On finite-time stability of linear singular large-scale systems with state delays in interconnection

Pham T. Huong[a], Vu N. Phat[b]

[a]*Department of Mathematics, Hanoi Pedagogical University 2, Vinh Phuc, Vietnam*

[b] *ICRTM, Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, Hanoi, Vietnam*

Abstract

In this paper, we provide an efficient approach based on combination of singular value decomposition (SVD) and Lyapunov function methods to finite-time stability of linear singular large-scale complex systems with interconnected delays. By representing the singular large-scale system as a differential-algebraic system and using linear matrix inequality technique, we provide new delay-dependent conditions for the system to be regular, impulse- free and robustly finite-time stable. The conditions are presented in the form of a feasibility problem involving linear matrix inequalities (LMIs). Finally, a numerical example is presented to show the validity of the proposed results.

Keywords:
Finite-time stability, Large-scale systems, Interacted delays, Singularity, Lyapunov functions, Linear matrix inequalities.

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1. Introduction

Many systems in the practical world such as social systems, economic systems, power systems, and transportation are studied through the mathematical large-scale complex models. Large-scale complex systems have complex structures including a large number of variables and interconnected subsystems [1, 2]. In the past decades, stability problem for large-scale systems with delay has received extensive attention from researchers [3-6]. It is noted that most existing studies are focused on Lyapunov asymptotic stability (LAS). However, in practice, the main concern is the behavior of the system over a finite-time interval, called finite-time stability (FTS) [7]. FTS involves dynamical systems whose solutions do not exceed some bounds during this time-interval, while LAS deals with the behavior of a system at infinity time, FTS concerns solution behavior within a finite short interval. Many important results on finite-time stability of dynamical systems with delays can be found in [8-10]. Besides, finite-time stability of singular systems was also attracted by many researchers due to a wide range of its applications in many practical models, such as chemical process, power systems, social systems, and so on [11-13]. The study of such systems is much more complicated than that for non-singular systems, since their structure consists of differential equations coupled with algebraic equations. By decomposing the system into differential (fast) and algebraic (slow) subsystems, one needs to prove the stability of the fast and slow subsystems.

Email address: `Corresponding author: vnphat@math.ac.vn` (Vu N. Phat)

To prove the stability of the fast subsystem, an approach based on graph theory to express and estimate the fast variables via the slow variables. This approach is rather complicated for application and further improvement. It is noted that there are few results concern with the finite-time stability of large-scale interconnected systems. Most of the results only indicate the Lyapunov stability of non-singular large-scale systems with time delays [14-18]. To the best of our knowledge, results on FTS of large-scale systems are few. The authors of [19-21] proposed some sufficient conditions for FTS of linear large-scale systems, however, either the time delays or the singularity were not considered there. Therefore, the problem of FTS for singular large-scale delay systems with delays still remains open, which motivates our present study.

In this paper, the problem of finite-time stability for linear singular large-scale complex systems subjected to state delays interacted with the subsystems. Our aim is to derive delay-dependent conditions to guarantee the robust FTS of such system. To do this we proceed as follows. Employing the singular value decomposition method, we first decompose the system into the slow and fast subsystems. The boundedness of the fast system is proved by constructing an augmented Lyapunov-Krasovskii functional. Then, we prove the boundedness of the slow system by some estimation techniques specifically developed in this paper. The delay-dependent sufficient conditions for robust FTS of the system are derived in terms of LMIs, which can be easily determined by utilizing MATLABs LMI Control Toolbox [22]. The last, a numerical example is given to illustrate the validity and effectiveness of the proposed results.

This paper is outlined as follows. In Section 2, problem formulation and some auxiliary results are presented. New delay-dependent sufficient conditions for robust finite-time stability of the systems are presented with an illustrated example in Section 3. Some conclusions are given in Section 4.

Notations. $\mathbb{R}^+$ denotes the set of all real positive numbers; $\mathbb{R}^k$ denotes the $k-$ dimensional Euclidean space; $\mathbb{R}^{a\times b}$ denotes the space of all $(a\times b)$- matrices; $X^\top$ denotes the transpose of X; $\lambda(X)$ denotes the set of all eigenvalues of X; $\lambda_{min}(X) = \min\{\mathrm{Re}\lambda :\ \lambda\in\lambda(X)\}$; $\lambda_{max}(X) = \max\{\mathrm{Re}\lambda :\ \lambda\in\lambda(X)\}$; $C([a_1,b_1],\mathbb{R}^k)$ denotes the set of all $\mathbb{R}^k$-valued continuous functions on $[a_1,b_1]$; ; $L_2([0,T],\mathbb{R}^k)$ denotes the set of all square-integrable $\mathbb{R}^k-$ valued functions on $[0,T]$. If $(Py,y)\geq 0$, for all $y\in\mathbb{R}^n$ then P is semi-positive definite $(P\geq 0)$; $C\geq D$ means $C-D\geq 0$. C is positive definite $(C>0)$ if $(Cy,y)>0$ for all $y\neq 0$; The segment of the trajectory $y(t)$ is denoted by $y_t = \{y(t+s) :\ s\in[-h,0]\}$ with the norm $||y_t|| = \sup\limits_{s\in[-h,0]} ||y(t+s)||$. $*$ denotes the symmetric terms in a matrix.

2. Preliminaries

Consider the following linear large-scale complex system with delay

$$\begin{cases} E_i\dot{x}_i(t) &= A_ix_i(t) + \sum\limits_{j=1,j\neq i}^{N} A_{ij}x_j(t-h_{ij}) + D_iw_i(t), \quad t\geq 0,\\ x_i(t) &= \varphi_i(t), \quad t\in[-h,0], \end{cases} \tag{1}$$

where $0<h_{ij}\leq h$; $i,j=1,2,...,N$; $x_i(t)\in\mathbb{R}^{n_i}$ is the state; $w_i(t)\in\mathbb{R}^{p_i}$ is the disturbance; E_i is a singular matrix, rank $E_i = r_i, i=1,2,...,N$; $A_i\in\mathbb{R}^{n_i\times n_i}, A_{ij}\in\mathbb{R}^{n_i\times n_j}, D_i\in\mathbb{R}^{n_i\times p_i}$ are constant

matrices of appropriate dimensions; $\varphi_i(.) \in C([-h,0];\mathbb{R}^{n_i})$ is the initial delay function; the disturbance $w_i(t)$ satisfies the following condition

$$\exists d > 0 : \max_{i=\overline{1,N}} \left\{ \sup_{t>0} \{w_i^\top(t) w_i(t)\} \right\} \le d. \tag{2}$$

Let us set

$$R = diag\{R_1, \cdots, R_N\}, x^\top(t) = [x_1(t)^\top, \ldots, x_N(t)^\top],$$
$$\varphi^\top(t) = [\varphi_1(t)^\top, \ldots, \varphi_N(t)^\top].$$

Definition 1. (i) System (1) is regular if $det(sE_i - A_i), i = \overline{1,N}$, for some $s \in \mathbb{C}$, is not identical zero. (ii) System (1) is impulse-free if $deg(det(sE_i - A_i)) = r_i = rankE_i, i = \overline{1,N}$ for some $s \in \mathbb{C}$.

Definition 2. (Robust FTS) Given positive numbers c_1, c_2, T and a symmetric matrix $R > 0$, system (1) is robustly finite-time stable w.r.t. (c_1, c_2, T, R) if it is regular, impulse-free and the following relation holds:

$$\sup_{s\in[-h,0]} \{\varphi^\top(s) R \varphi(s)\} \le c_1 \quad \to \quad x^\top(t) R x(t) < c_2, \quad \forall t \in [0,T],$$

for all disturbances $w_i(t)$ satisfying (2).

We introduce some technical propositions, which will be used in the next section.

Proposition 1. *(Schur Complement Lemma [22]) For any matrices A, B, D, such that $B = B^T > 0, A = A^\top$, we have*

$$A + D^\top B^{-1} D < 0 \Leftrightarrow \begin{bmatrix} A & D^\top \\ D & -B \end{bmatrix} < 0.$$

Proposition 2. *Given constant matrices M_1, M_2, U, V, Q with appropriate dimensions satisfying $M_1 = M_1^\top, Q = Q^\top > 0$ and $V = V^\top > 0$, we have*

$$\begin{pmatrix} M_1 + U^\top V^{-1} U & M_2^\top \\ M_2 & -Q \end{pmatrix} < 0 \Leftrightarrow \begin{pmatrix} M_1 & M_2^\top & U^\top \\ M_2 & -Q & 0 \\ U & 0 & -V \end{pmatrix} < 0.$$

The proof of Proposition 2 is easily derived from the Schur complement lemma, Proposition 1.

3. Main result

In this section, we give delay-dependent sufficient conditions for the robust finite-time stability of the system (1). Since rank $E_i = r_i < n_i$, there are two non-singular (invertible) matrices M_i, G_i, such that $\begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix} = M_i E_i G_i$. Let us set

$$M_i A_i G_i = \begin{pmatrix} \bar{A}_{11}^i & \bar{A}_{12}^i \\ \bar{A}_{21}^i & \bar{A}_{22}^i \end{pmatrix}; M_i D_i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix} \text{ for all } \quad i, j = \overline{1,N};$$

$$G_i^\top P_i M_i^{-1} = \begin{pmatrix} P_{11}^i & P_{12}^i \\ P_{21}^i & P_{22}^i \end{pmatrix}; M_i A_{ij} G_j = \begin{pmatrix} A_{11}^{ij} & A_{12}^{ij} \\ A_{21}^{ij} & A_{22}^{ij} \end{pmatrix}.$$

Under coordinate transformation $y_i(t) = G_i^{-1} x_i(t) := [y_i^1(t), y_i^2(t)]^\top, y_i^1 \in \mathbb{R}^{r_i}, y_i^2 \in \mathbb{R}^{n_i - r_i}$ for all $i = \overline{1,N}$, the system (1) is reduced to the system

$$\begin{cases} \dot{y}_i^1(t) & = \bar{A}_{11}^i y_i^1(t) + \bar{A}_{12}^i y_i^2(t) + \sum\limits_{j=1, j\neq i}^{N} \Big[A_{11}^{ij} y_j^1(t-h_{ij}) + A_{12}^{ij} y_j^2(t-h_{ij}) \Big] + D_1^i \omega_i(t), \\ 0 & = \bar{A}_{21}^i y_i^1(t) + \bar{A}_{22}^i y_i^2(t) + \sum\limits_{j=1, j\neq i}^{N} \Big[A_{21}^{ij} y_j^1(t-h_{ij}) + A_{22}^{ij} y_j^2(t-h_{ij}) \Big] + D_2^i \omega_i(t), \\ y_i(t) & = G_i^{-1} \varphi_i(t), \quad t \in [-h, 0]. \end{cases} \tag{3}$$

Before presenting the main theorem, some following notations of several matrices variables are introduced for simplicity.

$\Delta_{i,i}^i = P_i A_i + A_i^\top P_i^\top + (N-1) Q_i, \quad \Delta_{i,j}^i = -A_i^\top U_i^\top + P_i A_{ij}, \forall j \neq i, j = \overline{1,N};$
$\Delta_{i,(N+1)}^i = A_i^\top Q_i, \quad \Delta_{j,j}^i = -Q_j - U_i A_{ij} - A_{ij}^\top U_i^\top, \forall j \neq i, j = \overline{1,N};$
$\Delta_{j,k}^i = -A_{ij}^\top U_i^\top - U_i A_{ik}, j \neq k, k = \overline{1,N}, \quad \Delta_{j,(N+1)}^i = A_{ij}^\top Q_i + U_i;$
$\Delta_{(N+1),(N+1)}^i = -2Q_i, \Delta_{(N+1+j),(N+1+j)}^i = -I;$
$\Delta_{j,(N+1+j)}^i = U_i D_i, j \neq i; j = \overline{1,N}, \Delta_{(N+1+i),(N+1+i)}^i = -I;$
$\Delta_{i,(N+1+i)}^i = P_i D_i, \Delta_{(2N+2),(2N+2)}^i = -I, \Delta_{(N+1),(2N+2)}^i = Q_i D_i;$
$\Delta_{jk}^i = 0,$ for all other cases, $\quad \alpha_1 = \min\limits_{i=\overline{1,N}} \{\lambda_{min}(P_{11}^i)\};$

$$\alpha_2 = \max_{i=\overline{1,N}} \left\{ \frac{\lambda_{max}(P_i E_i)}{\lambda_{min}(R_i)} \right\} + (N-1) h \max_{i=\overline{1,N}} \left\{ \frac{\lambda_{max}(Q_i)}{\lambda_{min}(R_i)} \right\};$$

$$\rho = \max_{i=\overline{1,N}} \left\{ \frac{\lambda_{max}([G_i^{-1}]^\top [G_i^{-1}])}{\lambda_{min}(R_i)} \right\}, g = \max_{i=\overline{1,N}} \left\{ \lambda_{max}(G_i^\top R_i G_i) \right\};$$

$$f(c_1) = \max \left\{ \frac{\alpha_2 c_1 + N(N+1) dT}{\alpha_1}; \rho c_1 \right\}, h^1 = \min_{i,j=\overline{1,N}; i\neq j} \{h_{ij}\};$$

$$\alpha_3 = 2N \max_{i=\overline{1,N}} \{\| [\bar{A}_{22}^i]^{-1} \bar{A}_{21}^i \|^2\} + 4N(N-1) \max_{i,j=\overline{1,N}; i\neq j} \{\| [\bar{A}_{22}^i]^{-1} A_{21}^{ij} \|^2\};$$

$$\alpha_4 = \sum_{l=0}^{[\frac{T}{h^1}]} \Big[2N(N-1) \max_{i,j=\overline{1,N}; i\neq j} \Big\{ \| [\bar{A}_{22}^i]^{-1} A_{22}^{ij} \|^2 \Big\} \Big]^l;$$

$$a_1 = 1 + \alpha_3 \alpha_4 + 2N(N-1) \alpha_4 \max_{i\neq j; i,j=\overline{1,N}} \Big\{ [\bar{A}_{22}^i]^{-1} A_{22}^{ij} \|^2 \Big\};$$

$$a_2 = 2dN^2 \alpha_4 \max_{i=\overline{1,N}} \{\| [\bar{A}_{22}^i]^{-1} D_2^i \|^2\}.$$

Theorem 1. *For given positive numbers T, c_1, c_2, $c_2 > c_1$, and symmetric matrices $0 < R_i \in \mathbb{R}^{n_i \times n_i}, i \in \overline{1,N}$, system (1) is robustly finite-time stable w.r.t. (c_1, c_2, T, R) if there exist non-singular matrices P_i, symmetric matrices $Q_i > 0, i = \overline{1,N}$, matrices U_i, and a number $\beta > 0$ satisfying the following conditions:*

$$P_i E_i = E_i^\top P_i^\top \geq 0; \tag{4}$$

$$\begin{pmatrix} \Delta^i_{1,1} & \Delta^i_{1,2} & \cdots & \Delta^i_{1,(N+2)} & \cdots & \Delta^i_{1,(2N+2)} \\ * & \Delta^i_{2,2} & \cdots & \Delta^i_{2,(N+2)} & \cdots & \Delta^i_{2,(2N+2)} \\ \cdot & \cdot & \cdots & \cdot\cdot & \cdots & \\ * & * & \cdots & * & \cdots & \Delta^i_{(2N+2),(2N+2)} \end{pmatrix} < 0, \quad i = \overline{1,N}; \tag{5}$$

$$a_1 f(c_1)e^{\beta T} + a_2 < \frac{c_2}{g}. \tag{6}$$

Proof. We first prove the regularity and impulse-free of the system. For this, we derive from the condition $P_iE_i = E_i^\top P_i^\top \geq 0$ that

$$\begin{aligned} G_i^\top P_iE_iG_i = G_i^\top P_iM_i^{-1}M_iE_iG_i &= \begin{pmatrix} P^i_{11} & 0_{r_i\times(n_i-r_i)} \\ P^i_{21} & 0_{(n_i-r_i)\times(n_i-r_i)} \end{pmatrix} \\ = G_i^\top E_i^\top P_i^\top G_i &= \begin{pmatrix} [P^i_{11}]^\top & [P^i_{21}]^\top \\ 0_{(n_i-r_i)\times r_i} & 0_{(n_i-r_i)\times(n_i-r_i)} \end{pmatrix} \geq 0, \end{aligned}$$

which gives $P^i_{21} = 0, P^i_{11} = [P^i_{11}]^\top \geq 0$. Due to the non-singularity of P_i, the matrix $G_i^\top P_iM_i^{-1} = \begin{pmatrix} P^i_{11} & P^i_{12} \\ 0_{(n_i-r_i)\times r_i} & P^i_{22} \end{pmatrix}$ is invertible such that $det(P^i_{11}) \neq 0$, hence $P^i_{11} > 0$. Using LMI (5) and applying Proposition 1, we get $\Delta^i_{i,i} < 0$, which implies

$$P_iA_i + A_i^\top P_i^\top + (N-1)Q_i < 0.$$

Since G_i are non-singular, we have

$$\begin{aligned} 0 &> G_i^\top \Big[A_i^\top P_i^\top + P_iA_i\Big] G_i \\ &= [G_i^\top A_i^\top M_i^\top][M_i^{-1}]^\top P_i^\top G_i + [G_i^\top P_iM_i^{-1}][M_iA_iG_i] \\ &= \begin{pmatrix} [\bar{A}^i_{11}]^\top & [\bar{A}^i_{21}]^\top \\ [\bar{A}^i_{12}]^\top & [\bar{A}^i_{22}]^\top \end{pmatrix} \begin{pmatrix} [P^i_{11}]^\top & 0 \\ [P^i_{12}]^\top & [P^i_{22}]^\top \end{pmatrix} + \begin{pmatrix} P^i_{11} & P^i_{12} \\ 0 & P^i_{22} \end{pmatrix} \begin{pmatrix} \bar{A}^i_{11} & \bar{A}^i_{21} \\ \bar{A}^i_{12} & \bar{A}^i_{22} \end{pmatrix} \\ &= \begin{pmatrix} C^i_{11} & C^i_{12} \\ C^i_{21} & [\bar{A}^i_{22}]^\top [P^i_{22}]^\top + P^i_{22}\bar{A}^i_{22} \end{pmatrix}. \end{aligned}$$

Applying Proposition 1 again, we obtain that $[\bar{A}^i_{22}]^\top [P^i_{22}]^\top + P^i_{22}\bar{A}^i_{22} < 0$, which gives $det(\bar{A}^i_{22}) \neq 0$ for all $i = \overline{1,N}$. Therefore, the system is regular and impulse-free. We are now in position to show the FTS of system (1). For this, we consider the following quadratic function

$$V(t,x_t) = \sum_{i=1}^{N} \Big[V_{i1}(t,x_t) + V_{i2}(t,x_t)\Big],$$

where

$$V_{i1}(t,x_t) = e^{\beta t} x_i(t)^\top P_iE_ix_i(t), V_{i2}(t,x_t) = e^{\beta t} \sum_{j=1,\ j\neq i}^{N} \int_{t-h_{ji}}^{t} x_i(s)^\top Q_ix_i(s)ds.$$

Taking the derivative of $V(t,x_t)$ in t, we have

$$\dot{V}_{i1}(t,x_t) = \beta V_{i1}(t,x_t) + e^{\beta t} x_i(t)^\top [A_i^\top P_i^\top + P_iA_i]x_i(t)$$

$$+e^{\beta t}2x_i(t)^\top P_i\sum_{j=1,j\neq i}^{N}A_{ij}x_j(t-h_{ij})+e^{\beta t}2x_i(t)^\top P_iD_iw_i(t)$$

$$\dot{V}_{i2}(t,x_t)=\beta V_{i2}(t,x_t)+e^{\beta t}(N-1)x_i(t)^\top Q_ix_i(t)-e^{\beta t}\sum_{j=1,j\neq i}^{N}x_i(t-h_{ji})^\top Q_ix_i(t-h_{ji}).$$

We use the following identities for evaluating $V(t,x_t)$:

$$2x_i^\top(t)P_iD_iw_i(t)\le x_i^\top(t)P_iD_iD_i^\top P_i^\top x_i(t)+w_i^\top(t)w_i(t),$$
$$-2e^{\beta t}\dot{x}_i^\top(t)E_i^\top Q_i[E_i\dot{x}_i(t)-A_ix_i(t)-\sum_{j=1,j\neq i}^{N}A_{ij}x_j(t-h_{ij})-D_iw_i(t)]=0,$$
$$2e^{\beta t}\sum_{j=1,j\neq i}^{N}x_j^\top(t-h_{ij})U_i[E_i\dot{x}_i(t)-A_ix_i(t)-\sum_{j=1,j\neq i}^{N}A_{ij}x_j(t-h_{ij})-D_iw_i(t)]=0,$$
$$2\dot{x}_i^\top(t)E_i^\top Q_iD_iw_i(t)\le\dot{x}_i^\top(t)E_i^\top Q_iD_iD_i^\top Q_iE_i\dot{x}_i(t)+w_i(t)^\top w_i(t),$$
$$-2\sum_{j=1,j\neq i}^{N}x_j^\top(t-h_{ij})U_iD_iw_i(t)\le\sum_{j=1,j\neq i}^{N}x_j^\top(t-h_{ij})U_iD_iD_i^\top U_i^\top x_j(t-h_{ij})+(N-1)w_i^\top(t)w_i(t),$$

and noting that

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}x_j(t-h_{ij})^\top x_j(t-h_{ij})=\sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N}x_i(t-h_{ji})^\top x_i(t-h_{ji}),$$

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}x_j^\top(t-h_{ij})Q_jx_j(t-h_{ij})=\sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N}x_i^\top(t-h_{ji})Q_ix_i(t-h_{ji}),$$

we get

$$\begin{aligned}\dot{V}(t,x_t)-\beta V(t,x_t)\le&e^{\beta t}\sum_{i=1}^{N}x_i(t)^\top\Big[P_iA_i+A_i^\top P_i^\top+P_iD_iD_i^\top P_i^\top+(N-1)Q_i\Big]x_i(t)\\&+e^{\beta t}\sum_{i=1}^{N}[E_i\dot{x}_i(t)]^\top[-2Q_i+Q_iD_iD_i^\top Q_i^\top][E_i\dot{x}_i(t)]\\&+e^{\beta t}\sum_{i=1}^{N}[E_i\dot{x}_i(t)]^\top[2Q_iA_i]x_i(t)+(N+1)e^{\beta t}\sum_{i=1}^{N}w_i^\top(t)w_i(t)\\&+e^{\beta t}\sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N}x_j^\top(t-h_{ij})[-Q_j+U_iD_iD_i^\top U_i^\top]x_j(t-h_{ij})\end{aligned}$$

$$
\begin{aligned}
&+e^{\beta t}\sum_{i=1}^{N}[E_i\dot{x}_i(t)]^{\top}[2Q_i\sum_{j=1,i\neq j}^{N}A_{ij}x_j(t-h_{ij})]+e^{\beta t}\sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N}[x_j(t-h_{ij})]^{\top}[2U_i][E_i\dot{x}_i(t)]\\
&+e^{\beta t}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}x_j^{\top}(t-h_{ij})[-2U_iA_i]x_i(t)+e^{\beta t}\sum_{i=1}^{N}x_i(t)^{\top}[2P_i]\sum_{j=1,j\neq i}^{N}A_{ij}x_j(t-h_{ij})\\
&-e^{\beta t}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}x_j^{\top}(t-h_{ij})[2U_i]\sum_{j=1,j\neq i}^{N}A_{ij}x_j(t-h_{ij})\\
\leq& e^{\beta t}\sum_{i=1}^{N}\xi_i(t)^{T}\Pi^{i}\xi_i(t)+(N+1)e^{\beta t}\sum_{i=1}^{N}w_i^{\top}(t)w_i(t),
\end{aligned}
$$

where $\xi_i(t)^{\top}=\Big[v_i^1,v_i^2,...,v_i^N,[E_i\dot{x}_i(t)]^{\top}\Big],v_i^j=x_j(t-h_{ij})^{\top},j\neq i;\ v_i^i=x_i(t)^{\top}$ and

$$
\Pi^{i}=\begin{pmatrix}\Pi^{i}_{1,1} & \Pi^{i}_{1,2} & \cdots & \Pi^{i}_{1,(N+1)}\\ * & \Pi^{i}_{2,2} & \cdots & \Pi^{i}_{2,(N+1)}\\ . & . & \cdots & .\\ * & * & \cdots & \Pi^{i}_{(N+1),(N+1)}\end{pmatrix};\quad \forall i=\overline{1,N};
$$

$\Pi^{i}_{i,i}=A_iP_i+P_i^{\top}A_i^{\top}+P_iD_iD_i^{\top}P_i^{\top}+(N-1)Q_i,\ \Pi^{i}_{i,j}=-A_i^{\top}U_i^{\top}+P_iA_{ij},\forall j\neq i,j=\overline{1,N},$
$\Pi^{i}_{i,(N+1)}=A_i^{\top}Q_i;\ \Pi^{i}_{j,j}=-Q_j+U_iD_iD_i^{\top}U_i^{\top}-U_iA_{ij}-A_{ij}^{\top}U_i^{\top},\forall j\neq i,j=\overline{1,N};$
$\Pi^{i}_{j,k}=-A_{ij}^{\top}U_i^{\top}-U_iA_{ik},j\neq k,k=\overline{1,N},\ \Pi^{i}_{j,(N+1)}=A_{ij}^{\top}Q_i+U_i,j\neq i,j=\overline{1,N},$
$\Pi^{i}_{(N+1),(N+1)}=-2Q_i+Q_iD_iD_i^{\top}Q_i^{\top}.$

Applying Proposition 2, the LMI condition (5) is equivalent to $\Pi^i<0,\ \forall i=\overline{1,N}$, and then we obtain that

$$
\dot{V}(t,x_t)-\beta V(t,x_t)\leq(N+1)e^{\beta t}\sum_{i=1}^{N}w_i^{\top}(t)w_i(t). \tag{7}
$$

Integrating both side of (7) from 0 to t, we have

$$
e^{-\beta t}V(t,x_t)-V(0,x_0)\leq\int_0^t(N+1)\sum_{i=1}^{N}w_i^{\top}(s)w_i(s)ds,
$$

hence

$$
V(t,x_t)\leq\Big(V(0,x_0)+N(N+1)dT\Big)e^{\beta T},\ t\in[0,T]. \tag{8}
$$

On the other hand, we see that

$$
\begin{aligned}
V(0,x_0) =& \sum_{i=1}^{N} x_i(0)^\top P_i E_i x_i(0) + \sum_{i=1}^{N} \sum_{j=1,\ j\neq i}^{N} \int_{-h_{ji}}^{0} x_i(s)^\top Q_i x_i(s) ds \\
\leq& \sum_{i=1}^{N} \lambda_{max}(P_i E_i) x_i^\top(0) x_i(0) + \sum_{i=1}^{N} (N-1) h \lambda_{max}(Q_i) \sup_{s\in[-h,0]} \varphi_i^\top(s)\varphi_i(s) \\
\leq& \sum_{i=1}^{N} \frac{\lambda_{max}(P_i E_i)}{\lambda_{min}(R_i)} x_i^\top(0) R_i x_i(0) + \sum_{i=1}^{N} (N-1) \frac{h\lambda_{max}(Q_i)}{\lambda_{min}(R_i)} \sup_{s\in[-h,0]} \varphi_i^\top(s) R_i \varphi_i(s) \\
\leq& \alpha_2 \sup_{s\in[-h,0]} \{\varphi(s)^\top R \varphi(s)\} = \alpha_2 c_1.
\end{aligned} \tag{9}
$$

Combining (8) and (9) gives

$$
V(t,x_t) \leq (\alpha_2 c_1 + N(N+1)Td)e^{\beta T}. \tag{10}
$$

Moreover, note that

$$
\begin{aligned}
x(t)^\top R x(t) =& \sum_{i=1}^{N} x_i(t)^\top R_i x_i(t) = \sum_{i=1}^{N} y_i(t)^\top G_i^\top R_i G_i y_i(t) \\
\leq& \sum_{i=1}^{N} \lambda_{max}(G_i^\top R_i G_i) y_i(t)^\top y_i(t) \\
\leq& \max_{i=\overline{1,N}} \{\lambda_{max}(G_i^\top R_i G_i)\}[\| Y_1(t) \|^2 + \| Y_2(t) \|^2],
\end{aligned} \tag{11}
$$

where

$$
\| y(t) \|^2 = \sum_{i=1}^{N} \Big[\| y_i^1(t) \|^2 + \| y_i^2(t) \|^2 \Big] := \| Y_1(t) \|^2 + \| Y_2(t) \|^2 .
$$

To prove the FTS of the system we need to estimate the states $\| Y_1(t) \|^2, \| Y_2(t) \|^2$ as follows. We first see from the view of $V(.)$ that

$$
\begin{aligned}
V(t,x_t) \geq& \sum_{i=1}^{N} x_i(t)^\top P_i E_i x_i(t) = \sum_{i=1}^{N} y_i(t)^\top G_i^\top P_i E_i G_i y_i(t) \\
\geq& \sum_{i=1}^{N} \lambda_{min}(P_{11}^i)[y_i^1(t)]^\top [y_i^1(t)] = \alpha_1 Y_1(t)^\top Y_1(t).
\end{aligned}
$$

Then, from (10) it follows that

$$
\begin{aligned}
\| Y_1(t) \|^2 = \sum_{i=1}^{N} \| y_i^1(t) \|^2 \leq& \frac{1}{\alpha_1} e^{\beta T} \Big[\alpha_2 c_1 + N(N+1)dT\Big] \\
\leq& [f(c_1)]e^{\beta T}, \quad \forall t \in [0,T].
\end{aligned} \tag{12}
$$

To estimate the second state $\| Y_2(t) \|$, we remark that

$$
\| y_i^2(t) \| \leq \| [\bar{A}_{22}^i]^{-1} \bar{A}_{21}^i \| \| y_i^1(t) \| + \sum_{j=1, j\neq i}^{N} \| [\bar{A}_{22}^i]^{-1} A_{21}^{ij} \| \| y_j^1(t - h_{ij}) \|
$$

$$+\sum_{j=1,j\neq i}^{N} \| [\bar{A}_{22}^{i}]^{-1}A_{22}^{ij} \|\| y_j^2(t-h_{ij})) \| + \| [\bar{A}_{22}^{i}]^{-1}D_2^i \|\| \omega_i(t) \|,$$

hence

$$\sum_{i=1}^{N} \| y_i^2(t) \|^2 \leq 2N\sum_{i=1}^{N} \| [\bar{A}_{22}^{i}]^{-1}\bar{A}_{21}^{i} \|^2\| y_i^1(t) \|^2 + 2N\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \| [\bar{A}_{22}^{i}]^{-1}A_{21}^{ij} \|^2\| y_j^1(t-h_{ij}) \|^2$$
$$+ 2N\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \| [\bar{A}_{22}^{i}]^{-1}A_{22}^{ij} \|^2\| y_j^2(t-h_{ij}) \|^2 + 2N\sum_{i=1}^{N} \| [\bar{A}_{22}^{i}]^{-1}D_2^i \|^2\| \omega_i(t) \|^2 .$$

For brevity, let us set $h_{ij} = h_j^i; \forall i,j = \overline{1,N}$ and

$$\begin{aligned}
\| p(t) \|^2 := {} & 2N\sum_{i=1}^{N} \| [\bar{A}_{22}^{i}]^{-1}\bar{A}_{21}^{i} \|^2\| y_i^1(t) \|^2 + 2N\sum_{i=1}^{N} \| [\bar{A}_{22}^{i}]^{-1}D_2^i \|^2\| \omega_i(t) \|^2 \\
& + 2N\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \| [\bar{A}_{22}^{i}]^{-1}A_{21}^{ij} \|^2\| y_j^1(t-h_j^i) \|^2 \\
& \leq 2N\max_{i=\overline{1,N}} \| [\bar{A}_{22}^{i}]^{-1}\bar{A}_{21}^{i} \|^2\,[f(c_1)]e^{\beta T} + 2dN^2\max_{i=\overline{1,N}} \| [\bar{A}_{22}^{i}]^{-1}D_2^i \|^2 \\
& + 2N\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \| [\bar{A}_{22}^{i}]^{-1}A_{21}^{ij} \|^2\| y_i^1(t-h_i^j) \|^2 .
\end{aligned}$$

Now, we estimate the value $\| p(t) \|^2$ on $[0,T]$ as follows. We consider two cases:

- $t \in [0,h^1]$ gives $(t-h_i^j) \in [-h,0]$, we obtain

$$\begin{aligned}
\| y_i^1(t-h_i^j) \|^2 \leq \| y_i(t-h_i^j) \|^2 &= \varphi_i(t-h_i^j)^\top [G_i^{-1}]^\top [G_i^{-1}]\varphi_i(t-h_i^j) \\
&\leq \frac{\lambda_{max}\{[G_i^{-1}]^\top [G_i^{-1}]\}}{\lambda_{min}(R_i)}\varphi_i(t-h_i^j)^\top R_i\varphi_i(t-h_i^j),
\end{aligned}$$

hence

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \| y_i^1(t-h_i^j) \|^2 \leq (N-1)\rho c_1.$$

- $t \in [h^1,T]$ gives $(t-h_i^j) \in [0,T]$, from (12), we have

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \| y_i^1(t-h_i^j) \|^2 \leq (N-1)f(c_1)e^{\beta T}.$$

Thus, for $t \in [0,T]$ we get

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \| y_i^1(t-h_i^j) \|^2 &\leq (N-1)\rho c_1 + (N-1)f(c_1)e^{\beta T} \\
&\leq 2(N-1)f(c_1)e^{\beta T}.
\end{aligned}$$

Then $\| p(t) \|^2 \leq \beta_1[f(c_1)]e^{\beta T} + \beta_2 := \gamma_1$, and hence

$$\sum_{i=1}^{N} \| y_i^2(t) \|^2 \leq \gamma_1 + \gamma_2 \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \| y_j^2(t-h_j^i) \|^2, t \in [0,T],$$

where $\gamma_2 = 2N \max_{i\neq j; i,j=\overline{1,N}} \{\| [\bar{A}_{22}^i]^{-1} A_{22}^{ij} \|^2\}$. Setting $h^1 = \min_{i\neq j, i,j=\overline{1,N}} \{h_{ij}\}$.

Further, we estimate the sum $\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \| y_j^2(t-h_j^i) \|^2 = \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \| y_i^2(t-h_i^j) \|^2$ on $[0,T]$ as follows.

a) For $t \in [0,h^1] \Rightarrow t - h_i^j \in [-h,0]$, we have

$$\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \| y_i^2(t-h_i^j) \|^2 \leq (N-1)\rho c_1 \leq (N-1)[f(c_1)]e^{\beta T},$$

then, $\forall t \in [0,h^1]$, we have

$$\sum_{i=1}^{N} \| y_i^2(t) \|^2 \leq \gamma_1 + (N-1)\gamma_2[f(c_1)]e^{\beta T} \leq \gamma_1 + b.$$

b) For $t \in [0,2h^1]$, we have

$$\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \| y_i^2(t-h_i^j) \|^2 \leq \begin{cases} (N-1)[f(c_1)]e^{\beta T}, & \text{if} \quad t-h_i^j \in [-h,0], \\ (N-1)(\gamma_1+b), & \text{if} \quad t-h_i^j \in [0,h^1]. \end{cases}.$$

$$\Rightarrow \sum_{i=1}^{N} \| y_i^2(t) \|^2 \leq \gamma_1 + (N-1)\gamma_2[f(c_1)]e^{\beta T} + \gamma_2(N-1)(\gamma_1+b)$$
$$\leq \Big[1 + \gamma_2(N-1)\Big](\gamma_1+b).$$

c) Similarly, for $t \in [0;(k+1)h^1] \cap [0,T]; kh^1 \leq T, k = 0,1,...$, we have

$$\sum_{i=1}^{N} \| y_i^2(t) \|^2 \leq \sum_{l=0}^{k} \Big[\gamma_2(N-1)\Big]^l (\gamma_1+b).$$

Therefore, for $t \in [0,T]$, we obtain that

$$\begin{aligned} \| Y_2(t) \|^2 = \sum_{i=1}^{N} \| y_i^2(t) \|^2 &\leq \max_{k=0,1,...,[\frac{T}{h^1}]} \sum_{l=0}^{k} \Big[\gamma_2(N-1)\Big]^l (\gamma_1+b) \\ &\leq \sum_{l=0}^{[\frac{T}{h^1}]} \Big[\gamma_2(N-1)\Big]^l (\gamma_1+b) := \alpha_4(\gamma_1+b). \end{aligned} \tag{13}$$

Then, from (11), (12) and (13) we finally obtain that

$$x(t)^\top Rx(t) \le \max_{i=\overline{1,N}} \lambda_{max}(G_i^\top R_i G_i)][\| Y_1(t) \|^2 + \| Y_2(t) \|^2]$$
$$\le \max_{i=\overline{1,N}} [\lambda_{max}(G_i^\top R_i G_i)][(f(c_1))e^{\beta T} + \alpha_4(\gamma_1 + b)] < c_2.$$

The proof is completed. $\square$

Remark 1. We see that (4) is not an LMIs that cannot be solved by MATLABs LMI Toolbox. Given P_i be a non-singular matrix, so by changing P_i to $P_i := E_i^\top \overline{P}_i + P_{ii}\overline{M}_i$, in which P_{ii} be any matrix, $\overline{P}_i$ be a symmetric positive define matrix, $\overline{M}_i$ be a matrix satisfying $\overline{M}_i E_i = 0$, then $P_i E_i = E_i^\top P_i^\top = E_i^\top \overline{P}_i E_i \ge 0$ for all $i = \overline{1,N}$. Hence, combining (4) with (5), we can easily solve these strict LMIs by using MATLABs LMI Toolbox.

Example 1. Consider large-scale system (1), where $N = 3$ and

$$\begin{cases} E_1\dot{x}_1(t) & = A_1x_1(t) + A_{12}x_2(t-h_{12}) + A_{13}x_3(t-h_{13}) + D_1w_1(t), \quad t \ge 0, \\ x_1(t) & = \varphi_1(t), \quad t \in [-h, 0], \end{cases}$$

and

$$\begin{cases} E_2\dot{x}_2(t) & = A_2x_2(t) + A_{21}x_1(t-h_{21}) + A_{23}x_3(t-h_{23}) + D_2w_2(t), \quad t \ge 0, \\ x_2(t) & = \varphi_2(t), \quad t \in [-h, 0], \end{cases}$$

and

$$\begin{cases} E_3\dot{x}_3(t) & = A_3x_3(t) + A_{31}x_1(t-h_{31}) + A_{32}x_2(t-h_{32}) + D_3w_3(t), \quad t \ge 0, \\ x_3(t) & = \varphi_3(t), \quad t \in [-h, 0], \end{cases}$$

where

$$E_1 = \begin{pmatrix} 0.5 & 0 \\ 0.5 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, A_{12} = \begin{pmatrix} -0.1 & 0 \\ -0.1 & 0.02 \end{pmatrix}, A_{13} = \begin{pmatrix} 0 & 0.02 \\ -0.01 & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 0 \\ 0.5 & 0.1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0.2 \\ -1 & -1 \end{pmatrix}, A_{21} = \begin{pmatrix} 0 & 0.01 \\ -0.1 & 0.1 \end{pmatrix}, A_{23} = \begin{pmatrix} 0 & 0 \\ 0.2 & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} -0.2 & 0 \\ 0.5 & 1 \end{pmatrix}, A_{31} = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.05 \end{pmatrix}, A_{32} = \begin{pmatrix} 0.02 & 0.03 \\ 0 & -0.02 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -0.4 \\ 0.01 \end{pmatrix}, D_2 = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, D_3 = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}; \quad G_1 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} -2 & 0.1 \\ 0 & -0.5 \end{pmatrix}; \quad M_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad G_3 = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix},$$

$$h_{12} = 0.7; \quad h_{21} = 0.8; \quad h_{13} = 1; \quad h_{23} = 1.2,$$

$$h_{31} = 1.1; \quad h_{32} = 0.5; \quad h = 1.2; \quad d = 0.1.$$

We give $\beta = 0.01, c_1 = 1, c_2 = 97, T = 100$ and

$$R = diag(R_1, R_2, R_3), R_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.6 \end{pmatrix}, R_2 = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.4 \end{pmatrix}, R_3 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}.$$

Using LMI Toolbox in Matlab, the LMI (5) with condition (4) is feasible with

$$P_1 = \begin{pmatrix} 2.0172 & 0.8718 \\ -2.8019 & 2.8019 \end{pmatrix}, U_1 = \begin{pmatrix} 0.0279 & 0.0044 \\ 0.0060 & 0.0088 \end{pmatrix}, Q_1 = \begin{pmatrix} 1.2000 & 0.1665 \\ 0.1665 & 1.2456 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 12.1473 & 2.5246 \\ -10.2157 & 0.5049 \end{pmatrix}, U_2 = \begin{pmatrix} -0.0041 & -0.2563 \\ -0.0054 & 0.0856 \end{pmatrix}, Q_2 = \begin{pmatrix} 1.4700 & 0.0422 \\ 0.0422 & 1.0954 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 3.3953 & -1.0800 \\ 0.0000 & -2.8098 \end{pmatrix}, U_3 = \begin{pmatrix} -0.0661 & -0.0201 \\ -0.0712 & -0.0347 \end{pmatrix}, Q_3 = \begin{pmatrix} 0.7484 & 0.6485 \\ 0.6485 & 1.4474 \end{pmatrix}.$$

Moreover, we can verify that the condition (6) holds with the following defined numbers

$$\alpha_1 = 5.0492,\ \alpha_2 = 30.503,\ \alpha_3 = 0,\ \alpha_4 = 1.1364,\ \rho = 16.0427,$$

$$g = 1.0472,\ f(c_1) = 29.8072,\ a_1 = 1.1364,\ a_2 = 0.3438.$$

Therefore, the system is robustly finite-time stable w.r.t. $(1, 97, 100, R)$.

4. Conclusions

In this paper, we have studied the problem of finite-time stability for linear singular large-scale complex systems with interconneted delays. By using the SVD approach and Lyapunov function method, new suficient conditions for the solvability of this problem have been obtained in terms of tractable LMIs. A numerical example is included to illustrate the effectiveness of our results.

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