Constructing Efficient 2-level Foldover Designs from Hadamard Matrices

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Abstract

This paper introduces two algorithms for constructing efficient 2-level foldover designs (EFDs): one constructs EFDs from Hadamard matrices and one constructs EFDs from scratch. Some of the constructed designs are less D-efficient than the efficient 2-level foldover designs of Errore et al. (2017) but offer more degrees of orthogonality among the main effects (MEs) and do not require some 2-factor interactions (2FIs) to be fully aliased with each other. The algorithms also offer a mechanism to choose follow-up runs which consist of additional foldover pairs. A catalog of EFDs for up to 28 factors is given.

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1 Introduction

Consider a well-known experiment discussed in Box et al. "Choosing follow-up runs" (2005, Section 7.2) (hereafter abbreviated as BHH) which studied the effects of eight factors on percent shrinkage in an injection molding process: **A**, Mold Temperature; **B**, Moisture; **C**, Hold Press; **D**, Cavity Thickness; **E**, Booster Pressure; **F**, Cycle Time; **G**, Gate Size and **H**, Screw Speed. (See also Meyer et al., 1996). The design for this experiment is a 2^{8-4} fractional factorial design (FFD) of resolution IV. It can also be considered as a foldover design with eight runs in Figure 1, forming a half fraction design matrix (HFM). In this figure **A**, **B**, **C** form a factorial; $\mathbf{D} = \mathbf{AB}$, $\mathbf{E} = \mathbf{AC}$, $\mathbf{F} = \mathbf{BC}$, $\mathbf{G} = \mathbf{ABC}$ and **H** is a column of 1's. The analysis of the data can be found from the references mentioned above. The following questions related to the design and analysis of this experiment can be raised:

A	B	C	D	E	F	G	H
-1	-1	-1	1	1	1	-1	1
1	-1	-1	-1	-1	1	1	1
-1	1	-1	-1	1	-1	1	1
1	1	-1	1	-1	-1	-1	1
-1	-1	1	1	-1	-1	1	1
1	-1	1	-1	1	-1	-1	1
-1	1	1	-1	-1	1	-1	1
1	1	1	1	1	1	1	1

Figure 1: HFM of the injection molding experiment in BHH.

1. The MEs of the design whose HFM is in Figure 1 are orthogonal to each other and

to the 2FIs. However, the following 2FIs are aliased with each other: $\mathbf{AB} = \mathbf{CG} = \mathbf{DH} = \mathbf{EF}$, $\mathbf{AC} = \mathbf{BG} = \mathbf{DF} = \mathbf{EH}$, $\mathbf{AD} = \mathbf{BH} = \mathbf{CF} = \mathbf{EG}$, $\mathbf{AE} = \mathbf{BF} = \mathbf{CH} = \mathbf{DG}$, $\mathbf{AF} = \mathbf{BE} = \mathbf{CD} = \mathbf{GH}$, $\mathbf{AG} = \mathbf{BC} = \mathbf{DE} = \mathbf{FH}$, $\mathbf{AH} = \mathbf{BD} = \mathbf{CE} = \mathbf{FG}$. Is there another candidate design, having the same number of factors and runs with MEs being orthogonal to the 2FIs, but no 2FI is fully aliased with another 2FI.

2. The normal probability plot of the effect injection molding experiment shows that factors **A** and **C**, together with the 2-factor interaction chains $\mathbf{AE} = \mathbf{BF} = \mathbf{CH} = \mathbf{DG}$, stand out. Using two empirical principles, effect sparsity and effect heredity (Wu & Hamada, 2009 Section 4.6), and the fact that factors **B**, **D** , **F** and **C** are negligible, we could collapse the original experiment to the one with only four factors **A**, **C**, **E** and **H**. Can we augment the collapsed experiment with two foldover pairs of points, so that the practitioners can dealias the interactions associated with **AE**?

Before answering to questions like these, let us review some of the most recent works on 2-level FFDs. Errore et al. (2017), hereafter abbreviated as EJLN, pointed out four desirable features for a screening design such as the one for the above injection molding experiment: (i) orthogonality of MEs; (ii) orthogonality of MEs and 2FIs; (iii) orthogonality of 2FIs with one another; (iv) economic run size. The Plackett-Burman designs and resolution III FFDs have features (i) and (iv) but not (ii) and (iii). The resolution IV FFDs, such as the one for the injection molding experiment, have all desirable features except (iii). Finally, the resolution V FFDs have all desirable features except (iv). EJLN extended the work of Webb (1968), Margolin (1969), Miller & Sitter (2001) and Lin, Miller & Sitter (2008) and introduced a new class of efficient EFDs. EFDs are available for any number of runs equal to or greater than $2m$ where m is the number of factors. As expected, all EFDs have MEs orthogonal to 2FIs and can be constructed such that the fully aliased 2FIs can be eliminated.

The purposes of this papers are: (i) to introduce two algorithms for constructing EFDs: one uses Hadamard matrices (Heydayat & Wallis, 1978) or the ± 1 maximal-determinant matrices, and one constructs EFDs from scratch. Both algorithms use the minimum G_2 aberration (Tang & Deng, 1999) as a surrogate design criterion; (ii) to provide a mechanism to build follow-up experiments using the augmented foldover pair of points; and (iii) to construct a catalog of HFMs of EFDs with $m \leq 28$.

2 Our surrogate criterion for finding EFDs

Consider an FFD whose model includes the MEs and 2FIs constructed from an $n \times m$ design matrix $X = (x_{ui}),\ u = 1, \ldots, n; i = 1, \ldots, m$:

$$y_u = \beta_0 + \sum_{i=1}^{m} \beta_i x_{ui} + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \beta_{ij} x_{ui} x_{uj} + \epsilon_u \ u = 1, \ldots, n. \tag{1}$$

Here, y_u is the response at point u, β's are the unknown parameters, and ϵ_u (ϵ_u iid $\mathrm{N}(0, \sigma^2)$) is the error associated with point u. Note that the first $m+1$ terms in (1) form the ME model. In matrix notation, (1) can be written as $\mathbf{y} = \mathbf{X}\beta + \epsilon$ where $\mathbf{X}$ is the model matrix (also called the expanded design matrix) of size $n \times p$ where $p = 1 + m + \binom{m}{2}$. The uth row of $\mathbf{X}$ can be written as $(1,\ x_{u1}, \ldots,\ x_{um},\ x_{u1}x_{u2}, \ldots,\ x_{u(m-1)}x_{um},)$. The $\mathbf{X}'\mathbf{X}$ information matrix contains the following terms: (i) $\sum x_i,\ (i = 1, \ldots, m)$; (ii) $\sum x_i x_j,\ (i < j)$; (iii) $\sum x_i x_j x_k,\ (i < j < k)$; and (iv) $\sum x_i x_j x_k x_l\ (i < j < k < l)$ where the subscript $i,\ j,\ k,\ l = 1, \ldots, m$ and the summations are taken over n design points. There are m

summations in (i), $\binom{m}{2}$ in (ii), $\binom{m}{3}$ in (iii) and $\binom{m}{4}$ in (iv). For regular FFDs, i.e. FFDs for m factors in 2^{m-k} runs, the summations in (i)-(iv) will be either 0 or n. If all the summations in (i) and (ii) are zeros, it is called resolution III; if all in (i)-(iii) are zeros, it is called resolution IV and if all in (i)-(iv) are zeros, it is called resolution V. For non-regular FFDs in n runs where n can take any value, the summations in (i)-(iv) can take values between $-n$ and n.

We denote the averages of the sum of squares of the summations in (i)-(iv) by $A_1,\ A_2,\ A_3$ and A_4, respectively. As for a foldover design, meaning a design whose first HFM is X and the second is $-X$, A_1 and A_3 are zeros, we can then use the pair $(A_2,\ A_4)$ as the surrogate criteria for finding EFDs. The pair $(A_2^*,\ A_4^*)$ of an EFD d^* is said to be minimum if for a pair $(A_2,\ A_4)$ of any different EFD with the same number of factors and runs, $A_2^* < A_2$ or $A_2^* = A_2$ and $A_4^* \leq A_4$.

Remarks:

1. It can be seen that our computationally-cheap surrogate criterion is very closely allied to the minimum G_2 aberration criterion (Tang & Deng, 1999) for finding good non-regular FFDs. This criterion can also be considered as the refinement of the $E(s^2)$ criterion proposed by Booth & Cox (1962) for finding supersaturated designs (See also Nguyen, 1996) or the $S-$criterion suggested by Shah (1960) for finding efficient incomplete block designs (See also Eccleston & Heydayat, 1974). Note that minimizing the sum of squares of the elements of $\mathbf{X}'\mathbf{X}$ (or minimizing trace $(\mathbf{X}'\mathbf{X})^2$) given trace$(\mathbf{X}'\mathbf{X})$ = const. as in the case of 2-level designs is the same as minimizing $\sum \lambda_i^2$ given $\sum \lambda_i$ =const., where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of $\mathbf{X}'\mathbf{X}$. Clearly, smaller $\sum \lambda_i^2$ tends to give the smaller $\sum \lambda_i^{-1}$ and larger $\Pi\lambda_i$, which are related to the well-known A- and D-optimality criteria, respectively.

2. There is no guarantee that the most D-efficient design is the one found by the surrogate criterion in the previous paragraphs. At the same time, there is no guarantee that the most D-efficient design is the most desirable. In Section 4, we will give examples to illustrate this point.

3 The FOLD algorithms

In the following, we will describe two FOLD algorithms for constructing HFMs of size $n \times m$ $(m \leq n)$ from which an EFD for m factors in $2n$ runs can be constructed:

FOLD 1: This algorithm, which is mostly used in this paper, forms an HFM of size $n \times m$ by selecting m columns randomly from an input ± 1 square matrix of order $n \geq m$. If $n = 4,\ 8,\ 12 \ldots = 4t$, where t is an integer, a Hadamard matrix of order n is used. If $n = 3,\ 7,\ 11 \ldots = 4t - 1$, the core of a normalized Hadamard matrix of order $n + 1$ will be used. For other n's, a ± 1 maximal-determinant matrix of order n in `http://www.indiana.edu/~maxdet/` is used. Hadamard matrices of order up to $n = 32$ can be found in Heydayat & Wallis (1978). Hadamard matrices of order up to $n = 256$ is available at `http://neilsloane.com/hadamard/` and the GNU Octave software `https://www.gnu.org/software/octave/`.

For each parameter set $(m,\ n)$, $m \leq n$, several HFMs are constructed, each of which corresponds to a "try". For each try, the $(A_2,\ A_4)$ pair is calculated. Among all tries that result in the minimum $(A_2,\ A_4)$, we choose the one with the maximum $|\mathbf{X}_1'\mathbf{X}_1|$ where $\mathbf{X}_1$ is the model matrix corresponding to the MEs. Clearly, when $m = n$, FOLD 1 requires a single try.

Remarks:

1. A ± 1 square matrix $\mathbf{H}$ of order n is Hadamard if $\mathbf{H}'\mathbf{H} = n\mathbf{I}$, where I is the identity matrix.

2. To calculate A_2 and A_4, for each row u of X, $(u = 1, \ldots, n)$ calculate vector J_u of length $\binom{m}{2} + \binom{m}{4}$:

$$J_u = (x_{u1}x_{u2}, \ldots,\ x_{u(m-1)}x_{um},\ x_{u1}x_{u2}x_{u3}x_{u4}, \ldots,\ x_{u(m-3)}x_{u(m-2)}x_{u(m-1)}x_{um}). \quad (2)$$

   We then calculate $J = \sum_{u=1}^{n} J_u$ and set A_2 and A_4 equal to the averages of the sums of squares of the first $\binom{m}{2}$ elements of J and the last $\binom{m}{4}$ elements of J respectively.

3. The lower bound for A_2 is 0 when $n = 4t$, and 1 when n is odd. We use this lower bound as the stopping rule for the FOLD algorithms.

FOLD 2: This algorithm constructs HFMs from scratch. It has two steps:

1. Form an initial HFM X of size $n \times m$ by allocating ± 1 randomly to its elements. Calculate $(A_2,\ A_4)$.

2. In each row u of X, search for a pair of different elements such that swapping them results in the smallest pair $(A_2,\ A_4)$. If found, swap them and update X and J. Repeat this step until A_2 reaches its lower bound or no further update on X is required.

For each parameter set $(m,\ n)$, Steps 1-2 make up one "try". Among all tries which result in the minimum $(A_2,\ A_4)$, choose the one with the maximum $|\mathbf{X}_1'\mathbf{X}_1|$, where $\mathbf{X}_1$ is the model matrix corresponding to the MEs.

Remarks:

1. For a parameter set $(m,\ n)$, there is no guarantee that FOLD 2 can construct a EFD with no 2FIs fully aliased. To construct an HFM of an EFD with no 2FIs fully

aliased (or compound EFD using the terminology of EJLN), we add the requirement that $r_{\max}^{2FIs}$ be smaller than 1. So among the set of candidate designs with minimum $(A_2,\ A_4)$ and $r_{\max}^{2FIs}$ less than a threshold value (say 0.9), the one with maximum $|\mathbf{X}_1'\mathbf{X}_1|$ is selected.

2. FOLD 2 can also augment an HFM with additional rows (or columns).

4 Discussion

Table 1 displays the goodness statistics of 63 EFDs with m ranging from 3 to 28 and run sizes $\geq 2m$. The goodness statistics of these selected EFDs include $D_{\text{eff}},\ r_{\text{ave}},\ r_{\max},\ f(r_{\max})$ and $r_{\max}^{\text{2FIs}}$ where

$$D_{\text{eff}} = \frac{1}{2n}|\mathbf{X}_1'\mathbf{X}_1|^{\frac{1}{m+1}} \tag{3}$$

in which $\mathbf{X}_1$ is the ME model matrix formed by the first $m+1$ columns of $\mathbf{X}$; r_{ave} and $r_{\max}$ are the average and the maximum of the correlations (in terms of the absolute values) among the m main-effect columns of $\mathbf{X}$. $f(r_{\max})$ is the frequency of $r_{\max}$, and $r_{\max}^{\text{2FIs}}$ is the maximum of the correlations among the 2FI columns of $\mathbf{X}$. Clearly, $r_{\max}^{\text{2FIs}} = 1$ indicates that at least a pair of 2FIs is fully aliased.

All HFMs of the 63 EFDs in Table 1 were constructed by FOLD 1 using Hadamard matrices, cores of Hadamard matrices and ± 1 maximal-determinant matrices of order $n \geq m$. Details on the choices of these matrices are in the previous section. FOLD 1 is extremely fast. It constructed the HFMs of 63 EFDs in Table 1 in less than 20 seconds on a laptop with CORE$^{\text{TM}}$i7 (each EFD with $m < n$ was given 100 tries).

Out of 63 EFDs in Table 1, 48 have A_2 values equal to either 0 or 1. We call them EFD*'s. For 26 EFD*'s with $A_2 = 0$, the correlation among the ME columns is 0; for 22 EFD*'s with $A_2 = 1$, this correlation is $\pm\frac{1}{n}$.

Out of 27 EFDs in Table 1 of EJCN, 22 match ours in terms of the goodness statistics. For the remaining five parameter sets $(m,\ n)$=(7, 7), (9, 11), (10, 11), (11,11) and (13, 15), our solutions are EFD*s, while the ones of EJCN are not. While our EFD*'s are slightly less D-efficient than EJCN's, they have smaller r_{ave} and r_{max}. For $(m,\ n)$= (9, 11), (10, 11) and (11, 11), the $r^{\text{2FIs}}_{\text{max}}$ of our EFD*'s is 0.47 while EJCN's is 1. For $(m,\ n)$=(7, 7) and (13, 15), the $r^{\text{2FIs}}_{\text{max}}$ of our EFD*'s is 1, while EJCN's are 0.75 and 0.875, respectively. Like other EFD*'s in Table 1 with $n = 4t - 1$, these five EFD*'s were constructed from the core of a normalized Hadamard matrix of order $n+1$. If the ± 1 maximal-determinant matrices of order n are used as input matrices instead, EFDs similar to EJCN's will be obtained.

Figure 2 shows two HFMs of two EFDs for $(m,\ n)$ =(11, 11): one constructed from a ± 1 maximal-determinant of order 11 (Figure 2a) and one from a circulant matrix of order 11 (which forms a core of a Hadamard matrix of order 12) generated by the following generator (-1, -1, 1, -1, -1, -1, 1, 1, 1, -1, 1) (Figure 2b). Figure 3 shows the correlation cell plots (CCPs) of the two EFDs whose HFMs are in Figure 2. These plots, proposed by Jones & Nachtsheim (2011), display the magnitude of the correlation (in terms of the absolute values) between main effects and 2-factor interactions in screening designs. The color of each cell in these plots goes from white (no correlation) to dark (correlation of 1 or close to 1). As expected, both CCPs in Figures 3a and 3b show that the MEs are orthogonal to the 2FIs. Figure 3b shows that the correlation among MEs is constant and none of the 2FIs are fully aliased with the other 2FIs.

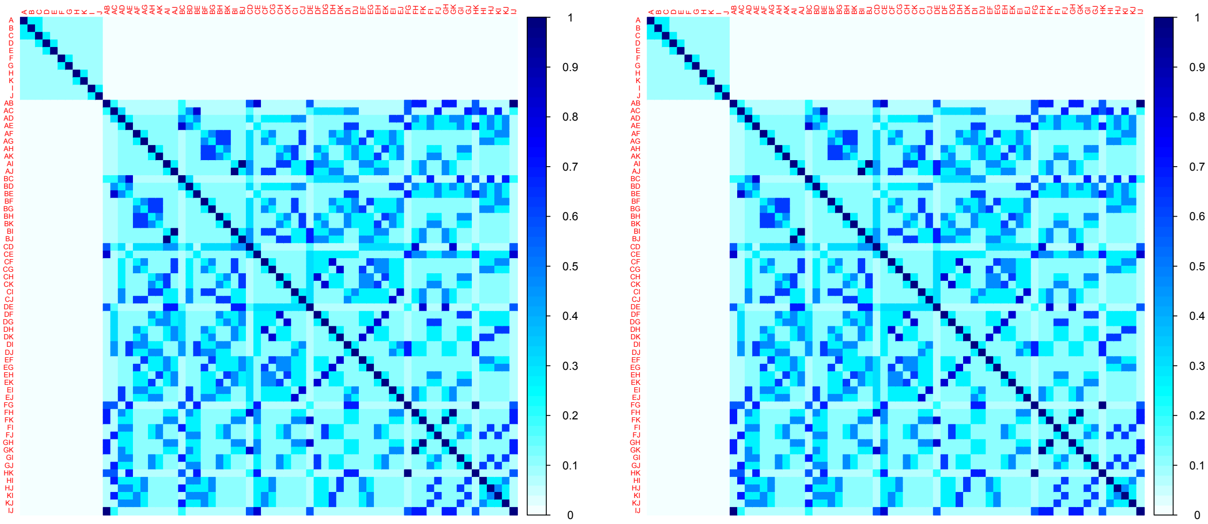
A	B	C	D	E	F	G	H	I	J	K
1	1	1	1	-1	-1	1	1	-1	-1	-1
1	1	1	1	-1	1	-1	-1	1	-1	-1
1	1	1	-1	1	-1	-1	-1	-1	-1	-1
1	1	-1	1	1	-1	-1	-1	-1	1	1
-1	-1	1	1	-1	-1	-1	-1	-1	1	1
-1	1	-1	-1	-1	-1	-1	1	1	1	-1
1	-1	-1	-1	-1	-1	-1	1	1	-1	1
1	-1	-1	-1	-1	1	1	-1	-1	1	-1
-1	1	-1	-1	-1	1	1	-1	-1	-1	1
-1	-1	-1	1	1	1	-1	1	-1	-1	-1
-1	-1	-1	1	1	-1	1	-1	1	-1	-1

(a)

A	B	C	D	E	F	G	H	I	J	K
-1	-1	1	-1	-1	-1	1	1	1	-1	1
1	-1	-1	1	-1	-1	-1	1	1	1	-1
-1	1	-1	-1	1	-1	-1	-1	1	1	1
1	-1	1	-1	-1	1	-1	-1	-1	1	1
1	1	-1	1	-1	-1	1	-1	-1	-1	1
1	1	1	-1	1	-1	-1	1	-1	-1	-1
-1	1	1	1	-1	1	-1	-1	1	-1	-1
-1	-1	1	1	1	-1	1	-1	-1	1	-1
-1	-1	-1	1	1	1	-1	1	-1	-1	1
1	-1	-1	-1	1	1	1	-1	1	-1	-1
-1	1	-1	-1	-1	1	1	1	-1	1	-1

(b)

Figure 2: Two HFMs of two EFDs for $(m,\ n)$ =(11, 11): (a) is a ± 1 maximal-determinant matrix of order 11; (b) is a core of a normalized matrix of order 12.



(a) (b)

Figure 3: CCPs of the two EFDs whose HFMs are in Figure 2.

EFDs in Table 1 with $r_{\max}^{\text{2FIs}} = 1$ have at least one pair of 2FIs being fully aliased with each other. Table 2 shows alternative EFDs for selected cases with fully aliased

2FIs eliminated. The HFMs of the EFDs with $m = n$ were either obtained from `http://www.indiana.edu/~maxdet/` (e.g. EFDs with $m = n = 15$) or constructed from scratch by FOLD 1 using a threshold of 0.9. For $m = n = 8$, we constructed an additional HMF using a threshold of 0.7. The remaining HFMs of EFDs with $m < n$ were constructed by FOLD 1 (using the HFMs of the EFDs with $m = n$ as inputs) in less than 10 seconds on the same laptop with CORE$^{\text{TM}}$i7 (each EFD was given 100 tries).

Note that the three sets of parameters $(m,\ n)$= (6, 8), (7, 8) and (8, 8) in Table 2 have two solutions. The one matching EJCN's has lower $r_{\max}^{\text{2FIs}}$ but higher $r_{\max}$ (and higher r_{ave}). Figure 4 shows two HFMs of two EFDs for $(m,\ n)$ =(8, 8): one was constructed by setting a threshold of 0.7 (Figure 4a) and the other a threshold of 0.9 (Figure 4b). The CCPs of these two EFDs are in Figure 5. These EFDs can be used as candidate designs for the injection molding experiment in Box et al. (2005) mentioned in the Introduction.

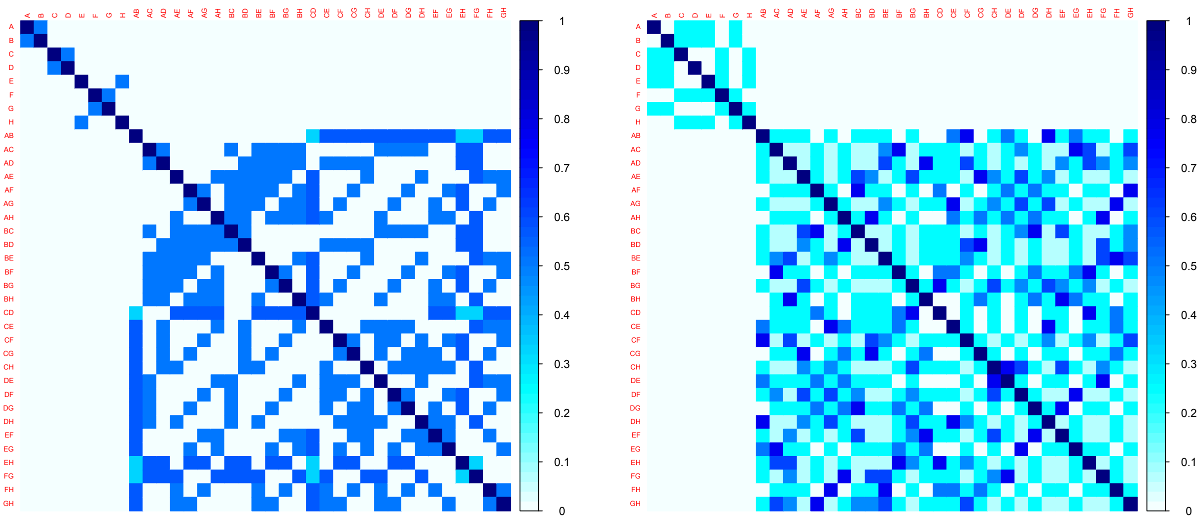
A	B	C	D	E	F	G	H
1	-1	1	1	-1	-1	1	-1
1	-1	-1	-1	-1	1	1	1
-1	-1	1	1	-1	1	-1	1
-1	1	1	-1	-1	-1	1	1
-1	-1	-1	-1	1	-1	1	-1
-1	1	-1	-1	-1	1	-1	-1
-1	1	1	1	1	1	1	-1
-1	1	-1	1	-1	-1	1	1

(a)

A	B	C	D	E	F	G	H
1	1	1	-1	-1	-1	-1	-1
-1	1	1	-1	1	1	-1	1
-1	1	-1	-1	1	-1	1	1
1	1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1
-1	-1	1	1	1	-1	-1	-1
1	-1	1	-1	1	1	1	1
1	-1	-1	-1	-1	-1	-1	1

(b)

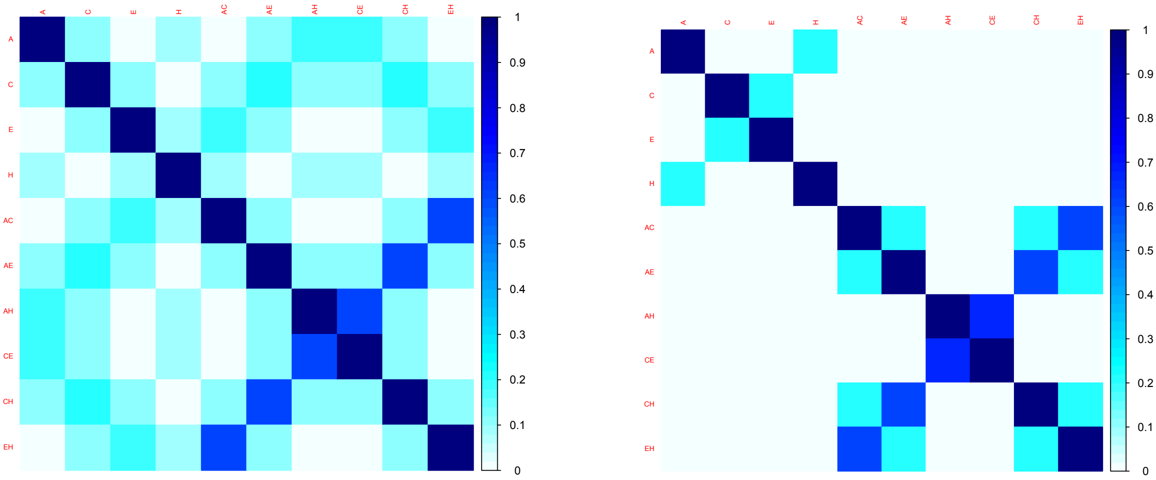
Figure 4: Two HFMs of two EFDs for $(m,\ n)$ =(8, 8): (a) uses a threshold value of 0.7; (b) uses a threshold value of 0.9.



(a) (b)

Figure 5: CCPs of the two EFDs whose HFMs are in Figure 4.

Let us return to the problem of augmenting the four columns **A**, **C**, **E** and **H** in the Introduction with four follow-up runs. While our purpose of choosing follow-up runs is to dealias the interactions associated with the interaction **AE**, the one of BHH is to allow maximum discrimination among the plausible model. Details about the model discrimination (MD) method and the MD criterion can be found in BHH. The four designs runs found by BHH are: (-1, -1, -1, +1), (-1, -1, -1, +1), (-1, +1, +1, +1) and (+1, +1, -1, +1). The foldover pairs obtained by FOLD 2 are: (1, 1, -1, 1) and (1, -1, 1, 1). It is interesting to compare the CCP of the 20-run design obtained by BHH (Figure 6a) and the one of our 20-run design (Figure 6b). It can be seen that neither augmented designs have 2FIs fully aliased with another 2FIs. However, unlike the FOLD design, the MEs of the BHH one are not orthogonal to the 2FIs.



(a) (b)

Figure 6: CCPs of the (a) augmented design of BHH, (b) augmented design constructed by FOLD 2.

5 Conclusion

Most popular designs for screening experiments up to this point are still regular FFDs of various resolutions. These designs have been popular because they are simple to analyze: the MEs are orthogonal to each other and the MEs and 2FIs are either orthogonal or fully aliased with other 2FIs. The cost of a regular FFD in a multifactor experiment is a huge number of runs if a resolution V design is used, or a follow-up experiment is required to disentangle the MEs from 2FIs or 2FIs from other 2FIs. Like the EFDs of EJLN, ours offer additional choices for experiments in terms of the flexible number of design runs. Some EFDs expect the practitioners to accept certain mild non-orthogonality among MEs to avoid any 2FI fully aliased. 48 of the new EFDs are the EFD*'s, meaning EFDs having any 2 MEs with the correlation 0 (when $A_2 = 0$) or $\pm\frac{1}{n}$ (when $A_2 = 1$). This desirable property will certainly helps the practitioners in the interpretation of the results and make EFD*'s particularly those with $r_{\max}^{\text{2FIs}} < 1$ popular.

The HFMs of the 63 EFDs in Table 1 and 23 designs in Table 2, as well as the Java program implementing the two FOLD algorithms in Section 3, and the corresponding input matrices for these two tables, are in the supplemental material.

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Table 1: Goodness statistics of constructed EFDs.

m	n	$2n$§	D_{eff}	r_{ave}	$r_{\max}$	$f(r_{\max})$	$r_{\max}^{\text{2FIs}}$
3	3	6	0.877	0.33	0.33†	3	0.50
3	4	8	1.000	0.00	0.00†	3	0.00
4	4	8	1.000	0.00	0.00†	6	1.00
5	5	10	0.950	0.20	0.20†	10	0.67
5	6	12	0.933	0.13	0.33	4	0.71
5	7	14	0.949	0.14	0.14†	10	1.00
5	8	16	1.000	0.00	0.00†	10	1.00
6	6	12	0.918	0.13	0.33	6	0.71
6	7	14	0.920	0.14	0.14†	15	1.00
6	8	16	1.000	0.00	0.00†	15	1.00
7‡	7	14	0.893	0.18	0.43	3	0.75
7	7	14	0.867	0.14	0.14†	21	1.00
7	8	16	1.000	0.00	0.00†	21	1.00
8	8	16	1.000	0.00	0.00†	28	1.00
9	9	18	0.939	0.12	0.56	1	1.00
9	10	20	0.951	0.09	0.20	16	1.00
9‡	11	22	0.946	0.11	0.27	3	1.00
9	11	22	0.941	0.09	0.09†	36	0.47
9	12	24	1.000	0.00	0.00†	36	0.33
10	10	20	0.946	0.09	0.20	20	1.00
10‡	11	22	0.938	0.11	0.27	4	1.00
10	11	22	0.920	0.09	0.09†	45	0.47
10	12	24	1.000	0.00	0.00†	45	0.33
11‡	11	22	0.922	0.12	0.27	8	1.00
11	11	22	0.880	0.09	0.09†	55	0.47
11	12	24	1.000	0.00	0.00†	55	0.33
12	12	24	1.000	0.00	0.00†	66	0.33
13	13	26	0.978	0.08	0.08†	78	0.86
13	14	28	0.962	0.07	0.14	36	0.87
13‡	15	30	0.952	0.09	0.20	11	0.88
13	15	30	0.942	0.07	0.07†	78	1.00
13	16	32	1.000	0.00	0.00†	78	1.00
14	14	28	0.960	0.07	0.14	42	0.87
14	15	30	0.925	0.07	0.07†	91	1.00
14	16	32	1.000	0.00	0.00†	91	1.00
15	15	30	0.893	0.07	0.07†	105	1.00
15	16	32	1.000	0.00	0.00†	105	1.00
16	16	32	1.000	0.00	0.00†	120	1.00

§run size.

‡EFDs from Table 1 of EJCN.

†$r_{\max}$ of EFD*'s.

Table 1: Goodness statistics of constructed EFDs (cont.)

m	n	$2n$§	D_{eff}	r_{ave}	$r_{\max}$	$f(r_{\max})$	$r_{\max}^{\text{2FIs}}$
17	17	34	0.968	0.07	0.18	16	1.00
17	18	36	0.970	0.05	0.11	64	1.00
17	19	38	0.945	0.05	0.05†	136	0.69
17	20	40	1.000	0.00	0.00†	136	0.60
18	18	36	0.968	0.05	0.11	72	1.00
18	19	38	0.930	0.05	0.05†	153	0.69
18	20	40	1.000	0.00	0.00†	153	0.60
19	19	38	0.904	0.05	0.05†	171	0.69
19	20	40	1.000	0.00	0.00†	171	0.60
20	20	40	1.000	0.00	0.00†	190	0.60
21	21	42	0.977	0.05	0.24	4	0.91
21	22	44	0.972	0.04	0.09	100	0.64
21	23	46	0.948	0.04	0.04†	210	0.39
21	24	48	1.000	0.00	0.00†	210	0.33
22	22	44	0.970	0.04	0.09	111	0.83
22	23	46	0.935	0.04	0.04†	231	0.39
22	24	48	1.000	0.00	0.00†	231	0.33
23	23	46	0.912	0.04	0.04†	253	0.39
23	24	48	1.000	0.00	0.00†	253	0.33
24	24	48	1.000	0.00	0.00†	276	0.33
25	25	50	0.988	0.04	0.04†	300	1.00
25	26	52	0.978	0.04	0.08	144	0.62
25	27	54	0.950	0.04	0.04†	300	0.78
25	28	56	1.000	0.00	0.00†	300	0.71
26	26	52	0.978	0.04	0.08	156	0.62
26	27	54	0.939	0.04	0.04†	325	0.78
26	28	56	1.000	0.00	0.00†	325	0.71
27	27	54	0.919	0.04	0.04†	351	0.78
27	28	56	1.000	0.00	0.00†	351	0.71
28	28	56	1.000	0.00	0.00†	378	0.71

§run size.

†$r_{\max}$ of EFD*'s.

Table 2: Some constructed EFDs with fully alised 2FIs eliminated

m	n	$2n$§	D_{eff}	r_{ave}	$r_{\max}$	$f(r_{\max})$	$r_{\max}^{\text{2FIs}}$
5	7	14	0.935	0.17	0.43	1	0.75
6	7	14	0.911	0.18	0.43	2	0.75
7	7	14	0.893	0.18	0.43	3	0.75
6‡	8	16	0.921	0.07	0.50	2	0.58
7‡	8	16	0.898	0.07	0.50	3	0.58
8‡	8	16	0.880	0.07	0.50	4	0.58
6	8	16	0.917	0.13	0.25	8	0.78
7	8	16	0.891	0.14	0.25	12	0.78
8	8	16	0.869	0.14	0.25	16	0.78
9	10	20	0.931	0.09	0.20	16	0.82
10	10	20	0.922	0.09	0.20	20	0.82
9	9	18	0.898	0.14	0.33	4	0.80
9	10	20	0.931	0.09	0.20	16	0.82
10	10	20	0.922	0.09	0.20	20	0.82
13	15	30	0.951	0.09	0.20	15	0.88
13	16	32	0.957	0.06	0.12	36	0.63
14	15	30	0.947	0.09	0.20	18	0.88
14	16	32	0.949	0.06	0.12	45	0.78
15	15	30	0.944	0.09	0.20	21	0.88
15	16	32	0.941	0.06	0.12	54	0.78
16	16	32	0.935	0.07	0.12	63	0.78
17	18	36	0.945	0.06	0.11	70	0.67
18	18	36	0.935	0.06	0.11	81	0.67

§run size.

‡Compound EFDs from Table 3 of EJCN.