RADIAL AND NON-RADIAL SOLUTIONS TO $\Delta^3 u + u^{-q} = 0$ IN $\mathbf{R}^3$

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ABSTRACT. Inspired by recent studies on the biharmonic equation $\Delta^2 u + u^{-q} = 0$ in $\mathbf{R}^3$, where q is a real number, we consider the higher-order analogous equation

$$\Delta^3 u + u^{-q} = 0 \quad \text{in } \mathbf{R}^3 .$$

It is known that this equation admits positive classical solutions that are radially symmetric if, and only if, $q > 1/2$. Besides, under the restriction $q > 1/2$, it is also known that there is a branch of radially symmetric solutions to the equation having the growth at infinity as that of $|x|^4$. In the first part of the paper, by a careful phase-plane analysis, we provide a complete description of possible growth at infinity for radially symmetric solutions to the equation. Having such a classification of growth, in the last part of the paper, we construct non-radial solutions to the equation via a fixed-point argument. To obtain these results, we borrow some ideas often used in the case of biharmonic equations. However, compared with the case of biharmonic equations, there are some differences leading to new difficulties. A typical example is that it is not clear if solutions to the equation enjoy the super polyharmonic property, which is often used to overcome the lack of maximum principles.

1. INTRODUCTION

In this paper, we are interested in positive C^6-solutions u to the following geometric interesting equation

$$\Delta^3 u + u^{-q} = 0 \tag{1.1}$$

in $\mathbf{R}^3$ with q being a real number. This equation can be rewritten in a traditional way as follows $(-\Delta)^3 u = u^{-q}$. Later on we shall see that we must have $q > 1/2$ but now let us discuss the significance of studying (1.1) and the reason why we work on this equation. Roughly speaking, the last three decades have witnessed the presence of significant advances in the theory of higher-order elliptic equations of the form

$$(-\Delta)^m u = f u^{-q} \quad \text{in } \mathbf{R}^n$$

starting from the work [WX99] by Wei and Xu in the case of constant functions f. These equations have their root in various branches of pure and applied mathematics. While the second-order case is very classical and often appears in many textbooks, the fourth-order case is of interest because it arises in the theory of elasticity. Higher-order cases such as (1.1) are also of interest because they serve as a tool for many problems in pure mathematics, very often in conformal geometry and geometric analysis. In this direction, equations of the form (1.1) come from the problem of prescribing Q-curvature on $\mathbb{S}^3$, which is associated with the conformally covariant GJMS operator of order six with the principle part $(-\Delta)^3$, discovered by Graham, Jenne, Mason, and Sparling in [GJMS92]. This operator is a high-order elliptic operator analogue with the well-known conformal Laplacian which is of second order and with the Paneitz operator which is of fourth order. Let us discuss this sixth order GJMS operator more precise.

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Suppose that f be a smooth function on the unit sphere $\mathbb{S}^3 \subset \mathbf{R}^4$ equipped with the standard metric $g_{\mathbb{S}^3}$. Let $\widetilde{g} = v^{-4/3} g_{\mathbb{S}^3}$ be a conformal metric, then one has the following equation with a critical exponent

$$\Big(-\Delta_{g_{\mathbb{S}^3}} - \frac{21}{4}\Big)\Big(-\Delta_{g_{\mathbb{S}^3}} - \frac{5}{4}\Big)\Big(-\Delta_{g_{\mathbb{S}^3}} + \frac{3}{4}\Big) v = f v^{-3} \quad \text{on } \mathbb{S}^3. \tag{1.2}$$

The geometric meaning of the above equation is that if $v > 0$ is a solution to (1.2), then the conformal metric $v^{-4/3} g_{\mathbb{S}^3}$ has Q-curvature f. Via the stereographic projection $\pi : \mathbb{S}^3 \to \mathbf{R}^3$, one should obtain from (1.2) the following equation

$$(-\Delta)^3 u = (f \circ \pi^{-1}) u^{-3} \quad \text{in } \mathbf{R}^3;$$

see [Han07]. Limiting ourselves to the case of constant function f, one further arrive at the following model equation

$$(-\Delta)^3 u = u^{-3} \quad \text{in } \mathbf{R}^3 . \tag{1.3}$$

It appears that (1.3) is exactly (1.1) with q replaced by -3 after using a simple scaling. Instead of considering (1.2) let us now consider its non-critical cases, namely we consider

$$\Big(-\Delta_{g_{\mathbb{S}^3}} - \frac{21}{4}\Big)\Big(-\Delta_{g_{\mathbb{S}^3}} - \frac{5}{4}\Big)\Big(-\Delta_{g_{\mathbb{S}^3}} + \frac{3}{4}\Big) v = f v^{-q} \quad \text{on } \mathbb{S}^3$$

for arbitrary $q \in \mathbf{R}$. Then still by the stereographic projection $\pi : \mathbb{S}^3 \to \mathbf{R}^3$, one should arrive at

$$(-\Delta)^3 u = \Big(\frac{1}{1+|x|^2}\Big)^{\frac{9-3q}{2}} (f \circ \pi^{-1}) u^{-q} \quad \text{in } \mathbf{R}^3 .$$

Hence, upon an appropriate choice for f to cancel out the term involving $1+|x|^2$ one should arrive at $(-\Delta)^3 u = u^{-q}$ which is exactly our equation (1.1). It should be mentioned that instead of using the sixth order GJMS operator on $\mathbb{S}^3$ one can use the fourth order Paneitz operator on $\mathbb{S}^3$ to obtain the following biharmonic equations $\Delta^2 u = \pm u^{-q}$ in $\mathbf{R}^3$.

Perhaps, we are motivated by the two works: first by Feng and Xu [FX13] and the other by Luo, Wei, and Zou [LWZ16] involving the triharmonic operator Δ^3. While the work by Luo, Wei, and Zou focuses on solutions to the triharmonic Lane–Emden equation

$$(-\Delta)^3 u = |u|^{p-1} u$$

in $\mathbf{R}^n$ with $p > 1$, the work of Feng and Xu focuses on non-negative solutions to

$$(-\Delta)^3 u = -u^{-q} \tag{1.4}$$

in $\mathbf{R}^5$ with $q > 0$. Clearly, equation (1.4) is different from equation (1.1) by a minus sign. In the work [FX13] this plays an important role because associated with (1.4) in $\mathbf{R}^5$ is the following integral equation

$$u(x) = \frac{1}{64\pi^2} \int_{\mathbf{R}^5} |x-y| u(y)^{-q} dy. \tag{1.5}$$

From this integral equation one can say more about non-negative, Lebesgue integrable solutions to (1.5); see [Li04, Xu05]. In fact, they are radially symmetric and therefore can be assumed $u(x) = (1+|x|^2)^{1/2}$. If we have (1.1) in $\mathbf{R}^5$ in hand, then it does not make sense to consider similar integral equations. Very recently, the authors in [DF22] further investigate solutions to (1.4) in $\mathbf{R}^n$ for any $n \geq 2$ and $q > 0$ and obtain various properties including the asymptotic behavior for radial solutions at infinity. Similar results were obtained earlier in [NNP18] under the condition $n \geq 1$ and $q \geq -1$. It is worth noting that there is no positive, C^6 solution to (1.4) if $q < -1$; see [NNPY20]. Therefore, the picture of radial solutions to (1.4) in $\mathbf{R}^n$ with $q \in \mathbf{R}$ is quite understood. For the case of non-radial solutions, as shown in [NNPY20], (1.4) in $\mathbf{R}^n$ with $n \geq 1$ admits solutions which are radial if, and only if, $q \geq -1$. Using these radial solutions one can quickly conclude the existence of non-radial solutions to (1.4) in $\mathbf{R}^n$ with $n \geq 2$ and $q \geq -1$. Therefore, we have quite clear picture of existence and non-existence results for solutions to (1.4). This motivates us to work on (1.1) in $\mathbf{R}^3$ since the existence of non-radial solutions is not obvious, see the paragraph after Theorem 1.5.

Let us now discuss (1.1) in $\mathbf{R}^3$. Unlike (1.4) in $\mathbf{R}^5$, the corresponding integral equation of (1.1) in $\mathbf{R}^3$ is

$$u(x) = \frac{1}{96\pi} \int_{\mathbf{R}^3} |x-y|^3 u(y)^{-q} dy. \tag{1.6}$$

From this, by the same beautiful classification of positive solutions to integral equations of the form (1.6), we deduce that $q = 3$ must hold and non-negative, Lebesgue integrable solutions to (1.6) are of the following form $u(x) = (1 + |x|^2)^{3/2}$, up to translations, dilations, and scalings. Although non-negative integrable solutions to the integral equations (1.6) and (1.5) are easily classified, we cannot expect that the structure of the solution set of the differential equations (1.1) and (1.4) is simple. This can be easily seen by comparing (1.4) and (1.5).

In the present paper, motivated by many interesting results for the case of biharmonic equations obtained in [CX09, Gue12], we initiate our study on the structure of solution set of (1.1) in $\mathbf{R}^n$. In view of the geometric meaning described above, the two cases $n = 3$ and $n = 5$ are of interest although one can consider (1.1) in $\mathbf{R}^n$ for arbitrary $n \geq 2$. By a general result of [NNPY20] we conclude that the equation (1.1) in $\mathbf{R}^n$ for any $n \geq 2$ admits one positive C^6-solution if, and only if, $q > 1/2$. In addition, it is proved that (1.1) with $q > 1/2$ admits at least one positive radial solution; see [NNPY20, Proposition A]. Moreover, such a radial solution has a growth as that of $|x|^4$ at infinity; see [KNS88]. Based on this point, we are interested in the structure of radial and non-radial solutions to (1.1). To achieve this goal, we split our analysis into two parts. In the first part, we consider radial solutions to (1.1). Although their existence is clear, it is not clear how they behave near infinity. In the second part, we make use of the behavior of radial solutions to construct non-radial solutions.

Let us discuss the two parts in details. In the first part of the paper, our aim is to *obtain a complete description of the asymptotic behavior at infinity of radial solutions to* (1.1) *for all* $q > 1/2$. Toward answering the above question completely, we start with a complete classification of exact growth of radial solutions to (1.1) for all $q > 1/2$.

Theorem 1.1. *Assume that* u *is a radial solution to* (1.1). *Then we have the following claims:*

(a) u *grows at least cubically and at most quartically.*
(b) u *grows either cubically or quartically if* $q > 1$.
(c) u *grows either like* $r^3\sqrt{\log r}$ *or quartically if* $q = 1$.
(d) u *grows either like* $r^{6/(q+1)}$ *or quartically if* $1/2 < q < 1$.

We prove Theorem 1.1 in section 3 below. This simply follows from Proposition 3.1 whose proof is done by a careful examination of the limit of $\Delta^2 u$ at infinity. It is worth noting that although the method used is more or less standard to experts, a few new ideas is required.

To be able to discuss our next result, one should notice that although Theorem 1.1 provides us a complete picture of growth at infinity, it does not tell us the precise asymptotic behavior at infinity. For example, if we know that the radial solution u grows quartically at infinity, then Theorem 1.1 does not give us the value of $|x|^{-4}u(x)$ at infinity. In the next result, we are able to prove the existence of radial solutions with prescribed asymptotic behavior at infinity for any $q > 1/2$. This, in particular, provides an affirmative answer for the question of existence raised earlier. Let us state our next result.

Theorem 1.2. *We have the following claims:*

(a) *For $q > 1$ and given any $\kappa > 0$, there exists a radially symmetric solution u to (1.1) such that*

$$\lim_{|x|\nearrow+\infty} \frac{u(x)}{|x|^3} = \kappa.$$

(b) *For $q = 1$, there exists a radially symmetric solution u to (1.1) such that*

$$\lim_{|x|\nearrow+\infty} \frac{u(x)}{|x|^3(\log|x|)^{1/2}} = \frac{1}{\sqrt{12}}.$$

(c) *For $1/2 < q < 1$, there exists a radially symmetric solution u to (1.1) such that u grows exactly between cubic and quartic in the sense that*

$$\lim_{|x|\nearrow+\infty} \frac{u(x)}{|x|^{6/(q+1)}} = (-K_0)^{-\frac{1}{q+1}},$$

where K_0 is a negative constant given by

$$K_0 = \frac{72(2q-1)(q-1)(q-2)(q-5)(q+7)}{(q+1)^6}.$$

(d) *For $q > 1/2$ and given any $\kappa > 0$, there exists infinitely many radial solution u to (1.1) such that*

$$\lim_{|x|\nearrow+\infty} \frac{u(x)}{|x|^4} = \kappa.$$

We prove Theorem 1.2 in section 5 (for case (a)), in section 6 (for cases (b) and (c)), and in section 7 (for case (d)). The idea of proof is to make use of the ODE version of (1.1) obtained via the shooting method; see the initial value (2.10). Such an approach is often used when working on biharmonic equations. For interested readers, we refer to [Gue12, ND17a] and the references there in.

Clearly, Theorem 1.2(d) indicates that (1.1) in the full range $q > 1/2$ always admits radial solutions with quartic growth at infinity. However, parts (a)–(c) imply that (1.1) admits another branch of radial solutions whose growth at infinity is strictly less than quartic.

Once we have radial solutions with the prescribing asymptotic behavior at infinity, see Theorem 1.2, we can compute lower order terms of the expansion at infinity. For simplicity, let us only treat radial solutions having either cubic or quartic growth at infinity. Let us consider radial solutions with cubic growth at infinity. Our next result concerns lower order terms in the expansion of these solutions.

Theorem 1.3. *Let $\kappa > 0$ be arbitrary and suppose that u is a radially symmetric solution with exactly cubic growth $\kappa > 0$ at infinity, namely,*

$$\lim_{|x|\nearrow+\infty} \frac{u(x)}{|x|^3} = \kappa.$$

Then we have the following further asymptotic behavior.

(a) *For $q > 4/3$,*

$$\lim_{|x|\nearrow+\infty} \frac{u(x) - \kappa|x|^3}{|x|^2} = \frac{\Delta u(0)}{6} - \frac{1}{12}\int_0^\infty |x|^3 u^{-q}(x)dx.$$

(b) *For $q = 4/3$,*

$$\lim_{|x|\nearrow+\infty} \frac{u(x) - \kappa|x|^3}{|x|^2\log|x|} = -\frac{1}{12\kappa^{4/3}}.$$

(c) *For $1 < q < 4/3$,*

$$\lim_{|x|\nearrow+\infty} \frac{u(x) - \kappa|x|^3}{|x|^{6-3q}} = \chi$$

with

$$\chi = \frac{1}{12\kappa^q}\begin{pmatrix} \dfrac{1}{2(3-3q)} - \dfrac{1}{10(2-3q)} - \dfrac{1}{4-3q} \\ -\dfrac{1}{2(6-3q)} + \dfrac{1}{5-3q} + \dfrac{1}{10(7-3q)} \end{pmatrix}. \tag{1.7}$$

We prove Theorem 1.3 in section 5. For radial solutions with quartic growth at infinity, our result concerning lower order terms of these solutions is as follows.

Theorem 1.4. *Let $\kappa > 0$ be arbitrary and suppose that u is a radially symmetric solution with exactly quartic growth $\kappa > 0$ at infinity, namely*

$$\lim_{|x| \nearrow +\infty} \frac{u(x)}{|x|^4} = \kappa.$$

Then we have the following further asymptotic behavior.

(a) *For $q > 3/4$,*
$$\lim_{|x| \nearrow +\infty} \frac{u(x) - \kappa|x|^4}{|x|^3} = \frac{1}{24}\int_0^\infty |x|^2 u^{-q}(x) dx.$$
(b) *For $q = 3/4$,*
$$\lim_{|x| \nearrow +\infty} \frac{u(x) - \kappa|x|^4}{|x|^3 \log|x|} = \frac{1}{24\kappa^{3/4}}.$$
(c) *For $1/2 < q < 3/4$,*
$$\lim_{|x| \nearrow +\infty} \frac{u(x) - \kappa|x|^4}{|x|^{6-4q}} = \chi$$
with
$$\chi = \frac{1}{12\kappa^q}\begin{pmatrix} \dfrac{1}{2(3-4q)} - \dfrac{1}{10(2-4q)} - \dfrac{1}{4-4q} \\ -\dfrac{1}{2(6-4q)} + \dfrac{1}{5-4q} + \dfrac{1}{10(7-4q)} \end{pmatrix}. \tag{1.8}$$

We prove Theorem 1.4 in section 7. Let us sketch how to prove Theorem 1.3 and Theorem 1.4. In fact, the proof of these two theorems simply follows from integral representations (5.4) and (7.1).

Motivated by a classification result due to [CX09] for biharmonic equations, it would be interesting to know the set of entire solutions to (1.1) when $q = 3$. In this scenario, by a direct calculation, it is easy to verify that the function $u(x) = 315^{-1/4}(1 + |x|^2)^{3/2}$ solves (1.3). We expect that up to dilations and translations, the only entire solutions to (1.3), which has an exact asymptotic behavior at infinity, is that above; see [GW07] for a similar case of lower order. Due to the limit of length, we leave this issue here and shall address it in a forthcoming paper.

Finally, let us discuss our last result, which is also the second part of the paper. So far we have discussed the existence of radial solutions to (1.1) as well as the their growth at infinity. From this one can ask whether or not a non-radial solution actually exists. In the context of biharmonic equations of the form $\Delta^2 u + u^{-q} = 0$ in $\mathbf{R}^3$, it is proved in [HW19] that such a non-radial solution indeed exists. Motivated by this interesting result, we establish a similar result in the context of triharmonic equation of the form (1.1). Our result, which is also the last, reads as follows.

Theorem 1.5. *Let $q > 1/2$, there exists a non-radial, positive, C^6 solution to (1.1).*

There is a simple trick to prove the existence of non-radial solutions. In the case of (1.1), it is proved in [NNPY20] that (1.1) in $\mathbf{R}^n$ with $n \geq 3$ and $q > 1/2$ always admits solutions. Therefore, any radial solution to (1.1) in $\mathbf{R}^n$ immediately becomes a non-radial solution to (1.1) in $\mathbf{R}^{n+1}$. This proves the existence of non-radial solutions to (1.1) in $\mathbf{R}^n$ with $n \geq 4$. However, it is also worth noting that (1.1) in $\mathbf{R}^2$ with arbitrary $q \in \mathbf{R}$ does not admit any C^6-solution; see [NNPY20], which makes the existence of non-radial solutions to (1.1) in $\mathbf{R}^3$ is of interest and non-trivial. To prove Theorem 1.5, we adopt the method used in [HW19], which is based on a fixed-point argument. However, to be able to handle the higher-order case, a few new idea is introduced.

From the above discussion, one can also consider the equation in (1.1) in $\mathbf{R}^5$. However, due to the limit of length, we leave this for future research. As the last comment, we should mention that an earlier version of this work is already available, see [ND17b]. The existence of non-radial solutions to (1.1) in Theorem 1.5 is the major difference between this version and the previous one.

Before closing this section, we briefly mention the organization of the present paper.

Contents

2. PRELIMINARIES

In this section, we collect some useful results. First we discuss spherical averages of functions in subsection 2.1. Then, we study growth of radial solutions to (1.1) in subsection 2.2. Finally, we spend a large portion of this section to study the initial value problem (2.10) obtained via the shooting method. These play an important role in our analysis.

2.1. **Spherical average and a comparison principle for radial functions.** To understand the structure of solutions to (1.1), we often rewrite (1.1) as the following system

$$\begin{cases} \Delta u = v & \text{in } \mathbf{R}^3, \\ \Delta v = w & \text{in } \mathbf{R}^3, \\ \Delta w = -u^{-q} & \text{in } \mathbf{R}^3. \end{cases} \tag{2.1}$$

For each function f, we denote by $\overline{f}_{x_0}(r)$ the spherical average of f centered at some point $x_0 \in \mathbf{R}^3$ with radius r, that is

$$\overline{f}_{x_0}(r) = \fint_{\partial B(x_0,r)} f d\sigma.$$

For simplicity, if either x_0 is the origin or no confusion occurs, we simply write $\overline{f}_{x_0}(r)$ as $\overline{f}(r)$. The spherical average has the following nice property $\overline{\Delta f} = \Delta \overline{f}$ which is easy to verify. By the Jensen inequality, there holds

$$\overline{f}_{x_0}^{-q}(r) \leq \overline{f_{x_0}^{-q}}(r)$$

for all r. Keep in mind that the following rule

$$\Delta^k f(r) = r^{-2k}(r^{2k} f^{(2k-1)})', \tag{2.2}$$

which holds only in $\mathbf{R}^3$, will be used frequently throughout the paper. Throughout this paper, we frequently apply the following well-known comparison principle for solutions of poly-harmonic equations; see [FF16, Proposition A.2].

Lemma 2.1. *Let $p \in \mathbb{N}$ with $p \geq 1$ and assume that $f : \mathbf{R} \to \mathbf{R}$ is locally Lipschitz continuous and monotonically increasing. Let also $\underline{u}, \overline{u} \in C^{2p}([0,R))$ be such that*

$$\Delta^p \overline{u}(r) - f(\overline{u}(r)) \geq \Delta^p \underline{u}(r) - f(\underline{u}(r))$$

for all $r \in [0,R)$ and that

$$\begin{cases} \overline{u}(0) \geq \underline{u}(0), \quad \overline{u}'(0) \geq \underline{u}'(0), \\ \Delta^k \overline{u}(0) \geq \Delta^k \underline{u}(0), \quad (\Delta^k \overline{u})'(0) \geq (\Delta^k \underline{u})'(0) \quad \text{for all } k = 1, 2, ..., p-1. \end{cases}$$

Then for any $r \in [0,R)$ and for all $k = 1, 2, ..., p-1$, we have

$$\begin{cases} \overline{u}(r) \geq \underline{u}(r), \quad \overline{u}'(r) \geq \underline{u}'(r), \\ \Delta^k \overline{u}(r) \geq \Delta^k \underline{u}(r), \quad (\Delta^k \overline{u})'(r) \geq (\Delta^k \underline{u})'(r). \end{cases}$$

Moreover, the initial point 0 can be replaced by any initial point $\rho > 0$ if all the $2p$ initial data are weakly ordered and a strict inequality in one of these initial data at ρ or in the differential inequality in (ρ, R) implies a strict ordering of $\overline{u}$, $\overline{u}'$, $\Delta^k \overline{u}$, $(\Delta^k \overline{u})'$ and $\underline{u}$, $\underline{u}'$, $\Delta^k \underline{u}$, $(\Delta^k \underline{u})'$ on (ρ, R) for any $k \in \{1, 2, ..., p-1\}$.

2.2. **A super poly-harmonic property and growth of radial solutions.** Now let $u > 0$ solve (1.1). Then we take the spherical average of (2.1) centered at some point $x_0 \in \mathbf{R}^3$ to be specified later to get

$$\Delta \overline{u} = \overline{v}, \quad \Delta \overline{v} = \overline{w}, \quad \text{and} \quad \Delta \overline{w} \leq -\overline{u}^{-q} \quad \text{in } \mathbf{R}^3. \tag{2.3}$$

(As mentioned earlier, we simply write $\overline{u}$ instead of $\overline{u}_{x_0}$.) Since the underlying equation is higher order, it is common to determine the sign of v and w. In the following lemma, we show that w has a sign, which is important in the rest of analysis.

Lemma 2.2. *If $u > 0$ is a C^6 positive solution in $\mathbf{R}^3$, then we necessarily have $w > 0$.*

This type of result is well-known for solutions to the biLaplace equation $\Delta^2 u = -u^{-q}$ in $\mathbf{R}^n$ with $n \geq 3$; see [CX09, Lemma 2.2] for the case $n = 3$ and [LY16, Lemma 4.1] for the case of arbitrary n. In the case of the triLaplace equation $\Delta^3 u = -u^{-q}$ in $\mathbf{R}^3$, we simply mimic the proof provided in [CX09, LY16]. Therefore, we omit it here and leave the detail for interested readers.

In the previous result, we have shown that w has a sign, which is good to control the sign of $\overline{w}$ and higher order derivatives of $\overline{u}$. To overcome the *lacking of the maximum principle* since the underlying equation is of higher order, it is commonly to make use of the sign of w and v; see [WX99]. Unfortunately, it seems to be difficult to capture the sign of v. Without having any sign control of v but w, we refer to the *partially super polyharmonic property* for solutions to (1.1); see [NY22] for further information.

Fortunately, inspire by [CX09, Lemma 2.3], in the following step, we can control the sign of derivatives of $\overline{w}$ and $\overline{v}$. We note that the result below is independent of the center x_0 that we are using to compute the average.

Lemma 2.3. *We have the following claims:*

$$\overline{w}'(r) < 0, \tag{2.4}$$

and

$$\overline{v}'(r) > 0, \quad \overline{v}''(r) > 0, \quad \overline{v}^{(3)}(r) < 0 \tag{2.5}$$

for all $r > 0$.

Proof. Recall that $w > 0$. Using $\Delta \overline{w} \leq -\overline{u}^{-q}$ in (2.3) we obtain

$$r^2 \overline{w}'(r) \leq -\int_0^r s^2 \overline{u}^{-q}(s) ds;$$

hence $\overline{w}'(r) < 0$ for any $r > 0$. This proves (2.4). Now we use $\Delta \overline{v} = \overline{w}$ in (2.3) to get

$$r^2 \overline{v}'(r) = \int_0^r s^2 \overline{w}(s) ds > 0$$

which implies $\overline{v}'(r) > 0$ for $r > 0$. To estimate higher order derivatives of $\overline{v}$, we note from (2.2) and $\Delta^2 \overline{v} = -\overline{u}^{-q}$ that

$$r^4 \overline{v}^{(3)}(r) \leq -\int_0^r s^4 \overline{u}^{-q}(s) ds.$$

Therefore, $\overline{v}^{(3)}(r) < 0$ for $r > 0$. Consequently, $\overline{v}''$ is monotone decreasing. Suppose $\overline{v}''(r) = 0$ at some $r = r_1 > 0$. Fix some $r_2 > r_1$. Then there exists some $\delta > 0$ such that $\overline{v}''(r) \leq -\delta$ for all $r \geq r_2$. Integrating both sides of the preceding inequality gives

$$\overline{v}'(r) \leqslant \overline{v}'(r_2) - \delta(r - r_2)$$

for all $r \geq r_2$. Sending $r \nearrow +\infty$, we deduce that $\overline{v}' < 0$ for some large r, which contradicts to the fact that $\overline{v}'(r) > 0$ for $r > 0$. Thus, (2.5) is proved. $\square$

Up to this position, we are left with the sign of $\overline{v}$ as well as the sign of derivatives of $\overline{u}$. In the following result, we show how to control the sign of higher order derivatives of $\overline{u}$.

Lemma 2.4. *We have the following claims:*

$$\overline{u}^{(3)}(r) > 0, \quad \overline{u}^{(4)}(r) > 0, \quad \overline{u}^{(5)}(r) < 0 \tag{2.6}$$

for all $r > 0$.

Proof. First, by using $\Delta^2 \overline{u} = r^{-4}(r^4 \overline{u}^{(3)})' > 0$, see (2.2), we obtain $\overline{u}^{(3)}(r) > 0$. Similarly, still by (2.2), namely $\Delta^3 \overline{u} = r^{-6}(r^6 \overline{u}^{(5)})' < 0$, we obtain $\overline{u}^{(5)}(r) < 0$. Consequently, $\overline{u}^{(4)}(r)$ is strictly decreasing. Suppose that $\overline{u}^{(4)}(r) = 0$ at some $r = r_3 > 0$. Fix some $r_4 > r_3$. Then there exists some $\delta > 0$ such that

$$\overline{u}^{(4)}(r) \leqslant -\delta$$

for all $r \geq r_4$. Integrating both sides of the preceding inequality gives

$$\overline{u}^{(3)}(r) \leqslant \overline{u}^{(3)}(r_4) - \delta(r - r_4)$$

for all $r \geq r_4$. From this we deduce that $\overline{u}^{(3)}(r) < 0$ for large r, which is a contradiction; thus showing $\overline{u}^{(4)}(r) > 0$. This finishes the proof of (2.6). □

An immediate consequently of Lemma 2.4 is the following.

Proposition 2.5. *Growth of any radial solution to* (1.1) *is at least cubic and at most quartic at infinity.*

Proof. Suppose that u is a radial solution to (1.1) about some point $x_0 \in \mathbf{R}^3$. As always, by $\overline{u}$, we mean the spherical average of u centered at the x_0. To bound the growth of u at infinity, it suffices to bound the growth of $\overline{u}$ at infinity. Since $\overline{u}^{(4)}(r) > 0$, we obtain $\overline{u}^{(3)}(r) \geq \overline{u}^{(3)}(1) > 0$ for any $r \geq 1$. From this we get

$$\overline{u}''(r) \geq \overline{u}^{(3)}(1)(r-1) + \overline{u}''(1) \tag{2.7}$$

for all $r \geq 1$. From this, simply integrating twice, we deduce that $\overline{u}$ grows at least cubic at infinity. To obtain the greatest growth at infinity for $\overline{u}$, we observe from (2.4) that

$$\Delta \overline{v}(r) = \overline{w}(r) \leq \overline{w}(0),$$

which by integration implies

$$\overline{u}(r) \leq \overline{u}(0) + \frac{\overline{v}(0)}{12} r^2 + \frac{\overline{w}(0)}{120} r^4 \tag{2.8}$$

and this is enough to conclude the assertion. □

Up to this point, we have not mention the sign of $\overline{v}$ as well as $\overline{u}'$ and $\overline{u}''$. The next lemma addresses this which will be used in the proof of Lemma 3.9.

Lemma 2.6. *There hold*

$$\overline{u}'(r) > 0, \quad \overline{u}''(r) > 0, \quad \text{and} \quad \overline{v}(r) > 0$$

provided $r > 0$ *is large enough.*

Proof. First in view of (2.7) we quickly conclude that $\overline{u}''(r) > 0$ if $r > 0$ is large enough. If we integrate both sides of (2.7) over $[1, r]$, then we get

$$\overline{u}'(r) \geq (\overline{u}^{(3)}(1)/2)(r-1)^2 + \overline{u}''(1) r + \overline{u}'(1)$$

for all $r \geq 1$. In particular, $\overline{u}'(r) > 0$ if $r > 0$ is large enough. For the function $\overline{v}$, making use of (2.5) yields $\overline{v}'(r) \geq \overline{v}'(1) > 0$ for all $r \geq 1$. From this we deduce that

$$\overline{v}(r) \geq \overline{v}'(1)(r-1) + \overline{v}(1) \tag{2.9}$$

for all $r \geq 1$. In particular, $\overline{v}(r) > 0$ provided r is large enough. The proof is now completes. $\square$

We conclude this section by the following remarks which give us some information on the sign of $\overline{v}$ for any $r \geq 0$.

Remark 2.7. Although it is not clear whether or not $\overline{v}(r) > 0$ for any $r \geq 0$. However, we can easily conclude that there is a point $x_0 \in \mathbf{R}^3$ such that $\overline{v}_{x_0}(r) > 0$ for all $r \geq 0$. Indeed, by way of contradiction, we know that with respect to origin $O \in \mathbf{R}^3$, there holds $\overline{v}_O(r) < 0$ for all $r \geq 0$. However, this contradicts (2.9) if we choose r large enough; hence proving the existence of some point x_0.

Remark 2.8. In view of Remark 2.7 above, we easily deduce that $\overline{u}'_{x_0}(r) > 0$ and that $\overline{u}''_{x_0}(r) > 0$ for any $r \geq 0$.

2.3. **An initial value problem.** Since we are only interested in radial solutions to (1.1), it is necessary to study the radial version of (1.1). In view of the shooting method, we shall keep $u(0)$ fixed. Therefore, suppose $\beta > 0$, we consider the following initial value problem:

$$\begin{cases} \Delta^3 U = -U^{-q}, & U > 0, \qquad r \in (0, R_{\max}(\beta)), \\ U(0) = 1, \quad \Delta U(0) = \beta, & \Delta^2 U(0) = \beta, \\ U'(0) = 0, \quad (\Delta U)'(0) = 0, & (\Delta^2 U)'(0) = 0, \end{cases} \tag{2.10}$$

where $[0, R_{\max}(\beta))$ is the maximal interval of existence of solutions. (Such an existence of solutions for (2.10) follows from standard ODE theory.) The main result of this subsection is Proposition 2.11. To obtain such a result, we follow closely the arguments used in [KR03] for radially symmetric solutions for the equation $\Delta^2 u + u^{-q} = 0$ in $\mathbf{R}^3$; see also [GW08] and [Gue12]. However, our analysis is rather involved due to the fact that we are dealing with higher order equations.

By the l'Hôpital rule, it is not hard to see that $(\Delta u)(0) = 3u''(0)$, $(\Delta u)'(0) = 2u^{(3)}(0)$, $(\Delta^2 u)(0) = 5u^{(4)}(0)$, and $(\Delta^2 u)'(0) = 3u^{(5)}(0)$. Therefore, the initial value problem (2.10) can be rewritten as follows

$$\begin{cases} \Delta^3 U = -U^{-q}, & U > 0, \qquad r \in (0, R_{\max}(\beta)), \\ U(0) = 1, \quad U''(0) = \dfrac{\beta}{3}, & U^{(4)}(0) = \dfrac{\beta}{5} \\ U'(0) = 0, \quad U^{(3)}(0) = 0, & U^{(5)}(0) = 0. \end{cases} \tag{2.11}$$

We note that although a local solution of (2.10) always exists for any given $\beta > 0$, such a solution may not entire in the sense that its maximum interval of existence could be finite. The statement Proposition 2.11(b) below indicates that whenever $q > 1/2$ we successfully obtain an entire solution of (2.10) if the parameter β is chosen appropriately. Consequently, we do have an existence result for entire solutions to (1.1) in $\mathbf{R}^3$.

In the sequel, we frequently apply Lemma 2.1 in the following way: Suppose two positive C^6-functions $\underline{u}(r)$ and $\overline{u}(r)$ are given with

$$\begin{cases} \Delta^3 \underline{u} + \underline{u}^{-q} \leq 0, \\ \underline{u}(0) \leq 1, \quad \underline{u}'(0) = 0, \quad \underline{u}''(0) \leq \delta, \\ \underline{u}^{(3)}(0) = 0, \quad \underline{u}^{(4)}(0) \leq \delta, \quad \underline{u}^{(5)}(0) = 0 \end{cases}$$

and

$$\begin{cases} \Delta^3 \overline{u} + \overline{u}^{-q} \geq 0, \\ \overline{u}(0) \geq 1, \quad \overline{u}'(0) = 0, \quad \overline{u}''(0) \geq \delta, \\ \overline{u}^{(3)}(0) = 0, \quad \overline{u}^{(4)}(0) \geq \delta, \quad \overline{u}^{(5)}(0) = 0. \end{cases}$$

Then $\underline{u}$ and $\overline{u}$ are called sub- and super-solutions relative to the initial value problem

$$\begin{cases} \Delta^3 u + u^{-q} = 0, \\ \qquad u(0) = 1, \quad u'(0) = 0, \quad u''(0) = \delta, \\ u^{(3)}(0) = 0, \quad u^{(4)}(0) = \delta, \quad u^{(5)}(0) = 0. \end{cases}$$

Lemma 2.1 applied to $\overline{u}, u, \underline{u}$ yields the conclusion that $\underline{u} \le u \le \overline{u}$, $\underline{u}' \le u' \le \overline{u}'$ on their common interval of existence. Moreover, strict inequality holds as soon as one strict inequality holds in the initial conditions for the function or its derivatives.

First, analogue to [KR03, Lemma 3.3], we prove that solutions to (2.11) cannot be entire if β is small.

Lemma 2.9. *There exists some $\underline{\beta} > 0$ such that for all $\beta \le \underline{\beta}$, any solution u to* (2.11) *with*

$$u(0) = 1, \quad \Delta u(0) = \beta, \quad \Delta^2 u(0) = \beta$$

has compact support.

Proof. Our aim is to construct a super-solution $\overline{U}$ to (2.11), which has compact support. Indeed, thanks to $q > 1/2$ we easily verify that $(2/3)(1 + 1/q) < 2$. Therefore, we can select a $\tau \in (2/3, 1)$ such that $\tau < (2/3)(1 + 1/q)$ and fix it. Now consider the following function

$$U(r) = \varepsilon r^2(\varepsilon^{-\tau} - r^4 + r^2) + 1$$

for some $\varepsilon > 0$ to be specified later. Clearly U is positive on $(0, 1/2 + \sqrt{1 + 4\varepsilon^{-\tau}}/2)^{1/2})$ and $U(r) \le 1$ for any $r \ge (1/2 + \sqrt{1 + 4\varepsilon^{-\tau}}/2)^{1/2}$. A direct calculation shows that

$$\Delta^3 U(r) = -720\varepsilon$$

and that $U(0) = 1$, $U'(0) = 0$, $U''(0) = 2\varepsilon^{1-\tau}$, $U^{(3)}(0) = 0$, $U^{(4)}(0) = 24\varepsilon$, and $U^{(5)}(0) = 0$. It is to determine ε in such a way that $\Delta^3 U \ge -U^{-q}$. We observe that the maximum of U over $(0, (1/2 + \sqrt{1 + 4\varepsilon^{-\tau}}/2)^{1/2})$ is obtained at

$$r = \frac{\sqrt{3 + 3\sqrt{1 + 3\varepsilon^{-\tau}}}}{3}.$$

In addition, the maximum value of U at this point is

$$\frac{1}{27}\big[(6\varepsilon^{-\tau} + 2)\sqrt{1 + 3\varepsilon^{-\tau}} + 9\varepsilon^{-\tau} + 2\big]\varepsilon + 1.$$

Therefore, to fulfill the differential inequality in $(0, (1/2 + \sqrt{1 + 4\varepsilon^{-\tau}}/2)^{1/2})$, we require

$$\big((6\varepsilon^{-\tau} + 2)\sqrt{1 + 3\varepsilon^{-\tau}} + 9\varepsilon^{-\tau} + 2\big)\varepsilon + 1 \le (720\varepsilon)^{-1/q}. \tag{2.12}$$

To find some suitable $\varepsilon > 0$ satisfying (2.12), we rewrite (2.12) as the following

$$6\varepsilon^{1-3\tau/2}\sqrt{\varepsilon^\tau + 3} + 2\sqrt{\varepsilon^2 + 3\varepsilon^{2-\tau}} + 9\varepsilon^{1-\tau} + 2\varepsilon + 1 \le (720\varepsilon)^{-1/q}. \tag{2.13}$$

Clearly, all three terms $2\sqrt{\varepsilon^2 + 3\varepsilon^{2-\tau}}$, 2ε, and $\varepsilon^{1-\tau}$ converge to 0 as $\varepsilon \to 0$. From our choice for τ, it is clear that $0 > 1 - 3\tau/2 > -1/q$. From this we know that $\varepsilon^{1-3\tau/2}\sqrt{\varepsilon^\tau + 3}$ grows slower than $\varepsilon^{-1/q}$ as $\varepsilon \to 0$. Hence, by choosing $\varepsilon > 0$ sufficiently small, the key estimate (2.13) is satisfied, which proves that

$$\Delta^3 U(r) \ge -U(r)^{-q}$$

for all $0 < r < (1/2 + \sqrt{1 + 4\varepsilon^{-\tau}}/2)^{1/2}$. For $r \ge (1/2 + \sqrt{1 + 4\varepsilon^{-\tau}}/2)^{1/2}$, recall that $U(r) \le 1$ which implies $-U(r)^{-q} \le -1$. Hence by selecting ε even smaller, if necessary, we also obtain

$$\Delta^3 U(r) \ge -U(r)^{-q}$$

for all $r > 0$. Finally, choose

$$\underline{\beta} < \min\left\{(2/3)\varepsilon^{1-\tau}, (24/5)\varepsilon\right\}$$

we set $\overline{U} = U$. Thus, we have just proved that $\overline{U}$ to (2.11) with $\overline{U}(0) = 1$, $\Delta\overline{U}(0) = \underline{\beta}$, $\Delta^2\overline{U}(0) = \underline{\beta}$ is a super-solution which has compact support. □

Then analogue to [KR03, Lemma 3.5], we prove that any solution to (2.11) is entire if β is large.

Lemma 2.10. *There exists some $\overline{\beta} > 0$ such that for all $\beta \geq \overline{\beta}$, there exists at least one entire solution of* (2.11) *satisfying*

$$u(0) = 1, \quad \Delta u(0) = \beta, \quad \Delta^2 u(0) = \beta.$$

Proof. The idea of proof is to construct an entire sub-solution $\underline{U}$ to (2.11) with the initial conditions $\underline{U}(0) = 1$, $\Delta\underline{U}(0) = \beta$, and $\Delta^2\underline{U}(0) = \beta$. Indeed, let $m \in \mathbf{R}$ be fixed and set

$$U(r) = (1 + r^2)^m.$$

It is easy to verify that

$$\Delta^3 U(r) = \sum_{3 \leq k \leq 6} A_k(m)(1 + r^2)^{m-k},$$

where

$$\begin{cases} A_3(m) = 8m(m-1)(m-2)(2m+1)(2m-1)(2m-3), \\ A_4(m) = -48m(m-1)(m-2)(m-3)(2m-1)(2m-3), \\ A_5(m) = 96m(m-1)(m-2)(m-3)(m-4)(2m-3), \\ A_6(m) = -64(m-5)(m-4)(m-3)m(m-1)(m-2). \end{cases}$$

For $m \in (3/2, 2)$, it is not hard to see that $A_3(m) < 0$, $A_4(m) < 0$, $A_5(m) < 0$, and $A_6(m) < 0$. Hence $\Delta^3 U(r) < 0$ provided $m \in (3/2, 2)$. Now we let $b > 0$ and set

$$\underline{U}(r) = U(br).$$

Clearly, $\Delta^3\underline{U}(r) = b^6\Delta^3 U(br)$ and $\underline{U}(0) = 1$, $\underline{U}''(0) = 2mb^2$, $\underline{U}^{(4)}(0) = 12m(m-1)b^4$, $\underline{U}'(0) = 0$, $\underline{U}^{(3)}(0) = 0$, and $\underline{U}^{(5)}(0) = 0$. Therefore, because $0 < (1 + b^2r^2)^{-mq} < 1$, there holds

$$\Delta^3\underline{U}(r) + \underline{U}^{-q}(r) = b^6(\Delta^3 U)(br) + (1 + b^2r^2)^{-mq} < 0$$

provided we choose b large enough. Hence, for b large enough and fix it, we have shown that $\underline{U}$ is an entire sub-solution to (2.11) with $\underline{U}(0) = 1$, $\Delta\underline{U}(0) = (2/3)mb^2$, and $\Delta^2\underline{U}(0) = (12/5)m(m-1)b^4$. Now we set

$$\overline{\beta} = \max\{(2/3)mb^2, (12/5)m(m-1)b^4\}$$

and the proof follows. □

In view of Lemmas 2.9–2.10, it is possible to conclude the existence of a threshold for β, denoted by $\beta^\star$, similar to a threshold obtained in [KR03, Theorem 3.1]; see also [Gue12, Proposition 2.1].

Proposition 2.11. *Assume that $q > 1/2$ and $\beta > 0$. Let U_β be the unique local solution of* (2.10) *above. Then there is a unique $\beta^\star > 0$ such that:*

(a) *If $\beta < \beta^\star$ then $R_{\max}(\beta) < \infty$.*
(b) *If $\beta \geq \beta^\star$ then $R_{\max}(\beta) = \infty$.*
(c) *If $\beta \geq \beta^\star$ then*

$$\lim_{r \nearrow +\infty} \Delta^2 U_\beta(r) \geq 0.$$

(d) *We have* $\beta = \beta^\star$ *if and only if*

$$\lim_{r \nearrow +\infty} \Delta^2 U_\beta(r) = 0.$$

Proof. First, we prove the uniqueness of $\beta^\star$. Indeed, we shall prove that $U_{\beta^\star}$ is the unique solution such that

$$\lim_{r \nearrow +\infty} \Delta^2 U_\beta(r) = 0.$$

By way of contradiction, we suppose that there are $0 < \beta^\star < \beta^{\star\star}$ such that

$$\lim_{r \nearrow +\infty} \Delta^2 U_{\beta^\star}(r) = \lim_{r \nearrow +\infty} \Delta^2 U_{\beta^{\star\star}}(r) = 0.$$

By the comparison principle, we know that $U_{\beta^{\star\star}} > U_{\beta^\star}$ on $(0, +\infty)$. Therefore,

$$-(U_{\beta^{\star\star}}^{-q} - U_{\beta^\star}^{-q}) = r^{-2}\big(r^2 (\Delta^2)'(U_{\beta^{\star\star}} - U_{\beta^\star})\big)' > 0$$

on $(0, +\infty)$, which implies that $(\Delta^2)'(U_{\beta^{\star\star}} - U_{\beta^\star}) > 0$ on $(0, +\infty)$. From this we obtain a contradiction since

$$\lim_{r \nearrow +\infty} \Delta^2 (U_{\beta^{\star\star}} - U_{\beta^\star})(r) = 0.$$

Part (a). By Lemma 2.9, there is a $\underline{\beta} > 0$ such that any solution u to (2.11) with $u(0) = 1$, $\Delta u(0) = \underline{\beta}$, and $\Delta^2 u(0) = \underline{\beta}$ has compact support. Using the comparison principle we see that for any $\beta < \underline{\beta}$ any solution u to (2.11) with $u(0) = 1$, $\Delta u(0) = \beta$, and $\Delta^2 u(0) = \beta$ also has compact support. Therefore, we may define

$$\beta^\star = \sup \left\{ \beta \;\middle|\; \begin{array}{l} \text{any solution } u \text{ of (2.11) with } u(0) = 1, \Delta u(0) = \beta, \\ \text{and } \Delta^2 u(0) = \beta \text{ has compact support} \end{array} \right\}.$$

Thanks to Lemma 2.10, we deduce that $\beta^\star$ is finite and positive. Consequently, any solution u to (2.11) with $u(0) = 1$, $\Delta u(0) = \beta$, and $\Delta^2 u(0) = \beta > \beta^\star$ is entire. This establishes Part (a).

Part (b). It suffices to prove that any solution u to (2.11) with $u(0) = 1$, $\Delta u(0) = \beta^\star$, and $\Delta^2 u(0) = \beta^\star$ is entire. Indeed, let $R(\beta)$ be the first zero of the solution u to (2.11) with $u(0) = 1$, $\Delta u(0) = \beta$, and $\Delta^2 u(0) = \beta < \beta^\star$. Since $\beta < \beta^\star$, the number $R(\beta)$ exists and is finite. By the comparison principle, the function $R(\beta)$ is non-decreasing in β. Moreover, for two solutions u_1 and u_2 of (2.11) with $u_i(0) = 1$, $\Delta u_i(0) = \beta$ with $i = 1, 2$, and

$$\Delta^2 u_1(0) = \beta_1 < \beta_2 = \Delta^2 u_2(0),$$

we find that

$$\Delta^2 u_1(r) < \Delta^2 u_2(r)$$

in their common interval of existence. From this we deduce that $(u_1 - u_2)^{(3)}(r) < 0$. In particular, this implies that $(u_1 - u_2)'(r) < 0$; hence the gap between two solutions is increasing. Consequently, $R(\beta)$ is strictly increasing in β. This and the continuous dependence of the solution on initial values tell us that $R(\beta)$ is in fact continuous. Now we assume for contradiction that $R(\beta) \to R$ for some finite number R as $\beta \nearrow \beta^\star$. However, this is impossible due to the continuous dependence of the solution on initial values. This proves Part (b).

Part (c). Let $\beta \geq \beta^\star$. Suppose that

$$\lim_{r \nearrow +\infty} \Delta^2 U_\beta(r) < 0.$$

Integrating gives

$$U_\beta(r) \leq -C_1 r^4 + C_2 r^3 + C_3 r^2 + C_4 r + C_5$$

for all $r \geq 0$ with $C_1 > 0$, which is impossible; so Part (c) follows.

Part (d). Suppose that

$$\lim_{r\nearrow+\infty}\Delta^2 U_\beta(r)>0.$$

It is easy to see that $U_\beta(r)\geq cr^4$, $U_\beta'(r)\geq cr^3$, $U_\beta''(r)\geq cr^2$, and $U_\beta'''(r)\geq cr$ for all $r>0$ and for some constant $c>0$. Using $r^{-2}(r^2(\Delta^2 U_\beta)'(r))'=-U_\beta^{-q}$, we deduce that

$$(\Delta^2 U_\beta)'(r)\geq -c\begin{cases} r^{-2} & \text{if } q>3/4,\\ r^{-2}\log r & \text{if } q=3/4,\\ r^{1-4q} & \text{if } q<3/4,\end{cases}$$

for all $r\geq 2$ and for some new constant $c>0$. Let $m\in(3/2,2)$ to be fixed and set

$$U(r)=(1+r^2)^m.$$

Recall from the proof of Lemma 2.10 that

$$\Delta^3 U(r)=\sum_{3\leq k\leq 6}A_k(m)(1+r^2)^{m-k},$$

where $A_k(m)<0$ for all $3\leq k\leq 6$ whenever $m\in(3/2,2)$. Let $b>0$ and set

$$\overline{U}(r)=U(br).$$

Also as in the proof of Lemma 2.10, if we choose b large enough, then we have $\Delta^3\overline{U}+\overline{U}^{-q}\leq 0$. In addition, there exists some r_0 such that

$$\begin{cases} U_\beta(r_0)>\overline{U}(r_0), \quad \Delta U_\beta(r_0)>\Delta\overline{U}(r_0), \quad \Delta^2 U_\beta(r_0)>\Delta^2\overline{U}(r_0),\\ U_\beta'(r_0)>\overline{U}'(r_0), \quad (\Delta U_\beta)'(r_0)>(\Delta\overline{U})'(r_0), \quad (\Delta^2 U_\beta)'(r_0)>(\Delta^2\overline{U})'(r_0).\end{cases}$$

By the continuous dependence of solutions and Lemma 2.1, there is $\beta_1<\beta$ such that $U_{\beta_1}\geq\overline{U}$ for all $r\geq r_0$. This shows that U_{β_1} exists for all $r\geq 0$; hence $\beta_1\geq\beta^\star$. From this we deduce that $\beta>\beta^\star$. Now we suppose that $\beta>\beta^\star$. Then there holds

$$(\Delta^2 U_\beta)'(r)>(\Delta^2 U_{\beta^\star})'(r)$$

for all $r\geq 0$. Using the initial data and by integration by parts, we obtain

$$\lim_{r\nearrow+\infty}\Delta^2 U_\beta(r)-\beta\geq\lim_{r\nearrow+\infty}\Delta^2 U_{\beta^\star}(r)-\beta^\star,$$

which implies

$$\lim_{r\nearrow+\infty}\Delta^2 U_\beta(r)\geq\beta-\beta^\star.$$

The proof of Part (d) is complete. □

3. Classification of growth at infinity for radial solutions to (1.1): proof of Theorem 1.1

In this section, we are interested in the exact growth at infinity for radial solutions to (1.1). We observe from Lemma 2.2 and (2.4) the limit $\lim_{r\nearrow+\infty}\Delta^2 u(r)$ exists and is non-negative. The main result of this section is the following:

Proposition 3.1. *Let u be a radial solution to (1.1). Then we have the following claims:*

(I) If

$$\lim_{r\nearrow+\infty}\Delta^2 u(r)>0, \tag{3.1}$$

then $u(r)\approx r^4$ for any $q>1/2$.

(II) If

$$\lim_{r\nearrow+\infty}\Delta^2 u(r)=0, \tag{3.2}$$

then $u(r)=o(r^4)$ at infinity. Furthermore, we have the following possibilities:

(a) $u(r) \approx r^3$ *if* $p > 1$;
(b) $u(r) \approx r^3\sqrt{\log r}$ *if* $p = 1$; *and*
(c) $u(r) \approx r^{6/(q+1)}$ *if* $1/2 < p < 1$.

Clearly, Theorem 1.1 follows from Proposition 3.1 above. Hence, in the rest of this section, we prove this proposition. From our point of view, the most difficult part is (IIc); to handle this case, new arguments are introduced, see the proof of Lemma 3.10.

3.1. **Solutions with quartic growth.** Now we prove Part (I) of Proposition 3.1. For simplicity, set

$$\wp = \lim_{r\nearrow+\infty} \Delta^2 u(r) > 0.$$

In view of (2.4), there holds $\Delta^2 u(r) \geq \wp$ for any $r \geq 0$. Integrating this differential inequality as in the preceding subsection, we deduce that u grows at least quartically at infinity and this is enough to conclude that $u(r) \approx r^4$. In the next two subsections, we prove Part (II) of Proposition 3.1.

3.2. **Solutions with non-quartic growth.** We start by observing that whenever

$$\lim_{r\nearrow+\infty} \Delta^2 u(r) = 0$$

there holds $u(r) = o(r^4)$ at infinity. This is elementary because the zero limit implies that

$$\lim_{r\nearrow+\infty} r^{-4}(r^4 u^{(3)}(r))' = 0,$$

which then implies, by the l'Hôpital rule, that

$$u^{(3)}(r) = o(r), \quad u''(r) = o(r^2), \quad u'(r) = o(r^3), \quad u(r) = o(r^4) \tag{3.3}$$

at infinity. We now examine the growth of u at infinity more closely because $u(r) = o(r^4)$ is not what we need.

3.2.1. *Proof of Proposition 3.1(IIa).* We now consider Part (II) under the case $q > 1$. By integration by parts, we obtain from $\Delta w = -u^{-q}$ the following

$$w(r) = w(0) + \frac{1}{r}\int_0^r s^2 u^{-q} ds - \int_0^r s u^{-q} ds. \tag{3.4}$$

Since u grows at least cubic at infinity and $q > 1$, it is easy to see that $r^2 u^{-q} \to 0$ as $r \nearrow +\infty$. In (3.4), we send r to $+\infty$ to obtain

$$w(0) = \int_0^{+\infty} s u^{-q} ds < +\infty.$$

Still by integration by parts, we obtain from (3.4) and $\Delta^2 u = w$ the identity

$$\begin{aligned} u(r) =& u(0) + \frac{\Delta u(0)}{6} r^2 + \frac{r^4}{120}\int_r^{+\infty} s u(s)^{-q} ds - \frac{r^2}{12}\int_0^r s^3 u(s)^{-q} ds \\ & - \frac{1}{24}\int_0^r s^5 u(s)^{-q} ds + \frac{r^3}{24}\int_0^r s^2 u(s)^{-q} ds \\ & + \frac{r}{12}\int_0^r s^4 u(s)^{-q} ds + \frac{1}{120r}\int_0^r s^6 u(s)^{-q} ds \end{aligned} \tag{3.5}$$

in $\mathbf{R}^3$. It is worth noting that the representation (3.5) is valid for all $q > 1$. Let us denote by Ξ all terms in (3.5) involving integrals, that is

$$\begin{aligned}\Xi(r) =& \frac{r^4}{120}\int_r^{+\infty} su(s)^{-q}ds - \frac{r^2}{12}\int_0^r s^3u(s)^{-q}ds - \frac{1}{24}\int_0^r s^5u(s)^{-q}ds\\ &+\frac{r^3}{24}\int_0^r s^2u(s)^{-q}ds + \frac{r}{12}\int_0^r s^4u(s)^{-q}ds + \frac{1}{120r}\int_0^r s^6u(s)^{-q}ds.\end{aligned}$$

Our aim is to show that $\Xi(r)$ has cubic growth at infinity. To achieve that goal, we use the l'Hôpital rule to see that

$$\lim_{r\nearrow+\infty}\Xi'''(r) = \frac{1}{4}\int_0^{+\infty} s^2u(s)^{-q}ds > 0.$$

Note that the preceding limit is also finite, thanks to $q > 1$. From this we conclude that u has exactly cubic growth at infinity.

Remark 3.2. Concerning Proposition 3.1(IIa), it is worth noticing that the reserve case also holds, that is, if u is a positive solution to (1.1) having cubic growth (uniformly) at infinity, then $q > 1$. The argument is similar to the one used in [CX09].

Next we consider Part (II) under the case $1/2 < q \leq 1$, which shows that radial solutions to (1.1) satisfying (3.2) grow between cubic and quartic at infinity. To obtain the desired result, we exploit several ODE techniques used in [DFG10] and in [Lai14]. The idea is to transform (1.1) into a high order ODE via the following change of variable

$$W(t) = e^{mt}u(e^t)$$

where $m = -6/(q+1)$ and $t = \log r$. By direct computation, we observe the following.

Lemma 3.3. *Let $q > 1/2$. If $u(r)$ is a positive radial solution to (1.1), then $W(t)$ solves*

$$Q_6(m-\partial)W = -W^{-q}, \tag{3.6}$$

where we formally define

$$Q_6(m-\partial) = \prod_{1\leq k\leq 6}(\partial - m - 5 + k)$$

with $\partial = d/dt$.

For simplicity, we put $\lambda_k = m+5-k$ with $1 \leq k \leq 6$, namely $\lambda_1 = 2(2q-1)/(q+1)$, $\lambda_2 = 3(q-1)/(q+1)$, $\lambda_3 = 2(q-2)/(q+1)$, $\lambda_4 = (q-5)/(q+1)$, $\lambda_5 = -6/(q+1)$, and $\lambda_6 = -(q+7)/(q+1)$. By the variation of parameters formula, the solution $W(t)$ to (3.6) has an integral representation given as follows.

Lemma 3.4. *Let $W(t)$ be a solution to (3.6). Given any t_0, then there exist constants α_i such that*

$$W(t) = \sum_{i=1}^{6}\Big(\alpha_i e^{\lambda_i t} - d_i\int_{t_0}^t e^{\lambda_i(t-s)}W^{-q}ds\Big), \tag{3.7}$$

where

$$d_i = \prod_{j=1,\, j\neq i}^{6}(\lambda_i - \lambda_j)^{-1}.$$

More precisely, we have $d_1 = 1/120$, $d_2 = -1/24$, $d_3 = 1/12$, $d_4 = -1/12$, $d_5 = 1/24$, and $d_6 = -1/120$.

We note that we also have an expansion form for (3.6) given by

$$W^{(6)} + \sum_{i=0}^{5} K_i W^{(i)} = -W^{-q},$$

where

$$\begin{cases} K_0 = \dfrac{72(2q-1)(q-1)(q-2)(q-5)(q+7)}{(q+1)^6} < 0, \\ K_1 = \dfrac{12(q-3)(2q^4 - 10q^3 - 249q^2 + 512q - 223)}{(q+1)^5}, \\ K_2 = -\dfrac{2(13q^4 + 187q^3 - 2217q^2 + 4777q - 2552)}{(q+1)^4}, \\ K_3 = -\dfrac{15(q-3)(q^2 - 34q + 37)}{(q+1)^3} > 0, \\ K_4 = \dfrac{5(5q^2 - 44q + 59)}{(q+1)^2} > 0, \\ K_5 = -\dfrac{9(q-3)}{q+1} > 0. \end{cases}$$

Note that in the range $(1/2, 1]$, the sign of K_1 and K_2 is not constant. The constant K_0 is already appeared in the statement of Theorem 1.2(c) and in terms of Q_6, we can express $K_0 = Q_6(m)$. With all ingredients above, we are now in position to prove Theorem 1.2(b, c).

3.2.2. *Proof of Proposition 3.1(IIb).* In this subsection, we restrict ourselves to the case $q = 1$, which leads to $K_0 = 0$ and $m = -3$. For clarity, we split our proof of Part (d) into three steps as follows.

Lemma 3.5. *$W(t)$ is unbounded.*

Proof. We prove by way of contradiction. Indeed, suppose that $W(t)$ is bounded, then there exists $A > 0$ and $t_0 > 0$ such that

$$\big(W^{(5)} + \sum_{i=1}^{5} K_i W^{(i-1)}\big)' = -W^{-q} \leq -A$$

for all $t \geq t_0$. Integrating both sides gives

$$\big(W^{(4)} + \sum_{i=2}^{5} K_i W^{(i-2)}\big)' \leq -A(t - t_0) + A_1$$

for all $t \geq t_0$. By continuing this process, we shall obtain the following estimate

$$W(t) \leq -\frac{A}{6!}(t - t_0)^6 + o((t - t_0)^6)$$

as $t \nearrow +\infty$. This contradicts to the fact that $W(t) > 0$ for all t. Hence $W(t)$ is unbounded on any $[t_0, +\infty)$ as claimed. $\square$

Lemma 3.6. *There holds $W(t) \nearrow +\infty$ as $t \nearrow +\infty$.*

Proof. For $q = 1$, (3.6) becomes

$$(\partial + 4)(\partial + 3)(\partial + 2)(\partial + 1)(\partial - 1)W' = -W^{-1} < 0.$$

Multiplying by e^{4s} and integrating over $(-\infty, s)$, we get

$$e^{4s}(\partial + 3)(\partial + 2)(\partial + 1)(\partial - 1)W' \leq 0$$

since $W^{-1}(s) \to 0$ as $s \to -\infty$. Therefore

$$(\partial+3)(\partial+2)(\partial+1)(\partial-1)W' \le 0.$$

Performing a similar argument, we eventually obtain $\big(e^{-s}W'(s)\big)' \le 0$. Therefore, the function $e^{-s}W'(s)$ is monotone decreasing. Thanks to (3.3), we deduce that

$$e^{-s}W'(s) = -3e^{-4s}u(e^s) + e^{-3s}u'(e^s) \to 0$$

as $s \nearrow +\infty$. Thus, there must hold $W'(s) > 0$. Combining with the fact that $W(s)$ is unbounded, we obtain $\lim_{s\nearrow+\infty} W(s) = +\infty$ as claimed. $\square$

Lemma 3.7. *There holds*

$$\lim_{r\nearrow+\infty} \frac{u(r)}{r^3(\log r)^{1/2}} = \frac{1}{\sqrt{12}}.$$

Proof. Note that when $q=1$, we have $m=3$, $\lambda_1=1$, $\lambda_2=0$, $\lambda_3=-1$, $\lambda_4=-2$, $\lambda_5=-3$, and $\lambda_6=-4$. Thanks to (3.3), in this case, $e^{-t}W(t) = r^{-4}u(r) = o(1)$ at infinity. Thus, multiplying both side of (3.7) by e^{-t} and sending $t \nearrow +\infty$ in the resulting equation gives

$$\alpha_1 = d_1 \int_{t_0}^{+\infty} e^{-s}W^{-1}ds.$$

Hence,

$$W(t) = d_1 \int_t^{+\infty} e^{(t-s)}W^{-1}ds + \sum_{i=2}^{6}\Big(\alpha_i e^{\lambda_i t} - d_i \int_{t_0}^{t} e^{\lambda_i(t-s)}W^{-1}ds\Big).$$

By direct computing and using the relation $\sum_{i=1}^6 d_i = 0$, we easily get

$$W'(t) = d_1 \int_t^{+\infty} e^{(t-s)}W^{-1}ds + \sum_{i=3}^{6}\lambda_i\Big(\alpha_i e^{\lambda_i t} - d_i \int_{t_0}^{t} e^{\lambda_i(t-s)}W^{-1}ds\Big).$$

Hence, by the Hôpital rule, we easily verify $W'(t) = o(1)$ at infinity. Furthermore, making use of the Hôpital rule, we can estimate

$$\begin{aligned}
\lim_{t\nearrow+\infty} W'(t)W(t) =& \lim_{t\nearrow+\infty} d_1 W(t)\int_t^{+\infty} e^{(t-s)}W^{-1}ds \\
&+ \lim_{t\nearrow+\infty}\sum_{i=3}^{6}\lambda_i\Big(\alpha_i W(t)e^{\lambda_i t} - d_i W(t)\int_{t_0}^{t} e^{\lambda_i(t-s)}W^{-1}ds\Big) \\
=& d_1 + \sum_{i=3}^{6} d_i = -d_2 = \frac{1}{24},
\end{aligned}$$

which immediately implies that

$$\lim_{t\nearrow+\infty} \frac{W^2(t)}{t} = 2\lim_{t\nearrow+\infty} W'(t)W(t) = \frac{1}{12}.$$

From this it is easy to obtain

$$\lim_{r\nearrow+\infty} \frac{u(r)}{r^3(\log r)^{1/2}} = \lim_{t\nearrow+\infty}\frac{W(t)}{\sqrt{t}} = \frac{1}{\sqrt{12}}.$$

The present proof is complete. $\square$

3.2.3. *Proof of Proposition 3.1(IIc).* We need a few steps to prove this part. First, we need a lemma which essentially says that W is bounded.

Lemma 3.8. *Let $W(t)$ be solution of (3.6). Then, the following assertions hold.*

(i) It cannot happen that $W(t) \nearrow +\infty$ as $t \nearrow +\infty$.
(ii) If the limit $\lim_{t\nearrow+\infty} W(t) = L$ exists, then

$$L = (-K_0)^{-1/(q+1)}.$$

Proof. For $1/2 < q < 1$, we have $\lambda_1 > 0$ and $\lambda_i < 0$ for $2 \leq i \leq 6$. Recall that $\lambda_1 = 2(2q-1)/(q+1)$, which implies that $e^{-\lambda_1 t} W(t) = u(r)/r^4 = o(1)$ at infinity. Suppose that $\lim_{t\nearrow+\infty} W(t) = +\infty$. By multiplying both side of (3.7) by $e^{-\lambda_1 t}$ and sending $t \nearrow +\infty$ in the resulting equation, we obtain

$$\alpha_1 = d_1 \int_{t_0}^{+\infty} e^{-\lambda_1 s} W^{-q} ds.$$

Hence, it is not hard to verify that

$$W(t) = d_1 \int_t^{+\infty} e^{\lambda_1 (t-s)} W^{-q} ds + \sum_{i=2}^{6} \Big(\alpha_i e^{\lambda_i t} - d_i \int_{t_0}^{t} e^{\lambda_i (t-s)} W^{-q} ds \Big).$$

Using the l'Hôpital rule, we can easily check that $W(t) \to 0$ as $t \nearrow +\infty$, which contradicts the assumption $\lim_{t\nearrow+\infty} W(t) = +\infty$. Therefore, $W(t)$ cannot diverge to $+\infty$ as $t \nearrow +\infty$. This establishes part (i).

We now prove part (ii). Assume that the limit $\lim_{t\nearrow+\infty} W(t) = L$ exists with $L \neq (-K_0)^{-1/(q+1)}$. Then

$$\alpha := \lim_{t\nearrow+\infty} \big(-K_0 W(t) - W^{-q}(t) \big) \neq 0.$$

Therefore, there exist two constants $M, T > 0$ such that either

$$-K_0 W(t) - W^{-q}(t) < -M \quad \text{for all } t \geq T$$

if $\alpha < 0$ or

$$-K_0 W(t) - W^{-q}(t) > M \quad \text{for all } t \geq T$$

if $\alpha > 0$. Putting

$$\delta = \sup_{t \geq T} |W(t) - W(T)| < +\infty.$$

Upon using the relation

$$\Big(W^{(5)} + \sum_{i=2}^{5} K_i W^{(i-1)} + K_1 W \Big)' = -K_0 W - W^{-q}$$

we obtain

$$W^{(5)} + \sum_{i=2}^{5} K_i W^{(i-1)} < -M(t-T) + |K_1|\delta + C \quad \text{for all } t \geq T$$

if $\alpha < 0$ and

$$W^{(5)} + \sum_{i=2}^{5} K_i W^{(i-1)} > M(t-T) - |K_1|\delta + C \quad \text{for all } t \geq T$$

if $\alpha > 0$. Here $C = C(T)$ is a constant containing all the terms $W^{(i)}(T)$ with $1 \leq i \leq 5$. Repeating the above process, we get

$$W(t) \leq -\frac{M}{6!}(t-T)^6 + O(t^5)$$

if $\alpha < 0$ and
$$W(t) \geq \frac{M}{6!}(t-T)^6 + O(t^5)$$
if $\alpha > 0$. This contradicts to our assumption that L is finite. Hence, there holds $L = (-K_0)^{-1/(q+1)}$ as claimed. □

Next, we rule out the case $W(t)$ approaching zero, which is important because our equation (3.6) involves a negative exponent.

Lemma 3.9. *Let $W(t)$ be a solution to (3.6). Then,*
$$\liminf_{t \nearrow +\infty} W(t) > 0.$$

Proof. We adopt the method used in [DFG10]. However, there is some improvement due to the fact that solutions to (1.1) only enjoys the partially super polyharmonic property, see Lemma 2.2. First we observe from Lemma 3.8(ii) and the inequality $W(t) > 0$ that
$$\limsup_{t \nearrow +\infty} W(t) > 0.$$
Suppose by contradiction that $\liminf_{t \nearrow +\infty} W(t) = 0$. Then, there is a sequence $(t_k)_k$ such that $t_k \nearrow +\infty$, $t_{k+1} \geq t_k + 1$, $W(t_k) \to 0$, $W'(t_k) = 0$, and $W''(t_k) \geq 0$. Put $R_k = e^{t_k}$ and define
$$U_k(r) = \frac{u(R_{k+1}r)}{W(t_{k+1})R_{k+1}^{-m}}.$$
Keep in mind that $m = -6/(q+1)$. Then U_k satisfies
$$\begin{cases} (-\Delta)^3 U_k = W(t_{k+1})^{-q-1} U_k^{-q}, \\ U_k(1) = 1, \\ U_k\left(\dfrac{R_k}{R_{k+1}}\right) = \dfrac{u(R_k)}{u(R_{k+1})}. \end{cases} \tag{3.8}$$
Since $u'(r) > 0$ and $R_{k+1} > R_k$, we conclude that $U_k(R_k/R_{k+1}) < 1$. Besides, as
$$\Delta^2 U_k(r) = \frac{R_{k+1}^{m+4}}{W(t_{k+1})}(\Delta^2 u)(R_{k+1}r), \quad \Delta U_k(r) = \frac{R_{k+1}^{m+2}}{W(t_{k+1})}(\Delta u)(R_{k+1}r)$$
and with help of Lemma 2.6, we deduce that
$$\begin{aligned} \Delta U_k(1) &= \frac{R_{k+1}^{m+2}}{W(t_{k+1})}(\Delta u)(R_{k+1}) \geq 0, \\ \Delta^2 U_k(1) &= \frac{R_{k+1}^{m+4}}{W(t_{k+1})}(\Delta^2 u)(R_{k+1}) \geq 0, \\ \Delta U_k\left(\frac{R_k}{R_{k+1}}\right) &= \frac{R_{k+1}^{m+2}}{W(t_{k+1})}(\Delta u)(R_k) \geq 0, \\ \Delta^2 U_k\left(\frac{R_k}{R_{k+1}}\right) &= \frac{R_{k+1}^{m+4}}{W(t_{k+1})}(\Delta^2 u)(R_k) \geq 0, \end{aligned}$$
for large k. Putting
$$D_k = B_1(0) \setminus \overline{B}_{R_k/R_{k+1}}(0), \quad D = B_{1/2}(0) \setminus \overline{B}_{1/e}(0).$$
Since $R_k/R_{k+1} < 1/e$, D is a proper subset of D_k for all k. Let λ be the first eigenvalue of the operator $-\Delta$ with the zero Dirichlet boundary condition in the annulus D and let $\varphi > 0$ be its associated eigenfunction, that is,
$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases}$$

Since φ is smooth up to the boundary ∂D and $D \subset D_k$, we extend φ smoothly to the whole $\overline{D}_k$ by setting $\varphi \equiv 0$ in $\overline{D}_k \setminus D$. Then we know that

$$(-\Delta)^2\varphi = \lambda^2\varphi, \quad (-\Delta)^3\varphi = \lambda^3\varphi \quad \text{in } D$$

and

$$\varphi = \Delta\varphi = \Delta^2\varphi = 0, \quad \partial_n\varphi = \partial_n\Delta\varphi = \partial_n\Delta^2\varphi = 0 \quad \text{on } \overline{D_k} \setminus D.$$

Multiplying both side of (3.8) by φ and integrating the resulting equation by parts to get

$$\begin{aligned}
W(t_{k+1})^{-q-1}\int_{D_k} U_k^{-q}\varphi dx = &-\int_{D_k} \varphi\Delta^3 U_k dx \\
=&-\int_{D_k} U_k\Delta^3\varphi dx - \int_{\partial D_k}[\varphi\frac{\partial\Delta^2 U_k}{\partial n} - \Delta^2 U_k\frac{\partial\varphi}{\partial n}] \\
&-\int_{\partial D_k}[\Delta\varphi\frac{\partial\Delta U_k}{\partial n} - \Delta U_k\frac{\partial\Delta\varphi}{\partial n}] \\
&-\int_{\partial D_k}[\Delta^2\varphi\frac{\partial U_k}{\partial n} - U_k\frac{\partial\Delta^2\varphi}{\partial n}] \\
=&\lambda^3\int_{D_k} U_k\varphi dx.
\end{aligned}$$

Thanks to $U_k \in (0,1)$ and $q > 0$, we get $U_k \leq U_k^{-q}$. Consequently, $\int_D U_k^{-q}\varphi dx \geq \int_D U_k\varphi dx$. Therefore,

$$W(t_{k+1})^{-q-1} \leq \lambda^3$$

for all k. This contradicts to our contradiction assumption that $W(t_{k+1}) \to 0$ as $k \nearrow +\infty$. Thus, $\liminf_{t\nearrow+\infty} W(t) > 0$ as claimed. ☐

Lemma 3.10. *There holds*

$$\lim_{t\nearrow+\infty} W(t) = (-K_0)^{-1/(q+1)}.$$

Proof. Our proof consists of two parts. First we prove that $W(t)$ is bounded on $[0,+\infty)$. For $q \in (1/2, 1)$, we observe that $\lambda_1 > 0$ and that $\lambda_j < 0$ for $2 \leq j \leq 6$. Set

$$\begin{cases} v_1(t) = (\partial - \lambda_1)W(t), \\ v_j(t) = (\partial - \lambda_j)v_{j-1}(t), \quad 2 \leq j \leq 5. \end{cases} \tag{3.9}$$

Then (3.6) becomes $(\partial - \lambda_6)v_5(t) = -W^{-q}(t)$ which implies

$$v_5(t) = e^{\lambda_6 t}v_5(0) + \int_0^t e^{\lambda_6(t-s)}(-W^{-q}(s))ds. \tag{3.10}$$

By Lemma 3.9, $W^{-q}(t)$ is bounded on $[0,+\infty)$. Combining this property with $\lambda_6 < 0$, we easily see that $v_5(t)$ is bounded on $[0,+\infty)$. Repeating the above argument we deduce that the functions v_i for $1 \leq i \leq 4$ are also bounded on $[0,+\infty)$. Now integrating the differential equation of W in the system (3.9) we obtain

$$W(t) = e^{\lambda_1 t}\Big[W(0) + \int_0^t e^{-\lambda_1 s}v_1(s)ds\Big].$$

On the other hand, as in the first paragraph of the proof of Lemma 3.8, we deduce that $W(t) = o(e^{\lambda_1 t})$ at infinity. This leads us to

$$W(0) = -\int_0^{+\infty} e^{-\lambda_1 s}v_1(s)ds.$$

In other words, there holds

$$W(t) = -e^{\lambda_1 t}\int_t^{+\infty} e^{-\lambda_1 s}v_1(s)ds, \tag{3.11}$$

which immediately implies that $W(t)$ is bounded on $[0,+\infty)$. Next we prove that $W(t)$ has limit at infinity. Indeed, the boundedness of $W(t)$ allows us to set

$$\liminf_{t\to+\infty} W(t) = B, \quad \limsup_{t\to+\infty} W(t) = A,$$

which are positive and finite. From this, given $\varepsilon > 0$, we exists some $T \gg 1$ such that

$$B-\varepsilon < W(t) < A+\varepsilon$$

for all $t \geq T$. Similar to (3.10), we use the equation for v_5 in (3.10) to get

$$v_5(t) = e^{\lambda_6(t-T)} v_5(T) + \int_T^t e^{\lambda_6(t-s)}(-W^{-q}(s))ds,$$

which implies that

$$-(B+\varepsilon)^{-q}\frac{1-e^{\lambda_6(t-T)}}{-\lambda_6} \leq v_5(t) - v_5(T)e^{\lambda_6(t-T)} \leq -(A+\varepsilon)^{-q}\frac{1-e^{\lambda_6(t-T)}}{-\lambda_6}$$

for all $t \geq T$. First appropriately sending t to infinity then sending ε down to zero, we deduce that

$$\frac{-B^{-q}}{-\lambda_6} \leq \liminf_{t\to+\infty} v_5(t) \leq \limsup_{t\to+\infty} v_5(t) \leq \frac{-A^{-q}}{-\lambda_6}.$$

Repeating this process for the functions v_{5-i} with $1 \leq i \leq 4$, we eventually obtain

$$\frac{-B^{-q}}{-\prod_{j=2}^6 \lambda_j} \leq \liminf_{t\to+\infty} v_1(t) \leq \limsup_{t\to+\infty} v_1(t) \leq \frac{-A^{-q}}{-\prod_{j=2}^6 \lambda_j}.$$

We now use (3.11) to deduce that

$$\frac{-B^{-q}}{-\prod_{j=1}^6 \lambda_j} \leq \liminf_{t\to+\infty}(-W(t)) \leq \limsup_{t\to+\infty}(-W(t)) \leq \frac{-A^{-q}}{-\prod_{j=1}^6 \lambda_j}.$$

Keep in mind that $\prod_{j=1}^6 \lambda_j = K_0 < 0$. Hence

$$\frac{-B^{-q}}{-K_0} \leq -A \leq -B \leq \frac{-A^{-q}}{-K_0}.$$

From the above inequalities we obtain $A = B = (-K_0)^{-1/(q+1)}$. This completes the present proof. $\square$

Clearly, Theorem 1.1 follows from Propositions 2.5 and 3.1 above. Hence we omit its proof.

4. An autonomous 6-dimensional system

To study the asymptotic behavior of (2.1), we follow the ideas in [HV96]. First, by the Emden–Fowler transformation we set

$$\begin{aligned} x_1(t) &= \frac{ru'(r)}{u(r)}, & x_2(t) &= \frac{rv'(r)}{v(r)}, & x_3(t) &= \frac{rw'(r)}{w(r)}, \\ x_4(t) &= \frac{r^2v(r)}{u(r)}, & x_5(t) &= \frac{r^2w(r)}{v(r)}, & x_6(t) &= \frac{r^2u^{-q}(r)}{w(r)}, \end{aligned}$$

with $t = \log r$. Clearly, the system (2.1) is transformed into a 6-dimensional system of the form

$$\begin{cases} x_1' = x_1(-1 - x_1) + x_4, \\ x_2' = x_2(-1 - x_2) + x_5, \\ x_3' = x_3(-1 - x_3) - x_6, \\ x_4' = x_4(2 - x_1 + x_2), \\ x_5' = x_5(2 - x_2 + x_3), \\ x_6' = x_6(2 - qx_1 - x_3), \end{cases} \tag{4.1}$$

where we denote $' = d/dt$. The linearization of (4.1) at the point $(x_1, x_2, x_3, x_4, x_5, x_6)$ is given by the following matrix

$$\begin{pmatrix} -1-2x_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1-2x_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1-2x_3 & 0 & 0 & -1 \\ -x_4 & x_4 & 0 & 2-x_1+x_2 & 0 & 0 \\ 0 & -x_5 & x_5 & 0 & 2-x_2+x_3 & 0 \\ -qx_6 & 0 & -x_6 & 0 & 0 & 2-qx_1-x_3 \end{pmatrix}.$$

The following proposition is the main result of this section.

Proposition 4.1. *Let*

$$\wp = \lim_{r \nearrow +\infty} w(r).$$

Suppose that $r^2 u^{-q}(r) \to 0$ as $r \nearrow +\infty$. Then we have the following identities

$$\frac{1}{x_1} = \frac{u(r)}{ru'(r)} = \frac{u(0)}{r^{-1}\int_0^r s^2 v(s)ds} + \frac{r\int_0^r sv(s)ds}{\int_0^r s^2 v(s)ds} - 1, \tag{4.2}$$

$$\frac{1}{x_2} = \frac{v(r)}{rv'(r)} = \frac{v(0)}{r^{-1}\int_0^r s^2 w(s)ds} + \frac{r\int_0^r sw(s)ds}{\int_0^r s^2 w(s)ds} - 1, \tag{4.3}$$

$$\frac{1}{x_3} = \frac{w(r)}{rw'(r)} = -\frac{\wp r}{\int_0^r s^2 u^{-q} ds} - \frac{r\int_r^{+\infty} su^{-q}ds}{\int_0^r s^2 u^{-q}ds} - 1, \tag{4.4}$$

$$\frac{1}{x_4} = \frac{u(r)}{r^2 v(r)} = \frac{u(0)}{r^2 v(r)} - \frac{1}{r^3 v(r)}\int_0^r s^2 v(s)ds + \frac{1}{r^2 v(r)}\int_0^r sv(s)ds, \tag{4.5}$$

and

$$\frac{1}{x_5} = \frac{v(r)}{r^2 w(r)} = \frac{v(0)}{r^2 w(r)} - \frac{1}{r^3 w(r)}\int_0^r s^2 w(s)ds + \frac{1}{r^2 w(r)}\int_0^r sw(s)ds. \tag{4.6}$$

Proof. In view of (3.4) and the fact that $r^2 u^{-q}(r) \to 0$ as $r \nearrow +\infty$, after sending $r \nearrow +\infty$, we obtain

$$w(0) = \wp + \int_0^{+\infty} su^{-q}ds < +\infty,$$

which implies that

$$w(r) = \wp + \frac{1}{r}\int_0^r s^2 u^{-q}ds + \int_r^{+\infty} su^{-q}ds.$$

Now an easy computation shows that$w(r)/(rw'(r))$ has the representation provided by (4.4). To obtain (4.6), we observe from $\Delta v = w$ that

$$v(r) = v(0) - \frac{1}{r}\int_0^r s^2 w(s)ds + \int_0^r sw(s)ds.$$

From this, it is not hard to realize (4.6). To obtain (4.3), we combine the representation for v and

$$v'(r) = \frac{1}{r^2}\int_0^r s^2 w(s)ds.$$

Finally, (4.5) can be obtained through

$$u(r) = u(0) - \frac{1}{r}\int_0^r s^2 v(s)ds + \int_0^r sv(s)ds,$$

while (4.2) follows from the preceding identity and

$$u'(r) = \frac{1}{r^2}\int_0^r s^2 v(s)ds.$$

The proof is complete. □

In view of Proposition 2.11, we shall easily see that if $\beta > \beta^\star$ then the radial solution $u = U_\beta$ to the ODE (2.10) grows at least quartically at infinity. Therefore, in view of Proposition 2.5, we must consider $u = U_{\beta^\star}$ in proof of Theorem 1.2(a, b, c).

5. Existence of radial solutions of cubic growth at infinity: proof of Theorems 1.2(a) and 1.3

This section is devoted to a proof of the existence of radial solutions to (1.1) having a prescribed cubic growth at infinity. Following the last paragraph in the previous section, we examine the entire radial solution $u = U_{\beta^\star}$ to the ODE (2.10) as a candidate.

5.1. **Asymptotic behavior of the transformed system (4.1).** In the first part of the proof, we study the asymptotic behavior of solution $(x_1, x_2, x_3, x_4, x_5, x_6)$ to (4.1) at infinity. For clarity, we divide our proof into several claims as follows. First, because

$$\wp = \lim_{r\nearrow+\infty} \Delta^2 U_{\beta^\star} = 0,$$

(4.4) becomes

$$\frac{w(r)}{rw'(r)} = -\frac{r\int_r^{+\infty} su^{-q}ds}{\int_0^r s^2u^{-q}ds} - 1.$$

First, we study x_3 and x_6. We establish the following:

Claim 1. There holds $w(r)/(rw'(r)) \to -1$ and $(r^2u^{-q})/w(r) \to 0$ as $r \nearrow +\infty$. In other words, $x_3(t) \to -1$ and $x_6(t) \to 0$ as $t \nearrow +\infty$.

Proof of Claim 1. Since $q > 1$ we apply the l'Hôpital rule to get

$$\lim_{r\nearrow+\infty} r\int_r^{+\infty} su^{-q}ds = \lim_{r\nearrow+\infty} r^3u^{-q} = 0$$

This helps us to conclude that $w(r)/(rw'(r)) \to -1$ as $r \nearrow +\infty$. Thus, we obtain

$$\lim_{r\nearrow+\infty}\frac{r^2u^{-q}}{w(r)} = \lim_{r\nearrow+\infty} -\frac{r^2u^{-q}}{rw'(r)} = \lim_{r\nearrow+\infty}\frac{r^3u^{-q}}{\int_0^r s^2u^{-q}ds} = 0.$$

The proof of Claim 1 is complete. □

Next we study x_2 and x_5. We prove the following:

Claim 2. There holds $v(r)/(r^2w(r)) \to 1/2$ and $v(r)/(rv'(r)) \to 1$ as $r \nearrow +\infty$. In other words, $x_5(t) \to 2$ and $x_2(t) \to 1$ as $t \nearrow +\infty$.

Proof of Claim 2. Recall $r^2 w(r) = r\int_0^r s^2 u^{-q} ds + r^2 \int_r^{+\infty} s u^{-q} ds$ which implies that $r^2 w(r) \nearrow +\infty$ as $r \nearrow +\infty$. Therefore, by the l'Hôpital rule, we obtain

$$\lim_{r\nearrow+\infty} \frac{1}{r^3 w(r)} \int_0^r s^2 w(s) ds = \lim_{r\nearrow+\infty} \frac{1}{3 + rw'(r)/w(r)} = \frac{1}{2}$$

and

$$\lim_{r\nearrow+\infty} \frac{1}{r^2 w(r)} \int_0^r s w(s) ds = \lim_{r\nearrow+\infty} \frac{1}{2 + rw'(r)/w(r)} = 1.$$

In view of (4.6), we know that $v(r)/(r^2 w(r)) \to 1/2$ as $r \nearrow +\infty$. On the other hand, we use the representation (4.3) to get $v(r)/(rv'(r)) \to 1$ as $r \nearrow +\infty$. The claim follows. $\square$

Finally we study x_1 and x_4. We show the following:

Claim 3. There holds $u(r)/(r^2 v(r)) \to 1/12$ and $u(r)/(ru'(r)) \to 1/3$ as $r \nearrow +\infty$. In other words, $x_4(t) \to 12$ and $x_1(t) \to 3$ as $t \nearrow +\infty$.

Proof of Claim 3. From $v(r)/(r^2 w(r)) \to 1/2$ and $r^2 w(r) \nearrow +\infty$ as $r \nearrow +\infty$, we obtain $v(r) \nearrow +\infty$ as $r \nearrow +\infty$. Therefore, by the l'Hôpital rule, we get

$$\lim_{r\nearrow+\infty} \frac{1}{r^3 v(r)} \int_0^r s^2 v(s) ds = \lim_{r\nearrow+\infty} \frac{1}{3 + rv'(r)/v(r)} = \frac{1}{4}$$

and

$$\lim_{r\nearrow+\infty} \frac{1}{r^2 v(r)} \int_0^r s v(s) ds = \lim_{r\nearrow+\infty} \frac{1}{2 + rv'(r)/v(r)} = \frac{1}{3}.$$

Again by (4.5), we have $v(r)/(r^2 w(r)) \to 1/12$ as $r \nearrow +\infty$. On the other hand, it follows from (4.2) that $u(r)/(ru'(r)) \to 1/3$ as $r \nearrow +\infty$. $\square$

From Claims 1, 2, and 3 above we see that the solution $(x_1, x_2, x_3, x_4, x_5, x_6)$ corresponding to the radially symmetric solution u is attracted to the fixed point

$$p_3 := (3, 1, -1, 12, 2, 0)$$

at infinity. Therefore, the asymptotic behavior is obtained by analyzing the asymptotic behavior of solutions about fixed point p_3 of the system (4.1). At p_3, the linearized matrix is

$$\begin{pmatrix} -7 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -12 & 12 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3-3q \end{pmatrix}$$

which has the following eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, $\lambda_4 = -4$, $\lambda_5 = 3-3q$, and $\lambda_6 = 1$. Since these eigenvalues are non-zero whenever $q > 1$, we conclude that there exists a constant $c_q \neq 0$ such that the following asymptotic behavior occurs: For $q > 4/3$

$$\frac{ru'(r)}{u(r)} = 3 + c_q e^{-t} + o(e^{-t})_{t\nearrow+\infty} \tag{5.1}$$

while for $q = 4/3$

$$\frac{ru'(r)}{u(r)} = 3 + c_q t e^{-t} + o(te^{-t})_{t\nearrow+\infty} \tag{5.2}$$

due to $\lambda_1 = \lambda_5$, and for $1 < q < 4/3$

$$\frac{ru'(r)}{u(r)} = 3 + c_q e^{-(3q-3)t} + o(e^{-(3q-3)t})_{t\nearrow+\infty}. \tag{5.3}$$

5.2. **Proof of Theorem 1.2(a).** Put $h(t) = \log u(r)$ with $r = e^t$. Then we obtain $h'(t) = (ru'(r))/u(r)$. Integrating both sides of (5.1), (5.2), and (5.3) gives

$$\frac{u(r)}{r^3} = \begin{cases} u(1)\exp \displaystyle\int_0^t \big(c_q e^{-s} + o(e^{-s})\big)ds & \text{if } q > 4/3 \\ u(1)\exp \displaystyle\int_0^t \big(c_q s e^{-s} + o(s e^{-s})\big)ds & \text{if } q = 4/3 \\ u(1)\exp \displaystyle\int_0^t \big(c_q e^{-(3q-3)s} + o(e^{-(3q-3)s})\big)ds & \text{if } 1 < q < 4/3. \end{cases}$$

From this, it is easy to see that the following limit $\lim_{|x|\nearrow+\infty} u(x)/|x|^3 = \kappa$ exists for some $\kappa > 0$. Given $\varpi > 0$, we scale the function u found in the preceding step as follows

$$\omega(r) = \Big(\frac{\varpi}{\kappa}\Big)^{\frac{2}{1-q}} u\Big(\big(\frac{\varpi}{\kappa}\big)^{\frac{q+1}{3q-3}} r\Big).$$

It is not hard to see that ω is also a solution of (1.1) and $\lim_{|x|\nearrow+\infty} \omega(x)/|x|^3 = \varpi$. The proof is complete.

5.3. **Proof of Theorem 1.3.** Using the presentation (3.5), it is not hard to verify that $u(r) \approx r^3$. To be more precise, we get

$$\kappa = \lim_{r\nearrow+\infty} \frac{u(r)}{r^3} = \frac{1}{24}\int_0^{+\infty} s^2 u(s)^{-q} ds.$$

Therefore,

$$\begin{aligned} u(r) - \kappa r^3 =& u(0) + \frac{\Delta u(0)}{6} r^2 - \frac{r^3}{24}\int_r^{+\infty} s^2 u(s)^{-q} ds + \frac{r^4}{120}\int_r^{+\infty} s u(s)^{-q} ds \\ & - \frac{r^2}{12}\int_0^r s^3 u(s)^{-q} ds - \frac{1}{24}\int_0^r s^5 u(s)^{-q} ds \\ & + \frac{r}{12}\int_0^r s^4 u(s)^{-q} ds + \frac{1}{120 r}\int_0^r s^6 u(s)^{-q} ds. \end{aligned} \tag{5.4}$$

Keep in mind that $q > 1$ and u has cubic growth at infinity. To prove the theorem, we make use of (5.1)–(5.3) plus the l'Hôpital rule to conclude the theorem. Indeed, when $q > 4/3$, there holds

$$\lim_{r\nearrow+\infty} \frac{u(r) - \kappa r^3}{r^2} = \frac{\Delta u(0)}{6} - \frac{1}{12}\int_0^{+\infty} s^3 u(s)^{-q} ds$$

due to the contribution of the second term and the third integral in (5.4). When $q = 4/3$,

$$\lim_{r\nearrow+\infty} \frac{u(r) - \kappa r^3}{r^2 \log r} = -\frac{1}{12\kappa^{4/3}}$$

due to the contribution of the third integral in (5.4) while in the case $1 < q < 4/3$, we obtain

$$\lim_{r\nearrow+\infty} \frac{u(r) - \kappa r^3}{r^{6-3q}} = \chi$$

due to the contribution of all six integrals in (5.4). (Here the constant χ is given in (1.7).) So, Theorem 1.3 is proved.

6. Existence of radial solutions of growth between cubic and quartic at infinity: Proof of Theorem 1.2(b,c)

This section is devoted to a proof of the existence of radial solutions to (1.1) having a prescribed growth between cubic and quartic at infinity. Following the last paragraph in the previous section, we examine the entire radial solution $u = U_{\beta^*}$ to the ODE (2.10) as a candidate.

Indeed, in view of Proposition 2.11(c), the radial solution u satisfies $\Delta^2 u(r) \searrow 0$ as $r \nearrow +\infty$. Therefore, we can apply Proposition 3.1(II) to get $u(r) \approx r^3\sqrt{\log r}$ at infinity when $q = 1$ while $u(r) \approx r^{6/(1+q)}$ at infinity when $1/2 < q < 1$. In fact, we can say more. This is because by Lemma 3.7 there holds

$$\lim_{r\nearrow+\infty} \frac{u(r)}{r^3(\log r)^{1/2}} = \frac{1}{\sqrt{12}}$$

when $q = 1$ and by Lemma 3.10 we know that

$$\lim_{t\nearrow+\infty} W(t) = (-K_0)^{-1/(1+q)}$$

when $1/2 < q < 1$. The proof of Theorem 1.2(b,c) is complete.

7. Existence of radial solutions of quartic growth at infinity: proof of Theorem 1.2(d) and 1.4

In view of Proposition 3.1(II), we have to consider q in the whole range, that is, $q > 1/2$. In this section, we are interested in radial solutions of quartic growth at infinity. Let $\beta > \beta^\star$ be arbitrary but fixed. Consider the solution $u = U_\beta$ to the ODE (2.10), which immediately yields that

$$\wp := \lim_{r\nearrow+\infty} \Delta^2 u(r) > 0.$$

Therefore, u grows at least quartically at infinity and $\lim_{r\nearrow+\infty} r^2 u^{-q} = 0$. Then, (4.4) becomes

$$\frac{w(r)}{rw'(r)} = -\frac{\gamma r}{\int_0^r s^2 u^{-q} ds} - \frac{r\int_r^{+\infty} s u^{-q} ds}{\int_0^r s^2 u^{-q} ds} - 1.$$

Now we study the asymptotic behavior of the point $(x_1, x_2, x_3, x_4, x_5, x_6)$ at infinity As always, we split our proof into several steps.

Claim 4. There holds $w(r)/(rw'(r)) \to -\infty$ and $w(r)/(r^2u^{-q}) \nearrow +\infty$ as $r \nearrow +\infty$. In other words, $x_3(t) \to 0$ and $x_6(t) \to 0$ as $t \nearrow +\infty$.

Proof of Claim 4. We observe that if $\int_0^{+\infty} s^2 u^{-q} ds$ converges, then $w(r)/(rw'(r)) \to -\infty$ as $r \nearrow +\infty$. Otherwise, we should have

$$\lim_{r\nearrow+\infty} \frac{\gamma r}{\int_0^r s^2 u^{-q} ds} = \lim_{r\nearrow+\infty} \frac{\gamma}{r^2 u^{-q}} = +\infty.$$

Hence $w(r)/(rw'(r)) \to -\infty$ as $r \nearrow +\infty$. On the other hand, we also have

$$\frac{w(r)}{r^2u^{-q}} = \frac{\gamma}{r^2u^{-q}} + \frac{1}{r^2u^{-q}} \int_r^{+\infty} s u^{-q} ds + \frac{1}{r^3 u^{-q}} \int_0^r s^2 u^{-q} ds.$$

Therefore, $w(r)/(r^2u^{-q}) \nearrow +\infty$ as $r \nearrow +\infty$. □

By the same arguments as in the proof of Claims 2, 3 in the previous section, we easily obtain the following claims.

Claim 5. There holds $v(r)/(r^2 w(r)) \to 1/6$ and $v(r)/(rv'(r)) \to 1/2$ as $r \nearrow +\infty$. In other words, $x_5(t) \to 6$ and $x_2(t) \to 2$ as $t \nearrow +\infty$.

Claim 6. There holds $u(r)/(r^2 v(r)) \to 1/20$ and $u(r)/(ru'(r)) \to 1/4$ as $r \nearrow +\infty$. In other words, $x_4(t) \to 20$ and $x_1(t) \to 4$ as $t \nearrow +\infty$.

From Claims 4, 5, and 6 above we know that the solution $(x_1, x_2, x_3, x_4, x_5, x_6)$ corresponding to the radially symmetric solution u is attracted to the fixed point

$$p_4 := (4, 2, 0, 20, 6, 0)$$

at infinity. Therefore, the asymptotic behavior is obtained by analyzing the asymptotic behavior of solutions about fixed point p_4 of the system (4.1). At p_4, the linearized matrix is

$$\begin{pmatrix} -9 & 0 & 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ -20 & 20 & 0 & 0 & 0 & 0 \\ 0 & -6 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2-4q \end{pmatrix}$$

which has the following eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, $\lambda_4 = -4$, $\lambda_5 = -5$, and $\lambda_6 = 2 - 4q$. So, these eigenvalues are non-zero whenever $q > 1/2$.

7.1. **Proof of Theorem 1.2(d).** Simply repeating the argument used in Section 5, the following limit $\lim_{|x|\nearrow+\infty} u(x)/|x|^4 = \kappa$ exists for some $\kappa > 0$. Given $\varpi > 0$, we scale the function u found in the previous step as follows

$$\omega(r) = \Big(\frac{\varpi}{\kappa}\Big)^{\frac{3}{1-2q}} u\Big(\big(\frac{\varpi}{\kappa}\big)^{\frac{q+1}{4q-2}} r\Big).$$

Then, ω is also a solution of (1.1) with $\lim_{|x|\nearrow+\infty} \omega(x)/|x|^4 = \varpi$. The proof is complete.

7.2. **Proof of Theorem 1.4.** Thanks to (3.5), we have

$$\begin{aligned} u(r) =& u(0) + \frac{\Delta u(0)}{6} r^2 + \frac{\wp}{120} r^4 \\ &+ \frac{r^4}{120} \int_r^{+\infty} s u(s)^{-q} ds - \frac{r^2}{12} \int_0^r s^3 u(s)^{-q} ds - \frac{1}{24} \int_0^r s^5 u(s)^{-q} ds \\ &+ \frac{r^3}{24} \int_0^r s^2 u(s)^{-q} ds + \frac{r}{12} \int_0^r s^4 u(s)^{-q} ds + \frac{1}{120 r} \int_0^r s^6 u(s)^{-q} ds \end{aligned}$$

in $\mathbf{R}^3$ where

$$\wp = (\Delta^2 u)(0) - \int_0^{+\infty} s u(s)^{-q} ds,$$

which is finite since $q > 1/2$ and u is of exactly quartic growth at infinity by our assumption. From the presentation above, we get

$$\kappa = \lim_{r\nearrow+\infty} \frac{u(r)}{r^4} = \frac{\wp}{120}.$$

Therefore,

$$\begin{aligned} u(r) - \kappa r^4 =& u(0) + \frac{\Delta u(0)}{6} r^2 + \frac{r^4}{120} \int_r^{+\infty} s u(s)^{-q} ds - \frac{r^2}{12} \int_0^r s^3 u(s)^{-q} ds \\ &- \frac{1}{24} \int_0^r s^5 u(s)^{-q} ds + \frac{r^3}{24} \int_0^r s^2 u(s)^{-q} ds \\ &+ \frac{r}{12} \int_0^r s^4 u(s)^{-q} ds + \frac{1}{120 r} \int_0^r s^6 u(s)^{-q} ds \end{aligned} \tag{7.1}$$

The next, we use the l'Hôpital rule to conclude the theorem. Indeed, when $q > 3/4$, there holds

$$\lim_{r\nearrow+\infty} \frac{u(r) - \kappa r^4}{r^3} = \frac{1}{24} \int_0^{+\infty} s^2 u(s)^{-q} ds$$

due to the contribution of the fourth integral in (7.1). When $q = 3/4$ we have

$$\lim_{r\nearrow+\infty} \frac{u(r) - \kappa r^4}{r^3 \log r} = \frac{1}{24 \kappa^{3/4}}$$

due to the contribution of the fourth integral in (7.1) while in the case $1/2 < q < 3/4$ we obtain

$$\lim_{r \nearrow +\infty} \frac{u(r) - \kappa r^4}{r^{6-4q}} = \chi$$

due to the contribution of all six integrals in (7.1). (Here the constant χ is given in (1.8).) So, Theorem 1.4 is proved.

8. Existence of non-radial solutions to (1.1): proof of Theorem 1.5

This section is devoted to proof of Theorem 1.5. As mentioned in Introduction, we adopt the method used in [HW19], see also [Alb21], namely, we look for a non-radial solution to (1.1) of the form $u = v + P_0$ where $P_0 \geq 1$ is a polynomial of degree 4. Note that $u = v + P_0$ satisfies (1.1) if, and only if, v satisfies $v + P_0 > 0$ and

$$\Delta^3 v = -(v + P_0)^{-q} \quad \text{in } \mathbf{R}^3 .$$

Note that in $\mathbf{R}^3$ the fundamental solution of the triharmonic equation $\Delta^3 w = 0$ is nothing but $-|x|^3/(96\pi)$. Hence, it suffices to find v solving the following integral equation

$$v(x) = \frac{1}{96\pi} \int_{\mathbf{R}^3} \frac{|x-y|^3}{\big(P_0(y) + v(y)\big)^q} dy \quad \text{in } \mathbf{R}^3,$$

namely, our desired solution to (1.1) satisfies

$$u(x) = \frac{1}{96\pi} \int_{\mathbf{R}^3} |x-y|^3 u(y)^{-q} dy + P_0(x) \quad \text{in } \mathbf{R}^3;$$

see Lemma 8.2 below. Compared with the work [HW19], our analysis is more involved because our polynomial P_0 has higher order and the kernel of the above inequality equation is $|x|^3$. To look for solutions to the above integral equation, we shall make use of the Schaefer Fixed Point Theorem. Let

$$X_{\rm ev} = \{ v \in C^0(\mathbf{R}^3) : v(x) = v(-x) \}.$$

Then, the space $X_{\rm ev}$, endowed with the norm $\| \cdot \|$ given by

$$\|v\| := \sup_{x \in \mathbf{R}^3} \frac{|v(x)|}{1 + |x|^3},$$

is a well-defined Banach space. At the first glance, it is not clear why we need the the symmetry in the definition of $X_{\rm ev}$. We shall soon see this in the proof of following proposition. We start with the following crucial result.

Proposition 8.1. *Let $q > 0$ and P be a smooth function on $\mathbf{R}^3$ such that $P(x) \geq 1$ and $P(x) = P(-x)$ for any x, and*

$$\int_{\mathbf{R}^3} \frac{|y|^3}{P(y)^q} dy < +\infty. \tag{8.1}$$

Then there exists some function $v \in X_{\rm ev}$ satisfying $\min_{\mathbf{R}^3} v = v(0) = 0$ and

$$v(x) = \frac{1}{96\pi} \int_{\mathbf{R}^3} \frac{|x-y|^3 - |y|^3}{\big(P(y) + v(y)\big)^q} dy \quad \text{in } \mathbf{R}^3 . \tag{8.2}$$

Moreover, there holds

$$\lim_{|x| \nearrow +\infty} \frac{v(x)}{|x|^3} = \frac{1}{96\pi} \int_{\mathbf{R}^3} \frac{dy}{\big(P(y) + v(y)\big)^q} . \tag{8.3}$$

Proof. We start our proof by observing

$$\begin{aligned}\big||x-y|^3-|y|^3\big| &= \big||x-y|-|y|\big|(|x-y|^2+|x-y||y|+|y|^2)\\ &\leq |x|(3|x|^2+|x||y|+3|y|^2).\end{aligned} \tag{8.4}$$

This and the integrability condition (8.1) tells us that the function $\widehat{v}$ defined by

$$\widehat{v}(x)=\frac{1}{96\pi}\int_{\mathbf{R}^3}\frac{|x-y|^3-|y|^3}{\big(P(y)+|v(y)|\big)^q}dy$$

is well-defined everywhere in $\mathbf{R}^3$. Besides, if $v\in C^0(\mathbf{R}^3)$, then $\widehat{v}$ is of class $C^5(\mathbf{R}^3)$. In addition, thanks to the symmetry $P(x)=P(-x)$ in $\mathbf{R}^3$ we can also conclude that if $v(x)=v(-x)$, then $\widehat{v}(x)=\widehat{v}(-x)$ everywhere. This allows us to define the following operator

$$\begin{aligned}T:X_{\rm ev}&\to X_{\rm ev},\\ v&\mapsto \widehat{v}.\end{aligned}$$

Our aim is to show that T has a fixed point in $X_{\rm ev}$. First we show that T is well-defined. To see this, it remains to verify $\|\widehat{v}\|<+\infty$. Thanks to (8.4), we can estimate

$$|\widehat{v}(x)|\leq \frac{3|x|^3}{96\pi}\int_{\mathbf{R}^3}\frac{dy}{P(y)^q}+\frac{|x|^2}{96\pi}\int_{\mathbf{R}^3}\frac{|y|dy}{P(y)^q}+\frac{3|x|}{96\pi}\int_{\mathbf{R}^3}\frac{|y|^2dy}{P(y))^q}.$$

Under the integrability (8.1) we deduce that the above three integrals are finite. Hence there is some $C>0$ such that

$$|\widehat{v}(x)|\leq C(|x|^3+|x|^2+|x|),$$

which is enough to guarantee $|\widehat{v}(x)|\leq 3C(1+|x|^3)$, namely $\|\widehat{v}\|<+\infty$. Differentiating under the integral sign gives

$$|\nabla\widehat{v}(x)|\leq\frac{1}{32\pi}\int_{\mathbf{R}^3}\frac{|x-y|^2}{\big(P(y)+|v(y)|\big)^q}dy \quad \text{in } \mathbf{R}^3.$$

Now we show that T is continuous. Indeed, let $(v_k)_k\subset X_{\rm ev}$ be such that $v_k\to v$ in $(X_{\rm ev},\|\cdot\|)$. In particular, $v_k\to v$ in $\mathbf{R}^3$ in the pointwise sense. Thanks to (8.4) we can argue as before to obtain

$$\|v_k-v\|\leq C\int_{\mathbf{R}^3}(3+|y|+3|y|^2)\Big|\frac{1}{\big(P(y)+|v_k(y)|\big)^q}-\frac{1}{\big(P(y)+|v(y)|\big)^q}\Big|dy$$

for some $C>0$. Hence, to conclude the continuity of T, it suffices to show that the right hand side of the preceding inequality goes to zero as $k\nearrow+\infty$. But this simply follows from dominated convergence, thanks to the integrability condition (8.1).

Now we show that T is compact. Let $(v_k)_k\subset X_{\rm ev}$ be an arbitrary sequence, we must show that $(T(v_k))_k$ has a convergent subsequence. The idea is to use the Arzelà–Ascoli Theorem on the sequence $(T_k)_k$ with

$$T_k:=\frac{T(v_k)}{1+|x|^3}.$$

First by the above estimate we deduce that the sequence $(T_k)_k$ is uniformly bounded. Now, for any $x,y,z\in\mathbf{R}^3$ and similar to (8.4) we observe that

$$\big||x-z|^3-|y-z|^3\big|\leq|x-y|(|x-z|+|y-z|)^2\leq 4(|x|^2+|y|^2+2|z|^2)|x-y| \tag{8.5}$$

and that

$$\big||x-z||y|-|y-z||x|\big|\leq(2|x|+2|y|+|z|)|x-y|$$

we deduce that

$$\begin{aligned}\Big|\frac{|x-z|^3}{1+|x|^3} - \frac{|y-z|^3}{1+|y|^3}\Big| &\leq \frac{\big||x-z|^3-|y-z|^3\big|}{(1+|x|^3)(1+|y|^3)} + \frac{\big||x-z|^3|y|^3-|y-z|^3|x|^3\big|}{(1+|x|^3)(1+|y|^3)}\\ &\leq \frac{4(|x|^2+|y|^2+2|z|^2)}{(1+|x|^3)(1+|y|^3)}|x-y|\\ &\quad + \frac{(2|x|+2|y|+|z|)(|x-z||y|+|y-z||x|)^2}{(1+|x|^3)(1+|y|^3)}|x-y|\\ &\leq C(1+|z|+|z|^2+|z|^3)|x-y|\end{aligned}$$

for some $C > 0$. This tells us that

$$\begin{aligned}\Big|\frac{T(v_k)(x)}{1+|x|^3} - \frac{T(v_k)(y)}{1+|y|^3}\Big| &\leq \frac{1}{96\pi}\int_{\mathbf{R}^3}\Big|\frac{|x-z|^3}{1+|x|^3} - \frac{|y-z|^3}{1+|y|^3}\Big|\frac{1}{\big(P(z)+|v(z)|\big)^q}dz\\ &\leq \frac{C}{96\pi}\Big(\int_{\mathbf{R}^3}\frac{1+|z|+|z|^2+|z|^3}{\big(P(z)+|v(z)|\big)^q}dz\Big)|x-y|\end{aligned}$$

for any $x, y \in \mathbf{R}^3$ and any k. Hence, the sequence $(T_k)_k$ is equicontinuous. By the Arzelà–Ascoli Theorem, the sequence $(T_k)_k$ admits a subsequence which converges uniformly. Thus, the sequence $(T(v_k))_k$ admits a converging subsequence in $(X_{\rm ev}, \|\cdot\|)$ and therefore T is a compact operator.

Next, we prove that T has a fixed point by means of the Schaefer Fixed Point Theorem, see [GT98, Theorem 11.6]. Let $v \in X_{\rm ev}$ be such that

$$v = tT(v) \quad \text{for some } t \in [0,1],$$

namely

$$v(x) = \frac{t}{96\pi}\int_{\mathbf{R}^3}\frac{|x-y|^3-|y|^3}{\big(P(y)+|v(y)|\big)^q}dy \quad \text{in } \mathbf{R}^3.$$

Then by following the argument for $\widehat{v}$ we should arrive at

$$|v(x)| \leq \frac{1}{96\pi}\int_{\mathbf{R}^3}\frac{\big||x-y|^3-|y|^3\big|}{\big(P(y)+|v(y)|\big)^q}dy \leq C(1+|x|^3)$$

for some $C > 0$ independent of v. Consequently, $\|v\| \leq C$. This means that the set

$$\{v \in X : v = tT(v) \text{ for some } t \in [0,1]\}$$

is bounded in $(X_{\rm ev}, \|\cdot\|)$. Thus, T has a fixed point in $X_{\rm ev}$, still denoted by v.

Next, we verify $\min_{\mathbf{R}^3} v = v(0) = 0$. As $v(0) = 0$ is trivial, it suffices to show $v \geq 0$. Indeed, by the symmetry of v we have

$$2v(x) = v(x) + v(-x) = \frac{1}{96\pi}\int_{\mathbf{R}^3}\frac{|x-y|^3+|x+y|^3-2|y|^3}{\big(P(y)+|v(y)|\big)^q}dy.$$

By the Jensen inequality we know that

$$|x-y|^3+|x+y|^3 \geq \frac{1}{4}(|x-y|+|x+y|)^3 \geq 2|y|^3.$$

Putting the above two facts together we arrive at $v \geq 0$. This also implies (8.2).

Let us now verify the limit (8.3). Clearly the integral on the right hand side of (8.3) converges and

$$\lim_{|x|\nearrow+\infty}\frac{v(x)}{|x|^3} = \frac{1}{96\pi}\lim_{|x|\nearrow+\infty}\int_{\mathbf{R}^3}\frac{|x-y|^3-|y|^3}{|x|^3}\frac{dy}{\big(P(y)+|v(y)|\big)^q}.$$

Under the integrability condition (8.1), we can apply the Lebesgue dominated convergence to obtain (8.3). □

Proposition 8.1 tells us that there is some continuous function v solving the integral equation (8.2). In fact, we can say more.

Lemma 8.2. *If $v \in X_{\rm ev}$ solves (8.2), then v solves*

$$\Delta^3 v = -(v+P)^{-q} \quad \text{in } \mathbf{R}^3 .$$

In fact, as P is smooth, v is smooth.

Proof. Let $v \in X_{\rm ev}$ solve (8.2). In particular, $v \in C^0(\mathbf{R}^3)$. Unfortunately, this is not enough because $v \in C^{5,\alpha}(\mathbf{R}^3)$ for any $\alpha \in (0,1)$ by elliptic estimates. However, it is not hard to see that v is actually locally Lipschitz continuous. Indeed, thanks to (8.5) for any x and y in any given compact set we can estimate

$$\begin{aligned}
|v(x)-v(y)| &\leq \frac{1}{96\pi}\int_{\mathbf{R}^3} \frac{\big||x-z|^3-|y-z|^3\big|}{\big(P(z)+v(z)\big)^q} dz \\
&\leq \frac{1}{24\pi}\Big(\int_{\mathbf{R}^3} \frac{|x|^2+|y|^2+2|z|^2}{\big(P(z)+v(z)\big)^q} dz\Big)|x-y|.
\end{aligned}$$

From this and the integrability condition (8.1) we obtain the locally Lipschitz continuity of v. Now it follows from elliptic estimates that $v \in C^{6,\alpha}(\mathbf{R}^3)$ for any $\alpha \in (0,1)$. This is enough to conclude that v solves the desired PDE. Finally, the smoothness of v can be easily obtained, for example, by induction. $\square$

We are now in position to prove the result. To do so, we first recall $q > 1/2$ (instead of $q > 0$ as in the statement of Proposition 8.1). For any $\varepsilon > 0$ but fixed we let

$$P_\varepsilon(x) = 1 + 3x_1^4 + 4(x_2^4 + x_3^4) + \varepsilon |x|^{12}, \quad x = (x_1, x_2, x_3) \in \mathbf{R}^3 .$$

Clearly, P_ε is smooth, $P_\varepsilon \geq 1 + |x|^4$, and P_ε enjoys the symmetry property $P_\varepsilon(x) = P_\varepsilon(-x)$ for any $x \in \mathbf{R}^3$. Furthermore, it is easy to see that the condition (8.1) with P replaced by P_ε is fulfilled. Then, by Proposition 8.1 there is some $v_\varepsilon \geq 0$ in $X_{\rm ev}$ solving (8.2), namely

$$v_\varepsilon(x) = \frac{1}{96\pi}\int_{\mathbf{R}^3} \frac{|x-y|^3-|y|^3}{\big(P_\varepsilon(y)+v_\varepsilon(y)\big)^q} dy \quad \text{in } \mathbf{R}^3 . \tag{8.6}$$

By Lemma 8.2 we know that $v_\varepsilon \in C^\infty(\mathbf{R}^3)$. Now we claim that the sequence (v_ε) is bounded in $C^9_{\rm loc}(\mathbf{R}^3)$. Indeed, by differentiating under the integral sign, we obtain from (8.6) the following

$$|D^\beta v_\varepsilon(x)| \leq C \int_{\mathbf{R}^3} \frac{1}{|x-y|} \frac{1}{\big(P_\varepsilon(y)+v_\varepsilon(y)\big)^q} dy.$$

for some $C > 0$ and for any multi-index β with $|\beta| = 4$. For $|x| \leq 2$ we can estimate

$$\begin{aligned}
|D^\beta v_\varepsilon(x)| &\leq C\Big(\int_{B_3} + \int_{\mathbf{R}^3 \setminus B_3}\Big) \frac{1}{|x-y|} \frac{1}{\big(P_\varepsilon(y)+v_\varepsilon(y)\big)^q} dy \\
&\leq C \int_{B_3} \frac{dy}{|x-y|} + 3C \int_{\mathbf{R}^3 \setminus B_3} \frac{dy}{|y|\big(P_\varepsilon(y)+v_\varepsilon(y)\big)^q}.
\end{aligned}$$

Keep in mind that $q > 1/2$ and $P_\varepsilon(x) + v_\varepsilon(x) \geq |x|^4$. Hence, the preceding estimate yields a bound of $|D^\beta v_\varepsilon|$, which is independent of ε, in the region $\{x \in \mathbf{R}^3 : |x| \leq 2\}$. Let us now consider the region $\{x \in \mathbf{R}^3 : |x| > 2\}$. Initially, we decompose the integral $\int_{\mathbf{R}^3}$ as follows

$$\begin{aligned}
|D^\beta v_\varepsilon(x)| &\leq C\Big(\int_{\{|y| \leq \frac{|x|}{2}\}} + \int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} + \int_{\{|y| \geq 2|x|\}}\Big) \frac{1}{|x-y|} \frac{1}{\big(P_\varepsilon(y)+v_\varepsilon(y)\big)^q} dy \\
&= C(I_1 + I_2 + I_3).
\end{aligned}$$

For the term I_1, as $|x-y| \geq |x| - |y| \geq |x|/2$ we can bound

$$I_1 \leq \frac{2}{|x|} \int_{\{|y| \leq \frac{|x|}{2}\}} \frac{dy}{(1+|y|^4)^q} \leq \frac{C}{|x|} \int_{\{|y| \leq \frac{|x|}{2}\}} \frac{dy}{(1+|y|)^{4q}}.$$

Note that

$$\int_{\{|y| \leq \frac{|x|}{2}\}} \frac{dy}{(1+|y|)^{4q}} = C \int_0^{|x|/2} \frac{r^2 dr}{(1+r)^{4q}} \leq C \int_0^{|x|/2} \frac{dr}{(1+r)^{4q-2}}.$$

Thus we get

$$I_1 \leq C \begin{cases} \dfrac{1}{|x|^{4q-2}} & \text{if } q \neq 3/4, \\ \dfrac{\log(1+|x|)}{|x|} & \text{if } q = 3/4. \end{cases}$$

For the term I_2, as $P_\varepsilon(y) + v_\varepsilon(y) \geq |y|^4 \geq (|x|/2)^4$ we can bound

$$I_2 \leq \Big(\frac{2}{|x|}\Big)^{4q} \int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} \frac{dy}{|x-y|} \leq \frac{C}{|x|^{4q-2}}.$$

Finally, for the term I_3 as $|x-y| \geq |y| - |x| \geq |y|/2$ we can estimate as follows

$$I_3 \leq 4 \int_{\{|y| \geq 2|x|\}} \frac{1}{|y|} \frac{1}{(1+|y|^4)^q} dy \leq \frac{C}{|x|^{4q-2}}.$$

Thus, we have just shown that, for some $C > 0$, $|D^\beta v_\varepsilon(x)| \leq C$ in B_2 and

$$|D^\beta v_\varepsilon(x)| \leq C \begin{cases} \dfrac{1}{|x|^{4q-2}} & \text{if } q \neq 3/4, \\ \dfrac{\log(1+|x|)}{|x|} & \text{if } q = 3/4, \end{cases}$$

in $\mathbf{R}^3 \setminus B_2$. As $q > 1/2$, this also implies that $|D^\beta v_\varepsilon(x)|$ with $|\beta| = 4$ is bounded. In fact, this is true for any β with $|\beta| \leq 4$. Hence, we have just shown that (v_ε) is bounded in $C^4_{\text{loc}}(\mathbf{R}^3)$. Using the integral equation (8.6) one can further show that (v_ε) is bounded in $C^9_{\text{loc}}(\mathbf{R}^3)$.

The boundedness in $C^9_{\text{loc}}(\mathbf{R}^3)$ allows us to select a convergence subsequence via a diagonal argument, namely for some $\varepsilon_k \searrow 0$ as $k \nearrow +\infty$ we have

$$v_{\varepsilon_k} \to v \quad \text{in } C^6_{\text{loc}}(\mathbf{R}^3)$$

for some $v \geq 0$ satisfying

$$\Delta^3 v = -(P_0 + v)^{-q} \quad \text{uniformly in } \mathbf{R}^3,$$

where $P_0(x) = 1 + x_1^4 + 2(x_2^4 + x_3^4)$. Indeed, this is standard and goes as follows. By the Arzelà–Ascoli Theorem, there is a subsequence $(v_{\varepsilon_k^{(1)}})_k$ of (v_ε) and a function $v^{(1)} \in C^6(\overline{B}_1)$ such that

$$v_{\varepsilon_k^{(1)}} \rightrightarrows v^{(1)}, \quad \Delta^3 v_{\varepsilon_k^{(1)}} \rightrightarrows \Delta^3 v^{(1)} \quad \text{uniformly in } \overline{B}_1.$$

Again by the Arzelà–Ascoli Theorem, there is a subsequence $(\varepsilon_k^{(2)})_k$ of $(\varepsilon_k^{(1)})_k$ and a function $v^{(2)} \in C^6(\overline{B}_2)$ such that

$$v_{\varepsilon_k^{(2)}} \rightrightarrows v^{(2)}, \quad \Delta^3 v_{\varepsilon_k^{(2)}} \rightrightarrows \Delta^3 v^{(2)} \quad \text{uniformly in } \overline{B}_2.$$

It is important to note that $v^{(2)} \equiv v^{(1)}$ in $\overline{B}_1$. Repeating this process, for each integer $i > 1$ we obtain a subsequence $(\varepsilon_k^{(i)})_k$ of $(\varepsilon_k^{(i-1)})_k$ and a function $v^{(i)} \in C^6(\overline{B}_i)$ such that

$$v_{\varepsilon_k^{(i)}} \rightrightarrows v^{(i)}, \quad \Delta^3 v_{\varepsilon_k^{(i)}} \rightrightarrows \Delta^3 v^{(i)} \quad \text{uniformly in } \overline{B}_i.$$

and

$$v^{(i)} \equiv v^{(i-1)} \quad \text{in } \overline{B}_{i-1}.$$

From this we simply consider the pointwise limit of $(v^{(i)})_i$ to get the desired function v and the diagonal sequence $(v_{\varepsilon_i^{(i)}})_i$ of $(v_{\varepsilon_k^{(i)}})_k$ to get the desired convergence.

Hence, by passing to the limit, we conclude that the function $u = P_0 + v$ solves our PDE. Moreover, thanks to (8.3) we know that $v = o(|x|^4)$ at infinity. Hence

$$\liminf_{|x| \nearrow +\infty} \frac{u(x)}{|x|^4} = \liminf_{|x| \nearrow +\infty} \frac{P_0(x)}{|x|^4} = 3 < 4 = \limsup_{|x| \nearrow +\infty} \frac{P_0(x)}{|x|^4} = \limsup_{|x| \nearrow +\infty} \frac{u(x)}{|x|^4}.$$

By the classification of growth in Theorem 1.1 we readily see that u cannot be radially symmetric.

Remark 8.3. It can be seen from the above proof that we have chosen P_ε precisely which is enough to serve our purpose. However, arguing similarly one can show that given any two constants $\kappa_2 > \kappa_1 > 0$ there is a solution $u > 0$ to (1.1) such that

$$\liminf_{|x| \nearrow +\infty} \frac{u(x)}{|x|^4} = \kappa_1, \quad \limsup_{|x| \nearrow +\infty} \frac{u(x)}{|x|^4} = \kappa_2,$$

giving a result similar to that in [HW19].

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