ON THE TOPOLOGY OF GEOMETRIC AND RATIONAL ORBITS FOR ALGEBRAIC GROUP ACTIONS OVER VALUED FIELDS, II

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ABSTRACT. The aim of this paper is twofold. Firtsly, we show that if G is a smooth nilpotent group acting on an algebraic variety V defined over an admissible valued field k and $v \in V(k)$, then the Zariski closedness of the geometric orbit $G(\overline{k}).v$ in $V(\overline{k})$ is equivalent to the Hausdorff closedness of the rational orbit $G(k).v$ in $V(k)$. Secondly, we provide some calculations for the fact that there is a bijection between the set of $G(k)$-orbits and the kernel of the natural map in flat cohomology. These results are obtained in the framework of studying the rational orbits.

INTRODUCTION

Let G be a linear algebraic group acting on an affine variety V, all defined over a field k, and let $v \in V(k)$ be a rational point. When k is a valued field (e.g., $k = \mathbb{Q}_p, \mathbb{F}_q((T))$, or henselian valued fields), we may endow $G(k)$ and $V(k)$ with the v-adic topology induced from that of the base field k. As indicated in [7], we are interested in the relationship between the Zariski closedness of geometric orbit $G(\overline{k}).v$ in $V(\overline{k})$ (for short, we say the Zariski closedness of $G(\overline{k}).v$) and the Hausdorff (or relative) closedness (closed in the topology induced from that of k) of the rational orbit $G(k).v$ in $V(k)$ (for short, we say the Hausdorff (or relative) closedness of $G(k).v$). We refer to [6], [7], and references therein for the discusssion on several recent relationships between these two types of closedness. For example, in [7, Theorem 4.1], we show a relative version of the Kostant-Rosenlicht Theorem saying that if $G = U$ is a unipotent group, then $G(k).v$ is always relative closed in $V(k)$ when $k = (k, v)$ is an admissible valued field. Furthermore, if G is commutative, then the Zariski closedness of $G(\overline{k}).v$ implies the relative closedness of $G(k).v$ (see [7, Theorem 4.3]). One of the main results of this note (Theorem 2.4(a)) shows that this fact is also true if we consider the action of any smooth nilpotent algebraic group G on a separated scheme of finite type defined over an admissible valued field $k = (k, v)$. Besides, if we restrict to the case that V is an affine variety, two above types of closedness for geometric orbits and rational orbits are equivalent (see Theorem 2.4(b)). It is worth noticing that by considering an example due to Gabber, Gille, and Moret-Bailly, Theorem 2.4(a) is false if we replace the nilpotency of G by that of solvability (Remarks 2.5(b)).

On the other hand, we note that, the main issue appears when we consider the relationship between two above types of closedness is that, generally, $G(k).v \subsetneq (G.v)(k)$. Furthermore, if the stabilizer G_v is a smooth subgroup scheme, then there is a bijection between the set of $G(k)$-orbits in $(G.v)(k)$ and the kernel $\mathrm{Ker}\left(\mathrm{H}^1(k, G_v) \to \mathrm{H}^1(k, G)\right)$ of the natural map between Galois cohomologies (see e.g. [2, p. 36]). In fact, this bijection was considered carefully to obtain some landmark results in the arithmetic of hyperelliptic curves (see e.g. [1, 2]). In Section 3, we are interested in studying the case that the stabilizer is not necessarily smooth. Here we need to replace the Galois cohomology by the flat cohomology and lead to an analogue version (see Proposition 3.1) of the above bijection. Some calculations for this fact

Date: August 20, 2021.

2010 *Mathematics Subject Classification.* Primary: 14L30 ; Secondary: 14L24, 20G15, 20G10, 11E72.

Key words and phrases. Geometric orbits, Rational orbits, Flat cohomology, Gabber's condition.

Supported by Grant number 101.04-2020.09 from NAFOSTED and The Vietnam Institute for Advanced Study in Mathematics (VIASM).

are provided. Section 3 can be considered as a complement to [2, Remark 2, p. 36] and [7, Example 2.4].

1. Preliminaries

1.1. **Some basic notions and Gabber's condition.** We use standard notions of algebraic group schemes and their actions on schemes as in [16]. An algebraic variety over k is a scheme of finite type over k that is both separated and geometrically reduced. So the set of geometric points $V(\overline{k})$ is a variety defined over k in the sense of [4]. All group schemes that we consider are affine algebraic group schemes, i.e. affine group schemes of finite type. We emphasize that for every affine group scheme G over k, its base change $G_{\overline{k}}$ is the $\overline{k}$-group scheme which is represented by $k[G]\otimes_k \overline{k}$. If G is geometrically reduced, i.e. the base change $G_{\overline{k}}$ is reduced, we say that G is a smooth k-group scheme. Similar to the above, if G is a smooth k-group scheme, then $G(\overline{k})$ is also a linear algebraic group over k in the sense of [4]. We denote by $\mathbb{G}_m$ (resp., $\mathbb{G}_a$) the k-group represented by $k[T, T^{-1}]$ (resp., $k[T]$). Thoughout the forthcoming, unless otherwise specified, algebraic groups are not assumed to be smooth.

Let G be an algebraic group acting on a separated algebraic scheme V. For $v \in V(k)$, we denote $G.v$ the orbit of v. This is the image of the orbit map $\mu_v : G \to G.v$, $g \mapsto g.v$ equipped with the structure of a reduced subscheme of V (see [16, Chap. 7, p. 139]). Furthermore, the geometric (resp., rational) orbit is denoted by $G(\overline{k}).v$ (resp., $G(k).v$). The stabilizer G_v is defined by the fibre of the orbit map $\mu_v : G \to G.v$ over v. So G_v is an algebraic subgroup (not necessarily smooth) of G. When G is smooth, by virtue of [16, Prop. 7.4], the orbit $G.v$ is stable under the action of G and the orbit map $\mu_v : G \to G.v$ is faithfully flat. It follows from [16, Prop. 7.11] that $G.v$ is the quotient G/G_v of G by the stabilizer G_v. Futhermore, since the smoothness of G, the induced canonical map $G/G_v \to V$ is an immersion (see [16, Prop. 1.65(c)]). So in the case that G is smooth, the above notion of orbit $G.v$ is compatible with the one in the sense of [9, Chapter III, Section 3]. More precisely, in [9, Chapter III, Section 3], the orbit is defined as the sheaf-image of G via the orbit map, which is also isomorphic to the quotient G/G_v (see [9, Prop. 1.6, p.325]).

Remark 1.1. We refer to [7, Section 1.2] for a presentation on the notion of fppf G-torsors. Note that in [16, Defn. 2.66], this notion is simply called G-torsor. As we have seen in the preceding paragraph, if G is smooth, then $G/G_v \cong G.v$. This implies that the orbit map $\mu_v : G \to G.v \cong G/G_v$, $g \mapsto g.v$ is also an fppf G_v-torsor (see e.g., [16, Corollary 5.27]).

To study the relations between the Zariski closedness of $G(\overline{k}).v$ in $V(\overline{k})$ and the Hausdorff closedness of $G(k).v$ in $V(k)$, we need to discuss the Gabber condition, sometimes called the $(*)$-condition, for the stabilizer. First, we recall the notion of largest smooth subgroup schemes. From [8, C.11], we note that for any affine k-group scheme G, there is a unique *largest smooth k-subgroup*, denoted by G^+. The Gabber condition is defined as follows.

Definition 1.2. (Gabber's condition, see [10], [11, Defn. 2.4.3]) We say that a k-group G satisfies the $(*)$-condition (or the Gabber condition) if all $\overline{k}$-tori of $G_{\overline{k}}$ are contained in $(G^+)_{\overline{k}}$.

Since an arbitrary unipotent k-group contains no nontrivial k-tori, all unipotent groups satisfy the $(*)$-condition. The conclusion is also true for all commutative groups (see a detailed proof in Section 2).

We say that a valued field $k = (k, v)$ is called admissible if it is henselian and the completion $\hat{k}$ of k is separable over k, e.g. any local field (since such fields are complete) or the algebraic closure of $\mathbb{F}_p(t)$ in $\mathbb{F}_p((t))$. The following theorem is an important result due to Gabber, Gille and Moret-Bailly.

Theorem 1.3. (Cf. [11, 1.2, 1.4], [13, Theorem 7.2.1]) *Suppose that $k = (k, v)$ is an admissible valued field, and let G be an affine algebraic k-group scheme. Let $f : X \to Y$ be a (fppf) G-torsor where X, Y are separated k-schemes of finite type. If G satisfies the $(*)$condition, then $I = f(X(k))$ is clopen (closed and open) in $Y(k)_{Top}$.*

Here, the notation $Y(k)_{Top}$ is used to denote the set of k-points equipped with the topology induced from k, as mentioned above. It is worth noticing that in [11], the authors mean an algebraic k-variety to be a separated k-scheme of finite type.

1.2. **Nilpotent Groups.** We refer to [9, Chap. 4, Section 4], [16, Sections 6.f, 16.f] for expositions of nilpotent group schemes. Namely, we say that an algebraic group scheme is nilpotent if it has a central normal series, i.e. a normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

such that each quotient G_i/G_{i+1} is contained in the centre of G/G_{i+1}. In particular, the last nontrivial term in such series is contained in the (schematic) centre $Z(G)$ of G. We know that a connected k-group G is nilpotent if and only if the abstract group $G(\overline{k})$ is also nilpotent (see [9, Corollary IV.4.1.5]). Furthermore, we have

Proposition 1.4. (see [9, Chap. 4, IV.4.1.4]) *Any k-group G is nilpotent if and only if $G_{\overline{k}}$ is also nilpotent.*

We recall that the class of nilpotent groups contains the class of commutative groups, as well as the class of unipotent groups, whereas it is contained in the class of solvable groups (the groups can be constructed from commutative algebraic groups by successive extensions). The following result provides some important properties of nilpotent group schemes.

Theorem 1.5. (cf. [16, Theorem 16.47]) *Let G be a connected nilpotent group scheme over k. Then*

(a) *The semisimple part $Z(G)_s$ of the centre $Z(G)$ is the largest subgroup of multiplicative type of G and $G/Z(G)_s$ is unipotent.*
(b) *If G is smooth, then $Z(G)_s$ is a torus.*

Since each torus is of multiplicative type, it follows directly that

Corollary 1.6. *Any subtorus T over k of a given connected nilpotent k-group scheme G is contained in the centre $Z(G)$.*

In the next section, we will show that Gabber's condition is also valid for all nilpotent groups (see Proposition 2.2).

1.3. **Flat cohomology.** To study the problem of parametrization the set of rational points $G(\overline{k}).v \cap V(k)$, we need to take into account the non-smoothness of the stabilizer G_v. For nonsmooth group schemes, the Galois cohomology is not good enough so we need in the sequel several facts concerning flat cohomology of affine algebraic groups (Cf. [15], [17], [18]). Let R be a commutative ring with identity, S a faithfully flat R-algebra, and let G be an algebraic group scheme over R. We consider two natural maps $d^0, d^1 : S \to S \otimes_R S$ given by

$$d^0 : S \to S \otimes_R S,\ a \mapsto 1 \otimes a,$$
$$d^1 : S \to S \otimes_R S,\ a \mapsto a \otimes 1.$$

The functoriality of G implies two induced maps $d^0, d^1 : G(S) \to G(S \otimes_R S)$. Similarly, we still use the above notation for the following natural maps

$$d^0 : S \otimes_R S \to S \otimes_R S \otimes_R S,\ a \otimes b \mapsto 1 \otimes a \otimes b,$$
$$d^1 : S \otimes_R S \to S \otimes_R S \otimes_R S,\ a \otimes b \mapsto a \otimes 1 \otimes b,$$
$$d^2 : S \otimes_R S \to S \otimes_R S \otimes_R S,\ a \otimes b \mapsto a \otimes b \otimes 1,$$

and they induce the corresponding maps $d^0, d^1, d^2 : G\left(S \otimes_R S\right) \to G\left(S \otimes_R S \otimes_R S\right)$.

Definition 1.7. We define the 0-th flat cohomology of G with respect to S/R is

$$\mathrm{H}^0_{flat}(S/R, G) = \{\lambda \in G(S) \mid d^0\lambda = d^1\lambda\} (= G(R)).$$

The first flat cohomology of G with respect to S/R is defined by

$$\begin{aligned}\mathrm{H}^1_{flat}(S/R, G) &= Z^1_{flat}(S/R, G)/\sim, \\ &= \{\varphi \in G\left(S \otimes_R S \mid d^1\varphi = (d^0\varphi)(d^2\varphi)\right)\}/\sim .\end{aligned}$$

Here the equivalence relation $\sim$ is given by $\varphi \sim \psi$ if and only if $\psi = (d^0\lambda)\varphi(d^1\lambda)^{-1}$ for some $\lambda \in G(S)$. For the extension $\overline{k}/k$, the flat cohomology $\mathrm{H}^1_{flat}(\overline{k}/k, G)$ is often simply denoted by $\mathrm{H}^1_{flat}(k, G)$.

Remark 1.8. Let $R = k$, $S = \overline{k}$. Then $\overline{k} \otimes_k \overline{k}$ is a k-algebra with the k-structure given by the commutative diagram

$$\begin{array}{ccc} k & \hookrightarrow & \overline{k} \\ \downarrow & & \downarrow d^0 \\ \overline{k} & \xrightarrow{d^1} & \overline{k} \otimes_k \overline{k} \end{array}$$

Since the tensor product is taken over k, we have $d^0\Big|_k = d^1\Big|_k$. Thus for $i = 1, 2$ the induced maps

$$G(k) \overset{id}{\hookrightarrow} G(\overline{k}) \overset{d^i}{\to} G(\overline{k} \otimes \overline{k})$$

satisfy that $d^0(g_{\overline{k}}) = d^1(g_{\overline{k}}) = g_{\overline{k}\otimes\overline{k}}$ for any k-point $g \in G(k)$. Here we denote $g_{\overline{k}}$ (resp., $g_{\overline{k}\otimes\overline{k}}$) the image of $g \in G(k)$ in $G(\overline{k})$ (resp., in $G(\overline{k} \otimes \overline{k})$). In particular, $G(k) \subseteq \mathrm{H}^0_{flat}(\overline{k}/k, G)$. Furthermore, since any algebraic group scheme G over a field k is faithfully flat and quasi-compact, the 0-th cohomology is exactly $G(k)$.

2. Actions of nilpotent groups

2.1. **Gabber's condition for nilpotent group schemes.** The following fact is well-known (see e.g. [11, Lemme 2.4.5]). Nevertheless, for the sake of completeness and self-containedness, we write down the proof in more detail.

Proposition 2.1. *Let G be a commutative algebraic group over k. Then G satisfies the Gabber condition.*

Proof. Since G is commutative, the base change $G_{\overline{k}}$ contains a unique maximal $\overline{k}$-torus T. On the other hand, if we choose a maximal k-torus T_1 of G, by [16, Lemma C.4.4], its base change $(T_1)_{\overline{k}}$ is also a maximal torus of $G_{\overline{k}}$. Since the uniqueness of the maximal torus T, we have $T = (T_1)_{\overline{k}}$. Moreover, any torus is geometrically reduced, the torus T_1 is smooth, it means that $T_1 \subseteq G^+$. This implies that $T = (T_1)_{\overline{k}}$ is a torus of $(G^+)_{\overline{k}}$. Therefore, every commutative group scheme G satisfies the Gabber condition. □

Now we show that the above result also holds true for nilpotent group schemes.

Proposition 2.2. *Let G be a nilpotent group scheme over k. Then G satisfies the Gabber condition.*

Proof. Without loss of generality, we may assume that G is connected. Indeed, assume that the desired conclusion holds for all connected nilpotent groups. We consider a torus T which is contained in $G_{\overline{k}}$. Since T is connected, it is contained in the connected component $\left(G_{\overline{k}}\right)^0$. Since the formation of connected component commutes with the base change (see [16, Propositions 1.34, 2.37(c)]), we have

$$T \leq \left(G_{\overline{k}}\right)^0 = \left(G^0\right)_{\overline{k}}.$$

Hence, by virtue of the assumption that the conclusion holds for all connected nilpotent groups, we have $T \leq \left(\left(G^0\right)^+\right)_{\overline{k}}$. This implies that $T \leq (G^+)_{\overline{k}}$ as desired.

For a connected nilpotent group scheme G, we consider the following diagram

(1)
$$\begin{array}{ccc} (G^+)_{\overline{k}} & \hookrightarrow & G_{\overline{k}} \\ | & & | \\ G^+ & \hookrightarrow & G \end{array}$$

Here G^+ is the largest smooth subgroup scheme over k, and $(G^+)_{\overline{k}}$ is its base change. Let T be an arbitrary $\overline{k}$-torus in $G_{\overline{k}}$. By Proposition 1.4, $G_{\overline{k}}$ is still connected and nilpotent. Thus, by Corollary 1.6, T is contained in the centre $Z(G_{\overline{k}})$. On the other hand, the formation of centralizer commutes with extensions of the base field (see e.g. [16, pages 34, 379]), particularly, $Z(G_{\overline{k}}) = Z(G)_{\overline{k}}$. Therefore, we have
$$T \leq Z(G_{\overline{k}}) = Z(G)_{\overline{k}}.$$
Applying Proposition 2.1 for the commutative group scheme $Z(G)$ yields $T \leq (Z(G)^+)_{\overline{k}}$. This implies that $T \leq (G^+)_{\overline{k}}$. Therefore, G satisfies the Gabber condition. □

Remark 2.3. We note that Proposition 2.2 is false if we only assume that G is solvable. An example is implied from the work of Gabber, Gille and Moret-Bailly (see [11, Example 7.1, p. 605]). More precisely, we choose
$$G = \{x, y) \mid x^p + (y-1)^p T = 0\} \leq \mathbb{G}_a \rtimes \mathbb{G}_m.$$
Then by the discusssion in [7, Remark 4.4], G^+ is trivial, but the base change $G_{\overline{k}}$ contains its reduced part $\left(G_{\overline{k}}\right)_{red}$ which is a nontrivial torus $\left(G_{\overline{k}}\right)_{red} \overset{\overline{k}}{\cong} \mathbb{G}_m$. Hence, G is solvable and does not satisfy the Gabber condition.

2.2. **Main result.**

Theorem 2.4. *Let $k = (k, v)$ be an admissible valued field, and let G be a smooth nilpotent group acting on a separated k-scheme of finite type V, all defined over k. Assume that $v \in V(k)$. Then*

(a) *The geometric orbit $G(\overline{k}).v$ is Zariski closed in $V(\overline{k})$ implies that $G(k).v$ is Hausdorff closed in $V(k)$.*
(b) *If V is an affine k-variety, then $G(\overline{k}).v$ is Zariski closed in $V(\overline{k})$ if and only if $G(k).v$ is Hausdorff closed in $V(k)$.*

Proof. (a): Since G is a nilpotent group, its subgroup G_v is also nilpotent (see [9, Chapter 4, Section 4.1.2]). It follows from Proposition 2.2 that G_v satisfies the Gabber condition. On the other hand, by Remark 1.1, the orbit map $\mu_v : G \to V$, $g \mapsto g.v$, is a G_v-torsor. Since G is an algebraic group, it is also separated of finite type (see [16, Prop. 1.22]). Combining with Theorem 1.3 yields $G(k).v$ is closed in $(G.v)(k) = (G.v)(\overline{k}) \cap V(k) = G(\overline{k}).v \cap V(k)$. By the assumption that $G(\overline{k}).v$ is Zariski closed in $V(\overline{k})$, the set $(G.v)(k) = G(\overline{k}).v \cap V(k)$ is closed in $V(k)$. The desired implication follows.

If we assume further that G is connected, we may argue directly as follows. First, we have the following diagram

(2)
$$\begin{array}{ccccc} (G_v^+)_{\overline{k}} & \hookrightarrow & (G_v)_{\overline{k}} & \hookrightarrow & G_{\overline{k}} \\ | & & | & & | \\ G_v^+ & \hookrightarrow & G_v & \hookrightarrow & G \end{array}$$

Let T be a $\overline{k}$-torus in $(G_v)_{\overline{k}}$. Since G is connected and nilpotent, so is its base change $G_{\overline{k}}$ by Proposition 1.4. Hence, it follows from Corollary 1.6 that $T \leq Z(G_{\overline{k}})$. Therefore, it implies that $T \leq Z\left((G_v)_{\overline{k}}\right)$. On the other hand, the formation of centralizer commutes with extensions of the base field, we have
$$T \subseteq Z\left((G_v)_{\overline{k}}\right) = (Z(G_v))_{\overline{k}}.$$

Applying Proposition 2.1 for the commutative group scheme $Z(G_v)$, we have

$$T \leq \big(Z(G_v)^+\big)_{\overline{k}}.$$

So $T \leq (G_v^+)_{\overline{k}}$. Then G_v satisfies the Gabber condition. So combining Theorem 1.3 and Remarks 1.1, it completes this implication.

(b): ($\Rightarrow$:) This is a special case of part (a).

($\Leftarrow$:) If the action of G on V is linear, i.e. V is a vector space over k and the action is given by a k-linear representation $\rho : G \to \mathrm{GL}(V)$, we use a result due to Birkes (see [3, Prop. 9.10]) and argue as in [5, Section 3] to get the implication. In fact, this part is true for any valued field which is not necessarily admissible. For a general action of G on an affine k-variety V, by using the exsistence of equivariant k-embeddings (see [4, Proposition 1.12]), we may take a closed k-embedding $\varphi : V \hookrightarrow E$ and a k-morphism $\mu : G \to \mathrm{GL}(E)$ such that $\varphi(g.v) = \mu(g)\,(\varphi(v))$ for all $g \in G(\overline{k})$ and $v \in V(\overline{k})$. Now we assume the contrary that $G(\overline{k}).v$ is not closed in $V(\overline{k})$. Then $Y := \overline{G(\overline{k}).v} \setminus G(\overline{k}).v \neq \emptyset$ is a closed $G(\overline{k})$-stable subset of $\overline{G(\overline{k}).v}$. Since $\varphi : V \hookrightarrow E$ is a closed k-embedding, $\varphi(Y) = \overline{G(\overline{k}).\varphi(v)} \setminus G(\overline{k}).\varphi(v) \neq \emptyset$. By Birkes' result (see [3, Prop. 9.10]), there exist $y_1 \in \varphi(Y) \cap E(k)$ and a k-cocharacter $\lambda : \mathbb{G}_m \to G$ such that $\lambda(\alpha).\varphi(v) \to y_1$ as $\alpha \to 0$. Put $y := \varphi^{-1}(y_1) \in Y$. By this choice, since $G(k).v$ is Hausdorff closed, $y \in G(k).v \subseteq G(\overline{k}).v$. This is a contradiction since $Y \cap G(\overline{k}).v = \emptyset$. It implies that $G(\overline{k}).v$ is closed as required. □

Remarks 2.5. (a) Theorem 2.4(b) is an extension of an our previous result in [5, Theorem, p. 1062] saying that the assertion holds for groups of multiplicative type defined over local function fields. Furthermore, [7, Remark 4.4(a)] shows that Theorem 2.4 does not hold if we only assume that k is henselian.

(b) Theorem 2.4 also provides an answer for the question proposed in [7, Remark 4.4(c)]. This result is optimal in the sense that if we replace the nilpotency by the solvability of G, then each implication in Theorem 2.4(b) should be false (see [7, Example 2.4, parts (1), (2)] and [5, Example 5.2]).

(c) As indicated in the proof, Theorem 2.4(a) also holds for the case that G_v is nilpotent (G is not necessarily nilpotent). Naturally, when the stabilizer G_v is nilpotent, we may consider the converse statement of Theorem 2.4(a). Nevertheless, this statement is not true by reconsidering the Example [5, Example 5.2]. Here we choose

$$G = B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = 1 \right\}$$

acting on $\mathbb{A}^3$ via the representation

$$\rho : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 0 & ad & 2bd \\ 0 & 0 & d^2 \end{pmatrix},$$

and let $v = (1, 0, 1)$. Then

$$\begin{aligned} G.v &= \{(a^2 + b^2, 2bd, d^2) \mid ad = 1\}, \\ &= \{(x, y, z) \mid 4xz = y^2 + 4\} \setminus \{z = 0\}. \end{aligned}$$

By a direct computation, we see that $G(\overline{k}).v$ is not Zariski closed in $V(\overline{k})$ but $G(k).v$ is Hausdorff closed in $V(k)$ if $k = \mathbb{R}$ or $k = \mathbb{Q}_p$ with $p = 2$ or $p \equiv 3 \pmod 4$. In this case, the stabilizer

$$G_v = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a^2 = 1, ad = 1 \right\} \cong \mathbb{Z}/2$$

is a commutative (in particular, nilpotent) smooth group scheme. Hence, this implies that the converse statement of Theorem 2.4(a) is not true. Furthermore, since the Hausdorff closedness of $G(k).v$, this example also shows that $G(k).v$ contains all limits $\lim_{\alpha\to 0}\lambda(\alpha).v$ corresponding to k-cocharacters $\lambda:\mathbb{G}_m\to G$. Thus for the closed $G(\overline{k})$-stable subset $Y:=\overline{G(\overline{k}).v}\setminus G(\overline{k}).v$ of $V(\overline{k})$, the set $Y(k)$ does not contain any limit $\lim_{\alpha\to 0}\lambda(\alpha).v$ of the above form. It means that Property (A) mentioned in Birkes' paper [3] is also false if we only assume the nilpotency of G_v.

3. Decomposition of the rational points of $G(\overline{k}).v$ into $G(k)$-orbits

Let G be a smooth algebraic group acting on a variety V defined over k. In this section, we introduce an analogue version of the bijection between the set of $G(k)$-orbits in $(G.v)(k)$ and the kernel $\operatorname{Ker}\left(\mathrm{H}^1(k,G_v)\to\mathrm{H}^1(k,G)\right)$ (see [2, Proposition 1, p. 36]) when G_v is not necessarily smooth.

Proposition 3.1. *There exists a bijection between the set of $G(k)$-orbits that lies in $G(\overline{k}).v\cap V(k)$ and the kernel of the map*

$$\gamma:\mathrm{H}^1_{flat}(k,G_v)\to\mathrm{H}^1_{flat}(k,G) \tag{3}$$

in flat cohomology.

Remark 3.2. In fact, Proposition 3.1 is a special case of [14, Chapter III, 3.2.3, p. 160] as well as [12, Prop. 2.4.3]. So we do not present the proof. Nevertheless, in the rest of the paper, we provide some concrete calculations illustrating this fact.

Now we go back to the example due to Gabber, Gille, and Moret-Bailly (see [11, Example 7.1, p. 605]) and investigate the behavior of the above bijection in this situation. The following computation is also a complement to [2, Remark, p. 35] and [7, Example 2.4].

Example 3.3. Let $k=\mathbb{F}_q((T))$ be the imperfect local function field with T-adic topology, i.e. the basis of open neighbourhoods of 0 is given by the sequence of ideals $\{\langle T^n\rangle\}_{n=1}^{\infty}$. Assume that $G=\mathbb{G}_a\rtimes\mathbb{G}_m$ is the semidirect product with the operation $(x,y)\cdot(x',y')=(x+yx',yy')$, and let G act on the affine line $V=\mathbb{A}^1$ by $(x,y)\cdot z=x^p+y^pz$.

(1) Assume that $v=T\in k\setminus k^p$ is a rational point of $\mathbb{A}^1$. Then we have the following that:
   (a) $G(\overline{k}).v=\overline{k}^p+(\overline{k}^{\times p})v=\overline{k}$ and then $G(\overline{k}).v\cap V(k)=k$.
   (b) Now we consider the kernel of the natural map $\mathrm{H}^1_{flat}(k,G_v)\xrightarrow{\gamma}\mathrm{H}^1_{flat}(k,G)$. From the short exact sequence
   $$1\to\mathbb{G}_a\to G=\mathbb{G}_a\rtimes\mathbb{G}_m\to\mathbb{G}_m\to 1,$$
   and from the triviality of $\mathrm{H}^1(k,\mathbb{G}_a)$ and $\mathrm{H}^1(k,\mathbb{G}_m)$ (the Hilbert 90 Theorem), we have $\mathrm{H}^1_{flat}(k,G)=1$. This implies the bijection between the set of $G(k)$-orbits that lies in $G(\overline{k}).v\cap V(k)$ and the first cohomology set $\mathrm{H}^1_{flat}(k,G_v)$. Next, the stabilizer of v is given by
   $$\begin{aligned}G_v&=\{(x,y)\in\mathbb{G}_a\rtimes\mathbb{G}_m\mid x^p+y^pT=T\}\\&=\{(x,y)\in\mathbb{G}_a\rtimes\mathbb{G}_m\mid x^p+(y-1)^pT=0\}.\end{aligned}\tag{4}$$
   Then the reduced stabilizer $(G_v)_{\overline{k},\mathrm{red}}=\{(x,y)\in\mathbb{G}_a\rtimes\mathbb{G}_m\mid x+(y-1)T^{\frac{1}{p}}=0\}$ is not defined over $k=\mathbb{F}_q((T))$. Let $\mathcal{C}$ be a representative set of the disjoint union
   $$\begin{aligned}G(\overline{k}).v\cap V(k)=k&=\bigsqcup_{w\in\mathcal{C}}G(k).w\\&=\bigsqcup_{w\in\mathcal{C}}\left(k^p+k^{\times p}w\right).\end{aligned}$$
   The set $\mathcal{C}$ is infinite since we may choose a subfamily of representatives as follows
   $$T,T+T^2,T+T^2+T^3,\ldots,T+T^2+\cdots+T^{p-1},T+T^2+\cdots+T^{p-1}+T^{p+1},\ldots,$$

or in other words, the set of sum $\sum_{i\in\{1,\dots,N\},p\nmid i} T^i$ with $N = 1, 2, \dots$. Consequently, the set $\mathrm{H}^1_{flat}(k, G_v)$ is infinite.

(2) If we choose $v' = 0$, then $(x, y).v' = x^p + y^p v' = x^p$. Thus $G(\overline{k}).v' = \overline{k}^p = \overline{k}$, and $G(\overline{k}).v \cap V(k) = k$. Besides, the stabilizer

$$G_{v'} = \{(x, y) \in \mathbb{G}_a \rtimes \mathbb{G}_m \mid x^p = 0\}$$
$$= \alpha_p \rtimes \mathbb{G}_m.$$

It follows from the short exact sequence

$$1 \to \alpha_p \to \alpha_p \rtimes \mathbb{G}_m \to \mathbb{G}_m \to 1$$

that

$$\mathrm{H}^1_{flat}(k, G_{v'}) = \mathrm{H}^1_{flat}(k, \alpha_p \rtimes \mathbb{G}_m) = \mathrm{H}^1_{flat}(k, \alpha_p) = k/k^p.$$

Applying Proposition 3.1, we have a bijection between the set of $G(k)$-orbits that lies in $G(\overline{k}).v' \cap V(k) = k$ and $\mathrm{H}^1_{flat}(k, G_{v'}) = k/k^p$. It implies that there are infinite $G(k)$-orbits in k and furthermore, there is a bijection between the above set $\mathcal{C}$ and k/k^p. Since $k = \mathbb{F}_q((T))$, the set k/k^p is uncountable. So the cohomological set $\mathrm{H}^1_{flat}(k, G_v)$ where G_v is given in (4), and the set of $G(k)$-orbits in k are not only infinite, but also uncountable.

Acknowledgement. The authors are indebted to the referee for very helpful comments and suggestions which help to improve the main result and the presentation of this paper. Thanks are also due to Professor Nguyen Quoc Thang for his useful remarks. One of versions of the paper was written while the first-named author was doing research at The Vietnam Institute for Advanced Study in Mathematics (VIASM). We would like to thank the VIASM for the support and the hospitality. This research is funded by the Grant number 101.04-2020.09 from NAFOSTED to which the authors express our sincerest thanks.

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