UNIFORM ANNIHILATORS OF SYSTEMS OF PARAMETERS

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Abstract. In this paper we give uniform annihilators for some relations of all systems of parameters in a local ring. Our results not only shed light to some classical results but also have potential applications.

1. Introduction

In this note, let $(R, \mathfrak{m})$ be a local ring of dimension d, and $\underline{x} = x_1, \ldots, x_d$ a system of parameters of R. For all $s \leq d$, set $Q_s = (x_1, \ldots, x_s)$, and $Q = Q_d$. It is well known that if R is Cohen-Macaulay then $x_1, \ldots, x_d$ is a regular sequence. In this case $x_1, \ldots, x_d$ act as independent variables. For all $0 \leq s \leq d-1$, any linear relation

$$a_1x_1 + \cdots + a_sx_s + a_{s+1}x_{s+1} = 0$$

is trivial, i.e. $a_{s+1} \in (x_1, \ldots, x_s) : x_{s+1} = (x_1, \ldots, x_s)$. In general, $(x_1, \ldots, x_s) : x_{s+1}$ often strictly contains $(x_1, \ldots, x_s)$ and the situation becomes complicated. Schenzel [15] defined the ideal

$$\mathfrak{b}(R) = \bigcap_{\underline{x}, s<d} \operatorname{Ann}\big(\frac{Q_s : x_{s+1}}{Q_s}\big),$$

where $\underline{x}$ runs over all systems of parameter of R. We can think $\mathfrak{b}(R)$ as the one that controls all linear relations of systems of parameters. By the definition we can not expect to determine $\mathfrak{b}(R)$ explicitly, but *is* $\mathfrak{b}(R)$ *a non-zero ideal*? For $i \leq d$ let $\mathfrak{a}_i(R) = \operatorname{Ann}(H^i_{\mathfrak{m}}(R))$, and $\mathfrak{a}(R) = \prod_{i=0}^{d-1} \mathfrak{a}_i(R)$. Schenzel [15, Satz 2.4.5] proved the following inclusions

$$\mathfrak{a}(R) \subseteq \mathfrak{b}(R) \subseteq \mathfrak{a}_0(R) \cap \cdots \cap \mathfrak{a}_{d-1}(R).$$

Therefore $\mathfrak{a}(R)$ and $\mathfrak{b}(R)$ have the same radical ideal. Importantly, if R is an image of a Cohen-Macaulay local ring then $\dim R/\mathfrak{a}(R) < d$, so $\mathfrak{b}(R) \neq 0$ in this case. Moreover we can choose parameter elements $x \in \mathfrak{b}(R)$. These elements admit several nice properties, see [5] for more details.

An other type of relation of system of parameters comes from the Hochster monomial conjecture. For all $s \leq d$ we define the *limit closure* $Q_s^{\lim}$ of Q_s as the formula

$$Q_s^{\lim} = \bigcup_{n \geq 1} \big((x_1^{n+1}, \ldots, x_s^{n+1}) : (x_1 \ldots, x_s)^n\big).$$

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Thank to Hochster in equicharacteristic [7] and André in mixed characteristic [1] we have $Q^{\lim} \subseteq \mathfrak{m}$.[1] One can check that (see [14, Lemma 5.6])

$$\mathfrak{b}(R)^s Q_s^{\lim} \subseteq Q_s$$

for all systems of parameters $\underline{x}$, and for all $s \leq d$. In particular we have

$$\mathfrak{a}(R)^d \subseteq \bigcap_{\underline{x}, s \leq d} \operatorname{Ann}\Big(\frac{Q_s^{\lim}}{Q_s}\Big).$$

In this paper we are interested in relations concerning powers of parameter ideals. Inspired by $\mathfrak{b}(R)$ it is natural to define the ideal

$$\mathfrak{c}(R) = \bigcap_{\underline{x}, s<d, n \geq 1} \operatorname{Ann}\Big(\frac{Q_s^n : x_{s+1}}{Q_s^n}\Big),$$

where $\underline{x}$ runs over all systems of parameter of R. It is clear that $\mathfrak{c}(R) \subseteq \mathfrak{b}(R)$.

Question 1. *Can we choose N, depending only on d, such that $\mathfrak{a}(R)^N \subseteq \mathfrak{c}(R)$?*

We are also interested in the *parametric decomposition* of powers of parameter ideals. The parametric decomposition was firstly studied by Heinzer, Ratliff and Shah [11], and followed by many researchers [9, 10, 6]. For each $1 \leq s \leq d$ and $n \geq 1$ we set

$$\Lambda_{s,n} = \{(\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s \mid \alpha_i \geq 1 \text{ for all } 1 \leq i \leq s, \sum_{i=1}^{s} \alpha_i = s + n - 1\}.$$

For all $s \leq d$ and all $\alpha \in \Lambda_{s,n}$, set $Q_s(\alpha) = (x_1^{\alpha_1}, \ldots, x_s^{\alpha_s})$. It is not hard to see that

$$Q_s^n \subseteq \bigcap_{\alpha \in \Lambda_{s,n}} Q_s(\alpha)$$

for all $n \geq 1$. Moreover the equalities hold for all s and n provided R is Cohen-Macaulay by [11, Theorem 2.4] (see Remark 3.3 for a short proof). In general, we consider the ideal

$$\mathfrak{d}(R) = \bigcap_{\underline{x}, s \leq d, n \geq 1} \operatorname{Ann}\Big(\frac{\bigcap_{\alpha \in \Lambda_{s,n}} Q_s(\alpha)}{Q_s^n}\Big),$$

where $\underline{x}$ runs over all systems of parameters of R.

Question 2. *Can we choose N, depending only on d, such that $\mathfrak{a}(R)^N \subseteq \mathfrak{d}(R)$?*

It is quite surprisingly that the two above questions are closely related. The aim of this paper is to give positive answers for both questions.

Main Theorem. *Let $(R, \mathfrak{m})$ be a local ring of dimension d. Then*

(1) $\mathfrak{a}(R) \subseteq \mathfrak{c}(R)$ if $d = 1$, and $\mathfrak{a}(R)^{2^{d-2}} \subseteq \mathfrak{c}(R)$ if $d \geq 2$.
(2) $\mathfrak{a}(R)^{2^{d-1}} \subseteq \mathfrak{d}(R)$.

[1] Recently, we [13] showed that $Q^{\lim} \subseteq \overline{Q}$, the integral closure of Q, for every parameter ideal Q provided R is quasi-unmixed.

1.1. **Some direct applications.** We present the usefulness of the Main Theorem in proving the colon capturing of integral closure and tight closure. The readers are encouraged to compare our proof with the old ones [12, Theorem 5.4.1] and [2, Theorem 10.1.9].

Corollary 1.1. *Let $(R, \mathfrak{m})$ be an equidimensional local ring of dimension d that is a homological image of a Cohen-Macaulay local ring. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R. Then for all $s < d$ and all $n \geq 1$ we have*

(1) $(x_1, \ldots, x_s)^n : x_{s+1} \subseteq \overline{(x_1, \ldots, x_s)^n} : x_{s+1} = \overline{(x_1, \ldots, x_s)^n}$, *where $\overline{I}$ denotes the integral closure of I.*[2]

(2) Suppose R contains a field of prime characteristic $p > 0$. Then we have

$$(x_1, \ldots, x_s)^n : x_{s+1} \subseteq ((x_1, \ldots, x_s)^n)^* : x_{s+1} = ((x_1, \ldots, x_s)^n)^*,$$

where I^* denotes the tight closure of I.

Proof. (1) The first inclusion is trivial. Let $r \in \overline{(x_1, \ldots, x_s)^n} : x_{s+1}$, we have $rx_{s+1} \in \overline{(x_1, \ldots, x_s)^n}$. By [12, Corollary 6.8.12] this condition is equivalent to the condition that we have an element $c \notin P$ for all $P \in \operatorname{minAss}(R)$ such that $c(rx_{s+1})^m \in Q_s^{nm}$ for large enough m. Therefore $cr^m \in Q_s^{nm} : x_{s+1}^m$. Since R is an image of a Cohen-Macaulay local ring, $\dim R/\mathfrak{a}(R) < d$. By the Main Theorem we can choose $c' \in \mathfrak{c}(R)$ such that $c' \notin P$ for all $P \in \operatorname{minAss}(R)$, reminding that R is equidimensional. Therefore $(cc')r^m \in Q_s^{nm}$ for all m large enough, and so $r \in \overline{(x_1, \ldots, x_s)^n}$ by [12, Corollary 6.8.12].
(2) We leave the proof to the readers. $\square$

We next prove that every power of parameter ideal has a parametric decomposition up to tight closure. The result can be proved by using [8, Section 7].

Corollary 1.2. *Let $(R, \mathfrak{m})$ be an equidimensional local ring of dimension d and of prime characteristic p that is a homological image of a Cohen-Macaulay local ring. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R. Then for all $s \leq d$ and all $n \geq 1$ we have*

$$\bigcap_{\alpha \in \Lambda_{s,n}} Q_s(\alpha)^* = (Q_s^n)^*.$$

Proof. Clearly, RHS $\subseteq$ LHS. Let $r \in \cap_{\alpha \in \Lambda_{s,n}} Q_s(\alpha)^*$. Then we can choose $c \notin P$ for all $P \in \operatorname{minAss}(R)$ such that for all $q = p^e \gg 0$ we have

$$cr^q \in \bigcap_{\alpha \in \Lambda_{s,n}} (Q_s(\alpha))^{[q]} = \bigcap_{\alpha \in \Lambda_{s,n}} (Q_s^{[q]}(\alpha))$$

By the Main Theorem (2) we have

$$\mathfrak{a}(R)^{2^{d-1}} \big(\bigcap_{\alpha \in \Lambda_{s,n}} (Q_s^{[q]}(\alpha)) \big) \subseteq (Q_s^{[q]})^n = (Q_s^n)^{[q]}$$

for all q. Therefore we can choose $c' \notin P$ for all $P \in \operatorname{minAss}(R)$ such that $(cc')r^q \in (Q_s^n)^{[q]}$ for all $q = p^e \gg 0$. Hence $r \in (Q_s^n)^*$. The proof is complete. $\square$

[2] This assertion was stated for quasi-unmixed local ring. However we can reduce to our assumption by using [12, Proposition 1.6.2].

At the time of writing we do not have an answer for the following question.

Question 3. *Let $(R,\mathfrak{m})$ be an equidimensional local ring of dimension d that is a homological image of a Cohen-Macaulay local ring. Let $\underline{x} = x_1,\ldots,x_d$ be a system of parameters of R. Then is it true that*

$$\bigcap_{\alpha\in\Lambda_{s,n}} Q_s(\alpha) \subseteq \overline{Q_s^n}$$

for all $s \le d$ and all $n \ge 1$.

The paper is organized as follows: We recall some notations of this paper in the next section. In Section 3 we prove a technical lemma which plays the key role in study the parametric decomposition. The Main Theorem will be proved in Section 4. We also give some applications of the Main Theorem about the uniform annihilator of local cohomology of R/Q_s^n.

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2. Preliminaries

Let $(R,\mathfrak{m})$ be a local ring of dimension d, and $\underline{x} = x_1,\ldots,x_d$ a system of parameters of R. For $s \le d$ we set $Q_s = (x_1,\ldots,x_s)$ with the convention $Q_0 = (0)$.

Notation 1. Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.

(1) For all $i < d$ we set $\mathfrak{a}_i(R) = \mathrm{Ann}H^i_{\mathfrak{m}}(R)$, and $\mathfrak{a}(R) = \mathfrak{a}_0(R)\ldots\mathfrak{a}_{d-1}(R)$.

(2) For all $s < d$ we set
$$\mathfrak{b}_s(R) = \bigcap_{\underline{x}} \mathrm{Ann}\Big(\frac{Q_s : x_{s+1}}{Q_s}\Big),$$
where $\underline{x} = x_1,\ldots,x_d$ runs over all systems of parameters of R, and
$$\mathfrak{b}(R) = \mathfrak{b}_0(R)\cap\cdots\cap\mathfrak{b}_{d-1}(R).$$

Remark 2.1. (1) Schenzel [15, Satz 2.4.5] proved that
$$\mathfrak{a}(R) \subseteq \mathfrak{b}(R) \subseteq \mathfrak{a}_0(R)\cap\cdots\cap\mathfrak{a}_{d-1}(R).$$
In particular, $\sqrt{\mathfrak{a}(R)} = \sqrt{\mathfrak{b}(R)}$.

(2) If R is a homomorphic image of a Cohen-Macaulay local ring, then $\dim R/\mathfrak{a}_i(R) \le i$ for all $i < d$. Furthermore, $\dim R/\mathfrak{a}_i(R) = i$ if and only if there exists $\mathfrak{p} \in \mathrm{Ass}R$ such that $\dim R/\mathfrak{p} = i$ (see [2, Theorem 8.1.1] and [3, 9.6.6]).

We next mention the main objects of this paper.

Notation 2. *For all $s < d$ we set*
$$\mathfrak{c}_s(R) = \bigcap_{\underline{x},n\ge 1} \mathrm{Ann}\Big(\frac{Q_s^n : x_{s+1}}{Q_s^n}\Big),$$
where $\underline{x} = x_1,\ldots,x_d$ runs over all systems of parameters of R, and
$$\mathfrak{c}(R) = \mathfrak{c}_0(R)\cap\cdots\cap\mathfrak{c}_{d-1}(R).$$

Clearly, $\mathfrak{c}_i(R) = \mathfrak{b}_i(R)$ when $i = 0, 1$, and $\mathfrak{c}_i(R) \subseteq \mathfrak{b}_i(R)$ in general.
For the next object we need the notation

$$\Lambda_{s,n} = \{(\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s \mid \alpha_i \geq 1 \,\text{for all}\, 1 \leq i \leq s, \sum_{i=1}^{s} \alpha_i = s + n - 1\}$$

for all $s, n \geq 1$. For all $s \leq d$ and $\alpha \in \Lambda_{s,n}$, set $Q_s(\alpha) = (x_1^{\alpha_1}, \ldots, x_s^{\alpha_s})$. It is not hard to see that

$$Q_s^n \subseteq \bigcap_{\alpha \in \Lambda_{s,n}} Q_s(\alpha)$$

for all $n \geq 1$. Moreover the equalities hold if R is Cohen-Macaulay by [11, Theorem 2.4] (see also [9, 10, 6]). In this case we say that Q_s^n admit the parametric decomposition for all $n \geq 1$. In general, it is natural to consider the following.

Notation 3. *For every $1 \leq s \leq d$, we set*

$$\mathfrak{d}_s(R) = \bigcap_{\underline{x}, n \geq 1} \operatorname{Ann}\big(\frac{\bigcap_{\alpha \in \Lambda_{s,n}} Q_s(\alpha)}{Q_s^n}\big),$$

where $\underline{x}$ runs over all systems of parameters of R, and

$$\mathfrak{d}(R) = \mathfrak{d}_1(R) \cap \cdots \cap \mathfrak{d}_d(R).$$

It is clear that $\mathfrak{d}_1(R) = R$. The aim of this paper is to bound both $\mathfrak{c}(R)$ and $\mathfrak{d}(R)$ by $\mathfrak{b}(R)$ (and $\mathfrak{a}(R)$). We will need the following relation between these ideals.

Lemma 2.2. *For every $1 \leq s \leq d - 1$ we have $\mathfrak{b}_s(R)\mathfrak{d}_s(R) \subseteq \mathfrak{c}_s(R)$.*

Proof. For all $n \geq 1$ we have

$$Q_s^n : x_{s+1} \subseteq \bigcap_{\alpha \in \Lambda_{s,n}} \big(Q_s(\alpha) : x_{s+1}\big).$$

By the definition of $\mathfrak{b}_s(R)$ we have $\mathfrak{b}_s(R)(Q_s(\alpha) : x_{s+1}) \subseteq Q_s(\alpha)$ for all $\alpha \in \Lambda_{s,n}$. Therefore

$$\begin{aligned}
\mathfrak{b}_s(R)\mathfrak{d}_s(R)(Q_s^n : x_{s+1}) &\subseteq \mathfrak{d}_s(R)\mathfrak{b}_s(R)\big(\bigcap_{\alpha \in \Lambda_{s,n}} (Q_s(\alpha) : x_{s+1})\big) \\
&\subseteq \mathfrak{d}_s(R)\big(\bigcap_{\alpha \in \Lambda_{s,n}} \mathfrak{b}_s(R)(Q_s(\alpha) : x_{s+1})\big) \\
&\subseteq \mathfrak{d}_s(R)\big(\bigcap_{\alpha \in \Lambda_{s,n}} Q_s(\alpha)\big) \\
&\subseteq Q_s^n.
\end{aligned}$$

The proof is complete. $\square$

3. A technical Lemma

Let I be an ideal, and x an element of R. We will use the notation $I : x^\infty = \cup_{n\geq 1}(I : x^n)$, and the conventions $I^0 = R$ and $x^0 = 1$.

Lemma 3.1. *Let I be an ideal, and x an element of R. Then*

(1) For all $n \geq m \geq 0$ we have $(x^n) \cap x^m(I : x^\infty) = x^n(I : x^\infty)$.

(2) For all $n+1 \geq \alpha > m \geq 1$ we have

$$(x^m) \cap \big(\sum_{i=0}^{m-2} x^i(I^{n-i} : x^\infty) + x^{m-1}(I^{n+1-\alpha} : x^\infty)\big) = x^m(I^{n+1-\alpha} : x^\infty).$$

Proof. (1) It is clear that $x^n(I : x^\infty) \subseteq (x^n) \cap x^m(I : x^\infty)$. For the converse, we have

$$(x^n) \cap x^m(I : x^\infty) \subseteq (x^n) \cap (I : x^\infty) = x^n((I : x^\infty) : x^n) = x^n(I : x^\infty).$$

(2) Similarly, it is enough to show LHS $\subseteq$ RHS. Since $\alpha > m$ we have $n+1-\alpha < n-i$ for all $i = 0, \ldots, m-2$. Therefore

$$\big(\sum_{i=0}^{m-2} x^i(I^{n-i} : x^\infty) + x^{m-1}(I^{n+1-\alpha} : x^\infty)\big) \subseteq (I^{n+1-\alpha} : x^\infty).$$

Hence

$$\text{LHS} \subseteq x^m \cap (I^{n+1-\alpha} : x^\infty) = x^m(I^{n+1-\alpha} : x^\infty).$$

The proof is complete. $\square$

The next lemma is the key ingredient of the proof of the main result.

Lemma 3.2. *Let I be an ideal, and x an element of R. Then for all $n \geq 1$ we have*

$$\bigcap_{\alpha=1}^{n}(x^\alpha, I^{n+1-\alpha} : x^\infty) = \sum_{\alpha=0}^{n} x^\alpha(I^{n-\alpha} : x^\infty).$$

Proof. We will prove by induction the following equalities

$$\text{LHS} = \bigcap_{\alpha=m}^{n} \big(x^\alpha, \sum_{i=0}^{m-2} x^i(I^{n-i} : x^\infty), x^{m-1}(I^{n+1-\alpha} : x^\infty)\big)$$

for all $m = 1, \ldots, n$. Notice that if we apply $m = n$ to above equality, we will obtain the desired. There is nothing to do in the case $m = 1$. Suppose we have the equality for some m, $1 \leq m \leq n-1$. Then

$$\begin{aligned}
\text{LHS} &= \bigcap_{\alpha=m+1}^{n} \Big[\big(x^m, \sum_{i=0}^{m-1} x^i(I^{n-i} : x^\infty)\big) \cap \big(x^\alpha, \sum_{i=0}^{m-2} x^i(I^{n-i} : x^\infty), x^{m-1}(I^{n+1-\alpha} : x^\infty)\big)\Big] \\
&= \bigcap_{\alpha=m+1}^{n} \Big[x^\alpha, \sum_{i=0}^{m-1} x^i(I^{n-i} : x^\infty), (x^m) \cap \big(\sum_{i=0}^{m-2} x^i(I^{n-i} : x^\infty), x^{m-1}(I^{n+1-\alpha} : x^\infty)\big)\Big] \\
&= \bigcap_{\alpha=m+1}^{n} \big(x^\alpha, \sum_{i=0}^{m-1} x^i(I^{n-i} : x^\infty), x^m(I^{n+1-\alpha} : x^\infty)\big).
\end{aligned}$$

The last equality follows from Lemma 3.1(2). The proof is complete. $\square$

Remark 3.3. The above Lemma give us a direct proof for the fact $Q_s^n = \cap_{\alpha\in\Lambda_{s,n}} Q_s(\alpha)$ provided $x_1,\ldots,x_s$ is a regular sequence. Indeed, we can assume the ring is local and will proceed by induction on s. The case $s=1$ is trivial. For $s>1$ we have

$$\bigcap_{\alpha\in\Lambda_{s,n}} Q_s(\alpha) = \bigcap_{\alpha_s=1}^{n}\Big(\bigcap_{\alpha'\in\Lambda_{s-1,n+1-\alpha_s}}(Q_{s-1}(\alpha'),x_s^{\alpha_s})\Big) = \bigcap_{\alpha_s=1}^{n}(x_s^{\alpha_s},Q_{s-1}^{n+1-\alpha_s}).$$

The last equality follows from the fact $x_1,\ldots,x_{s-1}$ is a regular sequence on $R/(x_s^{\alpha_s})$ and the induction. Now since x_s is an regular element of R/Q_{s-1}^m for all $m\ge 1$ we have

$$\bigcap_{\alpha_s=1}^{n}(x_s^{\alpha_s},Q_{s-1}^{n+1-\alpha_s}) = \bigcap_{\alpha_s=1}^{n}(x_s^{\alpha_s},Q_{s-1}^{n+1-\alpha_s}:x_s^{\infty}) = \sum_{\alpha=0}^{n} x_s^{\alpha_s}(Q_{s-1}^{n-\alpha_s}:x_s^{\infty}) = \sum_{\alpha=0}^{n} x_s^{\alpha_s}Q_{s-1}^{n-\alpha_s} = Q_s^n.$$

4. Proof of the main result

We need the following [5, Lemma 3.7].

Lemma 4.1. *For every part of system of parameters $x_1,\ldots,x_i, i<d$, we have $\mathfrak{b}(R)\subseteq \mathfrak{b}(R/(x_1,\ldots,x_i))$.*

Theorem 4.2. *Let $(R,\mathfrak{m})$ be a local ring of dimension d. Then*

(1) We have $\mathfrak{b}(R)^{2^{s-1}}\subseteq \mathfrak{c}_s(R)$ for all $i=1,\ldots,d-1$, and $\mathfrak{b}(R)\subseteq c_0(R)$.
(2) We have $\mathfrak{b}(R)^{2^{s-1}-1}\subseteq \mathfrak{d}_s(R)$ for all $i=1,\ldots,d$.

In particular for all systems of parameters $x_1,\ldots,x_d$ and for all $n\ge 1$ we have

$$\mathfrak{a}(R)^{2^{d-2}}\big((x_1,\ldots,x_s)^n : x_{s+1}\big)\subseteq (x_1,\ldots,x_s)^n$$

for all $s\le d-1$, and

$$\mathfrak{a}(R)^{2^{d-1}}\Big(\bigcap_{\alpha\in\Lambda_{s,n}} Q_s(\alpha)\Big)\subseteq (x_1,\ldots,x_s)^n$$

for all $s\le d$.

Proof. By the definition we have $\mathfrak{b}(R)\subseteq \mathfrak{c}_0(R)\cap\mathfrak{c}_1(R)$ and $\mathfrak{d}_1(R)=R$. We will prove both (1) and (2) by induction on s. The case $s=1$ is as above. By Lemma 2.2 we need only to prove that if $\mathfrak{b}(R)^{2^{s-1}-1}\subseteq\mathfrak{d}_s(R)$ and $\mathfrak{b}(R)^{2^{s-1}}\subseteq\mathfrak{c}_s(R)$ for some $s<d$, then $\mathfrak{b}(R)^{2^{s}-1}\subseteq\mathfrak{d}_{s+1}(R)$. For all systems of parameter $x_1,\ldots,x_d$ and all $n\ge 1$ we have

$$\mathfrak{b}(R)^{2^{s-1}-1}\Big(\bigcap_{\alpha\in\Lambda_{s+1,n}} Q_{s+1}(\alpha)\Big)\subseteq \bigcap_{\alpha_{s+1}=1}^{n}\Big(b(R)^{2^{s-1}-1}\bigcap_{\alpha'\in\Lambda_{s,n+1-\alpha_{s+1}}}(Q_s(\alpha'),x_{s+1}^{\alpha_{s+1}})\Big).$$

By Lemma 4.1 we have $\mathfrak{b}(R)\subseteq\mathfrak{b}(R/x_{s+1}^{\alpha_{s+1}})$ for all $1\le\alpha_{s+1}\le n$. Hence by using the induction for $R/x_{s+1}^{\alpha_{s+1}}, 1\le\alpha_{s+1}\le n$ we have

$$b(R)^{2^{s-1}-1}\subseteq\mathfrak{b}(R/x_{s+1}^{\alpha_{s+1}})^{2^{s-1}-1}\subseteq\mathfrak{d}_s(R/x_{s+1}^{\alpha_{s+1}}).$$

Therefore

$$\mathfrak{b}(R)^{2^{s-1}-1}\big(\bigcap_{\alpha\in\Lambda_{s+1,n}} Q_{s+1}(\alpha)\big) \subseteq \bigcap_{\alpha_{s+1}=1}^{n} (Q_s^{n+1-\alpha_{s+1}}, x_{s+1}^{\alpha_{s+1}}).$$

Hence

$$\begin{aligned}
\mathfrak{b}(R)^{2^{s}-1}\big(\bigcap_{\alpha\in\Lambda_{s+1,n}} Q_{s+1}(\alpha)\big) &\subseteq \mathfrak{b}(R)^{2^{s-1}}\big(\bigcap_{\alpha_{s+1}=1}^{n} (Q_s^{n+1-\alpha_{s+1}}, x_{s+1}^{\alpha_{s+1}})\big)\\
&\subseteq \mathfrak{b}(R)^{2^{s-1}}\big(\bigcap_{\alpha_{s+1}=1}^{n} (x_{s+1}^{\alpha_{s+1}}, Q_s^{n+1-\alpha_{s+1}} : x_{s+1}^{\infty})\big)\\
&= \mathfrak{b}(R)^{2^{s-1}}\big(\sum_{\alpha_{s+1}=1}^{n} x_{s+1}^{\alpha_{s+1}}(Q_s^{n+1-\alpha_{s+1}} : x_{s+1}^{\infty})\big) \quad \text{(By Lemma 3.2)}\\
&\subseteq \sum_{\alpha_{s+1}=0}^{n} x_{s+1}^{\alpha_{s+1}} Q_s^{n-\alpha_{s+1}} \qquad\qquad (\text{Since } \mathfrak{b}(R)^{2^{s-1}} \subseteq \mathfrak{c}_s(R))\\
&= Q_{s+1}^n.
\end{aligned}$$

We finish the inductive proof. The last claims are easy since $\mathfrak{a}(R) \subseteq \mathfrak{b}(R)$. Hence the proof is complete. ☐

We next give some applications of Theorem 4.2 to annihilator of local cohomology of quotient rings R/Q_s^n.

Corollary 4.3. *Let $(R, \mathfrak{m})$ be a local ring of dimension d, and $\underline{x} = x_1, \ldots, x_d$ a system of parameters of R. Then for every $1 \le s \le d-1$ we have $\mathfrak{b}(R)^{2^{s-1}} \subseteq \mathfrak{b}(R/Q_s^n)$ for all $n \ge 1$.*

Proof. Recalling that

$$\mathfrak{b}(R/Q_s^n) = \bigcap_{\underline{y}, i<d-s} \mathrm{Ann}\big(\frac{(Q_s^n, y_1, \ldots, y_i) : y_{i+1}}{(Q_s^n, y_1, \ldots, y_i)}\big),$$

where $\underline{y} = y_1, \ldots, y_{d-s}$ runs over all systems of parameter of R/Q_s^n. Applying Theorem 4.2 for $R/(y_1, \ldots, y_i)$ we have

$$\mathfrak{b}(R/(y_1, \ldots, y_i))^{2^{s-1}} \subseteq \mathfrak{c}_s(R/(y_1, \ldots, y_i)).$$

Therefore

$$\mathfrak{b}(R/(y_1, \ldots, y_i))^{2^{s-1}}\big(\frac{(Q_s^n, y_1, \ldots, y_i) : y_{i+1}}{(Q_s^n, y_1, \ldots, y_i)}\big) = 0.$$

On the other hand $\mathfrak{b}(R) \subseteq \mathfrak{b}(R/(y_1, \ldots, y_i))$ by Lemma 4.1. Hence

$$\mathfrak{b}(R)^{2^{s-1}}((Q_s^n, y_1, \ldots, y_i) : y_{i+1}) \subseteq (Q_s^n, y_1, \ldots, y_i)$$

for all $\underline{y}$ and all $i < d-s$. The proof is complete. ☐

Corollary 4.4. *Let $(R, \mathfrak{m})$ be a local ring of dimension d. For all $1 \leq s \leq d-1$ and for all systems of parameters $\underline{x} = x_1, \ldots, x_d$ we have*

$$\mathfrak{a}(R)^{2^{s-1}} H^i_{\mathfrak{m}}(R/Q^n_s) = 0$$

for all $n \geq 1$ and for all $i < d - s$.

Corollary 4.5. *Let $(R, \mathfrak{m})$ be a k-Buchsbaum local ring of dimension d, i.e. $\mathfrak{m}^k H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$. Then for all $1 \leq s \leq d-1$ and for all systems of parameters $\underline{x} = x_1, \ldots, x_d$ we have*

$$\mathfrak{m}^{kd2^{s-1}} H^i_{\mathfrak{m}}(R/Q^n_s) = 0$$

for all $n \geq 1$ and for all $i < d - s$.

The above result is bit surprisingly since $\ell(H^i_{\mathfrak{m}}(R/Q^n_s))$ may be a polynomial ring in n of degree $s-1$ [4, Theorem 3.10].

References

[1] Y. André, *La conjecture du facteur direct*, Publ. Math. I.H.E.S., **127** (2018), 71–93.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press (Revised edition), 1998.

[3] M. Brodmann and R.Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, 1998.

[4] D.T. Cuong, P.H. Nam and P.H. Quy, *On the Length Function of Saturations of Ideal Powers*, Acta Math. Vietnam. **43** (2018), 275–288.

[5] N.T. Cuong and P.H. Quy, *On the structure of finitely generated modules over quotients of Cohen-Macaulay local rings*, preprint, arXiv:1612.07638.

[6] N.T. Cuong and H.L. Truong, *Parametric decomposition of parameter ideals and sequentially Cohen-Macaulay modules*, Proc. Amer. Math. Soc., **137** (2009), 19–26.

[7] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J., **51** (1973), 25–43.

[8] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.

[9] S. Goto and Y. Shimoda, *Parametric decomposition of powers of ideals versus regularity of sequences*, Proc. Amer. Math. Soc., **132** (2004), 929–933.

[10] S. Goto and Y. Shimoda, *On the parametric decomposition of powers of parameter ideals in a Noetherian local ring*, Tokyo J. Math., **27** (2004), 125–135.

[11] W. Heinzer, L.J. Ratliff and K. Shah, *Parametric decomposition of monomial ideals I*, Houston J. Math., **21** (1995), 29–52.

[12] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, Cambridge University Press, 2006.

[13] L. Ma, P.H. Quy and I. Smirnov, *Colength, multiplicity, and ideal closure operations*, Comm. Algebra to appear, arXiv:1907.00951.

[14] T. Polstra and P.H. Quy, *Nilpotence of Frobenius actions on local cohomology and Frobenius closure of ideals*, J. Algebra, **529** (2019), 196–225.

[15] P. Schenzel, *Dualisierende komplexe in der lokalen algebra und Buchsbaum - ringe*, Lect. Notes in Math., Springer-Verlag Berlin - Heidelberg - New York, 1982.

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