The pluripolar parts of the Monge-Ampère measures of $\mathcal{F}$-plurisubharmonic functions

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Abstract

In this article we introduce the notion of the complex Monge-Ampère measures for a subclass of the class of unbounded $\mathcal{F}$-plurisubharmonic functions. This result generalizes classical results by Cegrell [4] who introduced the notion of complex Monge-Ampère operator for unbounded plurisubharmonic functions.

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1 Introduction and results

The complex Monge-Ampère operator is one of the important notions of pluripotential theory and complex variables. On the class of smooth plurisubharmonic functions, the operator is defined by

$$(dd^c u)^n = n!4^n det \left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) dV_{2n},$$

where dV_{2n} is the volume form in $\mathbb{C}^n$. Bedford and Taylor [2] introduced in 1982 the notion of complex Monge-Ampère operator as a positive Radon measure for the class of locally bounded plurisubharmonic functions. They showed that the operator is well-defined on this class, i.e. it is continuous on decreasing sequences of continuous plurisubharmonic functions. Later on, Cegrell [4] generalized in 2004 the notion to a subclass of unbounded plurisubharmonic functions.

The plurisubharmonic functions have been generalized in several directions. El Kadiri [7] introduced in 2003 the notion of $\mathcal{F}$-plurisubharmonic functions in an $\mathcal{F}$-open set which extends the notion of plurisubharmonic functions in a natural way. Later on, El Kadiri and Wiegerinck [10] gave in 2014 the notion of complex Monge-Ampère measure on the class of finite $\mathcal{F}$-plurisubharmonic functions as a non-negative measure. Recently, Trao, Viet and Hong [20] studied in 2017 the notion of Cegrell's classes for $\mathcal{F}$-plurisubharmonic functions. They introduced the following definition.

Definition 1.1. A bounded, connected, $\mathcal{F}$-open set Ω is called $\mathcal{F}$-hyperconvex if there exist a negative bounded plurisubharmonic function γ_Ω defined on a bounded hyperconvex domain $\Omega' \supset \Omega$ such that $\Omega = \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega$ is $\mathcal{F}$-plurisubharmonic in Ω.

Observe that a Euclidean open set is $\mathcal{F}$-hyperconvex if and only if it is hyperconvex. In [20], the authors give non obvious an example of $\mathcal{F}$-hyperconvex. We recall this in the following example.

Example 1.2. Let $\{a_j\}$ be a dense sequence in the close unit disk $\overline{D} \subset \mathbb{C}$. Theorem 4.14 in [1] deduce that there exist $h_j \in \mathcal{F}(D)$ such that $h_j(a_j) = -\infty$ and

$$dd^c h_j = \frac{1}{2^j} \delta_{a_j} \text{ in } D,$$

where δ_{a_j} denotes the Dirac measure at a_j. Then,

$$h = \sum_{k=1}^{+\infty} h_k$$

is a plurisubharmonic function in D. Assume that

$$\{h > -\frac{1}{2}\} \neq \emptyset.$$

Let Ω be a connected component of the $\mathcal{F}$-open set $D \cap \{h > -1\}$. Then, Ω is $\mathcal{F}$-hyperconvex.

The purpose of this paper is to study the pluripolar part of complex Monge-Ampère measures of $\mathcal{F}$-plurisubharmonic functions in bounded $\mathcal{F}$-hyperconvex domains. It is natural to expect a suitable definition which is an expansion of complex Monge-Ampère measure of unbounded plurisubharmonic functions. Notice that complex Monge-Ampère measure is $\mathcal{F}$-locally defined on the class of finite $\mathcal{F}$-plurisubharmonic functions. Hence, it is needed to find an another approach in studying this problem. The technique that we use in this article is taken from [6] (also see [12], [19]).

Some notions in our results can be found in Section 2. The first main result of this paper is as follows.

Theorem 1.3. *Let $\Omega \subset D \Subset \mathbb{C}^n$ be $\mathcal{F}$-hyperconvex domains. Assume that $a \geq 0$ and $u \in \mathcal{F}(\Omega)$. Then,*

$$\hat{u} := \sup\{\varphi \in \mathcal{F}\text{-}PSH^-(D) : \varphi \leq u + a \text{ on } \Omega\} \in \mathcal{F}(D).$$

and

$$\int_D (dd^c \max(\hat{u}, -1))^n \leq \int_\Omega (dd^c \max(u, -1)^n.$$

Moreover,

$$NP(dd^c\hat{u})^n \leq 1_{\Omega\cap\{\hat{u}=u+a\}} NP(dd^c u)^n \text{ on } QB(\Omega).$$

Now, assume that the domain D is hyperconvex, Cegrell [4] showed that $1_D(dd^c\hat{u})^n$ is a Radon measure on D. The second main result of this paper give the information on pluripolar part of the complex Monge-Ampère measures $(dd^c\hat{u})^n$.

Theorem 1.4. *Let D, Ω, a, u, $\hat{u}$ be as in Theorem 1.3. Assume that D is hyperconvex. Then, the Borel measure in $\mathbb{C}^n$ which is defined by*

$$P(dd^c u)^n := 1_{D\cap\{\hat{u}=-\infty\}}(dd^c\hat{u})^n$$

does not depend on a and D. Moreover,

$$(dd^c\hat{u})^n = 0 \text{ on } D\backslash\Omega$$

and

$$\int_D (dd^c\hat{u})^n \leq \int_{\Omega\cap\{u<-a\}} (dd^c \max(u, -a-t))^n, \quad \forall t > 0.$$

Observe that the above result tells us that $P(dd^c u)^n$ vanishes outside of Ω. To provide some more of its properties, we need the following definition from [20].

Definition 1.5. We say that a bounded $\mathcal{F}$-hyperconvex domain Ω has the $\mathcal{F}$-approximation property if there exists an increasing sequence of negative plurisubharmonic functions ρ_j defined on bounded hyperconvex domains Ω_j such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and $\rho_j \nearrow \rho \in \mathcal{E}_0(\Omega)$ a.e. on Ω as $j \nearrow +\infty$.

Example 1.6. In Example 1.2, the $\mathcal{F}$-hyperconvex domain Ω has the $\mathcal{F}$-approximation property.

Finally, we shall prove the following result.

Theorem 1.7. *Let $\Omega \Subset \mathbb{C}^n$ be a $\mathcal{F}$-hyperconvex domain and let $u \in \mathcal{F}(\Omega)$. Assume that Ω has the $\mathcal{F}$-approximation property. Then,*

$$\int_\Omega P(dd^c u)^n = \int_\Omega (dd^c \max(u, -1))^n - \int_\Omega NP(dd^c u)^n.$$

The remainder of this paper is organized as follows. In Section 2, we recall some notions of (plurifine) pluripotential theory and give the proof of Theorem 1.3. Section 3 is devoted to prove Theorem 1.4 and Theorem 1.7.

2 Subextensions of $\mathcal{F}$-plurisubharmonic functions

The elements of pluripotential theory (plurifine potential theory) that will be used in this paper can be found in [1]-[21]. Let $\Omega \subset \mathbb{C}^n$ be an $\mathcal{F}$-open set. Denote by $QB(\mathbb{C}^n)$ the measurable space on $\mathbb{C}^n$ generated by the Borel sets and the pluripolar subsets of $\mathbb{C}^n$. Let $QB(\Omega)$ be the trace of $QB(\mathbb{C}^n)$ on Ω. Firstly, we recall the notion of $\mathcal{F}$-plurisubharmonic functions from [7].

Definition 2.1. A function $u : \Omega \to [-\infty, +\infty)$ is called $\mathcal{F}$-plurisubharmonic (briefly, $u \in \mathcal{F}\text{-}PSH(\Omega)$) if u is $\mathcal{F}$-upper semicontinuous and for every complex line l in $\mathbb{C}^n$, the restriction of u to any $\mathcal{F}$-component of the finely open subset $l \cap \Omega$ of l is either finely subharmonic or $\equiv -\infty$.

Definition 2.2. Let $\Omega \subset \mathbb{C}^n$ be an $\mathcal{F}$-open set and $u \in \mathcal{F}\text{-}PSH(\Omega)$.

(i) If u is finite then there exist a pluripolar set $E \subset \Omega$, a sequence of $\mathcal{F}$-open subsets $\{O_j\}$ and plurisubharmonic functions f_j, g_j defined in Euclidean neighborhoods of $\overline{O}_j$ such that $\Omega = E \cup \bigcup_{j=1}^{\infty} O_j$ and $u = f_j - g_j$ on O_j. The Monge-Ampère measure $(dd^c u)^n$ on $QB(\Omega)$ is defined by

$$\int_A (dd^c u)^n := \sum_{j=1}^{\infty} \int_{A \cap (O_j \setminus \bigcup_{k=1}^{j-1} O_k)} (dd^c(f_j - g_j))^n, \quad A \in QB(\Omega).$$

(ii) The non-polar part $NP(dd^c u)^n$ is defined by

$$\int_A NP(dd^c u)^n = \lim_{j \to +\infty} \int_A (dd^c \max(u, -j))^n, \; A \in QB(\Omega).$$

This definition is independent of O_j, f_j, g_j and refer to [10].

Next, we recall the definition of the Cegrell's classes for $\mathcal{F}$-plurisubharmonic functions from [20].

Definition 2.3. Let $\Omega \Subset \mathbb{C}^n$ be a bounded $\mathcal{F}$-hyperconvex domain, Ω' and γ_Ω as in Definition 1.1, and let $\mathcal{F}\text{-}PSH^-(\Omega)$ be the set of negative $\mathcal{F}$-plurisubharmonic functions in Ω. We set

$$\begin{aligned}\mathcal{E}_0(\Omega) := \{u \in \mathcal{F}\text{-}PSH^-(\Omega) \cap L^\infty(\Omega) : \int_\Omega (dd^c u)^n < +\infty \\ \text{and } \forall \varepsilon > 0, \; \exists \delta > 0, \; \overline{\Omega \cap \{u < -\varepsilon\}} \subset \{\gamma_\Omega > -1 + \delta\}\}\end{aligned}$$

and

$$\mathcal{F}(\Omega) := \{u \in \mathcal{F}\text{-}PSH^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni u_j \searrow u, \sup_{j \geq 1} \int_\Omega (dd^c u_j)^n < +\infty\}.$$

Now, we prove the following proposition which generalizes Lemma 3.1 in [16].

Proposition 2.4. *Let $\Omega \subset D \Subset \mathbb{C}^n$ be $\mathcal{F}$-hyperconvex domains. Assume that $a > 0$, $u \in \mathcal{E}_0(\Omega)$ and define*

$$w := \sup\{\varphi \in \mathcal{F}\text{-}PSH^-(D) : \varphi \leq u + a \text{ on } \Omega\}.$$

Then $w \in \mathcal{E}_0(D)$ and satisfies

$$(dd^c w)^n \leq 1_{\Omega \cap \{w = u + a\}} (dd^c u)^n \text{ in } QB(D).$$

Proof. Without loss of generality we can assume that $-\frac{1}{2} \leq u < 0$ in Ω, and hence, $-\frac{1}{2} \leq w < 0$ in D. Because Ω is an $\mathcal{F}$-hyperconvex domain so there exists a bounded hyperconvex domain $\Omega' \supset \Omega$ in $\mathbb{C}^n$ and $\gamma_\Omega \in PSH^-(\Omega') \cap L^\infty(\Omega')$ such that $\Omega = \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega \in \mathcal{F}\text{-}PSH(\Omega)$. Since $u \in \mathcal{E}_0(\Omega)$, we can find $\delta > 0$ such that

$$\overline{\Omega \cap \{u < -a\}} \subset \{\gamma_\Omega > -1 + 2\delta\} \subset D. \tag{2.1}$$

Firstly, we claim that $w \in \mathcal{E}_0(D)$. Indeed, since $K := \overline{\Omega \cap \{u < -a\}}$ is a compact set, we can find $\rho \in \mathcal{E}_0(D)$ such that

$$\rho = -1 \text{ on } K.$$

This implies that $\rho \leq u+a$ on Ω, and hence, $w \geq \rho$. It follows that $w \in \mathcal{E}_0(D)$. This proves the claim.

Next, we claim that

$$(dd^c w)^n \leq (dd^c u)^n \text{ on } \Omega \cap \{w = u + a\}. \tag{2.2}$$

Indeed, let j be an integer number with $ja > 1$. Since $-a + \frac{1}{j} < 0$ and Ω' is a Euclidean open set, at $\Omega' \cap \partial_{\mathcal{F}} \Omega$ we have $u + \dfrac{1}{\delta}\gamma_\Omega < -\dfrac{1}{\delta}$, hence Proposition 2.3 in [9] and Proposition 2.14 in [8] tells us that

$$f := \begin{cases} \max(-\frac{1}{\delta}, u + \frac{1}{\delta}\gamma_\Omega) & \text{in } \Omega \\ -\frac{1}{\delta} & \text{in } \Omega' \backslash \Omega \end{cases}$$

and

$$f_j := \begin{cases} \max(-\frac{1}{\delta}, \max(u, w - a + \frac{1}{j}) + \frac{1}{\delta}\gamma_\Omega) & \text{in } \Omega \\ -\frac{1}{\delta} & \text{in } \Omega' \backslash \Omega \end{cases}$$

are plurisubharmonic in Ω'. Since $f_j \searrow f$ on Ω' and $u = f - \frac{1}{\delta}\gamma_\Omega$, $\max(u, w - a + \frac{1}{j}) = f_j - \frac{1}{\delta}\gamma_\Omega$ in $\{\gamma_\Omega > -1 + \delta\}$, we infer by Theorem 3.2 in [3] that

$$\lim_{j \to +\infty} \int_\Omega \chi(dd^c \max(u, w - a + \frac{1}{j}))^n = \int_\Omega \chi(dd^c u)^n \tag{2.3}$$

for every bounded $\mathcal{F}$-continuous function χ with compact support on $\{\gamma_\Omega > -1 + \delta\}$. Let $K \subset \Omega \cap \{w = u + a\}$ be a compact set. Since $w \leq 0$ on Ω, we obtain by (2.1) that

$$\begin{aligned} \Omega \cap \{w = u + a\} &\subset \Omega \cap \{u \leq -a\} \\ &\subset \{\gamma_\Omega > -1 + 2\delta\}. \end{aligned}$$

Hence, there exists a decreasing sequence of bounded $\mathcal{F}$-continuous functions $\{\chi_k\}$ with compact support on $\{\gamma_\Omega > -1+\delta\}$ such that $\chi_k \searrow 1_K$ as $k \nearrow +\infty$. Using Theorem 4.8 in [10] we obtain by (2.3) that

$$\begin{aligned} \int_K (dd^c w)^n &\leq \lim_{j \to +\infty} \int_\Omega \chi_k (dd^c \max(u, w - a + \frac{1}{j}))^n \\ &= \int_\Omega \chi_k (dd^c u)^n, \ \forall k \geq 1. \end{aligned}$$

Letting $k \to +\infty$, we arrive that

$$\int_K (dd^c w)^n \leq \int_K (dd^c u)^n.$$

Therefore, $(dd^c w)^n \leq (dd^c u)^n$ on $\Omega \cap \{w = u + a\}$. This proves the claim.

Now, since u is $\mathcal{F}$-continuous on Ω and $\lim_{z \ni \Omega \to \partial_{\mathcal{F}} \Omega} u = 0$, it follows that the function

$$h := \begin{cases} u + a & \text{on } \Omega \\ a & \text{in } D \backslash \Omega \end{cases}$$

is $\mathcal{F}$-continuous on D, and hence,

$$U := D \cap \{w < h\} \text{ is } \mathcal{F}\text{-open set.}$$

Let $z \in U$ and let $b \in \mathbb{R}$ be such that $w(z) < b < h(z)$. Let V be a connected component of the $\mathcal{F}$-open set $D \cap \{w < b\} \cap \{h > b\}$ which contains the point z.

We claim that w is $\mathcal{F}$-maximal in V. Indeed, let G be a bounded $\mathcal{F}$-open set in $\mathbb{C}^n$ with $\overline{G} \subset V$ and let $v \in \mathcal{F}\text{-}PSH(G)$ such that v is bounded from above on G, extends $\mathcal{F}$-upper semicontinuously to $\overline{G}^{\mathcal{F}}$ and

$$v \leq w \text{ on } \partial_{\mathcal{F}} G.$$

Proposition 2.3 in [9] states that the function

$$\varphi := \begin{cases} \max(w, v) & \text{on } G \\ w & \text{on } D \backslash G \end{cases}$$

is $\mathcal{F}$-plurisubharmonic in D. Because $\overline{G} \subset V \subset D \cap \{w < b\}$, we infer by Theorem 2.3 in [8] that

$$\varphi < b \text{ on } \overline{G},$$

and hence, $\varphi \leq h$ in D. This implies that $\varphi = w$ in D. Thus, $v \leq w$ in G, and therefore, w is $\mathcal{F}$-maximal in V. This proves the claim. Thus, w is $\mathcal{F}$-locally $\mathcal{F}$-maximal in U, and therefore, we deduce by Theorem 1 in [17] that

$$(dd^c w)^n = 0 \text{ on } QB(U).$$

Combining this with (2.2) we conclude that

$$(dd^c w)^n \leq 1_{\Omega \cap \{w = u + a\}} (dd^c u)^n \text{ in } QB(D).$$

The proof is complete. □

We are now able to give the proof of theorem 1.3.

Proof of Theorem 1.3. (i) Let $\{u_j\} \subset \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ in Ω as $j \nearrow +\infty$ and

$$\sup_{j\geq 1} \int_\Omega (dd^c u_j)^n < +\infty. \tag{2.4}$$

We define

$$\hat{u}_j := \sup\{\varphi \in \mathcal{F}\text{-}PSH^-(D) : \varphi \leq u_j + a + \frac{1}{j} \text{ on } \Omega\}.$$

It is clear that $\{\hat{u}_j\}$ is decreasing and converges to $\hat{u}$ in D. Proposition 2.4 tells us that $\hat{u}_j \in \mathcal{E}_0(D)$ and

$$(dd^c\hat{u}_j)^n \leqslant 1_{\Omega\cap\{\hat{u}_j = u_j + a + \frac{1}{j}\}}(dd^c u_j)^n \text{ in } QB(D). \tag{2.5}$$

This implies that

$$\sup_{j\geq 1} \int_D (dd^c\hat{u}_j)^n \leq \sup_{j\geq 1} \int_\Omega (dd^c u_j)^n.$$

Combining this with (2.4), we conclude by Proposition 2.4 in [15] that $\hat{u} \in \mathcal{F}(D)$ and

$$\begin{aligned}\int_D (dd^c \max(\hat{u}, -1))^n &= \sup_{j\geq 1} \int_D (dd^c\hat{u}_j)^n \\ &\leq \sup_{j\geq 1} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c \max(u, -1)^n.\end{aligned}$$

(ii) Thanks to Theorem 4.5 in [9] we infer by (2.5) that

$$(dd^c\hat{u})^n \leq (dd^c u)^n \text{ on } \Omega \cap \{\hat{u} > -\infty\}.$$

Fix $b, c \in \mathbb{R}$ with $b > c$. Set

$$U_j := \Omega \cap \{\hat{u} > c\} \cap \{u > b - a\} \cap \{\hat{u}_j < b\}.$$

Since $u_j \searrow u$ and $\hat{u}_j \searrow \hat{u}$ on Ω, we have

$$\begin{aligned} U_j &\subset \Omega \cap \{c < \hat{u}_j < b < u_j + a\} \\ &\subset \Omega \cap \{\hat{u}_j < u_j + a\}. \end{aligned}$$

Combining this with (2.5), we obtain

$$(dd^c\hat{u}_j)^n = 0 \text{ on } U_k, \forall j \geq k$$

because $U_j \supset U_k$, $\forall j \geq k$. Moreover, since $\hat{u}$ is bounded on U_k, Theorem 4.5 in [9] states that

$$(dd^c \hat{u})^n = 0 \text{ on } U_k, \ \forall k \geq 1.$$

This implies that

$$(dd^c \hat{u})^n = 0 \text{ on } \Omega \cap \{-\infty < \hat{u} < u + a\}.$$

Therefore,

$$NP(dd^c \hat{u})^n \leq 1_{\Omega \cap \{\hat{u} = u + a\}} NP(dd^c u)^n \text{ on } QB(\Omega).$$

This proves the theorem. □

3 Pluripolar part of complex Monge-Ampère measures

Firstly, we prove the following lemma.

Lemma 3.1. *Let $\Omega \Subset \mathbb{C}^n$ be a $\mathcal{F}$-hyperconvex domain and let $\{u_j\} \subset \mathcal{E}_0(\Omega)$ be a decreasing sequence such that*

$$\sup_{j \geq 1} \int_\Omega (dd^c u_j)^n < +\infty.$$

Then, for every $\varepsilon > 0$, there exists $v \in \mathcal{E}_0(\Omega)$ such that

$$\sup_{j \geq 1} \int_{\{v > -1\}} (dd^c u_j)^n < \varepsilon.$$

Proof. Fix $\varepsilon > 0$. By the hypotheses we can find $j_0 \in \mathbb{N}$ such that

$$\sup_{j \geq 1} \int_\Omega (dd^c u_j)^n \leq \int_\Omega (dd^c u_{j_0})^n + \frac{\varepsilon}{3}. \tag{3.1}$$

By the hypotheses we can find $\varphi \in \mathcal{E}_0(\Omega)$ such that $-1 \leq \varphi \leq 0$ in Ω and

$$\max_{1 \leq j \leq j_0} \int_\Omega (1 + \varphi)(dd^c u_j)^n < \frac{\varepsilon}{3}. \tag{3.2}$$

Since $\{u_j\}$ is decreasing, Proposition 3.4 in [20] tells us that

$$\int_\Omega (-\varphi)(dd^c u_j)^n \geq \int_\Omega (-\varphi)(dd^c u_{j_0})^n, \ \forall j \geq j_0.$$

Hence, we deduce by (3.1) that

$$\int_\Omega (1 + \varphi)(dd^c u_j)^n \leq \int_\Omega (1 + \varphi)(dd^c u_{j_0})^n + \frac{\varepsilon}{3}, \ \forall j \geq j_0.$$

Combining this with (3.2) we obtain that

$$\int_\Omega (1+\varphi)(dd^c u_j)^n \leq \frac{2\varepsilon}{3},\ \forall j \geq 1. \tag{3.3}$$

We put $v = 5\varphi$, it is easy to see that $v \in \mathcal{E}_0(\Omega)$ and

$$\frac{4(1+\varphi)}{3} > 1 \text{ on } \{\varphi > -\frac{1}{5}\}.$$

Hence, we conclude by (3.3) that

$$\begin{aligned}\sup_{j\geq 1} \int_{\{v>-1\}} (dd^c u_j)^n &= \sup_{j\geq 1} \int_{\{\varphi > -\frac{1}{5}\}} (dd^c u_j)^n \\ &\leq \frac{4}{3} \sup_{j\geq 1} \int_\Omega (1+\varphi)(dd^c u_j)^n < \varepsilon.\end{aligned}$$

This proves the lemma. □

Next, we can give the proof of theorem 1.4.

Proof of Theorem 1.4. (i) Let U be a bounded hyperconvex domain such that $D \Subset U$. Put

$$\varphi_1 := \sup\{\varphi \in PSH^-(U) : \varphi \leq u \text{ on } \Omega\}$$

and

$$\varphi_2 := \sup\{\varphi \in PSH^-(U) : \varphi \leq u + a \text{ on } \Omega\}.$$

It is easy to see that

$$\varphi_1 \leq \varphi_2 \leq \varphi_1 + a \text{ in } U. \tag{3.4}$$

Since $D \subset U$, it follows that

$$\varphi_2 \leq \hat{u} \text{ on } D, \tag{3.5}$$

and hence,

$$\varphi_2 = \sup\{\varphi \in PSH^-(U) : \varphi \leq \hat{u} \text{ on } D\}.$$

Lemma 4.5 in [19] tells us that

$$(dd^c\varphi_2)^n \leq 1_D (dd^c\hat{u})^n \text{ on } U. \tag{3.6}$$

Moreover, using Lemma 4.4 in [1] we infer by (3.5) that

$$(dd^c\hat{u})^n \leq (dd^c\varphi_2)^n \text{ on } D \cap \{\hat{u} = -\infty\}. \tag{3.7}$$

Note that the measure $1_{U\cap\{\hat{u}>-\infty\}}(dd^c\hat{u})^n$ vanishes on pluripolar sets of U. Hence,

$$(dd^c\hat{u})^n = 0 \text{ on } D\cap\{\varphi_2=-\infty\}\cap\{\hat{u}>-\infty\}.$$

Combining this with (3.5), (3.6) and (3.7) we arrive at

$$1_{D\cap\{\hat{u}=-\infty\}}(dd^c\hat{u})^n = 1_{U\cap\{\varphi_2=-\infty\}}(dd^c\varphi_2)^n. \tag{3.8}$$

On the other hand, using (3.4) and Lemma 4.4 in [1] we obtain that

$$1_{U\cap\{\varphi_1=-\infty\}}(dd^c\varphi_1)^n = 1_{U\cap\{\varphi_2=-\infty\}}(dd^c\varphi_2)^n.$$

Therefore, we conclude by (3.8) that

$$1_{D\cap\{\hat{u}=-\infty\}}(dd^c\hat{u})^n = 1_{U\cap\{\varphi_1=-\infty\}}(dd^c\varphi_1)^n.$$

This implies that the measure $1_{D\cap\{\hat{u}=-\infty\}}(dd^c\hat{u})^n$ is independent on a and D because the function φ_1 does not depend on a.

(ii) Let $\{u_j\}\subset\mathcal{E}_0(\Omega)$ such that $u_j\searrow u$ in Ω as $j\nearrow+\infty$ and

$$\sup_{j\geq 1}\int_\Omega (dd^c u_j)^n < +\infty.$$

Proposition 2.4 states that the functions

$$\hat{u}_j := \sup\{\varphi\in\mathcal{F}\text{-}PSH^-(D) : \varphi\leq u_j + a + \frac{1}{j} \text{ on } \Omega\}$$

belong to $\mathcal{E}_0(D)$ and

$$(dd^c\hat{u}_j)^n \leq 1_\Omega(dd^c u_j)^n \text{ in } QB(D). \tag{3.9}$$

Let ε be a positive real number. By Lemma 3.1 we can find $v\in\mathcal{E}_0(\Omega)$ such that

$$\sup_{j\geq 1}\int_{\Omega\cap\{v>-1\}} (dd^c u_j)^n < \varepsilon. \tag{3.10}$$

On the other hand, since $\hat{u}_j\searrow\hat{u}$ in D, we infer by Theorem in [5] that

$$(dd^c\hat{u}_j)^n \to (dd^c\hat{u})^n \text{ in } D.$$

Combining this with (3.9) and (3.10) we arrive at

$$\begin{aligned}\int_{D\setminus\Omega}(dd^c\hat{u})^n &\leq \int_{D\setminus\overline{\Omega\cap\{v\leq-1\}}}(dd^c\hat{u})^n\\ &\leq \liminf_{j\to+\infty}\int_{D\setminus\overline{\Omega\cap\{v\leq-1\}}}(dd^c\hat{u}_j)^n\end{aligned}$$

$$\leq \liminf_{j\to+\infty} \int_{\Omega\setminus\overline{\Omega\cap\{v\leq -1\}}} (dd^c u_j)^n$$
$$\leq \liminf_{j\to+\infty} \int_{\Omega\cap\{v>-1\}} (dd^c u_j)^n < \varepsilon.$$

This implies that

$$\int_{D\setminus\Omega} (dd^c \hat{u})^n = 0.$$

(iii) Let $j \geq a$ be an integer number and define

$$\hat{u}_j := \sup\{\varphi \in \mathcal{F}\text{-}PSH^-(D) : \varphi \leq \max(u, -j) + a \text{ on } \Omega\}.$$

In fact, Proposition 2.4 states that $\hat{u}_j \in \mathcal{F}(D) \cap L^\infty(D)$ and

$$(dd^c \hat{u}_j)^n \leq 1_{\Omega\cap\{\hat{u}_j=\max(u,-j)+a\}} (dd^c \max(u, -j))^n \text{ on } QB(D).$$

Combining this with (ii) we infer that

$$\int_D (dd^c \hat{u}_j)^n \leq \int_{\Omega\cap\{\hat{u}_j=\max(u,-j)+a\}} (dd^c \max(u, -j))^n$$
$$\leq \int_{\Omega\cap\{u<-a\}} (dd^c \max(u, -j))^n.$$

This implies that

$$\int_D (dd^c \hat{u})^n \leq \limsup_{j\to+\infty} \int_{\Omega\cap\{u<-a\}} (dd^c \max(u, -j))^n, \tag{3.11}$$

because $\hat{u}_j \searrow \hat{u}$ on D. On the other hand, since $(dd^c \max(u, -j))^n = (dd^c \max(u, -a-t))^n$ on $\Omega \cap \{u \geq -a\}$ we infer by Proposition 2.4. in [15] that

$$\int_{\Omega\cap\{u<-a\}} (dd^c \max(u, -j))^n$$
$$= \int_\Omega (dd^c \max(u, -j))^n - \int_{\Omega\cap\{u\geq -a\}} (dd^c \max(u, -j))^n$$
$$= \int_\Omega (dd^c \max(u, -a - t))^n - \int_{\Omega\cap\{u\geq -a\}} (dd^c \max(u, -a - t))^n$$
$$= \int_{\Omega\cap\{u<-a\}} (dd^c \max(u, -a - t))^n.$$

Combining this with (3.11) we conclude that

$$\int_D (dd^c \hat{u})^n \leq \int_{\Omega\cap\{u<-a\}} (dd^c \max(u, -a - t))^n.$$

The proof is complete. □

Finally, we give the proof of Theorem 1.7.

Proof of Theorem 1.7. Since Ω has the $\mathcal{F}$-approximation property, by Theorem 1.2 in [15] we can find a decreasing sequence of bounded hyperconvex domains $\{\Omega_j\}$ and a sequence of functions $\varphi_j \in PSH^-(\Omega_j)$ such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and $\varphi_j \nearrow u$ a.e. on Ω. Theorem 1.3 and Theorem 1.4 tell us that

$$u_j := \sup\{\varphi \in PSH^-(\Omega_j) : \varphi \leq u \text{ on } \Omega\}$$

belongs to $\mathcal{F}(\Omega_j)$ and satisfies

$$\int_{\Omega_j} (dd^c \max(u_j, -1))^n \leq \int_{\Omega} (dd^c \max(u, -1))^n \tag{3.12}$$

$$(dd^c u_j)^n \leq 1_\Omega (dd^c u)^n \text{ on } QB(\Omega_j \cap \{u_j > -\infty\}). \tag{3.13}$$

Observe that $u_j \geq \varphi_j$ on Ω_j, and hence, $u_j \nearrow u$ a.e. on Ω. Therefore, we deduce by Proposition 2.7 in [20] that

$$\int_{\Omega} (dd^c \max(u, -1))^n \leq \liminf_{j \to +\infty} \int_{\Omega} (dd^c \max(u_j, -1))^n$$

and

$$\int_{\Omega} NP(dd^c u)^n \leq \liminf_{j \to +\infty} \int_{\Omega} NP(dd^c u_j)^n.$$

Combining this with (3.12) and (3.13) we arrive at

$$\int_{\Omega} (dd^c \max(u, -1))^n = \lim_{j \to +\infty} \int_{\Omega_j} (dd^c \max(u_j, -1))^n = \lim_{j \to +\infty} \int_{\Omega_j} (dd^c u_j)^n$$

and

$$\int_{\Omega} NP(dd^c u)^n = \lim_{j \to +\infty} \int_{\Omega_j} NP(dd^c u_j)^n.$$

This implies that

$$\begin{aligned}\int_{\Omega} (dd^c \max(u, -1))^n &= \lim_{j \to +\infty} \left[\int_{\Omega_j} NP(dd^c u_j)^n + \int_{\Omega_j \cap \{u_j = -\infty\}} (dd^c u_j)^n \right] \\ &= \int_{\Omega} NP(dd^c u)^n + \int_{\Omega} P(dd^c u)^n\end{aligned}$$

because $P(dd^c u)^n = 1_{\Omega_j \cap \{u_j = -\infty\}} (dd^c u_j)^n$. This proves the theorem. □

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