Constructing 2-level Orthogonal Minimally Aliased Screening Designs from Hadamard Matrices with Two Circulant Cores

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Abstract

The traditional approach of designing a screening experiment is to start with a regular fractional factorial design (FFD) of resolution III or IV, or a subset of columns of a Plackett-Burman design. This experiment is then followed by the foldover of the design in stage one or follow-up runs. This paper introduces a class of 2-level orthogonal minimally aliased designs (OMADs) for screening experiments. These OMADs are constructed by selecting subsets of columns of the Hadamard matrices with two circulant cores using the minimum G-aberration criterion (Deng & Tang, 1999). Unlike the regular FFDs of resolution III and IV, most of our

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OMADs do not have fully aliased effects. As such, follow-up runs which are used to disentangle theses effects from one another become unnecessary. Our OMADs can also be easily divided into two blocks. The OMADs are compared with those of Deng & Tang (2002), Schoen & Mee (2012) and Schoen et al. (2017). A catalogue of OMADs for 16, 20, 24, 28, 32, 36, 40, 44 and 48 runs is then given.

Keywords: Circulant matrices; Fractional factorial designs; Foldover designs; Minimum G-aberration criterion; Screening designs; Interchange algorithm.

1 Introduction

Consider an experiment discussed in Section 7.2 "Choosing follow-up runs" of Box et al., 2005, hereafter abbreviated as BHH which studied the effects of eight factors on percent shrinkage in an injection molding process: **A** Mold Temperature, **B** Moisture, **C** Hold Press, **D** Cavity Thickness, **E** Booster Pressure, **F** Cycle Time, **G** Gate Size and **H** Screw Speed (see also Meyer et al., 1996). The design for this experiment is a 2^{8-4}_{IV} FFD (fractional factorial design of resolution IV) in Table 1. It can also be considered as a foldover design with the first eight runs in Table 1 forming the first half fraction. In this fraction, **A**, **B**, **C** form a factorial; **D** = **AB**, **E** = **AC**, **F** = **BC**, **G** = **ABC** and **H** is a column of 1's. The analysis of the data can be found from the references mentioned above.

The MEs of the 2^{8-4}_{IV} FFD in Table 1 are orthogonal to each other and to the 2FIs. However, the following 2FIs are fully aliased with other 2FIs: **AB** = **CG** = **DH** = **EF**, **AC** = **BG** = **DF** = **EH**, **AD** = **BH** = **CF** = **EG**, **AE** = **BF** = **CH** = **DG**, **AF** = **BE** = **CD** = **GH**, **AG** = **BC** = **DE** = **FH**, **AH** = **BD** = **CE** = **FG**. It is natural to ask whether there is an alternative design, having the same number of factors

Table 1: The 2^{8-4}_{IV} FFD for the injection molding experiment from BHH

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
-1	-1	-1	1	1	1	-1	1
1	-1	-1	-1	-1	1	1	1
-1	1	-1	-1	1	-1	1	1
1	1	-1	1	-1	-1	-1	1
-1	-1	1	1	-1	-1	1	1
1	-1	1	-1	1	-1	-1	1
-1	1	1	-1	-1	1	-1	1
1	1	1	1	1	1	1	1
1	1	1	-1	-1	-1	1	-1
-1	1	1	1	1	-1	-1	-1
1	-1	1	1	-1	1	-1	-1
-1	-1	1	-1	1	1	1	-1
1	1	-1	-1	1	1	-1	-1
-1	1	-1	1	-1	1	1	-1
1	-1	-1	1	1	-1	1	-1
-1	-1	-1	-1	-1	-1	-1	-1

and runs with no 2FI being fully aliased with each other.

There have been attempts to eliminate the fully aliased effects (Jones & Montgomerry, 2010; Errore et al., 2017, Nguyen et al. 2021) or to minimise the fully aliased effects (Schoen & Mee, 2012; Schoen et al., 2017; Nguyen et al., 2021). This paper is in this direction. Hereafter, Schoen & Mee (2012) will be abbreviated as SM and Schoen et al. (2017) will be abbreviated as SVG.

The aims of this paper are: (i) to introduce a class of 2-level orthogonal minimally aliased designs (OMADs) constructed from Hadamard matrices with two circulant cores using the minimum G-aberration (MIGA) criterion (Deng & Tang, 1999); (ii) to provide examples in which our OMADs can be used; (iii) to compare our OMADs with those constructed by Deng & Tang (2002), Ingram & Tang (2005), SM and SVG; (iv) to construct a catalog of OMADs for 16, 20, 24, 28, 32, 36, 40, 44 and 48 runs.

2 Criteria for Ranking 2-level Designs

In this paper, we use a simplified version of the MIGA criterion to (i) construct the Hadamard matrices with two circulant cores; (ii) construct the OMADs from the columns of these matrices. As such, it is necessary for us to review this criterion and the related criterion, the minimum G_2-aberration (Tang & Deng, 1999).

Consider a 2-level design for m factors in n runs using the model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, which includes the MEs and 2FIs, constructed from the design matrix $\mathbf{D}_{n\times m} = (d_{ui}),\ u = 1,\ldots,n;\ i = 1,\ldots,m$. Here, $\mathbf{y}_{n\times 1}$ is the vector of response; $\mathbf{X}_{n\times p}$ is the model matrix which contains the intercept column, m MEs columns and $\binom{m}{2}$ 2FI columns; $\boldsymbol{\beta}_{p\times 1}$'s are the unknown parameters; and $\boldsymbol{\epsilon}_{n\times 1}$ is a vector of residuals with $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $V(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$. The uth row of $\mathbf{X}$ can be written as $(1,\ d_{u1},\ldots,\ d_{um},\ d_{u1}d_{u2},\ldots,\ d_{u(m-1)}d_{um})$. The off-diagonal elements of the information matrix $\mathbf{X}'\mathbf{X}$ contain the following elements: (i) $\sum d_i,\ (i = 1,\ldots,m)$; (ii) $\sum d_i d_j,\ (i < j)$; (iii) $\sum d_i d_j d_k,\ (i < j < k)$; and (iv) $\sum d_i d_j d_k d_l\ (i < j < k < l)$, where $i,\ j,\ k,\ l = 1,\ldots,m$ and the summations are taken over the n design points. The number of summations of the types (i), (ii), (iii) and (iv) are m, $\binom{m}{2}$, $\binom{m}{3}$ and $\binom{m}{4}$, respectively.

For regular FFDs - i.e. designs constructed by the generators such as the 2^{8-4}_{IV} FFD in Table 1 - the summations in (i)-(iv) are either 0 or $\pm n$. For this 2^{8-4}_{IV} FFD, all summations of type (i), (ii) and (iii) are 0. However, 14 type (iv) summations involving factors **ABCG**, **ABDH**, **ABEF**, **ACDF**, **ACEH**, **ADEG**, **AFGH**, **BCDE**, **BCFH, BDFG**, **BEGH**, **CDGH**, **CEFG**, and **DEFH** are 16. For nonregular designs such as the OMADs in this paper, the summations of the types (i)-(iv) could take a value between $-n$ and n. This means that, unlike regular FFDs, nonregular designs might possess effects that are partially aliased, i.e. they are neither orthogonal nor fully aliased.

We use $A_1,\ A_2,\ A_3$ and A_4 to denote the sums of squares of the summations of the types (i)-(iv) divided by n^2, respectively. It can be seen that for the 2^{8-4}_{IV} FFD in Table 1, the elements of the quadruple (A_1, A_2, A_3, A_4) is $(0,\ 0,\ 0,\ 14)$. For regular FFDs, we call the vector $(A_1, A_2, A_3, A_4, \ldots)$ the *word length pattern* and use it to rank FFDs. The FFD that sequentially minimises the elements of this vector is called the *minimum aberration design* (see e.g. Section 5.2.5 of Mee, 2009). The FFD is said to be of resolution III if $A_1 = A_2 = 0$ but $A_3 \neq 0$; of resolution IV if $A_1 = A_2 = A_3 = 0$ but $A_4 \neq 0$; and of resolution V if $A_1 = A_2 = A_3 = A_4 = 0$. For nonregular designs, the vector $(A_1, A_2, A_3, A_4, \ldots)$ is called the *generalised word length pattern*, and a design that sequentially minimise the elements of this vector is call the *minimum G_2-aberration* design (Tang & Deng, 1999; Section 6.3.2 of Mee, 2009).

To calculate the elements of the quadruple (A_1, A_2, A_3, A_4), we calculate vector J_u of length $\sum_{i=1}^{4}\binom{m}{i}$ for row u of $\mathbf{D}$ $(u = 1, \ldots, n)$ as:

$$J_u = (d_{u1}, \ldots,\ d_{u1}d_{u2}, \ldots,\ d_{u1}d_{u2}d_{u3} \ldots,\ d_{u1}d_{u2}d_{u3}d_{u4}, \ldots). \tag{1}$$

We then calculate $J = \sum_{u=1}^{n} J_u$ and set $A_1, \ldots, A_4$ equal to the sums of squares of the first m, and the next $\binom{m}{2}, \binom{m}{3}, \binom{m}{4}$ elements of J, divided by n^2, respectively.

Let M_i be the maximums (in terms of the absolute value) and f_i $(i = 1, \ldots, 4)$ be the frequencies of these maximums of the first m, and the next $\binom{m}{2}, \binom{m}{3}, \binom{m}{4}$ elements of J respectively. In this paper, we call a design that sequentially minimise the elements of the octuple $(M_1, f_1, \ldots, M_4, f_4)$ a *minimum G-aberration* design or MIGA. Note that for regular FFDs and 2-level orthogonal designs, whose factors are columns of a Hadamard matrix like those in this paper, $A_1 = A_2 = 0$ and $M_1 = M_2 = 0$. Also, for a foldover design, i.e. a design whose first half-fraction design matrix is $\mathbf{D}$ and the second is -$\mathbf{D}$,

$A_1 = A_3 = 0$ and similarly $M_1 = M_3 = 0$. Also, its A_2 and A_4 values will be the same as those of $\mathbf{D}$.

The minimum G_2-aberration criterion is a handy surrogate criterion for optimality criteria such as the D- and A-criteria. If we restrict ourselves to designs with equal-occurrence, i.e. with $A_1 = 0$, minimising the remaining A's are equivalent to minimising the off-diagonal of the information matrix $\mathbf{X}'\mathbf{X}$. This criterion, however, is not always practical for ranking 2-level designs. Both the Plackett-Burman design (Plackett & Burman, 1946) and a Hadamard design (a Hadamard matrix with the first column of 1's removed) constructed by two circulant cores in the next Section for 15 factors in 16 runs have (A_3, A_4)=(35, 105). However, the M_3 and M_4 values (and their frequencies) of these two designs are 16 (35) and 16 (105) vs 16 (7) and 16 (21). Similarly, for 31 factors in 32 runs have (A_3, A_4)= (155, 1085). However, the M_3 and M_4 values (and their frequencies) of these two designs are 32 (155) and 32 (1085) vs 8 (2480) and 8 (17360). Table 5 of SM show two strength-3 OAs for 12 factors in 48 runs (designs `12.0-541920` and `12.5-76810`). The A_4 values of these two OAs are 15.33 and 15. However, the M_4 values of these two OAs are 16 and 48.

The above examples show that the MIGA criterion appears to be more successful in identifying the minimally aliased designs. Therefore, we will use the MIGA criterion as our main design selection criterion in this paper. In addition, we will use df(2FI) of the designs as the second criterion. This is the rank of $\mathbf{X}_2$, the model matrix for 2FIs (see SM).

This paper uses of a quality measure called r_{worst}, the worst correlation among two effects in the model matrix $\mathbf{X}$ (see SM). For orthogonal designs, r_{worst} is calculated as $\max(M_3, M_4)/n$. Consider two designs for $(n, m) = (32, 13)$ `13.0` in Table 5 of SM and

`13.1.1` in Table 10 of SVG. The $[(M_3, f_3), (M_4, f_4)]$ values of these two designs are [(0, 286), (32, 10)] and [(8, 144), (8, 396)]. The r_{worst} of these two designs are 1 and 0.25 respectively and the second design is therefore considered more *minimally aliased* than the first.

3 Hadamard matrices with two circulant cores

A ± 1 square matrix $\mathbf{H}$ of order n is a Hadamard matrix (see Hedayat & Wallis, 1978) if $\mathbf{H}'\mathbf{H} = \mathbf{H}\mathbf{H}' = n\mathbf{I}_n$ where I_n is the identity matrix of order n. A Hadamard matrix H_{l+1} with a single circulant core can be written as $\begin{pmatrix} 1 & -\mathbf{1}' \\ \mathbf{1} & \mathbf{A} \end{pmatrix}$ or $\begin{pmatrix} 1 & \mathbf{1}' \\ \mathbf{1} & \mathbf{A} \end{pmatrix}$ where $\mathbf{1}$ is a vector of 1's and $\mathbf{A} = (a_{ij})$ is a circulant matrix of order l, i.e. $a_{ij} = a_{1,j-i+1(\text{mod l})}$. Many Plackett-Burman designs are of this form. A Hadamard matrix $\mathbf{H}_{2l+2}$ with two circulant cores (Fletcher et al., 2001; Kotsireas et al., 2006) can be written as:

$$\begin{pmatrix} 1 & 1 & \mathbf{1}' & \mathbf{1}' \\ 1 & -1 & \mathbf{1}' & -\mathbf{1}' \\ \mathbf{1} & \mathbf{1} & \mathbf{A} & \mathbf{B}' \\ \mathbf{1} & -\mathbf{1} & \mathbf{B} & -\mathbf{A}' \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & \mathbf{1}' & \mathbf{1}' \\ \mathbf{1} & \mathbf{1} & \mathbf{A} & \mathbf{B}' \\ 1 & -1 & \mathbf{1}' & -\mathbf{1}' \\ \mathbf{1} & -\mathbf{1} & \mathbf{B} & -\mathbf{A}' \end{pmatrix} \tag{2}$$

where $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are two circulant matrices of order l. For $\mathbf{H}$ in (2) to be a Hadamard matrix, $\mathbf{A}$ and $\mathbf{B}$ should satisfy the condition $\mathbf{A}'\mathbf{A} + \mathbf{B}'\mathbf{B} = (2l+2)\mathbf{I}_l - 2\mathbf{J}_l$, where $\mathbf{J}$ is a square matrix of order l of 1's. The Hadamard matrix in (2) is equivalent to the one in equation (1) of Kotsireas et al. (2006). The following is an example of the Hadamard matrix of order 16 in the form of equation (2), without the first column of 1's:

Table 2: Generating vectors for core OMADs in n runs and $\frac{1}{2}n-1$ (and $\frac{1}{2}n$) factors§

n	m	M_3	M_4	M_3‡	M_4‡	Cyclic generators
8	3	8 (1)	0 (1)	8 (1)	0 (1)	$+--$ $+--$
12	5	4 (10)	4 (5)	4 (20)	4 (15)	$+---+$ $-+-+-$
16†	7	8 (14)	8 (14)	8 (14)	8 (28)	$-++---+$ $-++-+--$
20	9	12 (9)	12 (9)	12 (9)	12 (12)	$++-+---+-$ $-+++--+--$
24	11	8 (55)	8 (220)	8 (55)	8 (330)	$+---+--+-++$ $++-+--+---+$
28†	13	4 (286)	12 (950)	4 (364)	12 (273)	$-+-+-++----++$ $--+--++++--+-$
32	15	8 (260)	8 (780)	8 (320)	8 (1020)	$-+--++-+-++---+$ $--+--+---+++++-$
36†	17	4 (680)	12 (952)	4 (816)	12 (1224)	$-+-++-++-+---++--$ $+++-+-----+-+++--$
40	19	8 (285)	16 (456)	8 (285)	16 (570)	$+++--+--++----+-+-+$ $---++--+--++++-+-+-$
44	21	12 (357)	12 (1533)	12 (399)	12 (1834)	$++--+++-+---++--+--+-$ $+-+++-------+++-+-++-$
48†	23	8 (506)	16 (2530)	8 (506)	16 (3036)	$--++++-+-++--++--+-+---$ $+--++-+-++++-----+-+--+$

†Used as core OMADs in this study.
‡M's of OMADs for $m+1$ factors formed by adding a column of half $-$ and half $+$ to OMADs for m factors
§The ID of the core OMAD for m factors in n runs in this paper and the supplemental material is `tcnxm`.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1
\end{pmatrix}. \tag{3}$$

It can be seen that the first rows of the two circulant matrices **A** and **B** used in the construction of (3) are $(-++---+)$ and $(-++-+--)$.

Table 2 and Table 3 contain the generators for OMADs for $\frac{1}{2}n-1$ and $n-2$ factors. OMADs for $\frac{1}{2}n$ and $n-1$ factors are constructed by adding a column with half 1's and half -1's to the OMAD for $\frac{1}{2}n-1$ and $n-2$ factors respectively. Table 2 and Table 3

Table 3: Generating vectors for core OMADs in n runs and $n-2$ (and $n-1$) factors§

n	m	M_3	M_4	M_3†	M_4†	Cyclic generators
8	6	8 (4)	8 (3)	8 (7)	8 (7)	Same as Table 2
12	10	4 (120)	4 (210))	4 (165)	4 (330)	Same as Table 2
16†	14	8 (112)	16 (21)	16 (7)	16 (21)	Same as Table 2
20†	18	12 (48)	12 (180)	12 (57)	12 (228)	Same as Table 2
24†	22	8 (660)	8 (3135)	8 (759)	8 (3795)	− + − + − + − − + + − + + − − − − + − − + +
28†	26	12 (312)	12 (1794)	12 (351)	12 (2106)	+ − + − + − + + − − − − + + − + + − − + − − − − + +
32†	30	8 (2240)	8 (15120)	8 (2480)	8 (17360)	Same as Table 2
36	34	12 (1080)	20 (272)	12 (1190)	20 (272)	− + + − − − + + + − + − − + − + − − − + − − − + + + + + − + − − − +
40†	38	16 (323)	24 (171)	16 (361)	24 (171)	+ − + − + − − − − + + − + + − − + + − − + + − − − − + + − + + + + − + − − −
44†	42	12 (2800)	12 (27300)	12 (3010)	12 (30100)	Same as Table 2
48†	46	16 (1012)	16 (10879)	16 (1081)	16 (11891)	+ − + − + − − + − − + − + − + + + − − − − + + + + − − + + + + + − + − − + − − − − − + + − −

†Used as core OMADs in this study.
‡M's of OMADs for $m+1$ factors formed by adding a column of half − and half + to OMADs for m factors
§The ID of the core OMAD for m factors in n runs in this paper and the supplemental material is `tcnxm`.

also report the values of M_3 and M_4 and the corresponding frequencies. Most OMADs in this paper for $m < \frac{1}{2}n - 1$ factors are projections from core OMADs for $\frac{1}{2}n$ factors in 2. Similarly, most OMADs for m factors ($\frac{1}{2}n < m < n-2$) are projections from core OMADs for $n-2$ factors in Table 3.

The algorithm used for generating the vectors in Table 2 and Table 3 are closely aliased to the one in Nguyen (1996) in the construction of the supersaturated designs (SSDs). Note that for $n = 12$, 16 and 20, the generators in Table 2 and Table 3 are identical to the ones for first three SSDs in Table 1 of Nguyen (1996). For $n = 24$, 28, 36, 40 and 48, the generators in Table 2 and Table 3 are different because the set of generators that produce the good OMADs in Table 2 might not do so in Table 3 and vice versa.

4 The MAD algorithm

MAD is an algorithm for (i) finding MIGA projections (subsets of columns) from a core OMAD or a Hadamard design and (ii) constructing an OMAD from scratch or augment a base design with new columns (factors). With (i), MAD picks a random sample of m distinct columns from core OMADs constructed in Table 2 or Table 3. Each sample makes up one "try". The best try is then selected. MAD is closely aliased to the FOLD algorithm reported in Nguyen et al. (2021). Unlike FOLD, MAD is not restricted to foldover designs. Below are two steps in MAD to construct a design from scratch using the column-wise interchange method:

1. Assign -1 to half of the number of elements of columns j of $\mathbf{D}$ ($j = 1, \ldots, m$), and 1 to the remaining. Randomise the positions of ± 1's. If the equal-occurrence constraint is not required, randomly assign ± 1's to n elements of column j. Calculate vector J $(= \sum_{u=1}^{n} J_u)$ in (1) and the octuple $(M_1, f_1, \ldots, M_4, f_4)$.

2. For column j of $\mathbf{D}$ ($j = 1, \ldots, m$), search for a pair of elements having different values such that swapping them results in the smallest octuple $(M_1, f_1, \ldots, M_4, f_4)$. If found, swap them and update $\mathbf{D}$ and J. This step is repeated until no further reduction can be made.

For each set $(m,\ n)$, Steps 1-2 make up one "try" . Among a subset S of tries which result in a best design with respect to the MIGA criterion, select the one with the maximum df(2FI), which is the rank of $\mathbf{X}_2$, the model matrix for the 2FIs.

Remarks

1. An equal-occurrence design has $A_1 = 0$ and the length of vector J_u in (1) shortened to $\binom{m}{2} + \binom{m}{3} + \binom{m}{4}$. A design whose columns are subset of columns of a Hadamard design like those in this study has $A_1 = A_2 = 0$ and the length of vector J_u shortened to $\binom{m}{3} + \binom{m}{4}$ (see e.g. the design in Table 4 (a)).

2. A foldover design has $A_1 = A_3 = 0$ and the length of vector J_u in (1) shortened to $\binom{m}{2} + \binom{m}{4}$ (see e.g the designs Table 4 (c) and Table 4 (d)). To construct a foldover design, we only need to construct its half fraction.

3. There are situations when the experimenter wish to eliminate all fully aliased effects. For example they may want to eliminate all fully aliased 2FIs in the 2^{8-4}_{IV} FFD mentioned in the Introduction or to set the M_3 or M_4 values to be smaller or equal to a specified value (see e.g the designs in Table 4 (b) and Table 4 (d)).

Table 4: Four 2-level designs for five factors in 12 runs

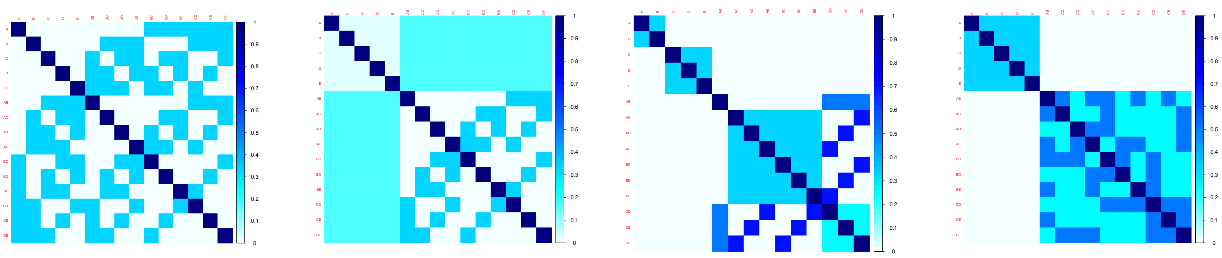
A	B	C	D	E	A	B	C	D	E	A	B	C	D	E	A	B	C	D	E
1	1	1	1	1	-1	-1	1	1	1	1	1	-1	-1	1	-1	-1	1	-1	-1
1	-1	-1	-1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	-1	1	1	1
1	1	-1	-1	-1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	-1
-1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	-1	1	1	-1	-1	-1	-1
-1	-1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	-1	1	1	1	-1
-1	-1	-1	1	1	-1	1	1	1	-1	-1	1	1	1	1	-1	1	-1	1	1
1	1	1	1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	1	-1	1	1
-1	1	-1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	-1	-1
-1	-1	1	-1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	-1	-1	1
1	-1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	1	1	1
-1	1	-1	-1	1	-1	1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	1
1	-1	1	-1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1

(a) (b) (c) (d)

Table 5: Quality measures of four 2-level designs for five factors in 12 runs in Table 4

Design	A_1	A_2	A_3	A_4	M_1	M_2	M_3	M_4	df(2FI)	r_{worst}	D-eff†
4a	0	0	1.11	0.56	0	0	4 (10)	4 (5)	10	0.33	1
4b	0.14	0	0.28	0.56	2 (5)	0	2 (10)	4 (5)	10	0.33	0.97
4c	0	0.44	0	1.22	0	4 (4)	0	8 (2)	6	0.71	0.93
4d	0	1.11	0	0.56	0	4 (10)	0	4 (5)	6	0.5	0.76

$\dagger \frac{1}{n}|\mathbf{X}_1'\mathbf{X}_1|^{\frac{1}{m+1}}$ where $\mathbf{X}_1$ is the model matrix corresponding to the MEs.



(a) (b) (c) (d)

Figure 1: CCPs of four 2-level designs for five factors in 12 runs in Table 4

Table 4 displays four 2-level designs for five factors in 12 runs. The design `tc12x5` in Table 4 (a) was constructed by two generating vectors $(+ - - - +)$ and $(- + - + -)$ in Table 2. The design in Table 4 (b) was constructed by setting $M_3 \leq 2$ and relaxing the equal-occurrence constraint. The design in Table 4 (c) was constructed as a foldover design. The design in Table 4 (d) was constructed as a foldover design with $M_4 \leq 4$. The quality measures of these four designs, including their D-efficiencies (D-eff) are displayed in Table 5. The quality measures of the first three designs in Table 4 match the ones for five factors in 12 runs in Tables 8-10 of Jones & Nachtsheim (2011). Figure 1 displays the correlation cell plots (CCPs) of the four designs in Table 4. These plots, proposed by Jones & Nachtsheim (2011), display the magnitude of the correlation (in terms of the absolute values) between the columns of the model matrix **X**. The color of each cell ranges from white (no correlation) to dark (correlation of 1 or close to 1).

5 OMADs for 16, 20, 24, 28, 32, 36, 40, 44 and 48 runs

Most OMADs in this paper were constructed by using a core OMAD in either Table 2 or Table 3. The exception is the 36-run OMADs for 19-35 factors, where we have to use a Hadamard design generated with a single core. OMADs with a small number of runs or factors were constructed from scratch. OMADs from core OMADs with $\frac{1}{2}n - 1$ or $n - 2$ factors do not include a column of half -1's and 1's. As such, for these OMADs, we can use this column as an additional factor or as a blocking factor without increasing r_{worst}. As all OMADs are orthogonal designs, their quality measures displayed in the Appendix only include A_3, A_4, M_3, M_4 and the frequencies M_3 and M_4 of as well as the df(2FI).

5.1 16 runs (Appendix A-1)

All OMADs for 16 runs were constructed from scratch. For 6-8 factors, we have three MIGA solutions with $r_{\text{worst}} = 1$ and three minimally aliased solutions with $r_{\text{worst}} = 0.5$ (=8/16). The latter, like the non-confounding designs of Jones, B. & Montgomery (2010) do not totally confound the 2FIs. The quality measures for the solutions for 8 factors and for 12-15 factors match the projections from core OMADs for 16 runs in Table 2 and Table 3. These designs have also been reported in Table 2 Deng & Tang (2002), who used columns of selected Hadamard matrices. Although the OMADs for 9-14 factors and the corresponding FFDs of resolution III (Mee, 2009, Table G.2) have the same A_3 and A_4 values, none of these OMADs confound the MEs and the 2FIs as the FFDs.

5.2 20 runs (Appendix A-2)

OMADs for 4-10 factors were constructed from scratch. OMADs for 11-13 factors were built up from smaller OMADs. The remaining OMADs are MIGA projections from core OMADs for 20 runs in Table 3. For 6 and 7 factors, our results slightly improve the ones in Table 4 of Deng & Tang (2002) in terms of the MIGA criterion (see also Table 6.32 of Mee, 2009).

5.3 24 runs (Appendix A-3)

OMADs for 4-12 factors were constructed from scratch. These OMADs are all strength-3 OAs and match the ones in Table 1 of Ingram & Tang (2005) and Table 2 of SVG, in terms of M_3 and M_4 and their frequencies. These designs are also foldover. Our OMADs for 13-23 factors are projections from core OMADs in Table 3. They are not as good as the designs in Table 2 of Ingram & Tang (2005) and Table 3 of SVG in terms of the MIGA criterion. However, while the r_{worst} of our OMADs is 0.333 (=8/24), theirs range from 0.667 (16/24) to 1 (=24/24). Table 6.34 of Mee (2009) displays the best-known 20-factor design, which is a projection of Sloan's `Had.24.59` with respect to the MIGA criterion. While the M_3 and M_4 values of this design and their frequencies are 8 (480) and 24 (5), the ones of our 20-factor OMADs are 8 (488) and 8 (2077).

5.4 28 runs (Appendix A-4)

OMADs for 4-8 factors were constructed from scratch. OMADs for 9 factors were constructed by three circulant matrices generated by three vectors $(++-++-+-+)$, $(---+-++--)$, $(---+---+++)$ and a row of 1's. The OMADs for 10-14 factors are projections from the core OMAD for 28 runs in Table 2. The OMADs for 15-27 factors are projections from core OMADs for 28 runs in Table 3. With the exception of the OMAD for 10 factors, ours compare quite well with the 28-run designs in Tables 6-7 of SVG with respect to the MIGA criterion. Actually, ur OMADs for 17 and 18 runs slightly improve the corresponding designs of these authors with respect to the MIGA criterion.

5.5 32 runs (Appendix A-5)

OMADs for 4-6 factors were constructed from scratch. OMADs for 7-31 factors are projections of the core OMADs for 32 factors in Table 3. For 10-11 factors, the 32-run designs in Table 10 of SVG slightly improve our OMADs. For 14-15 factors, the reverse is true. For 7-16 factors, our OMADs and SVG 32-run designs are not as good as the strength-3 OAs in Table 3 of SM (or the FFD of resolution IV in Table G.3 of Mee, 2009) with respect to the MIGA criterion. However, while the r_{worst} of the former is 0.25 (8/24), the one of the latter is 1 (32/32), meaning some pair of 2FIs of these designs are fully aliased. In addition, the df(2FI)'s of OMADs and SVG designs substantially increase the ones of the strength-3 OAs. For 17-31 our OMADs and SVG designs do not confound MEs and 2FIs and pairs of 2FIs as the FFD of resolution III in Table G.3. of Mee (2009).

5.6 36 runs (Appendix A-6)

OMADs for 4-8 factors were constructed from scratch. OMADs for 9-17 are the projections of the core OMAD for 36 runs in Table 2. OMADs for 19-35 are projections of the Hadamard design generated with a single core. The generator for this matrix is $(-+--++-++-+---+++------+---++-++++-+)$. Our OMADs from the 7-8 and 12-18 factors are as good as the 36-run designs in Table 12 of SVG in terms of the MIGA criterion.

5.7 40 runs (Appendix A-7)

OMADs for 4-10 factors were constructed sequentially from scratch (the one for m factors was constructed by adding a column to the one with $m-1$ factors). The quality measures of these OMADs are identical to those of the corresponding strength-3 OAs in Table 4 of SM. OMADs for 11-18 factors are projections of the core OMADs for 40 runs in Table 2. These OMADs are not strength-3 OAs like those of the designs in Table 4 of SM. However, while the r_{worst} of these OMADs is 0.4 (=16/40), the one of the corresponding strength-3 OAs is 0.6 (=24/40). The worst correlation between a ME and a 2FI of these OMAD is 0.2 (=8/40). OMADs for 20-37 factors are projections of core OMADs for 40 runs in Table 3. The r_{worst} of these OMADs is 0.6 (=24/40).

5.8 44 runs (Appendix A-8)

OMADs for 4-12 factors were constructed sequentially from scratch. The remaining OMADs for 12-42 factors are projections of the core OMADs for 44 runs in Table 3. The r_{worst} of these OMADs is 0.272 =(12/44).

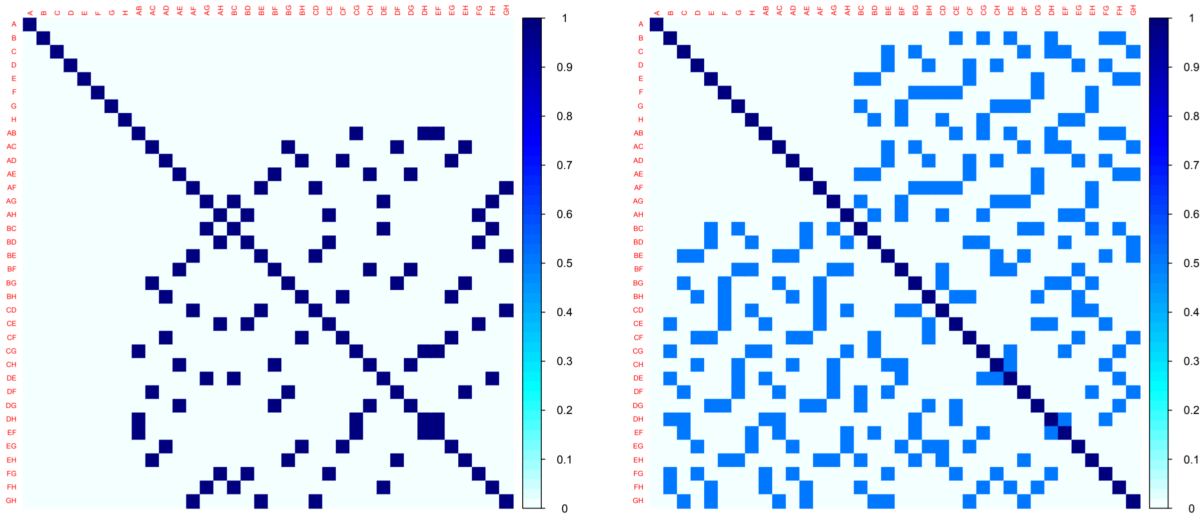
5.9 48 runs (Appendix A-9)

OMADs for 4-13 factors were constructed sequentially from scratch. The quality measures of these OMADs are identical to those of the corresponding strength-3 OAs in Table 5 of SM. We have three solutions for OMADs for eight factors. OMADs for 14-22 factors are projections of the core OMADs for 48 runs in Table 2. Like our OMADs for 40 runs, these OMADs are not strength-3 OAs. However, the worst correlation between a MEs and a 2FI is 0.167 (=8/48). OMADs for 25-45 factors are projections of the core OMADs for 48 runs in Table 3. The r_{worst} of these designs is 0.333 (=16/48).

6 Examples of the use of OMADs

Let us compare the 2^{8-4}_{IV} FFD for the injection molding of in Table 1 and our corresponding 16-run OMAD for eight factors generated by two generating vectors $(-++----+)$ and$(-++-+--)$ (`tc16x8` in Table 2). The M_3 and M_4 values (and their frequencies) are: 0 (56) and 16 (14) vs 8 (14) and 8 (28). The (r_{worst}, df(2FI), PIC_5) of these two designs are: (1, 7, 0) vs (0.5, 13, 0.1928). PIC5, used in SVG, is the projection information capacity for five factors. This value is the average D-efficiency with which all interaction models in five factors can be estimated. Clearly, for the experimenters who do not wish to spend extra time and resources on follow-up runs to disentangle the fully aliased effects, the OMAD alternative is a much preferred choice.

Figure 2 displays the CCPs of two 16-run designs for eight factors discussed in the previous paragraph. It can be seen that the MEs in the CCP in Figure 2 (a) are orthogonal to the 2FIs. This is not true for the MEs of CCP in Figure 2 (b). At the same time, unlike the 2FIs in Figure 2 (a), none of the 2FI in Figure 2 (b) is fully aliased with another 2FI.



(a) (b)

Figure 2: CCPs of two 16-runs design for eight factors in : (a) the 2_{IV}^{8-4} FFD of BHH p. 296 and (b) our corresponding OMAD

We now use the experiment requiring a 2-level design for the diamond turning of aluminum mirrors reported by SM as a second example. The objective of this experiment was to identify factors among 13 factors that affect the smoothness of mirrors produced under various conditions. There are two blocking factors (**A** machine, **B** operator), four rake related factors (**C** angle, **D** face orientation in deg, **E** nose radius in μm, **F** rake sharpness), two workpiece related factors (**G** material, **H** shape), two lubricant related factors (**I** amount, **J** pressure), three factors controlling the mechanical conditions of the diamond turning process (**K** feed rate, **L** depth of cut in μm, **M** spindle speed in rpm). Suitable designs for this experiment are the strength-3 OAs for 32, 40 and 48 runs (Designs `13.10`, `13.55` and `13.0-594498` in Tables 3-5 of SM) and OMADs for 13 factors in 28, 32 and 36 runs (see Appendix A-4, A-5 and A-6). The quality measures of these six designs are displayed in Table 6. All six are better than the FFD 2_{IV}^{13-8} (see design 13.8.1 in Table G.3 of Mee, 2009).

Let us now compare the strength-3 OA for 32 runs of SM and the 28-run OMAD (`tc28x13`). This OMAD was generated by two generating factors $(-+-+-++----++)$ $(--+--++++--+-)$ (see Table 2). While the MEs of the strength-3 OA are orthogonal to the 2FIs, several 2FIs of this OA are fully aliased with the other 2FIs. The MEs of the OMAD are slightly correlated with the 2FIs ($r = 0.143$)(=4/28) but the 2FIs

Table 6: Six candidate designs for the diamond turning experiment

Design	n	A_3	A_4	M_3	M_4	df(2FI)	r_{worst}	PIC5
`13.10`†	32	0	55	0	32 (10)	15	1	0.9174
`13.55`†	40	0	41.72	0	24 (41)	19	0.6	0.9336
`13.0-594498`†	48	0	23	0	16 (207)	34	0.33	0.9655
`tc28x13`	28	5.84	46.43	4 (286)	12 (195)	26	0.43	0.8791
`tc32x13`	32	8.94	24.88	8 (143)	8 (398)	31	0.25	0.8936
`tc36x13`	36	3.53	36.88	4 (286)	12 (284)	30	0.33	0.9176

†These strength-3 OAs form from Tables 3-5 of SM.

Table 7: Two halves of the OMAD recommended for the diamond turning of mirrors experiment

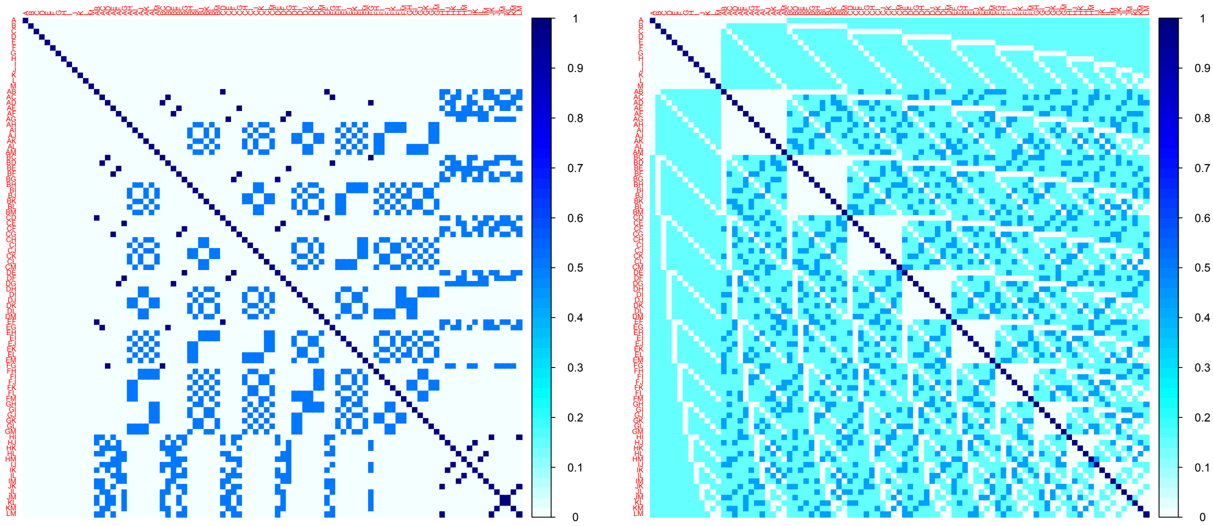
A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
1	1	1	1	1	1	1	1	1	1	1	1	1
-1	1	-1	1	-1	1	1	-1	-1	-1	-1	1	1
1	-1	1	-1	1	-1	1	1	-1	-1	-1	-1	1
1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	-1
-1	1	1	-1	1	-1	1	-1	1	1	-1	-1	-1
-1	-1	1	1	-1	1	-1	1	-1	1	1	-1	-1
-1	-1	-1	1	1	-1	1	-1	1	-1	1	1	-1
-1	-1	-1	-1	1	1	-1	1	-1	1	-1	1	1
1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1	1
1	1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1
-1	1	1	-1	-1	-1	-1	1	1	-1	1	-1	1
1	-1	1	1	-1	-1	-1	-1	1	1	-1	1	-1
-1	1	-1	1	1	-1	-1	-1	-1	1	1	-1	1
1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	-1

(a)

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**N**	**J**	**K**	**L**	**M**
1	1	1	1	1	1	1	1	1	1	1	1	1
-1	-1	1	-1	-1	1	1	1	1	-1	-1	1	-1
-1	-1	-1	1	-1	-1	1	1	1	1	-1	-1	1
1	-1	-1	-1	1	-1	-1	1	1	1	1	-1	-1
-1	1	-1	-1	-1	1	-1	-1	1	1	1	1	-1
-1	-1	1	-1	-1	-1	1	-1	-1	1	1	1	1
1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1	1
1	1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1
1	1	1	-1	-1	1	-1	-1	-1	1	-1	-1	1
1	1	1	1	-1	-1	1	-1	-1	-1	1	-1	-1
-1	1	1	1	1	-1	-1	1	-1	-1	-1	1	-1
-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	1
1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1
-1	1	-1	-1	1	1	1	1	-1	-1	1	-1	-1

(b)

are not fully aliased with each other. The two halves of this OMAD are displayed in Table 7. These two halves can be treated as two blocks, and the new design, which include the blocking factor, become a 28-run OMAD for 14 factors in Table 2. The CCPs of the two mentioned candidate designs are in Figures 3 (a) and 3 (b).



(a) (b)

Figure 3: CCPs of two 2-level designs for 13 factors: (a) the 32-run strength-3 OA 13.1 in Table 3 of Schoem & Mee (2012) and (b) our corresponding 28-run OMAD

7 Conclusion

Several combinatorial structures are related to the balance incomplete block design (BIBDs). Hedayat & Wallis (1978) show that the existence of a Hadamard matrix implies the existence of five different BIBDs. Several Box-Behnken designs (Box & Behnken (1960) are constructed from BIBDs or near-BIBDs. Nguyen (1996) shows that several $E(s^2)$-optimal supersaturated designs can be constructed from BIBDs. Identification of the simpler structure helps us to reduce computing tasks. Consider the core OMAD for five factors in 12 runs in Table 4 (a). If we remove the two rows of 1's and change the -1's into 0's, we will have the incidence matrix of a 2-resolvable cyclic BIBD with $(v,\ b,\ r,\ k, \lambda)$=(5, 10, 12, 4, 1), where v is the number of varieties, b the number of blocks, r the number of replications of each varieties, k the block size and λ the number of blocks containing any two distinct varieties. The blocks of this BIBD are (0, 4), (0, 1), (1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (0,3), (1, 4) and (0, 2). Similarly, the core OMAD in Table 7 is related to the 6-resolvable cyclic BIBD with $(v,\ b,\ r,\ k, \lambda)$=(13, 26, 12, 6, 5). Since the BIBDs, which are related to the OMADs in this paper, also have cyclic solutions, instead of generating the initial blocks of these cyclic BIBDs and convert the incidence matrices to OMADs, we can generate the cyclic generators in Table 2 and Table 3, which produce the core OMADs directly.

The OMADs presented in this paper, like the designs of Jones & Montgomery (2010), and those of SM and SVG, are considered economic alternatives to resolution IV FFDs.

The supplemental material includes (i) core OMADs in Table 2 and Table 3 of Section 3; (ii) the Java program which implements the MAD algorithm in Section 4; (iii) OMADs for 16, 20, 24, 28, 32, 36, 40, 44 and 48 runs discussed in Section 5.

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Appendix A: OMADs with 16, 20, 24, 28, 32, 36, 40, 44 and 48 runs

Appendix A1: *n*=16

m	A_3	A_4	M_3	M_4	df(2FI)
4	0	0	0	0	6
5	0	0	0	0	10
6	0	3	0	16 (3)	7
7	0	7	0	16 (7)	7
8	0	14	0	16 (14)	7
6	1	1	8 (4)	8 (4)	14
7	2	3	8 (8)	8 (12)	14
8†	3.5	7	8 (14)	8 (28)	14
9	4	14	8 (16)	16 (14)	15
10	8	18	8 (32)	16 (10)	15
11	12	26	8 (48)	16 (8)	15
12	16	39	8 (64)	16 (15)	15
13	22	55	8 (88)	16 (15)	15
14†	28	77	8 (112)	16 (21)	15
15†	35	105	16 (7)	16 (21)	15

†Core OMAD.

Appendix A2: *n*=20

m	A_3	A_4	M_3	M_4	df(2FI)
4	0.16	0.04	4 (4)	4 (1)	6
5	0.4	0.2	4 (10)	4 (5)	10
6	0.8	0.6	4 (20)	4 (15)	15
7	1.4	2.04	4 (35)	12 (2)	19
8	2.24	4.72	4 (56)	12 (6)	19
9	3.36	10.8	4 (84)	12 (18)	18
10	4.8	18	4 (120)	12 (30)	19
11	8.2	22.8	12 (5)	12 (30)	19
12	11.36	32.28	12 (8)	12 (39)	19
13	15.92	43.64	12 (14)	12 (47)	19
14	20.96	59.24	12 (20)	12 (60)	19
15	26.52	80.52	12 (26)	12 (81)	19
16	32.64	107.36	12 (32)	12 (108)	19
17	40	140	12 (40)	12 (140)	19
18†	48	180	12 (48)	12 (180)	19
19†	57	228	12 (57)	12 (228)	19

†Core OMAD.

Appendix A3: *n*=24

Appendix A3: *n*=24

m	A_3	A_4	M_3	M_4	df(2FI)
4	0	0.11	0	8 (1)	6
5	0	0.56	0	8 (5)	10
6	0	1.67	0	8 (15)	11
7	0	3.89	0	8 (35)	11
8	0	7.78	0	8 (70)	11
9	0	14	0	8 (126)	11
10	0	23.33	0	8 (210)	11
11	0	36.67	0	8 (330)	11
12	0	55	0	8 (495)	11
13	12.67	34.78	8 (114)	8 (313)	23
14	16.56	48	8 (149)	8 (432)	23
15	21	65.44	8 (189)	8 (589)	23
16	26.11	87	8 (235)	8 (783)	23
17	31.89	113.78	8 (287)	8 (1024)	23
18	38.56	145.89	8 (347)	8 (1313)	23
19	46	184.67	8 (414)	8 (1662)	23
20	54.22	230.78	8 (488)	8 (2077)	23
21	63.33	285	8 (570)	8 (2565)	23
22†	73.33	348.33	8 (660)	8 (3135)	23
23†	84.33	421.67	8 (759)	8 (3795)	23

†Core OMAD.

Appendix A4: *n*=28

m	A_3	A_4	M_3	M_4	df(2FI)
4	0.08	0.02	4 (4)	4 (1)	6
5	0.2	0.1	4 (10)	4 (5)	10
6	0.41	0.31	4 (20)	4 (15)	15
7	0.71	0.88	4 (35)	12 (1)	21
8	1.14	2.9	4 (56)	12 (9)	27
9	1.71	5.51	4 (84)	12 (18)	27
10	2.45	13.59	4 (120)	12 (57)	23
11	3.37	21.43	4 (165)	12 (90)	24
12	4.49	32.14	4 (220)	12 (135)	25
13†	5.84	46.43	4 (286)	12 (195)	26
14†	7.43	65	4 (364)	12 (273)	27
15	15.98	57.41	12 (41)	12 (181)	27
16	20.73	74.69	12 (57)	12 (230)	27
17	25.96	96.24	12 (74)	12 (292)	27
18	31.35	124.16	12 (90)	12 (378)	27
19	37.9	156	12 (111)	12 (471)	27
20	44.82	195.04	12 (132)	12 (589)	27
21	52.61	240.18	12 (156)	12 (723)	27
22	61.31	292.8	12 (183)	12 (879)	27
23	70.59	354.43	12 (211)	12 (1064)	27
24	80.82	425.18	12 (242)	12 (1276)	27
25	92	506	12 (276)	12 (1518)	27
26†	104	598	12 (312)	12 (1794)	27
27†	117	702	12 (351)	12 (2106)	27

†Core OMAD.

Appendix A5: n=32

m	A_3	A_4	M_3	M_4	df(2FI)
4	0	0	0	0	6
5	0	0	0	0	10
6	0	0	0	0	15
7	0.81	1.12	8 (13)	8 (18)	21
8	1.38	2.5	8 (22)	8 (40)	28
9	2.25	4.75	8 (36)	8 (76)	30
10	3.44	6.81	8 (55)	8 (109)	31
11	4.88	11.5	8 (78)	8 (184)	31
12	6.69	17.31	8 (107)	8 (277)	31
13	8.94	24.88	8 (143)	8 (398)	31
14	11.62	34.38	8 (186)	8 (550)	31
15	14.75	47.88	8 (236)	8 (766)	31
16	18.38	63.12	8 (294)	8 (1010)	31
17	22.5	82.12	8 (360)	8 (1314)	31
18	27.25	105.38	8 (436)	8 (1686)	31
19	32.56	133.38	8 (521)	8 (2134)	31
20	38.44	167.44	8 (615)	8 (2679)	31
21	45	206.62	8 (720)	8 (3306)	31
22	52.31	252.56	8 (837)	8 (4041)	31
23	60.38	305.88	8 (966)	8 (4894)	31
24	69.25	366.88	8 (1108)	8 (5870)	31
25	78.94	436.5	8 (1263)	8 (6984)	31
26	89.38	515.62	8 (1430)	8 (8250)	31
27	100.75	605.25	8 (1612)	8 (9684)	31
28	112.94	706.06	8 (1807)	8 (11297)	31
29	126	819	8 (2016)	8 (13104)	31
30†	140	945	8 (2240)	8 (15120)	31
31†	155	1085	8 (2480)	8 (17360)	31

†Core OMAD.

Appendix A6: n=36

m	A_3	A_4	M_3	M_4	df(2FI)
4	0.05	0.01	4 (4)	4 (1)	6
5	0.12	0.06	4 (10)	4 (5)	10
6	0.25	0.19	4 (20)	4 (15)	15
7	0.43	0.43	4 (35)	4 (35)	21
8	0.69	1.75	4 (56)	12 (9)	28
9	1.04	6	4 (84)	12 (45)	26
10	1.48	10.4	4 (120)	12 (79)	27
11	2.04	16.72	4 (165)	12 (128)	28
12	2.72	25.07	4 (220)	12 (192)	29
13	3.53	36.88	4 (286)	12 (284)	30
14	4.49	51.86	4 (364)	12 (400)	31
15	5.62	70.78	4 (455)	12 (546)	32
16	6.91	94.37	4 (560)	12 (728)	33
17	8.4	123.41	4 (680)	12 (952)	34
18	10.07	158.67	4 (816)	12 (1224)	35
19	27.77	120.25	12 (160)	12 (733)	35
20	32.94	148.9	12 (191)	12 (902)	35
21	38.74	182.63	12 (226)	12 (1101)	35
22	45.19	224.04	12 (265)	12 (1354)	35
23	51.99	271.79	12 (305)	12 (1645)	35
24	59.95	323.28	12 (354)	12 (1945)	35
25	68.4	385.01	12 (405)	12 (2317)	36
26	77.53	455.09	12 (460)	12 (2739)	35
27	87.47	533.8	12 (520)	12 (3211)	35
28	98.12	622.46	12 (584)	12 (3743)	35
29	109.8	721.17	12 (655)	12 (4333)	35
30	122.22	831.67	12 (730)	12 (4995)	35
31	135.89	953.89	12 (814)	12 (5725)	35
32	150.22	1089.78	12 (901)	12 (6539)	35
33	165.33	1240	12 (992)	12 (7440)	35
34	181.33	1405.33	12 (1088)	12 (8432)	35
35	198.33	1586.67	12 (1190)	12 (9520)	35

Appendix A6: n=36

†Core OMAD.

Appendix A7: n=40

m	A_3	A_4	M_3	M_4	df(2FI)
4	0	0.04	0 (4)	8 (1)	6
5	0	0.2	0 (10)	8 (5)	10
6	0	0.6	0 (20)	8 (15)	15
7	0	1.4	0 (35)	8 (35)	21
8	0	2.8	0 (56)	8 (70)	25
9	0	5.04	0 (84)	8 (126)	27
10	0	8.4	0 (120)	8 (210)	27
11	1.8	15.2	8 (45)	16 (37)	29
12	2.48	23	8 (62)	16 (56)	30
13	3.28	33.4	8 (82)	16 (82)	31
14	4.2	47.12	8 (105)	16 (118)	32
15	5.32	64.12	8 (133)	16 (160)	33
16	6.56	85.6	8 (164)	16 (214)	34
17	8	112	8 (200)	16 (280)	35
18	9.6	144	8 (240)	16 (360)	36
19†	11.4	182.4	8 (285)	16 (456)	37
20†	11.4	228	8 (285)	16 (570)	38
21	35.32	161.96	16 (38)	24 (14)	39
22	40.8	198.2	16 (43)	24 (17)	39
23	47.16	239.08	16 (54)	24 (20)	39
24	53.76	288.64	16 (62)	24 (24)	39
25	61.92	342.48	16 (72)	24 (31)	39
26	70.36	404.04	16 (84)	24 (33)	39
27	79.16	474.72	16 (97)	24 (40)	39
28	88.08	553.72	16 (109)	24 (49)	39
29	98.72	641.32	16 (121)	24 (53)	39
30	109.92	740.52	16 (139)	24 (63)	39
31	121.24	850.52	16 (154)	24 (72)	39
32	134	971.68	16 (173)	24 (82)	39
33	147.36	1105.92	16 (194)	24 (93)	39

†Core OMAD.

Appendix A7: n=40

m	A_3	A_4	M_3	M_4	df(2FI)
34	161.64	1253.48	16 (216)	24 (107)	39
35	176.92	1415.12	16 (240)	24 (123)	39
36	192.96	1592.04	16 (266)	24 (137)	39
37	210	1785	16 (294)	24 (153)	39
38†	228	1995	16 (323)	24 (171)	39
39†	247	2223	16 (361)	24 (171)	39

†Core OMAD.

Appendix A8: n=44

m	A_3	A_4	M_3	M_4	df(2FI)
4	0.03	0.01	4 (4)	4 (1)	6
5	0.08	0.04	4 (10)	4 (5)	10
6	0.17	0.12	4 (20)	4 (15)	15
7	0.29	0.29	4 (35)	4 (35)	21
8	0.46	1.04	4 (56)	12 (7)	28
9	0.69	2.63	4 (84)	12 (24)	36
10	1.12	5.11	12 (2)	12 (51)	43
11	1.89	8.41	12 (8)	12 (86)	43
12	4.2	12.16	12 (36)	12 (122)	43
13	5.67	18.6	12 (50)	12 (192)	43
14	7.5	25.33	12 (68)	12 (258)	43
15	9.58	34.36	12 (88)	12 (349)	43
16	12.1	45.59	12 (113)	12 (462)	43
17	15.01	59.6	12 (142)	12 (604)	43
18	18.38	73.95	12 (176)	12 (736)	43
19	22.29	96.36	12 (216)	12 (973)	43
20	26.21	119.31	12 (254)	12 (1199)	43
21†	30.83	149.23	12 (300)	12 (1509)	43
22†	36.07	180.19	12 (353)	12 (1811)	43
23	41.61	216.59	12 (408)	12 (2169)	43
24	47.8	261.04	12 (470)	12 (2620)	43
25	54.71	309.37	12 (540)	12 (3098)	43
26	62.08	366.07	12 (614)	12 (3668)	43
27	69.86	430.66	12 (691)	12 (4320)	43
28	78.71	500.26	12 (781)	12 (5007)	43
29	87.92	580.75	12 (873)	12 (5815)	43
30	97.95	669.07	12 (974)	12 (6694)	43
31	108.55	768.47	12 (1080)	12 (7690)	43
32	120.07	878.02	12 (1196)	12 (8785)	43
33	132.3	999.27	12 (1319)	12 (9999)	43
34	145.26	1131.83	12 (1449)	12 (11322)	43
35	159.02	1277.69	12 (1587)	12 (12780)	43
36	173.65	1437.17	12 (1734)	12 (14374)	43
37	189.11	1611.35	12 (1889)	12 (16116)	43
38	205.52	1800.52	12 (2054)	12 (18006)	43
39	222.77	2006.24	12 (2227)	12 (20063)	43
40	240.93	2229.07	12 (2409)	12 (22291)	43
41	260	2470	12 (2600)	12 (24700)	43
42†	280	2730	12 (2800)	12 (27300)	43
43†	301	3010	12 (3010)	12 (30100)	43

†Core OMAD.

Appendix A9: n=48

m	A_3	A_4	M_3	M_4	df(2FI)
4	0	0	0	0	6
5	0	0	0	0	10
6	0	0.11	0	16 (1)	15
7	0	0.33	0	16 (3)	21
8	0	1.22	0	16 (11)	27
9	0	2.44	0	16 (22)	29
10	0	5.33	0	16 (48)	31
11	0	9.11	0	16 (82)	32
12	0	15.33	0	16 (138)	33
13	0	23	0	16 (207)	34
14	2.72	40.06	8 (98)	16 (289)	36
15	3.5	53.72	8 (126)	16 (385)	37
16	4.33	71.94	8 (156)	16 (517)	38
17	5.33	94.06	8 (192)	16 (676)	39
18	6.42	121.31	8 (231)	16 (873)	40
19	7.67	153.67	8 (276)	16 (1106)	41
20	9.03	192.25	8 (325)	16 (1384)	42
21	10.56	237.5	8 (380)	16 (1710)	43
22	12.22	290.28	8 (440)	16 (2090)	44
23†	14.06	351.39	8 (506)	16 (2530)	45
24†	14.06	421.67	8 (506)	16 (3036)	46
25	49.39	281.67	16 (135)	16 (852)	47
26	56.22	334.61	16 (156)	16 (1016)	47
27	63.22	393.5	16 (176)	16 (1201)	47
28	70.78	457.56	16 (197)	16 (1387)	47
29	79.44	530.39	16 (227)	16 (1600)	47
30	88.42	612.36	16 (255)	16 (1852)	47
31	98.5	700.78	16 (285)	16 (2108)	47
32	109.03	800.58	16 (317)	16 (2412)	47
33	119.83	911.33	16 (350)	16 (2743)	47
34	131.58	1032.08	16 (386)	16 (3104)	47
35	144.44	1164.44	16 (425)	16 (3499)	47
36	157.64	1310.08	16 (465)	16 (3938)	47
37	171.81	1468.56	16 (509)	16 (4413)	47
38	186.72	1641	16 (554)	16 (4929)	47
39	202.08	1828.94	16 (602)	16 (5490)	47
40	219.06	2031.28	16 (653)	16 (6098)	47
41	236.47	2250.94	16 (706)	16 (6757)	47
42	254.78	2487.83	16 (762)	16 (7467)	47
43	274.06	2742.61	16 (821)	16 (8229)	47
44	294.22	3016.78	16 (882)	16 (9051)	47
45	315.33	3311	16 (946)	16 (9933)	47
46†	337.33	3626.33	16 (1012)	16 (10879)	47
47†	360.33	3963.67	16 (1081)	16 (11891)	47

†Core OMAD.