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On the nodal length of Gaussian fields from spinodal decomposition

Viet-Hung Pham

Abstract In this paper we consider the nodal length of a Gaussian fields derived from the spinodal decomposition of stochastic Cahn-Hilliard-Cook equation. Using the celebrated Kac-Rice formula, we prove that the behavior of the expected length is $\epsilon^{-1}/4$ as the parameter ϵ tends to 0.

Keywords Gaussian fields, Stochastic Cahn-Hilliard equation, spinodal decomposition, Kac-Rice formula.

Mathematics Subject Classification (2000) 60G15 · 60G60 · 60H15.

1 Introduction

Study of the geometric properties of level sets of random fields is an important branch in probability theory. Its applications can be found in many areas such that biology, neuroscience, physics and so on. For more detail on this topic, we refer the readers to two standard monographs by Adler and Taylor [1] and Azaïs and Wschebor [3].

In this paper, we are interested in the following random field model, defined as a random two-variable function with the parameter ϵ,

$$f_\epsilon(x,y) = \sum_{(k,l)\in R_\epsilon} a_{k,l}\cos(k\pi x)\cos(l\pi y), \tag{1}$$

where for a given positive constant $\gamma \in (0,1)$,

$$R_\epsilon = \left\{(k,l)\in \mathbb{N}^2 :\ \frac{1-\sqrt{1-\gamma}}{2\pi^2} < (k\epsilon)^2 + (l\epsilon)^2 < \frac{1+\sqrt{1-\gamma}}{2\pi^2}\right\}, \tag{2}$$

and the random coefficients $a_{k,l}$'s are independent and identically distributed (i.i.d.) standard normal random variables. More precisely, we will study the nodal curve

$$L_\epsilon = \{(x,y)\in[0,1]^2 :\ f_\epsilon(x,y) = 0\} \tag{3}$$

and consider the length of this curve $\lambda_1(L_\epsilon)$.

Viet-Hung Pham*
Institute of Mathematics, Vietnam Academy of Science and Technology (VAST), Hanoi, Vietnam
E-mail: pgviethung@gmail.com, pvhung@math.ac.vn

Our motivation to investigate this model is from the study of the stochastic Cahn-Hilliard-Cook equation, proposed by Blömker, Maier-Paape and Wanner [4, 5, 6],

$$\partial_t u = - \triangle \left(\epsilon^2 \triangle u + \mathcal{F}(u) \right) + \partial_t W, \tag{4}$$

where W_t is the Wiener process and $\mathcal{F}$ is the derivative of a good potential with a standard choice $\mathcal{F}(u) = u - u^3$. Restricted on the unit square with the orthogonal L^2-basis $e_{k,l}(x, y) = \cos(k\pi x)\cos(l\pi y)$, the solution to this equation is presented as a random combination of cosine functions. By the spinodal decomposition (see [12, 13]), the truncation of the solution on the domain R_ϵ dominates the dynamics of the equation. A simplified approximation of this truncation is reduced to the considering random field in (1).

In practical applications, the nodal curve of the solution to the Cahn-Hilliard-Cook equation models the patterns in coats of animals like pandas, tigers and zebras [14, 16, 17, 18]. Therefore, it is natural to study the geometric properties of the nodal curve. In [7], the authors examined the expected number of the intersections between the nodal curve (3) and a given segment. This number represents the thickness of the pattern structures along this segment. They found that the expectation is of the order ϵ^{-1} with the scale depending on the slope of the segment. Here they used a formula given by Edelman and Kostlan [10] that can be seen as the one-dimensional Kac-Rice formula.

In this paper, we address to the expectation of the nodal length. Our main result is the following.

Theorem 1 *Consider the Gaussian field given in (1) and define its nodal curve as in (3). As ϵ tends to 0, we have*

$$\lim_{\epsilon \to 0} \epsilon E(\lambda_1(L_\epsilon)) = \frac{1}{4}. \tag{5}$$

The detailed proof is presented in Section 2. Our main tool is the general Kac-Rice formula, see [3].

The readers that are familiar with random polynomials theory can see the similarity between the considering random field model in this paper and the random bivariate trigonometric polynomials in [2]. The difference between two models stays in the index sets of random coefficients. In [2], the author proved a universal result for the expected nodal length in the sense that the asymptotic behavior of this expectation as in (5) not only is true for the Gaussian case, but also holds for general distributions of the coefficients under some mild conditions. It is interesting to see whether the universality holds true for our Gaussian fields from spinodal decomposition?

Besides nodal length, we could investigate other geometric quantities of the nodal curve such as the number of connected components and the Betti number. Some numerical works on these questions are given by [8, 9, 11]. Some rigorous theoretical treatments are provided by Narazov and Sodin [15] and recently by Wigman [19]. However their methods only works for isotropic Gaussian fields, that does not fit to the considering model. In conclusion, study of geometry of solution to stochastic Cahn-Hilliard-Cook equation is still a challenge and it poses man interesting questions for future research.

2 Proof of the main theorem

Before proving the main theorem, we need some auxiliary lemmas. The first lemma states the distribution of the considering fields and its partial derivatives.

Lemma 1 *We have*

$$\left(f_\epsilon, \frac{\partial f_\epsilon}{\partial x}, \frac{\partial f_\epsilon}{\partial y} \right) \sim \mathcal{N}\left(0, (\Sigma_{ij})_{i,j=0,1,2} \right),$$

where

$$\begin{aligned}
\Sigma_{00} &= \sum_{(k,l)\in R_\epsilon} \cos^2(k\pi x)\cos^2(l\pi y),\\
\Sigma_{11} &= \pi^2 \sum_{(k,l)\in R_\epsilon} k^2\sin^2(k\pi x)\cos^2(l\pi y),\\
\Sigma_{22} &= \pi^2 \sum_{(k,l)\in R_\epsilon} \cos^2(k\pi x) l^2 \sin^2(l\pi y),\\
\Sigma_{01} &= -\pi \sum_{(k,l)\in R_\epsilon} k\cos(k\pi x)\sin(k\pi x)\cos^2(l\pi y),\\
\Sigma_{02} &= -\pi \sum_{(k,l)\in R_\epsilon} \cos^2(k\pi x) l\cos(l\pi y)\sin(l\pi y),\\
\Sigma_{12} &= \pi^2 \sum_{(k,l)\in R_\epsilon} kl\cos(k\pi x)\sin(k\pi x)\cos(l\pi y)\sin(l\pi y).
\end{aligned}$$

The next lemma provides the conditional distribution of the gradient vector. This conditional will be used in Kac-Rice formula later on.

Lemma 2 *Given $f_\epsilon = 0$, the conditional distribution of $\left(\frac{\partial f_\epsilon}{\partial x}, \frac{\partial f_\epsilon}{\partial y}\right)$ is centered normal with covariance matrix*

$$\frac{1}{\Sigma_{00}}\begin{pmatrix} \Sigma_{00}\Sigma_{11} - \Sigma_{01}^2 & \Sigma_{00}\Sigma_{12} - \Sigma_{01}\Sigma_{02} \\ \Sigma_{00}\Sigma_{12} - \Sigma_{01}\Sigma_{02} & \Sigma_{00}\Sigma_{22} - \Sigma_{02}^2 \end{pmatrix}.$$

Proof It is clear from the that under the condition $f_\epsilon = 0$, the conditional covariance matrix of the gradient vector $f'_\epsilon = \left(\frac{\partial f_\epsilon}{\partial x}, \frac{\partial f_\epsilon}{\partial y}\right)$ is given by

$$\mathrm{Var}(f'_\epsilon) - \mathrm{Cov}(f'_\epsilon, f_\epsilon)[\mathrm{Var}(f_\epsilon)]^{-1}[Cov(f'_\epsilon, f_\epsilon)]^T.$$

The next lemma can be found in [7].

Lemma 3 *For irrational numbers $x, y \in (0,1)$, we have*

$$\lim_{\epsilon\to 0} \frac{\Sigma_{00}}{|R_\epsilon|} = \frac{1}{4}, \tag{6}$$

$$\lim_{\epsilon\to 0} \epsilon^2 \frac{\Sigma_{11}}{\Sigma_{00}} = \lim_{\epsilon\to 0} \epsilon^2 \frac{\Sigma_{22}}{\Sigma_{00}} = \frac{1}{4}, \tag{7}$$

$$\lim_{\epsilon\to 0} \epsilon \frac{\Sigma_{01}}{\Sigma_{00}} = \lim_{\epsilon\to 0} \epsilon \frac{\Sigma_{02}}{\Sigma_{11}} = 0, \tag{8}$$

$$\lim_{\epsilon\to 0} \epsilon^2 \frac{\Sigma_{12}}{\Sigma_{00}} = 0, \tag{9}$$

Lemma 4 *For ϵ small enough, uniformly in $(x,y) \in [0,1]^2$, there exist a universal constant $C > 0$ such that*

$$0 \le \epsilon^2 \left(\frac{\Sigma_{11}}{\Sigma_{00}} + \frac{\Sigma_{22}}{\Sigma_{00}}\right) \le C.$$

Proof We have

$$\begin{aligned} 0 \le \epsilon^2 \left(\frac{\Sigma_{11}}{\Sigma_{00}} + \frac{\Sigma_{22}}{\Sigma_{00}}\right) &= \frac{\epsilon^4\pi^2 \sum_{(k,l)\in R_\epsilon} k^2 \sin^2(k\pi x)\cos^2(l\pi y) + \cos^2(k\pi x) l^2 \sin^2(l\pi y)}{\epsilon^4 \sum_{(k,l)\in R_\epsilon} \cos^2(k\pi x)\cos^2(l\pi y)} \\ &\le \frac{\epsilon^4\pi^2 \sum_{(k,l)\in R_\epsilon} k^2 + l^2}{\epsilon^2 \sum_{(k,l)\in R_\epsilon} \cos^2(k\pi x)\cos^2(l\pi y)} \le \frac{\epsilon^2 \sum_{(k,l)\in R_\epsilon} (1+\sqrt{1-\gamma})}{\epsilon^2 \sum_{(k,l)\in R_\epsilon} \cos^2(k\pi x)\cos^2(l\pi y)} \\ &\le \frac{const}{\epsilon^2 \sum_{k,l\in R_\epsilon} \cos^2(k\pi x)\cos^2(l\pi y)}, \end{aligned}$$

where the second line follows from the fact that

$$(k\epsilon)^2 + (l\epsilon)^2 < \frac{1+\sqrt{1-\gamma}}{2\pi^2},$$

and the third line follows from the fact that

$$\lim_{\epsilon\to 0} \epsilon^2 \times \text{card}\left\{(k,l)\in\mathbb{N}^2 : \frac{1-\sqrt{1-\gamma}}{2\pi^2} < (k\epsilon)^2 + (l\epsilon)^2 < \frac{1+\sqrt{1-\gamma}}{2\pi^2}\right\} = \frac{1}{4}.$$

As in [7], let us denote

$$\alpha_\ominus = \sqrt{\frac{1-\sqrt{1-\gamma}}{2\pi^2}}, \quad \alpha_\oplus = \sqrt{\frac{1+\sqrt{1-\gamma}}{2\pi^2}}.$$

Then

$$\begin{aligned} \epsilon^2 \sum_{k,l\in R_\epsilon} \cos^2(k\pi x)\cos^2(l\pi y) &\ge \epsilon^2 \sum_{k,l=\lceil \alpha_\ominus/(\sqrt{2}\epsilon)\rceil}^{\lfloor \alpha_\oplus/(\sqrt{2}\epsilon)\rfloor} \cos^2(k\pi x)\cos^2(l\pi y) \\ &= \left(\epsilon \sum_{k=\lceil \alpha_\ominus/(\sqrt{2}\epsilon)\rceil}^{\lfloor \alpha_\oplus/(\sqrt{2}\epsilon)\rfloor} \cos^2(k\pi x)\right)\left(\epsilon \sum_{l=\lceil \alpha_\ominus/(\sqrt{2}\epsilon)\rceil}^{\lfloor \alpha_\oplus/(\sqrt{2}\epsilon)\rfloor} \cos^2(l\pi y)\right). \end{aligned} \tag{10}$$

To complete the proof of the lemma, we will show that for ϵ small enough, uniformly in $x \in [0,1]$, there exist a universal constant $D > 0$ such that

$$\epsilon \sum_{k=\lceil \alpha_\ominus/(\sqrt{2}\epsilon)\rceil}^{\lfloor \alpha_\oplus/(\sqrt{2}\epsilon)\rfloor} \cos^2(k\pi x) \ge D.$$

Indeed, it is clear that there exist two natural numbers t and h such that

$$\left[\frac{t}{2^h}, \frac{(t+1)(t+2)}{t.2^h}\right] \subset \left(\frac{\alpha_\ominus}{\sqrt{2}}, \frac{\alpha_\oplus}{\sqrt{2}}\right),$$

then we have

$$\begin{aligned} \epsilon \sum_{k=\lceil \alpha_\ominus/(\sqrt{2}\epsilon)\rceil}^{\lfloor \alpha_\oplus/(\sqrt{2}\epsilon)\rfloor} \cos^2(k\pi x) &\ge \epsilon \sum_{k=\lceil t/(2^h\epsilon)\rceil}^{\lfloor (t+1)(t+2)/(t.2^h\epsilon)\rfloor} \cos^2(k\pi x) \\ &\ge \frac{\epsilon}{3} \sum_{k=\lceil t/(2^h\epsilon)\rceil, t|k}^{\lfloor (t+1)/(2^h\epsilon)\rfloor} \left(\cos^2(k\pi x) + \cos^2\left(\frac{k(t+1)}{t}\pi x\right) + \cos^2\left(\frac{k(t+2)}{t}\pi x\right)\right). \end{aligned} \tag{11}$$

For each $x \in [0,1]$, by Cauchy-Schwarz inequality,

$$\begin{aligned} &\cos^2(k\pi x) + \cos^2\left(\frac{k(t+1)}{t}\pi x\right) + \cos^2\left(\frac{k(t+2)}{t}\pi x\right) \\ &\ge \cos^2\left(\frac{k(t+1)}{t}\pi x\right) + \frac{1}{2}\left[\cos\left(\frac{k(t+2)}{t}\pi x\right) - \cos(k\pi x)\right]^2 \\ &= \cos^2\left(\frac{k(t+1)}{t}\pi x\right) + 2\sin^2\left(\frac{k(t+1)}{t}\pi x\right)\sin^2\left(\frac{k}{t}\pi x\right) \end{aligned}$$

It is clear that there exists a small enough positive constant $1/4 > \tau > 0$ such that if $\cos^2 x \geq 1-\tau$ then $\cos^2[(t+1)x] \geq 1/2$. And if $\cos^2 x \leq 1-\tau$, then $\sin^2 x \geq \tau$ and in this case

$$\cos^2[(t+1)x] + 2\sin^2[(t+1)x]\sin^2 x \geq \cos^2[(t+1)x] + 2\tau\sin^2[(t+1)x] \geq 2\tau.$$

It means that we always have

$$\cos^2(k\pi x) + \cos^2\left(\frac{k(t+1)}{t}\pi x\right) + \cos^2\left(\frac{k(t+2)}{t}\pi x\right) \geq 2\tau.$$

Using this fact in (11),

$$\epsilon \sum_{k=\lceil \alpha_\ominus/(\sqrt{2}\epsilon)\rceil}^{\lfloor \alpha_\oplus/(\sqrt{2}\epsilon)\rfloor} \cos^2(k\pi x) \geq \frac{\epsilon}{3}\frac{(t+1)-t}{2^h \epsilon t}2\tau = \frac{2\tau}{3.2^h.t}.$$

Then the result follows easily.

Now using these lemmas, we are able to calculate the expected length of the nodal curve of $f_\epsilon(x,y)$.

Proof ***(Proof of the main theorem)*** By Kac-Rice formula (see [3]), the rescaled expectation of the length of the nodal curve is equal to

$$\begin{aligned}\epsilon \mathrm{E}(\lambda_1(L_\epsilon)) &= \epsilon \iint_{[0,1]^2} \mathrm{E}\left(\sqrt{\left(\frac{\partial f_\epsilon}{\partial x}\right)^2 + \left(\frac{\partial f_\epsilon}{\partial y}\right)^2} \,|f_\epsilon(x,y)=0\right) p_{f_\epsilon(x,y)}(0)dxdy \\ &= \frac{1}{\sqrt{2\pi}} \iint_{[0,1]^2} \frac{\epsilon}{\Sigma_{00}^{1/2}} \mathrm{E}\left(\sqrt{\left(\frac{\partial f_\epsilon}{\partial x}\right)^2 + \left(\frac{\partial f_\epsilon}{\partial y}\right)^2} \,|f_\epsilon(x,y)=0\right) dxdy, \end{aligned} \tag{12}$$

where $p_{f_\epsilon(x,y)}$ is the density function of $f_\epsilon(x,y)$.

From Lemma 2,

$$\frac{\epsilon}{\Sigma_{00}^{1/2}} \mathrm{E}\left(\sqrt{\left(\frac{\partial f_n}{\partial x}\right)^2 + \left(\frac{\partial f_n}{\partial y}\right)^2} \,|f_n(x,y)=0\right) = \mathrm{E}\sqrt{X_{1,\epsilon}^2(x,y) + X_{2,\epsilon}^2(x,y)},$$

where

$$(X_{1,\epsilon}(x,y)+, X_{2,\epsilon}(x,y)) \sim \mathcal{N}\left(0, \frac{\epsilon^2}{\Sigma_{00}^2}\begin{pmatrix} \Sigma_{00}\Sigma_{11} - \Sigma_{01}^2 & \Sigma_{00}\Sigma_{12} - \Sigma_{01}\Sigma_{02} \\ \Sigma_{00}\Sigma_{12} - \Sigma_{01}\Sigma_{02} & \Sigma_{00}\Sigma_{22} - \Sigma_{02}^2 \end{pmatrix}\right).$$

From the Cauchy-Schwarz inequality and Lemma 4, it is clear that uniformly in $(x,y) \in [0,1]^2$,

$$\mathrm{E}\sqrt{X_{1,\epsilon}^2(x,y) + X_{2,\epsilon}^2(x,y)} \leq \sqrt{\mathrm{E}\left(X_{1,\epsilon}^2(x,y) + X_{2,\epsilon}^2(x,y)\right)} \leq \sqrt{\epsilon^2\left(\frac{\Sigma_{11}}{\Sigma_{00}} + \frac{\Sigma_{22}}{\Sigma_{00}}\right)} \leq \sqrt{C}.$$

As a consequence of Lemma 3, for any irrational numbers $x, y \in (0,1)$, as ϵ tends to 0, the Gaussian vector $((X_{1,\epsilon}(x,y)+, X_{2,\epsilon}(x,y)))$ converges in distribution to a centered Gaussian vector (X_1, X_2) with covariance matrix $1/4 \times I_2$. Then it is easy to check that

$$\lim_{\epsilon\to 0}\mathrm{E}\sqrt{X_{1,\epsilon}^2(x,y) + X_{2,\epsilon}^2(x,y)} = \mathrm{E}\sqrt{X_1^2 + X_2^2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

To complete the proof, using the dominated convergence theorem, we have

$$\lim_{\epsilon\to 0}\epsilon\mathrm{E}(\lambda_1(L_\epsilon)) = \frac{1}{\sqrt{2\pi}}\iint_{[0,1]^2}\frac{1}{2}\sqrt{\frac{\pi}{2}}dxdy = \frac{1}{4}.$$

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pgviethung@gmail.com