MODIFIED DEFECT RELATIONS OF THE GAUSS MAP OF A COMPLETE MINIMAL SURFACE ON ANNULAR ENDS

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Abstract. In this article, we study the modified defect relations of the Gauss map of complete minimal surfaces in $\mathbb{R}^m (m \geq 3)$ on annular ends.

1. Introduction

Let M be a non-flat minimal surface in $\mathbb{R}^3$, or more precisely, a non-flat connected oriented minimal surface in $\mathbb{R}^3$. By definition, the Gauss map G of M is a map which maps each point $p \in M$ to the unit normal vector $G(p) \in S^2$ of M at p. Instead of G, we study the map $g := \pi \circ G : M \to \mathbb{P}^1(\mathbb{C})$, where $\pi : S^2 \to \mathbb{P}^1(\mathbb{C})$ is the stereographic projection. By associating a holomorphic local coordinate $z = u + \sqrt{-1}v$ with each positive isothermal coordinate system (u, v), M is considered as an open Riemann surface with a conformal metric ds^2 and by the assumption of minimality of M, g is a meromorphic function on M.

In 1988, H. Fujimoto [4] proved that if M is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most 4 points, and the bound is sharp. After that, M. Ru [12] and H. Fujimoto [5], [6] also that result by studying the ramifications and the modified defect relations of the Gauss map of a complete minimal surface in $\mathbb{R}^m (m \geq 3)$. H. Fujimoto proved the following.

Theorem A. *Let M be a complete minimal surface in $\mathbb{R}^m$. If the Gauss map G of M is nondegenerate then*

$$\sum_{j=1}^{q} \delta^H_{G,M}(H_j) \leq \frac{m(m+1)}{2}.$$

for arbitrary q hyperplanes $H_1, ..., H_q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ located in general position.

On the other hand, in 1991, S. J. Kao [11] used the ideas of Fujimoto [4] to show that the Gauss map of an end of a non-flat complete minimal surface in $\mathbb{R}^3$ that is conformally an annulus $\{z : 0 < 1/r < |z| < r\}$ must also assume every value, with at most 4 exceptions. In 2007, L. Jin and M. Ru [10] extended Kao's result to minimal surfaces in $\mathbb{R}^m$. Recently, Dethloff-Ha [2], Dethloff et al. [3] gave some improvements for the results of Kao and Jin-Ru by studying the Gauss maps with ramification properties. They proved:

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Theorem B. *Let M be a non-flat complete minimal surface in $\mathbb{R}^m$ and let A be an annular end of M which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where z is a conformal coordinate. Assume that the generalized Gauss map g of M is $k-$non-degenerate on A (that is $g(A)$ is contained in a $k-$dimensional linear subspace in $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension), $1 \leq k \leq m-1$. If there are q hyperplanes $\{H_j\}_{j=1}^q$ in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ $(N \geq m-1)$ such that g is ramified over H_j with multiplicity at least m_j on A for each j, then*

$$\sum_{j=1}^{q}(1-\frac{k}{m_j}) \leq (k+1)(N-\frac{k}{2})+(N+1).$$

Moreover, the above inequality still holds if we replace, for all $j = 1, ..., q$, m_j by the limit inferior of the orders of the zeros of the function $(g, H_j) := \overline{c}_{j0}g_1 + \cdots + \overline{c}_{jm-1}g_{m-1}$ on A (where $g = (g_0 : \cdots : g_{m-1})$ is a reduced representation and, for all $1 \leq j \leq q$, the hyperplane H_j in $\mathbb{P}^{m-1}(\mathbb{C})$ is given by $H_j : \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jm-1}\omega_{m-1} = 0$, where we assume that $\sum_{i=0}^{m-1} |c_{ji}|^2 = 1$) or by ∞ if g intersects H_j only a finite number of times on A.

A natural question is whether a result for the modified defect relations of the Gauss map still holds on an annular end of a non-flat complete minimal surface in $\mathbb{R}^m (m \geq 3)$. In this paper we give the affirmative answers for this question. In particular, we give some results on the modified defect relations of the Gauss map of complete minimal surfaces in $\mathbb{R}^m$ on annular ends which are similar to Theorem A. We thus give some improvements of the previous results on annular ends of complete minimal surfaces of Kao [11], Jin-Ru [10], Dethloff-Ha [2] and Dethloff et al. [3].

2. Statements of the main results

Let M be an open Riemann surface and f a nonconstant holomorphic map of M into $\mathbb{P}^k(\mathbb{C})$. Assume that f has reduced representation $f = (f_0 : \cdots : f_k)$. Set $||f|| = (|f_0|^2 + \cdots + |f_k|^2)^{1/2}$ and, for each a hyperplane $H : \overline{a}_0w_0 + \cdots + \overline{a}_kw_k = 0$ in $\mathbb{P}^k(\mathbb{C})$ with $|a_0|^2 + \cdots + |a_k|^2 = 1$, we define $f(H) := \overline{a}_0f_0 + \cdots + \overline{a}_kf_k$.

Definition 2.1. We define the $S-$defect of H for f by

$$\delta^S_{f,M}(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_S\}.$$

Here, condition $(*)_S$ means that there exists a $[-\infty, \infty)-$valued continuous subharmonic function u $(\not\equiv -\infty)$ on M satisfying the following conditions:
(C1) $e^u \leq ||f||^\eta$,
(C2) for each $\xi \in f^{-1}(H)$, there exists the limit

$$\lim_{z\to\xi}(u(z) - \min(\nu_{f(H)}(\xi), k)\log|z - \xi|) \in [-\infty, \infty),$$

where z is a holomorphic local coordinate around ξ and $\nu_{f(H)}$ is the divisor of $f(H)$.

Remark 2.2. We always have that $\eta = 1$ satisfies condition $(*)_S$ with $u = \log|f(H)|$.

Definition 2.3. We define the $H-$defect of H for f by

$$\delta^H_{f,M}(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_H\}.$$

Here, condition $(*)_H$ means that there exists a $[-\infty, \infty)-$valued continuous subharmonic function u on M which is harmonic on $M - f^{-1}(H)$ and satisfies the conditions (C1) and (C2).

Definition 2.4. We define the $O-$defect of H for f by

$$\delta^O_{f,M}(H) := 1 - \inf\{\ \frac{1}{n};\quad f(H) \text{ has no zero of order less than } n\}.$$

Remark 2.5. We always have $0 \leq \delta^O_{f,M}(H) \leq \delta^H_{f,M}(H) \leq \delta^S_{f,M}(H) \leq 1$.

Definition 2.6. One says that f is ramified over a hyperplane H in $\mathbb{P}^k(\mathbb{C})$ *with multiplicity at least* e if all the zeros of the function $f(H)$ have orders at least e. If the image of f omits H, one will say that f is *ramified over H with multiplicity* ∞.

Remark 2.7. If f is ramified over a hyperplane H in $\mathbb{P}^k(\mathbb{C})$ with multiplicity at least n, then $\delta^S_{f,M}(H) \geq \delta^H_{f,M}(H) \geq \delta^O_{f,M}(H) \geq 1 - \dfrac{1}{n}$. In particular, if $f^{-1}(H) = \emptyset$, then $\delta^O_{f,M}(H) = 1$.

Let $x = (x_0, \cdots, x_{m-1}) : M \to \mathbb{R}^m (m \geq 3)$ be a (smooth, oriented) minimal surface immersed in $\mathbb{R}^m$. Then M has the structure of a Riemann surface and any local isothermal coordinate (ξ_1, ξ_2) of M gives a local holomorphic coordinate $z = \xi_1 + \sqrt{-1}\xi_2$. The (generalized) Gauss map of x is defined to be

$$g : M \to Q_{m-2}(\mathbb{C}) \subset \mathbb{P}^{m-1}(\mathbb{C}), g(z) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}),$$

where

$$Q_{m-2}(\mathbb{C}) = \{(w_0 : \cdots : w_{m-1}) | w_0^2 + \cdots + w_{m-1}^2 = 0\} \subset \mathbb{P}^{m-1}(\mathbb{C}).$$

By the assumption of minimality of M, g is a holomorphic map of M into $Q_{m-2}(\mathbb{C})$.

In this article, we would like to study the relations between $H-$ defect relations for the Gauss maps of minimal surfaces in $\mathbb{R}^m$ on annular ends. In particular, we firstly prove the following theorem.

Theorem 2.8. *Let M be a non-flat complete minimal surface in $\mathbb{R}^m$ with the Gauss map G and let $A \subset M$ be an annular end of M which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where z is a conformal coordinate. For arbitrary q hyperplanes $H_1, ..., H_q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ in $N-$subgeneral position. Assume that G is $k-$non-degenerate on A (that is $G(A)$ is* *contained in a $k-$dimensional linear subspace in $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension),* $1 \leq k \leq m-1$, *then*

$$\sum_{j=1}^{q} \delta_{G,A}^{H}(H_j) \leq (k+1)(N - \frac{k}{2}) + (N+1).$$

It is easy to see that Theorem 2.8 is an improvement of Theorem B.
When $m = 3$, we can identify $\mathbb{Q}_1(\mathbb{C})$ with $\mathbb{P}^1(\mathbb{C})$. So we can get a better result as the following.

Theorem 2.9. *Let M be a non-flat complete minimal surface in $\mathbb{R}^3$ and let $A \subset M$ be an annular end of M which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where z is a conformal coordinate. For arbitrary q distinct points $a^1, ..., a^q$ in $\mathbb{P}^1(\mathbb{C})$, then*

$$\sum_{j=1}^{q} \delta_{g,A}^{H}(a^j) \leq 4.$$

Moreover, we also would like to consider the Gauss map of complete minimal surfaces M immersed in $\mathbb{R}^4$, this case has been investigated by various authors (see, for examples Osserman [13], Chen [1], Fujimoto [5], Dethloff-Ha [2] and Ha-Trao [9]. In this case, the Gauss map of M may be identified with a pair of meromorphic functions $g = (g^1, g^2)$. We prove the following result of modified defect relations of the Gauss map restricted on an annular end.

Theorem 2.10. *Suppose that M is a complete non-flat minimal surface in $\mathbb{R}^4$ and $g = (g^1, g^2)$ is the Gauss map of M. Let A be an annular end of M which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where z is a conformal coordinate. Let $a^{11}, ..., a^{1q_1}, a^{21}, ..., a^{2q_2}$ be $q_1 + q_2$ $(q_1, q_2 > 2)$ distinct points in $\mathbb{P}^1(\mathbb{C})$.*

(i) *In the case $g^l \not\equiv constant$ $(l = 1, 2)$, then $\gamma_1 = \sum_{j=1}^{q_1} \delta_{g^1,A}^{H}(a^{1j}) \leq 2$, or $\gamma_2 = \sum_{j=1}^{q_2} \delta_{g^2,A}^{H}(a^{2j}) \leq 2$, or*

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} \geq 1.$$

(ii) *In the case where one of g^1 and g^2 is constant, say $g^2 \equiv constant$, we have the following*

$$\gamma_1 \leq 3.$$

The main idea to prove the main theorems is to construct and to compare explicit singular flat and negatively curved complete metrics on annular ends. This generalizes previous work of Dethloff-Ha [2] and Dethloff et al. [3] (which itself was a refinement of ideas of Fujimoto [5], [6]. After that we use arguments similar to those used by Fujimoto [5], [6] , Kao [11] , Jin-Ru [10], Dethloff-Ha [2] and Dethloff et al. [3] and Ha [8] to finish the proofs.

3. Auxiliary lemmas

Let f be a linearly non-degenerate holomorphic map of $\Delta_R := \{z \in \mathbb{C} : |z| < R\}$ into $\mathbb{P}^k(\mathbb{C})$, where $0 < R \leq +\infty$. Take a reduced representation $f = (f_0 : \cdots : f_k)$. Then $F := (f_0, \cdots, f_k) : \Delta_R \to \mathbb{C}^{k+1} \setminus \{0\}$ is a holomorphic map with $\mathbb{P}(F) = f$. Consider the holomorphic map

$$F_p = (F_p)_z := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(p)} : \Delta_R \longrightarrow \wedge^{p+1}\mathbb{C}^{k+1}$$

for $0 \leq p \leq k$, where $F^{(0)} := F = (f_0, \cdots, f_k)$ and $F^{(l)} = (F^{(l)})_z := (f_0^{(l)}, \cdots, f_k^{(l)})$ for each $l = 0, 1, \cdots, k$, and where the l-th derivatives $f_i^{(l)} = (f_i^{(l)})_z$, $i = 0, ..., k$, are taken with respect to z. (Here and for the rest of this paper the index $|_z$ means that the corresponding term is defined by using differentiation with respect to the variable z, and in order to keep notations simple, we usually drop this index if no confusion is possible.) The norm of F_p is given by

$$|F_p| := \left(\sum_{0 \leq i_0 < \cdots < i_p \leq k} \left| W(f_{i_0}, \cdots, f_{i_p}) \right|^2 \right)^{\frac{1}{2}},$$

where $W(f_{i_0}, \cdots, f_{i_p}) = W_z(f_{i_0}, \cdots, f_{i_p})$ denotes the Wronskian of $f_{i_0}, \cdots, f_{i_p}$ with respect to z.

Proposition 3.1. *([7, Proposition 2.1.6]).*
For two holomorphic local coordinates z and ξ and a holomorphic function $h : \Delta_R \to \mathbb{C}$, the following holds :
(a) $W_\xi(f_0, \cdots, f_p) = W_z(f_0, \cdots, f_p) \cdot (\frac{dz}{d\xi})^{p(p+1)/2}$.
(b) $W_z(hf_0, \cdots, hf_p) = W_z(f_0, \cdots, f_p) \cdot (h)^{p+1}$.

Proposition 3.2. *([7, Proposition 2.1.7]).*
For holomorphic functions $f_0, \cdots, f_p : \Delta_R \to \mathbb{C}$ the following conditions are equivalent:
(i) $f_0, \cdots, f_p$ *are linearly dependent over* $\mathbb{C}$.
(ii) $W_z(f_0, \cdots, f_p) \equiv 0$ *for some (or all) holomorphic local coordinate* z.

We now take a hyperplane H in $\mathbb{P}^k(\mathbb{C})$ given by

$$H : \bar{c}_0\omega_0 + \cdots + \bar{c}_k\omega_k = 0\,,$$

with $\sum_{i=0}^k |c_i|^2 = 1$. We set

$$F_0(H) := F(H) := \bar{c}_0 f_0 + \cdots + \bar{c}_k f_k$$

and

$$|F_p(H)| = |(F_p)_z(H)| := \left(\sum_{0 \leq i_1 < \cdots < i_p \leq k} \left| \sum_{l \neq i_1, \ldots, i_p} \bar{c}_l W(f_l, f_{i_1}, \cdots, f_{i_p}) \right|^2 \right)^{\frac{1}{2}},$$

for $1 \leq p \leq k$. We note that by using Proposition 3.1, $|(F_p)_z(H)|$ is multiplied by a factor $|\frac{dz}{d\xi}|^{p(p+1)/2}$ if we choose another holomorphic local coordinate ξ, and it is multiplied by

$|h|^{p+1}$ if we choose another reduced representation $f = (hf_0 : \cdots : hf_k)$ with a nowhere zero holomorphic function h. Finally, for $0 \leq p \leq k$, set the p-th contact function of f for H to be $\phi_p(H) := \dfrac{|F_p(H)|^2}{|F_p|^2} = \dfrac{|(F_p)_z(H)|^2}{|(F_p)_z|^2}$.

We next consider q hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^k(\mathbb{C})$ given by

$$H_j : \langle \omega, A_j \rangle \equiv \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jk}\omega_k \quad (1 \leq j \leq q)$$

where $A_j := (c_{j0}, \cdots, c_{jk})$ with $\sum_{i=0}^{k} |c_{ji}|^2 = 1$.

Assume now $N \geq k$ and $q \geq N+1$. For $R \subseteq Q := \{1, 2, \cdots, q\}$, denote by $d(R)$ the dimension of the vector subspace of $\mathbb{C}^{k+1}$ generated by $\{A_j; j \in R\}$.

The hyperplanes $H_1, \cdots, H_q$ are said to be in N-subgeneral position if $d(R) = k+1$ for all $R \subseteq Q$ with $\sharp(R) \geq N+1$, where $\sharp(A)$ means the number of elements of a set A. In the particular case $N = k$, these are said to be in general position.

Theorem 3.3. *([7, Theorem 2.4.11]) For given hyperplanes $H_1, \cdots, H_q$ $(q > 2N-k+1)$ in $\mathbb{P}^k(\mathbb{C})$ located in N-subgeneral position, there are some rational numbers $\omega(1), \cdots, \omega(q)$ and θ satisfying the following conditions:*

(i) $0 < \omega(j) \leq \theta \leq 1 \quad (1 \leq j \leq q)$,

(ii) $\sum_{j=1}^{q} \omega(j) = k+1+\theta(q-2N+k-1)$,

(iii) $\frac{k+1}{2N-k+1} \leq \theta \leq \frac{k+1}{N+1}$,

(iv) *If $R \subset Q$ and $0 < \sharp(R) \leq n+1$, then $\sum_{j\in R} \omega(j) \leq d(R)$.*

Constants $\omega(j)$ $(1 \leq j \leq q)$ and θ with the properties of Theorem 2.8 are called Nochka weights and a Nochka constant for $H_1, \cdots, H_q$ respectively.

Proposition 3.4. *([7, Lemma 3.2.13]) Let f be a non-degenerate holomorphic map of a domain in $\mathbb{C}$ into $\mathbb{P}^k(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_k)$ and let $H_1, \cdots, H_q$ be hyperplanes located in N-subgeneral position $(q > 2N-k+1)$ with Nochka weights $\omega(1), \cdots, \omega(q)$ respectively. Then,*

$$\nu_\phi + \sum_{j=1}^{q} \omega(j) \cdot \min(\nu_{(f,H_j)}, k) \geq 0,$$

where $\phi = \dfrac{|F_k|}{\Pi_{j=1}^{q} \mid F(H_j) \mid^{\omega(j)}}$.

Lemma 3.5. [8, Lemma 3.10] *Let $f = (f_0 : \cdots : f_k) : \Delta_R \to \mathbb{P}^k(\mathbb{C})$ be a non-degenerate holomorphic map, $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in $N-$subgeneral position ($N \geq k$ and $q > 2N-k+1$), and $\omega(j)(1 \leq j \leq q)$ be their Nochka weights. Assume that there are positive real numbers $\eta_j(1 \leq j \leq q)$ and $[-\infty, \infty)-$valued continuous subharmonic functions u_j sastifying conditions (C1), (C2). Then for an arbitrarily given ϵ satisfying*

$\gamma = \sum_{j=1}^{q} \omega(j)(1-\eta_j) - (k+1) > \epsilon\sigma_{k+1} > 0$, *there exists a positive constant* C, *depending only on* $\epsilon, H_j, \eta_j, \omega(j)(1 \le j \le q)$, *such that*

$$\frac{|F|^{\gamma-\epsilon\sigma_{k+1}} e^{\sum_{j=1}^{q}\omega(j)u_j}.|F_k|^{1+\epsilon}.\Pi_{j=1}^{q}\Pi_{p=0}^{k-1}|F_p(H_j)|^{\epsilon/q}}{\Pi_{j=1}^{q}|F(H_j)|^{\omega(j)}} \leqslant C\left(\frac{2R}{R^2-|z|^2}\right)^{\sigma_k+\epsilon\tau_k},$$

where $\sigma_p = p(p+1)/2$ *for* $0 \le p \le k$ *and* $\tau_k = \sum_{p=0}^{k}\sigma_p$.

For the case $k = 1$, we have the following lemma of Fujimoto as a corollary of Lemma 3.5.

Lemma 3.6. ([5, Main lemma]) *Let* $f = (f_0 : \cdots : f_k) : \Delta_R \to \mathbb{P}^1(\mathbb{C})$ *be a non-degenerate holomorphic map,* $a_1, ..., a_q$ *be distinct points in* $\mathbb{P}^1(\mathbb{C})$. *Assume that there are positive real number* $\eta_j (1 \le j \le q)$ *and* $[-\infty, \infty)-$*valued continuous subharmonic functions* u_j *sastifying conditions (C1), (C2) and* $\sum_{j=1}^{q}(1-\eta_j) - 2 > 0$. *There exists a positive constant* C *such that*

$$\frac{||f||^{q-2-\sum_{j=1}^{q}\eta_j - q\delta} e^{\sum_{j=1}^{q} u_j}|W(f_0, f_1)|}{\Pi_{j=1}^{q}|F(a_j)|^{1-\delta}} \le C\frac{2R}{R^2-|z|^2}.$$

Lemma 3.7. ([7, Lemma 1.6.7]). *Let* $d\sigma^2$ *be a conformal flat metric on an open Riemann surface* M. *Then for every point* $p \in M$, *there is a holomorphic and locally biholomorphic map* Φ *of a disk (possibly with radius* ∞*)* $\Delta_{R_0} := \{w : |w| < R_0\}$ $(0 < R_0 \le \infty)$ *onto an open neighborhood of* p *with* $\Phi(0) = p$ *such that* Φ *is a local isometry, namely the pull-back* $\Phi^*(d\sigma^2)$ *is equal to the standard (flat) metric on* Δ_{R_0}, *and for some point* a_0 *with* $|a_0| = 1$, *the* Φ*-image of the curve*

$$L_{a_0} : w := a_0 \cdot s \ (0 \le s < R_0)$$

is divergent in M *(i.e. for any compact set* $K \subset M$, *there exists an* $s_0 < R_0$ *such that the* Φ*-image of the curve* $L_{a_0} : w := a_0 \cdot s$ $(s_0 \le s < R_0)$ *does not intersect* K*).*

4. The proof of Theorem 2.8

Proof. For the convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^m$. Let M be a complete immersed minimal surface in $\mathbb{R}^m$. Take an immersion $x = (x_0, ..., x_{m-1}) : M \to \mathbb{R}^m$. Then M has the structure of a Riemann surface and any local isothermal coordinate (x, y) of M gives a local holomorphic coordinate $z = x + \sqrt{-1}y$. The generalized Gauss map of x is defined to be

$$g : M \to \mathbb{P}^{m-1}(\mathbb{C}), g = \mathbb{P}(\frac{\partial x}{\partial z}) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}).$$

Since $x : M \to \mathbb{R}^m$ is immersed,

$$G = G_z := (g_0, ..., g_{m-1}) = ((g_0)_z, ..., (g_{m-1})_z) = (\frac{\partial x_0}{\partial z}, \cdots, \frac{\partial x_{m-1}}{\partial z})$$

is a (local) reduced representation of g, and since for another local holomorphic coordinate ξ on M we have $G_\xi = G_z \cdot (\dfrac{dz}{d\xi})$, g is well defined (independently of the (local) holomorphic coordinate). Moreover, if ds^2 is the metric on M induced by the standard metric on $\mathbb{R}^m$, we have

$$ds^2 = 2|G_z|^2|dz|^2. \tag{1}$$

Finally since M is minimal, g is a holomorphic map.

Since by hypothesis of Theorem 2.8, g is k-non-degenerate $(1 \leq k \leq m-1)$ without loss of generality, we may assume that $g(M) \subset \mathbb{P}^k(\mathbb{C})$; then

$$g : M \to \mathbb{P}^k(\mathbb{C}), g = \mathbb{P}(\frac{\partial x}{\partial z}) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_k}{\partial z}).$$

is linearly non-degenerate in $\mathbb{P}^k(\mathbb{C})$ (so in particular g is not constant) and the other facts mentioned above still hold.

Let $H_j (j = 1, ..., q)$ be $q (\geq N+1)$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in N-subgeneral position $(N \geq m-1 \geq k)$. Then $H_j \cap \mathbb{P}^k(\mathbb{C}) (j = 1, ..., q)$ are q hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in N-subgeneral position. Let each $H_j \cap \mathbb{P}^k(\mathbb{C})$ be represented as

$$H_j \cap \mathbb{P}^k(\mathbb{C}) : \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jk}\omega_k = 0$$

with $\sum_{i=0}^k |c_{ji}|^2 = 1$.
Set

$$G(H_j) = G_z(H_j) := \overline{c}_{j0} g_0 + \cdots + \overline{c}_{jk} g_k.$$

We will now, for each contact function $\phi_p(H_j)$ for each of our hyperplanes H_j, choose one of the components of the numerator $|((G_z)_p)_z(H_j)|$ which is not identically zero: More precisely, for each j, p $(1 \leq j \leq q, 1 \leq p \leq k)$, we can choose $i_1, \cdots, i_p$ with $0 \leq i_1 < \cdots < i_p \leq k$ such that

$$\psi(G)_{jp} = (\psi(G_z)_{jp})_z := \sum_{l \neq i_1, .., i_p} \overline{c}_{jl} W_z(g_l, g_{i_1}, \cdots, g_{i_p}) \not\equiv 0,$$

(indeed, otherwise, we have $\sum_{l \neq i_1, .., i_p} \overline{c}_{jl} W(g_l, g_{i_1}, \cdots, g_{i_p}) \equiv 0$ for all $i_1, ..., i_p$, so $W(\sum_{l \neq i_1, .., i_p} \overline{c}_{jl} g_l, g_{i_1}, \cdots, g_{i_p}) \equiv 0$ for all $i_1, ..., i_p$, which contradicts the non-degeneracy of g in $\mathbb{P}^k(\mathbb{C})$. Alternatively we simply can observe that in our situation none of the contact functions vanishes identically.) We still set $\psi(G)_{j0} = \psi(G_z)_{j0} := G(H_j) (\not\equiv 0)$, and we also note that $\psi(G)_{jk} = ((G_z)_k)_z$. Since the $\psi(G)_{jp}$ are holomorphic, so they have only isolated zeros.

Finally we put for later use the transformation formulas for all the terms defined above, which are obtained by using Proposition 3.1 : For holomorphic coordinates z and ξ on A we have :

$$((G_\xi)_k)_\xi = ((G_\xi)_k)_z \cdot (\frac{dz}{d\xi})^{\frac{k(k+1)}{2}} = ((G_\xi)_k)_z (\frac{dz}{d\xi})^{\sigma_k}, \tag{2}$$

$$(\psi(G_\xi)_{jp})_\xi = (\psi(G_\xi)_{jp})_z \cdot (\frac{dz}{d\xi})^{\frac{p(p+1)}{2}} = (\psi(G_\xi)_{jp})_z \cdot (\frac{dz}{d\xi})^{\sigma_p}, \ (0 \leq p \leq k)\,. \tag{3}$$

Now we prove Theorem 2.8 in four steps:

Step 1: We fix notations on the annular end $A \subset M$. Moreover, by passing to a sub-annular end of $A \subset M$ we simplify the geometry of Theorem 2.8 .

Let $A \subset M$ be an annular end of M, that is, $A = \{z : 0 < 1/r < |z| < r < \infty\}$, where z is a (global) conformal coordinate of A. Since M is complete with respect to ds^2, we may assume that the restriction of ds^2 to A is complete on the set $\{z : |z| = r\}$, i.e., the set $\{z : |z| = r\}$ is at infinite distance from any point of A.

It is easy to see $\delta^H_{G,B}(H_j) \geq \delta^H_{G,A}(H_j)$ for all subsets $B \subset A$. So without loss of generality we may prove our theorem only on a sub-annular end, i.e., a subset $A_t := \{z : 0 < t \leq |z| < r < \infty\} \subset A$ with some t such that $1/r < t < r$. (We trivially observe that for $c := tr > 1$, $s := r/\sqrt{c}$, $\xi := z/\sqrt{c}$, we have $A_t = \{\xi : 0 < 1/s \leq |\xi| < s < \infty\}$.)

By passing to such a sub-annular end will be able to extend the construction of a metric in step 2 below to the set $\{z : |z| = 1/r\}$, and, moreover, we may assume that for all $j = 1, ..., q$:

Step 2: On the annular end $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$ minus a discrete subset $S \subset A$ we construct a flat metric $d\tau^2$ on $A \backslash S$ which is complete on the set $\{z : |z| = r\} \cup S$, i.e., the set $\{z : |z| = r\} \cup S$ is at infinite distance from any point of $A \setminus S$. We may assume that

$$\sum_{j=1}^{q} \delta^H_{G,A}(H_j) > (k+1)(N - \frac{k}{2}) + (N+1)\,, \tag{4}$$

otherwise our Theorem 2.8 is already proved. By definition, there exist constants $\eta_j \geq 0 (1 \leq j \leq q)$ such that $\gamma := q - \sum_{j=1}^{q} \eta_j > (k+1)(N - \frac{k}{2}) + (N+1)$ and $[-\infty, \infty)-$valued continuous subharmonic functions u_j $(\not\equiv -\infty), 1 \leq j \leq q$, on M such that each u_j is harmonic on $M \setminus g^{-1}(H_j)$ and satisfies conditions (C1) and (C2).
Then,

$$\sum_{j=1}^{q} (1 - \eta_j) - 2N + k - 1 > \frac{(2N - k + 1)k}{2} > 0\,, \tag{5}$$

and this implies in particular

$$q > 2N - k + 1 \geq N + 1 \geq k + 1. \tag{6}$$

By Theorem 3.3, we have

$$(q - 2N + k - 1)\theta = \sum_{j=1}^{q} \omega(j) - k - 1, \ \theta \geq \omega(j) > 0 \text{ and } \theta \geq \frac{k+1}{2N - k + 1},$$

so

$$\begin{aligned}2\Big(\sum_{j=1}^{q}\omega(j)(1-\eta_j)-k-1\Big) &= 2(\sum_{j=1}^{q}\omega(j)-k-1)-2\sum_{j=1}^{q}\omega(j)\eta_j\\ &= 2(q-2N+k-1)\theta-2\sum_{j=1}^{q}\omega(j)\eta_j\\ &\geq 2(q-2N+k-1)\theta-2\sum_{j=1}^{q}\theta\eta_j\\ &= 2\theta\Big(\sum_{j=1}^{q}(1-\eta_j)-2N+k-1\Big)\\ &\geq 2\frac{(k+1)\Big(\sum_{j=1}^{q}(1-\eta_j)-2N+k-1\Big)}{2N-k+1}.\end{aligned}$$

Thus, we now can conclude with (5) that

$$\begin{aligned}&2\Big(\sum_{j=1}^{q}\omega(j)(1-\eta_j)-k-1\Big)>k(k+1)\\ &\Rightarrow \sum_{j=1}^{q}\omega(j)(1-\eta_j)-k-1-\frac{k(k+1)}{2}>0.\end{aligned}\tag{7}$$

By (7), we can choose a number $\epsilon(>0)\in\mathbb{Q}$ such that

$$\frac{\sum_{j=1}^{q}\omega(j)(1-\eta_j)-(k+1)-\frac{k(k+1)}{2}}{\tau_{k+1}}>\epsilon>\frac{\sum_{j=1}^{q}\omega(j)(1-\eta_j)-(k+1)-\frac{k(k+1)}{2}}{\frac{1}{q}+\tau_{k+1}}.$$

So

$$h:=\sum_{j=1}^{q}\omega(j)(1-\eta_j)-(k+1)-\epsilon\sigma_{k+1}>\frac{k(k+1)}{2}+\epsilon\tau_k\tag{8}$$

and

$$\frac{\epsilon}{q}>\sum_{j=1}^{q}\omega(j)(1-\eta_j)-(k+1)-\frac{k(k+1)}{2}-\epsilon\tau_{k+1}.\tag{9}$$

We now consider the number

$$\rho:=\frac{1}{h}\Big(\frac{k(k+1)}{2}+\epsilon\tau_k\Big)=\frac{1}{h}\Big(\sigma_k+\epsilon\tau_k\Big).$$

Then, by (8), we have

$$0<\rho<1.\tag{10}$$

Set

$$\rho^* := \frac{1}{(1-\rho)h} = \frac{1}{(\sum_{j=1}^{q}\omega(j)(1-\eta_j)) - (k+1) - \frac{k(k+1)}{2} - \epsilon\tau_{k+1}}. \tag{11}$$

Using (9) we get

$$\frac{\epsilon\rho^*}{q} > 1. \tag{12}$$

Fix a coordinate on A. Consider the open subset

$$A_1 = Int(A) - \cup_{j=\overline{1,q},p=\overline{0,k}}\{z|\psi(G)_{jp}(z)=0\}$$

of A. We define a new pseudo metric

$$d\tau^2 = \left(\frac{\Pi_{j=1}^{q}|G_z(H_j)|^{\omega(j)}}{|((G_z)_k)_z|^{1+\epsilon}e^{\sum_{j=1}^{q}\omega(j)u_j}\Pi_{j=1}^{q}\Pi_{p=0}^{k-1}|(\psi(G_z)_{jp})_z|^{\epsilon/q}}\right)^{2\rho^*}|dz|^2 \tag{13}$$

on A_1.

Claim 4.1. *$d\tau$ is continuous and nowhere vanishing on A_1.*

Indeed, for $z_0 \in A_1$ with $\Pi_{j=1}^{q}G(H_j)(z_0) \neq 0$, $d\tau$ is continuous and not vanishing at z_0. Now assume that there exists $z_0 \in A_1$ such that $G(H_i)(z_0)=0$ for some i. Consider the function

$$\Gamma(z) = \frac{|((G_z)_k)_z|^{1+\epsilon}e^{\sum_{j=1}^{q}\omega(j)u_j}\Pi_{j=1}^{q}\Pi_{p=0}^{k-1}|(\psi(G_z)_{jp})_z|^{\epsilon/q}}{\Pi_{j=1}^{q}|G_z(H_j)|^{\omega(j)}}.$$

Combining this with Proposition 3.4, we obtain

$$\begin{aligned}\nu_\Gamma(z_0) &\geq \nu_{G_k}(z_0) - \sum_{j=1}^{q}\omega(j)\nu_{G(H_j)}(z_0) + \sum_{j=1}^{q}\omega(j)\min\{\nu_{G(H_j)}(z_0),k\}\\ &\quad + \sum_{j=1}^{q}\omega(j)\nu_{e^{u_j(z)}|z-z_0|^{-\min(\nu_{G(H_j)}(z_0),k)}}(z_0)\\ &\geq 0.\end{aligned}$$

This contradicts to $z_0 \in A_1$. Claim 4.1 is proved.

The key point is now to prove following claim.

Claim 4.2. *$d\tau$ is complete on the set $\{z:|z|=r\}\cup_{j=\overline{1,q},p=\overline{0,k}}\{z:\psi(G)_{jp}(z)=0\}$, i.e., set $\{z:|z|=r\}\cup_{j=\overline{1,q},p=\overline{0,k}}\{z:\psi(G)_{jp}(z)=0\}$ is at infinite distance from any interior point in A_1.*

First, assume that $\Pi_{p=0}^{k}\Pi_{j=1}^{q}|\psi(G)_{jp}|(z_0)=0$. Let γ be a path joining z_0 with an interior point in A_1. By (13) and Proposition 3.4 we have

$$\begin{aligned}\nu_{d\tau}(z_0) = -\Big(&\nu_{G_k}(z_0) - \sum_{j=1}^{q}\omega(j)\nu_{G(H_j)}(z_0) + \sum_{j=1}^{q}\omega(j)\min\{\nu_{G(H_j)}(z_0),k\}\\ &+ \sum_{j=1}^{q}\omega(j)\nu_{e^{u_j(z)}|z-z_0|^{-\min(\nu_{G(H_j)}(z_0),k)}}(z_0) + (\epsilon\nu_{G_k}(z_0) + \frac{\epsilon}{q}\sum_{j=1}^{q}\sum_{p=0}^{k-1}\nu_{\psi(G)_{jp}}(z_0))\Big)\rho^*\\ &\leq -\epsilon\rho^*\nu_{G_k}(z_0) - \frac{\epsilon\rho^*}{q}\sum_{j=1}^{q}\sum_{p=0}^{k-1}\nu_{\psi(G)_{jp}}(z_0) \leq -\frac{\epsilon\rho^*}{q}.\end{aligned}$$

Thus we can find a positive constant C such that

$$|d\tau| \geq \frac{C}{|z-z_0|^{\frac{\epsilon\rho^*}{q}}}|dz|$$

in a neighborhood of z_0 and then, combining with (12), we thus have

$$\int_{\gamma} d\tau = \infty.$$

Therefore, $d\tau$ is complete on $\{z : \Pi_{p=0}^{k}\Pi_{j=1}^{q}\psi(G)_{jp}(z)=0\}$.

Now assume that $d\tau$ is not complete on $\{z : |z|=r\}$. Then there exists $\gamma : [0,1) \to A_1$, where $\gamma(1) \in \{z : |z| = r\}$, so that $|\gamma| < \infty$. Furthermore, we may also assume that $dist(\gamma(0); \{z : |z| = 1/r\}) > 2|\gamma|$. Consider a small disk Δ with center at $\gamma(0)$. Since $d\tau$ is flat, Δ is isometric to an ordinary disk in the plane (cf. e.g. Lemma 3.7). Let $\Phi : \{w : |w| < \eta\} \to \Delta$ be this isometry. Extend Φ, as a local isometry into A_1, to the largest disk $\{w : |w| < R\} = \Delta_R$ possible. Then $R \leq |\gamma|$. The reason that Φ cannot be extended to a larger disk is that the image goes to the outside boundary $\{z : |z| = r\}$ of A_1 (it cannot go to points z of A with $\Pi_{j=\overline{1,q},p=\overline{0,k}}\psi(G)_{jp}(z)=0$ since we have shown already the completeness of A_1 with respect to these points). More precisely, there exists a point w_0 with $|w_0| = R$ so that $\Phi(\overline{0,w_0}) = \Gamma_0$ is a divergent curve on A.

We now want to use Lemma 3.5 to finish up Claim 4.2 by showing that Γ_0 has finite length in the original ds^2 on M, contradicting the completeness of the M. For the rest of the proof of Claim 4.2 we consider $G_z = ((g_0)_z, ..., (g_k)_z)$ as a *fixed globally defined reduced representation of* g by means of the global coordinate z of $A \supset A_1$. If again $\Phi : \{w : |w| < R\} \to A_1$ is our maximal local isometry, it is in particular holomorphic and locally biholomorphic. So $f := g \circ \Phi : \{w : |w| < R\} \to \mathbb{P}^k(\mathbb{C})$ is a linearly non-degenerate holomorphic map with fixed global reduced representation

$$F := G_z \circ \Phi = ((g_0)_z \circ \Phi, \cdots, (g_k)_z \circ \Phi) = (f_0, \cdots, f_k).$$

Since Φ is locally biholomorphic, the metric on Δ_R induced from ds^2 (cf. (1)) through Φ is given by

$$\Phi^* ds^2 = 2|G_z \circ \Phi|^2 |\Phi^* dz|^2 = 2|F|^2 |\frac{dz}{dw}|^2 |dw|^2 . \tag{14}$$

On the other hand, Φ is locally isometric, so we have

$$|dw| = |\Phi^* d\tau| = \left(\frac{\Pi_{j=1}^{q} |G_z(H_j) \circ \Phi|^{\omega(j)}}{|((G_z)_k)_z \circ \Phi|^{1+\epsilon} e^{\sum_{j=1}^{q} \omega(j) u_j \circ \Phi} \Pi_{p=0}^{k-1} \Pi_{j=1}^{q} |(\psi(G_z)_{jp})_z \circ \Phi|^{\epsilon/q}} \right)^{\rho^*} |\frac{dz}{dw}| |dw| .$$

By (2) and (3) we have

$$((G_z)_k)_z \circ \Phi = ((G_z \circ \Phi)_k)_w (\frac{dw}{dz})^{\sigma_k} = (F_k)_w (\frac{dw}{dz})^{\sigma_k} ,$$

$$(\psi(G_z)_{jp})_z \circ \Phi = (\psi(G_z \circ \Phi)_{jp})_w \cdot (\frac{dw}{dz})^{\sigma_p} = (\psi(F)_{jp})_w \cdot (\frac{dw}{dz})^{\sigma_p} , \ (0 \leq p \leq k) \ .$$

Hence, by definition of ρ^* in (11), we have

$$\begin{aligned} |\frac{dw}{dz}| &= \left(\frac{\Pi_{j=1}^{q} |G_z(H_j) \circ \Phi|^{\omega(j)}}{|((G_z)_k)_z \circ \Phi|^{1+\epsilon} e^{\sum_{j=1}^{q} \omega(j) u_j \circ \Phi} \Pi_{p=0}^{k-1} \Pi_{j=1}^{q} |(\psi(G_z)_{jp})_z \circ \Phi|^{\epsilon/q}} \right)^{\rho^*} \\ &= \left(\frac{\Pi_{j=1}^{q} |F(H_j)|^{\omega(j)}}{|(F_k)_w|^{1+\epsilon} e^{\sum_{j=1}^{q} \omega(j) u_j \circ \Phi} \Pi_{p=0}^{k-1} \Pi_{j=1}^{q} |(\psi(F)_{jp})_w|^{\epsilon/q}} \right)^{\rho^*} \frac{1}{|\frac{dw}{dz}|^{h\rho\rho^*}}. \end{aligned}$$

So we get

$$\begin{aligned} |\frac{dz}{dw}| &= \left(\frac{|(F_k)_w|^{1+\epsilon} e^{\sum_{j=1}^{q} \omega(j) u_j \circ \Phi} \Pi_{p=0}^{k-1} \Pi_{j=1}^{q} |(\psi(F)_{jp})_w|^{\epsilon/q}}{\Pi_{j=1}^{q} |F(H_j)|^{\omega(j)}} \right)^{\frac{\rho^*}{1+h\rho\rho^*}} \\ &= \left(\frac{|(F_k)_w|^{1+\epsilon} e^{\sum_{j=1}^{q} \omega(j) u_j \circ \Phi} \Pi_{p=0}^{k-1} \Pi_{j=1}^{q} |(\psi(F)_{jp})_w|^{\epsilon/q}}{\Pi_{j=1}^{q} |F(H_j)|^{\omega(j)}} \right)^{\frac{1}{h}} . \end{aligned}$$

Moreover, $|(\psi(F)_{jp})_w| \leq |(F_p)_w(H_j)|$ by the definitions, so we obtain

$$|\frac{dz}{dw}| \leq \left(\frac{|(F_k)_w|^{1+\epsilon} e^{\sum_{j=1}^{q} \omega(j) u_j \circ \Phi} \Pi_{p=0}^{k-1} \Pi_{j=1}^{q} |(F_p)_w(H_j)|^{\epsilon/q}}{\Pi_{j=1}^{q} |F(H_j)|^{\omega(j)}} \right)^{\frac{1}{h}} . \tag{15}$$

By (14) and (15), we have

$$\Phi^* ds \leqslant \sqrt{2} |F| \left(\frac{|(F_k)_w|^{1+\epsilon} e^{\sum_{j=1}^{q} \omega(j) u_j \circ \Phi} \Pi_{p=0}^{k-1} \Pi_{j=1}^{q} |(F_p)_w(H_j)|^{\epsilon/q}}{\Pi_{j=1}^{q} |F(H_j)|^{\omega(j)}} \right)^{\frac{1}{h}} |dw|.$$

By (8), all the conditions of Lemma 3.5 are satisfied. So we obtain the following from Lemma 3.5 :

$$\Phi^* ds \leqslant C (\frac{2R}{R^2 - |w|^2})^{\rho} |dw|$$

for some constant C. It follows from (10) that $0 < \rho < 1$. Then

$$d_{\Gamma_0} \leqslant \int_{\Gamma_0} ds = \int_{\overline{0, w_0}} \Phi^* ds \leqslant C \cdot \int_0^R (\frac{2R}{R^2 - |w|^2})^{\rho} |dw| < +\infty,$$

where d_{Γ_0} denotes the length of the divergent curve Γ_0 in M, contradicting the assumption of completeness of M. Thus, we complete Claim 4.2.

To summarize, in step 2 we have constructed, for $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$, a continuous and nowhere vanishing metric $d\tau^2$ on A_1 which is flat and complete with respect to the points of $S = \cup_{j=\overline{1,q},p=\overline{0,k}}\{z|\psi(G)_{jp}(z) = 0\}$ and with respect to the (outside) boundary $\{z : |z| = r\}$.

Step 3: We now define

$$\begin{aligned} d\tilde{\tau}^2 &= \left(\frac{\Pi_{j=1}^{q}|G_z(H_j)(z)G_z(H_j)(1/z)|^{\omega(j)(1-\eta_j)}}{|((G_z)_k)_z(z)((G_z)_k)_z(1/z)|^{1+\epsilon}\Pi_{p=0}^{k-1}\Pi_{j=1}^{q}|(\psi(G_z)_{jp})_z(z)(\psi(G_z)_{jp})_z(1/z)|^{\epsilon/q}}\right)^{2\rho^*}|dz|^2 \\ &= \lambda^2(z)|dz|^2, \end{aligned}$$

on $\tilde{A}_1 := \{z : 1/r < |z| < r\} \setminus \{z : \Pi_{p=0}^{k}\Pi_{j=1}^{q}(\psi(G_z)_{jp})_z(z)(\psi(G_z)_{jp})_z(1/z) = 0\}$. Then $d\tilde{\tau}^2$ is complete on $\tilde{A}_1 :$ In fact by what we showed above we have: Towards any point of the boundary $\partial\tilde{A}_1 := \{z : 1/r = |z|\}\cup\{z : |z| = r\}\cup\{z : \Pi_{p=0}^{k}\Pi_{j=1}^{q}(\psi(G_z)_{jp})_z(z)(\psi(G_z)_{jp})_z(1/z) = 0\}$ of $\tilde{A}_1$, one of the factors of $\lambda^2(z)$ is bounded from below away from zero, and the other factor is the one of a complete metric with respect of this part of the boundary. Moreover by the corresponding properties of the two factors of $\lambda^2(z)$ it is trivial that $d\tilde{\tau}^2$ is a continuous nowhere vanishing and flat metric on $\tilde{A}_1$.

Step 4 : We produce a contradiction by using Lemma 3.7 to the open Riemann surface $(\tilde{A}_1, d\tilde{\tau}^2)$:
In fact, we apply Lemma 3.7 to any point $p \in \tilde{A}_1$. Since $d\tilde{\tau}^2$ is complete, there cannot exist a divergent curve from p to the boundary $\partial\tilde{A}_1$ with finite length with respect to $d\tilde{\tau}^2$. Since $\Phi : \Delta_{R_0} \to \tilde{A}_1$ is a local isometry, we necessarily have $R_0 = \infty$. So $\Phi : \mathbb{C} \to \tilde{A}_1 \subset \{z : |z| < r\}$ is a non-constant holomorphic map, which contradicts to Liouville's theorem. So our assumption (4) was wrong. This proves Theorem 2.8. $\square$

5. The proof of Theorem 2.9

Proof. For convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be a non-flat complete minimal surface and $g : M \to \mathbb{P}^1(\mathbb{C})$ its Gauss map. Let z be a local holomorphic coordinate. Set $\phi_i := \partial x_i/\partial z$ $(i = 1, 2, 3)$ and $\phi := \phi_1 - \sqrt{-1}\phi_2$. Then, the (classical) Gauss map $g : M \to \mathbb{P}^1(\mathbb{C})$ is given by

$$g = \frac{\phi_3}{\phi_1 - \sqrt{-1}\phi_2},$$

and the metric on M induced from $\mathbb{R}^3$ is given by

$$ds^2 = |\phi|^2(1 + |g|^2)^2|dz|^2 \text{ (see Fujimoto ([7]))}.$$

We remark that although the ϕ_i, $(i = 1, 2, 3)$ and ϕ depend on z, g and ds^2 do not. Next we take a reduced representation $g = (g_0 : g_1)$ on M and set $||g|| = (|g_0|^2 + |g_1|^2)^{1/2}$. Then we can rewrite

$$ds^2 = |h|^2 ||g||^4 |dz|^2 \,,$$

where $h := \phi/g_0^2$. In particular, h is a holomorphic map without zeros. We remark that h depends on z, however, the reduced representation $g = (g_0 : g_1)$ is globally defined on M and independent of z. Finally we observe that by the assumption that M is not flat, g is not constant.

Now the proof of Theorem 2.9 also will be given in four steps :

Step 1: We may fix notations on the annular end $A \subset M$,that is, $A = \{z : 0 < 1/r < |z| < r < \infty\}$, where z is a (global) conformal coordinate of A. By the same arguments as in Step 1 of the proof of Theorem 2.8, we may prove our theorem only on a sub-annular end, i.e., a subset $A_t := \{z : 0 < t \leq |z| < r < \infty\} \subset A$ with some t such that $1/r < t < r$.

Step 2: Let a^j $(1 \leq j \leq q)$ be $q > 4$ distinct points in $\mathbb{P}^1(\mathbb{C})$. We may assume $a^j = (a_0^j : a_1^j)$ with $|a_0^j|^2 + |a_1^j|^2 = 1$ $(1 \leq j \leq q)$, and we set $G_j := a_0^j g_1 - a_1^j g_0$ $(1 \leq j \leq q)$ for the reduced representation $g = (g_0 : g_1)$ of the Gauss map. By the identity theorem, the G_j have at most countable many zeros.
On the annular end $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$ minus a discrete subset $S \subset A$ we construct a flat metric $d\tau^2$ on $A \setminus S$ which is complete on the set $\{z : |z| = r\} \cup S$, i.e., the set $\{z : |z| = r\} \cup S$ is at infinite distance from any point of $A \setminus S$.

We may assume that

$$\sum_{j=1}^{q} \delta_{g,A}^H(a^j) > 4,$$

since otherwise Theorem 2.9 is already proved.

By definition, there exist constants $\eta_j \geq 0 (1 \leq j \leq q)$ such that $\gamma := q - 2 - \sum_{j=1}^q \eta_j > 2$ and continous functions $u_j (1 \leq j \leq q)$ on M such that each u_j is harmonic on $M \setminus f^{-1}(a^j)$ and satisfies conditions (C1) and (C2). Take δ with

$$\frac{\gamma - 2}{q} > \delta > \frac{\gamma - 2}{q + 2},$$

and set $p = 2/(\gamma - q\delta)$. Then

$$0 < p < 1, \ \frac{p}{1-p} > \frac{\delta p}{1-p} > 1 \ .$$

Fix a coordinate on A. Consider the subset

$$A_1 = A \setminus \{z : W_z(g_0, g_1)(z) = 0\}$$

of A. We define a new metric

$$d\tau^2 = |h|^{\frac{2}{1-p}} \left(\frac{\Pi_{j=1}^q |G_j|^{1-\delta}}{e^{\sum_{j=1}^q u_j} |W(g_0, g_1)|} \right)^{\frac{2p}{1-p}} |dz|^2$$

on A_1 (where again $G_j := a_0^j g_1 - a_1^j g_0$ and h is defined with respect to the coordinate z on $A_1 \subset A$ and $W(g_0, g_1) = W_z(g_0, g_1)$).

Repeating the same arguments as Step 2 in the proof of Theorem 2.8, we give that $d\tau^2$ is a continuous and nowhere vanishing metric on A_1 , flat and complete with respect to the points of $S = \{z : W_z(g_0, g_1)(z) = 0\}$ and with respect to the (outside) boundary $\{z : |z| = r\}$.

Step 3 and 4: We now use the same arguments as Step 3 and 4 in the proof of Theorem 2.8 to finish the proof of Theorem 2.9. □

6. The proof of Theorem 2.10

Proof. For convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^4$. Let $x = (x_1, x_2, x_3, x_4) : M \to \mathbb{R}^4$ be a non-flat complete minimal surface in $\mathbb{R}^4$. As is well-known, the set of all oriented 2-planes in $\mathbb{R}^4$ is canonically identified with the quadric

$$Q_2(\mathbb{C}) := \{(w_1 : ... : w_4) | w_1^2 + ... + w_4^2 = 0\}$$

in $\mathbb{P}^3(\mathbb{C})$. By definition, the Gauss map $g : M \to Q_2(\mathbb{C})$ is the map which maps each point p of M to the point of $Q_2(\mathbb{C})$ corresponding to the oriented tangent plane of M at p. The quadric $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. By suitable identifications we may regard g as a pair of meromorphic functions $g = (g^1, g^2)$ on M. Let z be a local holomorphic coordinate. Set $\phi_i := \partial x_i/dz$ for $i = 1, ..., 4$. Then, g^1 and g^2 are given by

$$g^1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2},\ g^2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}$$

and the metric on M induced from $\mathbb{R}^4$ is given by

$$ds^2 = |\phi|^2(1 + |g^1|^2)(1 + |g^2|^2)|dz|^2,$$

where $\phi := \phi_1 - \sqrt{-1}\phi_2$. We remark that although the ϕ_i, $(i = 1, 2, 3, 4)$ and ϕ depend on z, $g = (g^1, g^2)$ and ds^2 do not. Next we take reduced representations $g^l = (g_0^l : g_1^l)$ on M and set $||g^l|| = (|g_0^l|^2 + |g_1^l|^2)^{1/2}$ for $l = 1, 2$. Then we can rewrite

$$ds^2 = |h|^2||g^1||^2||g^2||^2|dz|^2\,,$$

where $h := \phi/(g_0^1 g_0^2)$. In particular, h is a holomorphic map without zeros. We remark that h depends on z, however, the reduced representations $g^l = (g_0^l : g_1^l)$ are globally defined on M and independent of z. Finally we observe that by the assumption that M is not flat, g is not constant.

Now the proof of Theorem 2.10 will be given in four steps :

Step 1: This step is completely analogue to step 1 in the proof of Theorem 2.8. We get : By passing to a sub-annular end we may assume that the annular end is $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$, where z is a (global) conformal coordinate of A, that the restriction of ds^2 to A is complete on the set $\{z : |z| = r\}$, i.e., the set $\{z : |z| = r\}$ is at infinite distance from any point of A, and, moreover, that for all $j = 1, ..., q_l$, $l = 1, 2$ (case (i)) respectively for all $j = 1, ..., q_1$, $l = 1$ (case (ii)), we have .

Step 2: Our strategy is the same as for step 2 in the proof of Theorem 2.8. We may assume that $\gamma_1 = \sum_{j=1}^{q_1} \delta_{g^1}^H(a^{1j}) > 2$, $\gamma_2 = \sum_{j=1}^{q_2} \delta_{g^2}^H(a^{2j}) > 2$, and

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} < 1\,, \tag{16}$$

since otherwise case (i) of Theorem 2.10 is already proved.

Choose $\delta_0 (> 0)$ such that $\gamma_l - 2 - q_l\delta_0 > 0$ for all $l = 1, 2$, and

$$\frac{1}{\gamma_1 - 2 - q_1\delta_0} + \frac{1}{\gamma_2 - 2 - q_2\delta_0} = 1.$$

If we choose a positive constant $\delta (< \delta_0)$ sufficiently near to δ_0 and set

$$p_l := 1/(\gamma_l - 2 - q_l\delta), (l = 1, 2),$$

we have

$$0 < p_1 + p_2 < 1, \ \frac{\delta p_l}{1 - p_1 - p_2} > 1 \ (l = 1, 2)\ . \tag{17}$$

Fix a coordinate on A. Consider the subset

$$A_2 = A \setminus \{z : W_z(g_0^1, g_1^1)(z) \cdot W_z(g_0^2, g_1^2)(z) = 0\}$$

of A. We define a new metric

$$d\tau^2 = \left(|h| \frac{\Pi_{j=1}^{q_1} |G_j^1|^{(1-\delta)p_1} \Pi_{j=1}^{q_2} |G_j^2|^{(1-\delta)p_2}}{e^{p_1 \sum_{j=1}^{q_1} u_j^1} |W(g_0^1, g_1^1)|^{p_1} e^{p_2 \sum_{j=1}^{q_2} u_j^2} |W(g_0^2, g_1^2)|^{p_2}} \right)^{\frac{2}{1-p_1-p_2}} |dz|^2$$

on A_2 (where again $G_j^l := a_0^{lj} g_1^l - a_1^{lj} g_0^l$ $(l = 1, 2)$ and h is defined with respect to the coordinate z on $A_2 \subset A$ and $W(g_0^l, g_1^l) = W_z(g_0^l, g_1^l)$).

It is easy to see that by the same arguments as in step 2 of the proof of Theorem 2.9 (applied for each $l = 1, 2$), we get that $d\tau$ is a continuous nowhere vanishing and flat metric on A_2, which is moreover independant of the choice of the coordinate z.

The key point in this step is to prove the following claim :

Claim 6.1. *$d\tau^2$ is complete on the set $\{z : |z| = r\} \cup \{z : \Pi_{l=1,2} W(g_0^l, g_1^l)(z) = 0\}$, i.e., the set $\{z : |z| = r\} \cup \{z : \Pi_{l=1,2} W(g_0^l, g_1^l)(z) = 0\}$ is at infinite distance from any interior point in A_2.*

It is easy to see that by the same method as in the proof of Claim 4.2 in the proof of Theorem 2.8, we may show that $d\tau$ is complete on $\{z : \Pi_{l=1,2} W(g_0^l, g_1^l)(z) = 0\}$.

Now assume $d\tau$ is not complete on $\{z : |z| = r\}$. Then there exists $\rho : [0, 1) \to A_2$, where $\rho(1) \in \{z : |z| = r\}$, so that $|\rho| < \infty$. Furthermore, we may also assume that $dist(\rho(0), \{z : |z| = 1/r\}) > 2|\rho|$. Consider a small disk Δ with center at $\rho(0)$. Since $d\tau$ is flat, Δ is isometric to an ordinary disk in the plane. Let $\Phi : \{|w| < \eta\} \to \Delta$ be this isometry. Extend Φ, as a local isometry into A_2, to the largest disk $\{|w| < R\} = \Delta_R$

possible. Then $R \leq |\rho|$. The reason that Φ cannot be extended to a larger disk is that the image goes to the outside boundary $\{z : |z| = r\}$ of A_2. More precisely, there exists a point w_0 with $|w_0| = R$ so that $\Phi(\overline{0, w_0}) = \Gamma_0$ is a divergent curve on A.
The map $\Phi(w)$ is locally biholomorphic, and the metric on Δ_R induced from ds^2 through Φ is given by

$$\Phi^* ds^2 = |h \circ \Phi|^2 ||g^1 \circ \Phi||^2 ||g^2 \circ \Phi||^2 |\frac{dz}{dw}|^2 |dw|^2 \ . \tag{18}$$

On the other hand, Φ is isometric, so we have

$$|dw| = |d\tau| = \Big(|h| \frac{\Pi_{j=1}^{q_1} |G_j^1|^{(1-\delta)p_1} \Pi_{j=1}^{q_2} |G_j^2|^{(1-\delta)p_2}}{e^{p_1 \sum_{j=1}^{q_1} u_j^1} |W(g_0^1, g_1^1)|^{p_1} e^{p_2 \sum_{j=1}^{q_2} u_j^2} |W(g_0^2, g_1^2)|^{p_2}} \Big)^{\frac{1}{1-p_1-p_2}} |dz|$$

$$\Rightarrow |\frac{dw}{dz}|^{1-p_1-p_2} = |h| \frac{\Pi_{j=1}^{q_1} |G_j^1|^{(1-\delta)p_1} \Pi_{j=1}^{q_2} |G_j^2|^{(1-\delta)p_2}}{e^{p_1 \sum_{j=1}^{q_1} u_j^1} |W(g_0^1, g_1^1)|^{p_1} e^{p_2 \sum_{j=1}^{q_2} u_j^2} |W(g_0^2, g_1^2)|^{p_2}}.$$

For each $l = 1, 2$, we set $f^l := g^l(\Phi), f_0^l := g_0^l(\Phi), f_1^l := g_1^l(\Phi), u_j^l := u_j^l(\Phi)$ and $F_j^l := G_j^l(\Phi)$. Since

$$W_w(f_0^l, f_1^l) = (W_z(g_0^l, g_1^l) \circ \Phi) \frac{dz}{dw}, (l = 1, 2),$$

we obtain

$$|\frac{dz}{dw}| = \frac{\Pi_{l=1,2} (e^{p_l \sum_{j=1}^{q_l} u_j^l} |W(f_0^l, f_1^l)|^{p_l})}{|h(\Phi)| \Pi_{l=1,2} \Pi_{j=1}^{q_l} |F_j^l|^{(1-\delta)p_l}} \ . \tag{19}$$

By (18) and (19), we get

$$\begin{aligned} \Phi^* ds^2 &= \Big(\Pi_{l=1,2} \frac{||f^l|| (e^{p_l \sum_{j=1}^{q_l} u_j^l} |W(f_0^l, f_1^l)|)^{p_l}}{\Pi_{j=1}^{q_l} |F_j^l|^{(1-\delta)p_l}} \Big)^2 |dw|^2 \\ &= \Pi_{l=1,2} \Big(\frac{||f^l||^{\gamma_l - q_l \delta} |W(f_0^l, f_1^l)|}{\Pi_{j=1}^{q} |F_j^l|^{1-\delta}} \Big)^{2p_l} |dw|^2. \end{aligned}$$

Using the Lemma 3.6, we obtain

$$\Phi^* ds^2 \leqslant C_0^{2(p_1+p_2)} . (\frac{2R}{R^2 - |w|^2})^{2(p_1+p_2)} |dw|^2.$$

Since $0 < p_1 + p_2 < 1$ by (17), it then follows that

$$d_{\Gamma_0} \leqslant \int_{\Gamma_0} ds = \int_{\overline{0, w_0}} \Phi^* ds \leqslant C_0^{p_1+p_2} . \int_0^R (\frac{2R}{R^2 - |w|^2})^{p_1+p_2} |dw| < +\infty,$$

where d_{Γ_0} denotes the length of the divergent curve Γ_0 in M, contradicting the assumption of completeness of M. Claim 6.1 is proved.

Steps 2 and 3 for the case (i): These steps are analogue to the corresponding steps in the proof of Theorem 2.8. Define $d\tilde{\tau}^2 = \lambda^2(z)|dz|^2$ on

$$\tilde{A}_2 := \{z : 1/r < |z| < r\} \setminus$$

$$\setminus \{z : W_z(g_0^1, g_1^1)(z) \cdot W_z(g_0^2, g_1^2)(z) \cdot W_z(g_0^1, g_1^1)(1/z) \cdot W_z(g_0^2, g_1^2)(1/z) = 0\} \, ,$$

where

$$\lambda(z) = \left(|h(z)|\frac{\Pi_{j=1}^{q_1}|G_j^1(z)|^{(1-\delta)p_1}\Pi_{j=1}^{q_2}|G_j^2(z)|^{(1-\delta)p_2}}{e^{p_1\sum_{j=1}^{q_1}u_j^1(z)}|W_z(g_0^1,g_1^1)(z)|^{p_1}e^{p_2\sum_{j=1}^{q_2}u_j^2(z)}|W_z(g_0^2,g_1^2)(z)|^{p_2}}\right)^{\frac{1}{1-p_1-p_2}}$$
$$\times\left(|h(1/z)|\frac{\Pi_{j=1}^{q_1}|G_j^1(1/z)|^{(1-\delta)p_1}\Pi_{j=1}^{q_2}|G_j^2(1/z)|^{(1-\delta)p_2}}{e^{p_1\sum_{j=1}^{q_1}u_j^1(1/z)}|W_z(g_0^1,g_1^1)(1/z)|^{p_1}e^{p_2\sum_{j=1}^{q_2}u_j^2(1/z)}|W_z(g_0^2,g_1^2)(1/z)|^{p_2}}\right)^{\frac{1}{1-p_1-p_2}}.$$

By using Claim 6.1, the continuous nowhere vanishing and flat metric $d\tau$ on A_2 is also complete. Using the identical argument of step 3 in the proof of Theorem 2.8 to the open Riemann surface $(\tilde{A}_2, d\tilde{\tau})$ produces a contradiction, so assumption (16) was wrong. This implies case (i) of the Theorem 2.10.

We finally consider the case (ii) of Theorem 2.10 (where $g^2 \equiv constant$ and $g^1 \not\equiv constant$). Suppose that $\gamma_1 > 3$. We can choose δ with

$$\frac{\gamma_1 - 3}{q_1} > \delta > \frac{\gamma_1 - 3}{q_1 + 1},$$

and set $p = 1/(\gamma_1 - 2 - q_1\delta)$. Then

$$0 < p < 1, \ \frac{p}{1-p} > \frac{\delta p}{1-p} > 1.$$

Set

$$d\tau^2 = |h|^{\frac{2}{1-p}}\left(\frac{\Pi_{j=1}^{q_1}|G_j^1|^{1-\delta}}{|W(g_0^1, g_1^1)|}\right)^{\frac{2p}{1-p}}|dz|^2.$$

Using this metric, by the analogue arguments as in step 1 to step 3 of the proof of Theorem 2.8, we get the case (ii) of Theorem 2.10. □

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