$. Then by (i),
$$v(z)\le V_{P,\overline{P}(0,1)}(z)=H_P(z),$$
thus we get (ii).

(iii) Assume that $u^*(a)<+\infty$ for some a. Then there exists a polydisc $P(a,r)$ such that $C:=\sup_{\overline{P}(a,r)} u<+\infty$. From (ii) we infer that for every $\alpha\in I$ we have
$$u_\alpha(z)\le C+H_P(\frac{z-a}{r}), \forall z\in\mathbb{C}^d.$$
Hence for $z\in\mathbb{C}^d$ we obtain
$$u(z)\le C+H_P(\frac{z-a}{r})\le C'+H_P(z),$$
for some constant $C'>0$ depends only on C,a,r. We are done. □

We list below basic properties of $P-$global extremal functions that will be used throughout our work. The following properties of the global extremal functions remain valid for P-extremal functions (see also [5], discussion after Proposition 2.1 and 2.3).

Proposition 3.5. *Let E be a bounded Borel set in $\mathbb{C}^d$ and K be a compact set. Then we have the following assertions:*

(i) If $F\subset E$ then $V_{P,F}\ge V_{P,E}$;

(ii) $V^*_{P,E}\equiv+\infty$ if and only if E is pluripolar and when E is non-pluripolar then $V^*_{P,E}\in\mathscr{L}_P$.

(ii) If E is pluripolar if and only if E is $PL-$pluripolar.

(iv) If $K_j\downarrow K$ and if K_j are compact then $V_{P,K_j}\uparrow V_{P,K}$;

(v) If $E_j \uparrow E$ then $V^*_{P,E_j} \downarrow V^*_{P,E}$;

(vi) $V^*_{P,E\setminus F} = V^*_{P,E}$ if F is pluripolar.

(vii) If $V^*_{P,K} \equiv 0$ on K then $V_{P,K}$ is continuous on $\mathbb{C}^d$.

Proof. The assertion (i) is trivial while (ii) and (vii) can be proved by adapting the standard proofs for the case $P = \Sigma$.

(iii) We proceed by contradiction as in the classical case $P = \Sigma$. Assume that E is not *PL*-pluripolar. Then by (ii) $V^*_{P,E} \in \mathscr{L}_P$ and therefore $M := \sup_E V^*_{P,E} < +\infty$. Since E is bounded, there is a polydisc $P(0,r)$ such that $E \subset P(0,r)$. Then from Proposition 3.4 we infer

$$V^*_{P,E}(z) \geq V^*_{P,\overline{P}(0,r)} = \sup_{J\in P} \log^+ \frac{|z|^J}{r}, \ \ z \in \mathbb{C}^d.$$

Thus we can find $R > r$ such that $\inf_{\partial P((0,R)} V^*_{P,E} \geq 2M+1$. Now we choose $u \in \mathrm{PSH}(\mathbb{C}^d)$ such that $u = -\infty$ on E and $u < 0$ on $P(0,R)$. For each positive integer $j \geq 1$ we set

$$v_j := \begin{cases} \max\{\frac{1}{j}u+1, \frac{1}{2M+1}V^*_{P,E}\}, & \text{in } P(0,R) \\ \frac{1}{2M+1}V^*_{P,E}, & \text{otherwise.} \end{cases}$$

Then $(2M+1)v_j \in \mathscr{L}_P$ and on E we have $(2M+1)v_j \leq M$. Hence $(2M+1)v_j - M \leq V_{P,E}$ on $\mathbb{C}^d$. In particular

$$(2M+1)(\frac{1}{j}u+1) \leq M + V_{P,E} \ \text{ in } P(0,R)$$

for all $j \geq 1$. By letting $j \to \infty$ we obtain $V^*_{P,E} \geq M+1$ on E. This yields a contradiction to the fatc that $V^*_{P,E} \leq M$ on E.

(iv), (v), (vi) now follows from the same reasoning as in [7] and (iii). □

From Proposition 3.4 (iii) and repeating the proof Theorem 3.5 in [9] we have the following property of upper envelope of a family in $\mathscr{L}_P$.

Proposition 3.6. *Given any $\{u_\alpha\}_{\alpha\in I} \subset \mathscr{L}_P$ and put $u = \sup_{\alpha\in I} u_\alpha$. Then $u^* \in \mathscr{L}_P$ if and only if the set $A_u := \{z \in \mathbb{C}^d : u(z) < +\infty\}$ is non-pluripolar.*

Theorem 3.7. *Let $\{K_j\}$ be a sequence of Borel subsets of K. Consider the following assertions:*

(i) $V^*_{P,K_j} \to 0$ q.e on K.

(ii) $V^*_{P,K_j} \to V^*_{P,K}$ pointwise on $\mathbb{C}^d$;

(iii) $V^*_{P,K_j} \to V^*_{P,K}$ uniformly on $\mathbb{C}^d$;

(iv) $V^*_{K_j} \to 0$ q.e. on K.

(v) $V^*_{K_j} \to V^*_K$ pointwise on $\mathbb{C}^d$;

(vi) $V^*_{K_j} \to V^*_K$ uniformly on $\mathbb{C}^d$.

Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ if K is PL-regular, $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ if K is L-regular, and $(i) \Leftrightarrow (iv)$ if K is an admissible convex body.

Proof. First we consider the case K is *PL*-regular.

(i) $\Rightarrow$ (ii) We can assume that K_j is non-pluripolar for all $j \geq 1$. Then $V^*_{P,K_j} \in \mathscr{L}_{P,+}, \forall j \geq 1$. For $s \geq 1$, define

$$v_{P,s}(z) := \sup_{j \geq s} V^*_{P,K_j}(z), z \in \mathbb{C}^d.$$

Then the set $\{v_{P,1} < +\infty\}$ contains a non-pluripolar subset of K. Proposition 3.6 implies that $v^*_{P,s} \in \mathscr{L}_P$ for every $s \geq 1$. Therefore

$$V^*_{P,K} \leq v_P := \lim \downarrow v^*_{P,s}.$$

In particular $v_P \in \mathscr{L}_P, v_P(z) = 0$ q.e. on K. Here the latter equality follows from the fact that $v_{P,s} = v^*_{P,s}$ q.e. on $\mathbb{C}^d$. By Proposition 3.5 (v) we obtain $v_P \leq V^*_{P,K}$ on $\mathbb{C}^d$. Moreover, since $K_j \subset K$ we have

$$v_P \leq V^*_{P,K} \leq V^*_{P,K_j} \ \forall j \geq 1.$$

Putting all this together we concludes that

$$\lim_{j \to \infty} V^*_{P,K_j}(z) = V^*_{P,K}(z), \forall z \in \mathbb{C}^d.$$

(ii) $\Rightarrow$ (iii) Since K is PL-regular it follows that $V^*_{P,K_j} \to V^*_{P,K} = 0$ on K. On the other hand, by Proposition 3.6, the sequence V^*_{P,K_j} is locally uniformly bounded on $\mathbb{C}^d$. Then using Hartogs' lemma we infer that $V^*_{P,K_j} \to 0$ uniformly on K. By the definition we deduce easily that $V^*_{P,K_j} \to V^*_{P,K}$ uniformly on $\mathbb{C}^d$.

(iii) $\Rightarrow$ (i) is trivial.

If K is $L-$regular then by setting $P = \Sigma$ in the above proof we have $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$.

Finally, in case K is an admissible convex body we may apply the comparison lemma (Lemma 3.3) to see that $(i) \Leftrightarrow (iv)$. □

Remark 3.8. 1. We do not need $PL-$ regularity of K for the implication $(i) \Rightarrow (ii)$.

2. The assumption $V^*_{K_j} \to 0$ q.e. on K does not imply $L-$regularity of K. For a simple example we let K be the union of a closed disk Δ and an isolated point a while K_j is taken to be a sequence of closed disks increasing to Δ.

3. Under the assumptions that P is an admissible convex body and $V^*_{K_j} \to 0$ *pointwise* on K then by adapting the proof of the implication $(i) \Rightarrow (ii)$ to the case $P = \Sigma$ we can show that K is indeed $L-$ regular. So in this case all the equivalent conditions in Theorem 3.8 holds true.

4. BERNSTEIN-MARKOV PROPERTIES

Definition 4.1. *The triple (P, K, μ) is said to have:*

(a) the strong Bernstein-Markov property if for each $\varepsilon > 0$, there exists a positive constant $C = C_\varepsilon > 0$ such that

$$\|p\|_K \leq C e^{n\varepsilon} \|p\|_{L^2(\mu)}, \ \ \forall p \in Poly(nP), \ \ n \geq 1; \tag{4.1}$$

(b) the weak Bernstein-Markov property if there exists a constant $\lambda \geq 0$ such that for each $\varepsilon > 0$, there exists a positive constant $C = C_\varepsilon > 0$ such that

$$\|p\|_K \leq C e^{n(\lambda+\varepsilon)} \|p\|_{L^2(\mu)}, \ \ \forall p \in Poly(nP), \ \ n \geq 1. \tag{4.2}$$

Remark 4.2. (a) We present a class of pairs (K, μ) having the weak Bernstein-Markov property. Let

$$K := \{z \in \mathbb{C} : |z| = 1\} \bigcup \{z \in \mathbb{C} : z| = 2\}$$

and μ be any finite positive Borel measure on K whose support coincides with K such that $\mu|_{\partial\Delta}$ is the normalized Lebesgue measure where $\Delta := \{z \in \mathbb{C} : |z| = 1\}$. Consider a polynomial $p(z) := a_0 + a_1 z + \cdots + a_n z^n$. By Cauchy-Schwarz's inequality we obtain

$$\|p\|_K^2 \le \frac{4^{n+1}-1}{3}(|a_0|^2 + \cdots + |a_n|^2) \le \frac{4^{n+1}}{3} \int_{\partial\Delta} |p|^2 d\mu.$$

Thus (K,μ) enjoy the weak Bernstein-Markov property. It is not clear to us if we could also choose μ on the out circle $\{z : |z| = 2\}$ such that (K,μ) does *not* enjoy the strong Bernstein-Markov property.

(b) If $P = \Sigma$ then (4.1) becomes (1.1). Note that in general the exponent n in (4.1) may be less than $\deg p$.

We will give a sufficient condition, in terms of convergence of certain $P-$global extremal functions, for the triple (P,K,μ) to have the strong Berstein-Markov property. For this purpose, we first introduce the following type of function.

Definition 4.3. *A measurable function $f : (0,\infty) \to (0,\infty)$ is said to have the $(BM)-$property if for every $\varepsilon > 0$ there exists a sequence $\{r_n\} \downarrow 0$ and $\varepsilon' > 0$ satisfying the following conditions:*

(i) $\inf_{n\ge 1} f(r_n) e^{n(\varepsilon-\varepsilon')} > 0$;

(ii) $\lim_{n\to\infty} r_n e^{n\varepsilon'} = 0$.

Theorem 4.4. *Let K be a compact PL-regular set in $\mathbb{C}^d$ and μ be a finite positive Borel measure on K. Let $f : (0,\infty) \to (0,\infty)$ be a function satisfying the $(BM)-$ property. Assume that $V^*_{P,E_r} \to 0$ q.e on K as $r \downarrow 0$, where*

$$E_r := \{z \in K : \mu(K \cap B(z,r)) \ge f(r)\}.$$

Then the triple (P,K,μ) has the strong Bernstein-Markov property.

Remark 4.5. Observe that for any $T > 0$ the function $f(r) = r^T$ has the $(BM)-$property. Indeed, given $\varepsilon > 0$, we choose $\varepsilon' := \lambda\varepsilon, r_n := e^{\frac{n\varepsilon(\lambda-1)}{T}}$ where $\lambda \in (0, \frac{1}{T+1})$.

Our proof relies on Bloom-Levenberg's methods.

Proof. Fix $0 < \varepsilon < 1$. Then we choose ε' and a sequence $\{r_n\}$ satisfying the condition given on f.

Step 1. Then we claim that there exists $\delta > 0$ such that for $r \in (0,\delta)$ we have

$$\|p\|_{K_\delta} \le \|p\|_{E_r} e^{n\varepsilon'}, \tag{4.3}$$

where $K_\delta := \{z \in \mathbb{C}^d : d(z,K) \le \delta\}$. To see this, we first apply Proposition 3.5 to see that $V_{P,K_\delta} \downarrow V_{P,K}$ on $\mathbb{C}^d$. Since K is PL-regular, $V_{P,K}$ is continuous on $\mathbb{C}^d$. By Dini's theorem we can choose $\delta = \delta(\varepsilon')$ such that

$$|V_{P,K}(z) - V_{P,K_\delta}(z)| < \frac{\varepsilon'}{2}, \ \ \forall z \in K_\delta.$$

In particular, since $V_{P,K_\delta} = 0$ on K_δ we get

$$V_{P,K}(z) \le \frac{\varepsilon'}{2}, \forall z \in K_\delta. \tag{4.4}$$

The Berstein-Walsh inequality (Proposition 3.1) now implies that for any $n \geq 1$ and $p \in \text{Poly}(nP)$ we have

$$\|p\|_{K_\delta} \leq \|p\|_K e^{n\varepsilon'/2}. \tag{4.5}$$

On the other hand, by the hypothesis $V^*_{P,E_r} \to 0$ q.e on K, so by Proposition 2.5 we see that the family V^*_{P,E_r} is locally uniformly bounded from above on $\mathbb{C}^d$. So by shrinking δ and using Hartogs' lemma we may obtain that

$$V^*_{P,E_r}(z) \leq \frac{\varepsilon'}{2}\ \forall z \in K, \forall 0 < r < \delta.$$

Using again the Berstein-Walsh inequality for E_r we have

$$\|p\|_K \leq \|p\|_{E_r} e^{n\varepsilon'/2}. \tag{4.6}$$

Combining these last estimates we obtain (4.3).

Step 2. We will show for all n large enough and all $w \in E_{r_n}$

$$|p(z)| \geq |p(w)| - \frac{1}{2}\|p\|_{E_{r_n}}, \forall |z-w| < r_n. \tag{4.7}$$

For $z \neq w$ we put $e = \frac{z-w}{\|z-w\|} = (e_1, ..., e_d)$. Put $q(t) := q(w_1 + e_1 t, ..., w_d + e_d t)$. Then q is a polynomial of one complex variable t with $p(z) = q(\|z-w\|)$ and $p(w) = q(0)$. Then

$$p(z) - p(w) = q(\|z-w\|) - q(0) = \int_0^{\|z-w\|} q'(t)dt.$$

So for $r' > r > 0$ we have

$$|p(z) - p(w)| \leq r\|q'\|_{|t|<r} \leq r\frac{\|q\|_{|t|<r'}}{r'-r} \leq \frac{r}{r'-r}\|p\|_{K_{r'}}. \tag{4.8}$$

Here we use Cauchy's inequality in the last estimate. Choose $r := r_n, r' := r_n(1 + 2e^{n\varepsilon'})$, by Step 1 we obtain for n large enough the following estimate

$$|p(z)| \geq |p(w)| - \frac{\|p\|_{K_{r'_n}}}{2e^{n\varepsilon'}} \geq |p(w)| - \frac{1}{2}\|p\|_{E_{r_n}}.$$

We finish the proof of this step.

Step 3. Completion of the proof. Fix $p \in \text{Poly}(nP)$. Then for each $w \in E_{r_n}$, from (4.7) we obtain the following chain of estimates

$$\begin{aligned}
\|p\|_{L^2(\mu)} &= \Big(\int_K |p|^2 d\mu\Big)^{\frac{1}{2}} \geq \Big(\int_{B(w,r_n)\cap K} |p|^2 d\mu\Big)^{\frac{1}{2}} \\
&\geq \mu(B(w,r_n))^{1/2} \inf_{B(w,r_n)} |p(z)| \\
&\geq f(r_n)^{1/2}\big(|p(w)| - \frac{1}{2}\|p\|_{E_{r_n}}\big).
\end{aligned}$$

Taking supremum over $w \in E_{r_n}$ and using (4.6) we get

$$\|p\|_{L^2(\mu)} \geq \frac{1}{2} f(r_n)^{1/2}\|p\|_{E_{r_n}} \geq \frac{1}{2} f(r_n)^{1/2} e^{-n\varepsilon'/2}\|p\|_K.$$

So in view of the property (ii) of f, there exists a constant $C > 0$ such that for $n \geq n_0$ large enough we have

$$Ce^{n\varepsilon/2}\|p\|_{L^2(\mu)} \geq \|p\|_K,$$

Finally, since $\text{Poly}(n_0 P)$ is a finite dimension space, the norm $\|.\|_{L^2(\mu)}$ and the sup-norm are equivalent. The proof is thereby completed. □

Theorem 4.6. *Let K be a compact non-pluripolar subset of $\mathbb{C}^d$ and μ be a finite positive Borel measure on K. Let $f : (0,\infty) \to (0,\infty)$ be a function satisfying the $(BM)-$ property. Assume that the set $\{z \in \mathbb{C}^d : \sup\limits_{0<r<r_0} V^*_{P,E_r}(z) < \infty\}$ is non-pluripolar for some $r_0 > 0$, where*

$$E_r := \{z \in K : \mu(K \cap B(z,r)) \geq f(r)\}.$$

Then the triple (P,K,μ) has the weak Bernstein-Markov property.

Proof. By the assumption and Proposition 3.6 we infer that the family V^*_{P,E_r} is locally uniformly bounded on $\mathbb{C}^d$. Moreover, since K is non-pluripolar we have

$$\lambda := \max\{\limsup_{r\to 0}(\sup_K V^*_{P,E_r}), \limsup_{\delta\to 0}(\sup_K V^*_{P,K_\delta})\} < \infty,$$

where $K_\delta := \{z \in \mathbb{C}^d : d(z,K) \leq \delta\}$. Fix $0 < \varepsilon < 1$. Then we choose ε' and a sequence $\{r_n\}$ satisfying the condition given on f. Now by the same reasoning as in Step 1 of Theorem 4.3 we can find $\delta > 0$ such that for $r \in (0,\delta)$ we have

$$\|p\|_{K_\delta} \leq \|p\|_{E_r} e^{n(\lambda+\varepsilon')} \text{ and } \|p\|_K \leq \|p\|_{E_r} e^{n(\lambda+\varepsilon')/2}. \tag{4.9}$$

By Step 2 in Theorem 4.3 for n large enough and $w \in E_{r_n}$ we have the following estimate

$$|p(z)| \geq |p(w)| - \frac{1}{2}\|p\|_{E_{r_n}}, \forall |z-w| < r_n. \tag{4.10}$$

Finally we fix $p \in \text{Poly}(nP)$. Then by repeating the argument given in Step 3 and using (4.9) and (4.10) we obtain

$$\|p\|_{L^2(\mu)} \geq \frac{1}{2} f(r_n)^{1/2} \|p\|_{E_{r_n}} \geq \frac{1}{2} f(r_n)^{1/2} e^{-n(\lambda+\varepsilon')/2} \|p\|_K.$$

So in view of the property (ii) of f, we see that there exists a constant $C > 0$ such that for $n \geq n_0$ large enough we have

$$Ce^{n(\lambda+\varepsilon)/2} \|p\|_{L^2(\mu)} \geq \|p\|_K.$$

Finally, since $\text{Poly}(n_0P)$ is a finite dimension space, the norm $\|.\|_{L^2(\mu)}$ and the sup-norm are equivalent. The proof is thereby completed. □

Now, we deal with the following notation which is relevant to the Bernstein-Markov property that was introduced by Siciak [10].

Definition 4.7. *A measure μ is called P-determining for a compact $K \subset \mathbb{C}^d$ if for every Borel $E \subset K$ such that $\mu(E) = \mu(K)$ we have $V^*_{P,E} = V^*_{P,K}$.*

Example 4.8. (a) Let D be a bounded open set in $\mathbb{C}^d$ such that ∂D is C^1 smooth. Then the Lebesgue measure λ_{2d} is P-determining for $K = \overline{D}$ and the surface measure σ_{2d-1} is P-determining for $K' = \partial D$. These facts are easy consequences of basics facts that K (resp. K') is non-plurithin at every point of K (resp. K').

(b) By the same proof as Proposition 2.4 in [8] we conclude that if K is non-pluripolar compact then the measure $\mu = (dd^c V^*_{P,K})^d$ is P-determining for K.

In the case $P = \Sigma$, Siciak showed in [10] (see also Proposition 2.5 in [8]) that if K is compact L-regular and μ is determining for K then (K,μ) satisfies the Bernstein-Markov inequality (1.1). This result is expanded in [6] for the case K is compact non-pluripolar. The following is analogue to Proposition 4.8 in [6] and for the reader's convenience we give here the proof.

Theorem 4.9. *Let K be a $L-$ regular (resp. non-pluripolar) compact subset of $\mathbb{C}^d$. Assume that μ is a P-determining measure for K. Then (P,K,μ) has the strong (resp. weak) Bernstein-Markov property.*

Proof. We only give the proof for the weak Bernstein-Markov property, the other case is somewhat easier. Let $\lambda := \sup_K V^*_{P,K}$ and $E := \{z \in K : V^*_{P,K}(z) > 0\}$. Then E is pluripolar and so there exists a plurisubharmonic functions φ on $\mathbb{C}^d$ such that

$$E \subset E' := \{z \in K : \varphi(z) = -\infty\}.$$

Let $E_j := \{z \in K : \varphi(z) \geq -j\}$ and $\varepsilon' := \varepsilon/2$. Then $\{E_j\}$ is an increasing sequence of compact subsets of K and $E_j \uparrow K \setminus E'$. By Proposition 3.5 we have

$$V^*_{P,E_j} \downarrow V^*_{P,K\setminus E'} = V^*_{P,K}.$$

Then $\sup_K V^*_{P,E_j} \downarrow \sup_K V^*_{P,K}$, thus we can find $j(\varepsilon)$ sufficient large such that

$$V^*_{P,E_{j(\varepsilon)}}(z) \leq \lambda + \varepsilon' \ \forall z \in K. \tag{4.11}$$

We claim that there exists $C > 0$ such that for any $n \geq 1$ and any $p \in \text{Poly}(nP)$ we have

$$\|p\|_{E_{j(\varepsilon)}} \leq Ce^{n\varepsilon'}\|p\|_{L^2(\mu)}. \tag{4.12}$$

We proceed by contradiction. Suppose that there exists a sequence $\{n_k\}$ and $p_{n_k} \in \text{Poly}(n_kP)$ such that

$$\|p_{n_k}\|_{E_{j(\varepsilon)}} \geq k(1+\varepsilon')^{n_k}, \quad \|p_{n_k}\|_{L^2(\mu)} = \frac{1}{k}. \tag{4.13}$$

For each $m \geq 1$, define

$$K_m := \{z \in K : \sup_{k\geq 1}|p_{n_k}(z)| \leq m\} \ \text{ and } \ K' := \bigcup_{m\geq 1} K_m.$$

Then $K_m \uparrow K'$, hence $V^*_{P,K_m} \downarrow V^*_{P,K}$. We will show that

$$V^*_{P,K'} = V^*_{P,K} \text{ on } \mathbb{C}^d. \tag{4.14}$$

Since μ is P-determining for K, it suffices to check that $\mu(K \setminus K') = 0$. Indeed, we infer from (4.13) that $\sum_{k\geq 1} |p_{n_k}(z)|^2$ converges in $L^1(\mu)$ and hence $|p_{n_k}(z)| \to 0$ μ-a.e as $k \to \infty$, thus $\sup_k |p_{n_k}(z)| < +\infty$ μ-a.e. This means $\mu(K \setminus K') = 0$. Thus (4.14) is proved. Then it follows from (4.14) that $V^*_{P,K_m} \downarrow V^*_{P,K}$ on $\mathbb{C}^d$. In particular, $V^*_{P,K_m} \downarrow 0$ on $E_{j(\varepsilon)}$. By Dini's theorem we can find m_0 such that $V^*_{P,K_{m_0}} \leq \varepsilon'$ on $E_{j(\varepsilon)}$. It follows that

$$\frac{1}{n_k}\log\frac{|p_{n_k}(z)|}{m_0} \leq V^*_{P,K_{m_0}}(z) \leq \varepsilon', \quad \forall k \geq 1, \forall z \in E_{j(\varepsilon)}.$$

This yields a contradiction to (4.13) if k is large enough. Finally, combining (4.11), (4.12) and applying Bernstein-Walsh inequality to $E_{j(\varepsilon)}$ we obtain

$$\|p\|_K \leq \|p\|_{E_{j(\varepsilon)}} e^{(\lambda+\varepsilon')n} \leq Ce^{(\lambda+\varepsilon)n}\|p\|_{L^2(\mu)}, \ \forall p \in \text{Poly}(nP), \ \ n \geq 1.$$

The proof is thereby completed. □

We have the following result which gives examples of measures satisfying the condition of Theorem 4.4 and Theorem 4.6.

Proposition 4.10. *Let K be a compact set in $\mathbb{C}^d$ and μ be a finite positive Borel measure on K. Let $f:(0,\infty)\to(0,\infty)$ be a function satisfying the $(BM)-$ property. Set*

$$G := \{z\in K : \liminf_{r\to 0}\frac{\mu(B(z,r)\cap K)}{f(r)} > 1\}.$$

Then the following assertions hold true:

(i) If G is non-pluripolar then (K,P,μ) has the weak Bernstein-Markov property;

(ii) If K is PL-regular and if $V^*_{P,G} = V^*_{P,K}$ then (K,P,μ) has the strong Bernstein-Markov property.

Proof. For $r>0$ we set

$$f_r(z) := \frac{\mu(B(z,r))}{f(r)}, E_r := \{z\in K : f_r(z)\geq 1\}.$$

Then we have

$$\begin{aligned} G &= \{z\in K : \liminf_{r\to 0} f_r(z) > 1\} \subset \{z\in K : \sup_{r>0}\inf_{s\geq 0} f_{r+s}(z) > 1\} \\ &\subset \bigcup_{r>0}\{z\in K : \inf_{s\geq 0} f_{r+s}(z)\geq 1\} \subset \bigcup_{r>0}\bigcap_{s\geq 0}\{z\in K : f_{r+s}(z)\geq 1\} \\ &= \bigcup_{r>0}\bigcap_{s\geq 0} E_{r+s} = \bigcup_{r>0} F_r \end{aligned}$$

where $F_r := \bigcap\limits_{s\geq 0} E_{r+s}$. Note that $F_r\subset E_r$ and by the above reasoning $\{F_r\}_{r>0}\uparrow G$. Thus, if G is non-pluripolar then so is F_{r_0} for some r_0 close enough to 0. Since

$$F_{r_0}\subset \{z\in\mathbb{C}^d : \sup_{0<r<r_0} V^*_{P,E_r}(z) < \infty\}.$$

So the set on the right hand side is non-pluripolar, by Theorem 4.4 we conclude the assertion (i). For (ii), it suffices to use Proposition 3.5 (iii) to get

$$V^*_{P,F_r}\downarrow V^*_{P,G} = V^*_{P,E} \text{ on } \mathbb{C}^d.$$

Since $V^*_{P,E_r}\leq V^*_{P,F_r}$ we infer $V^*_{P,E_r}\to 0$ pointwise on K as $r\to 0$. By Theorem 4.4 we obtain the desired conclusion (ii). □

In case (K,P,μ) has the strong Bernstein-Markov property and P is an admissible convex body, we can express the $P-$global extremal function $V_{P,K}$ by a sequence of Szëgo kernels (see [1] and [4]). It's natural to see what may occur if (K,P,μ) only has the weak Bernstein-Markov property. We only has the following very partial result.

Proposition 4.11. *Let P be a convex compact subset of $\mathbb{R}^d$. Assume that (K,P,μ) has the weak Bernstein-Markov property. For $n\geq 1$ we let $\{f_j\}_{1\leq j\leq d_n}$ be an orthonormal basis for Poly(nP) with respect to the inner product in $L^2(\mu)$. Set*

$$S_n(z,w) := \sum_{1\leq j\leq d_n} f_j(z)\overline{f_j(w)}.$$

Then there exists $\lambda\geq 0$ such that

$$u_{P,K}(z) := \limsup_{n\to\infty}\frac{1}{2n}\log S_n(z,z)\leq \lambda + V_{P,K}(z)\ \forall z\in\mathbb{C}^d.$$

In particular $u_{P,K}\in L_P(\mathbb{C}^d)$. Furthermore, if K is PL$-$regular then $u_{P,K}\geq V_{P,K}$.

Proof. Set

$$\Phi_n(z) = \sup\{|p_n(z)| : p_n \in \mathrm{Poly}(nP), \|p_n\|_K \le 1\}.$$

Then it's clear that

$$\frac{1}{n}\log|\Phi_n| \le V_{P,K} \text{ on } \mathbb{C}^d.$$

Moreover, since P is convex, we infer that $\Phi_n\Phi_m \le \Phi_{n+m}$ on $\mathbb{C}^d$. It follows that

$$\exists \lim_{n\to\infty} \frac{1}{n}\log\Phi_n(z) := v(z) \le V_{P,K}(z)\ \forall z \in \mathbb{C}^d.$$

On the other hand, since (K,P,μ) has the weak Bernstein-Markov property, there exists $\lambda \ge 0$ such that for $n \ge 1, 1 \le j \le d_n$ we have

$$\|p_j\|_K \le C_\varepsilon e^{(\lambda+\varepsilon)n}.$$

By the arguments of Bloom and Shiffman we get the following key estimates

$$\frac{1}{d_n} \le \frac{S_n(z,z)}{\Phi_n(z)} \le C_\varepsilon e^{(\lambda+\varepsilon)n} d_n.$$

Putting all this together we obtain the desired conclusions. □

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