ON ALMOST P-STANDARD SYSTEM OF PARAMETERS
OF IDEALIZATION AND APPLICATIONS

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In memory of Professor Shiro Goto

Abstract. Let $(R, m)$ be a Noetherian local ring and $M$ a finitely generated $R$-module. In this paper, we construct almost p-standard systems of parameters (a very strict subclass of d-sequences) of the idealization $R \ltimes M$ of $M$ over $R$. As applications, we build Cohen-Macaulay Rees algebras for idealizations, Cohen-Macaulay Rees modules for unmixed modules, then give precise formulas computing all the Hilbert coefficients of the idealization with respect to an almost p-standard system of parameters.

1 Introduction

Throughout this paper, $(R, m)$ denotes a Noetherian local ring of dimension $r$. Let $M$ be a finitely generated $R$-module with $\dim_R(M) = d$. The notion of d-sequence introduced by C. Huneke [15] makes a useful mean to study the powers of ideals [14, 15] and have important applications in the theory of Buchsbaum modules and generalized Cohen-Macaulay modules. In [6], N.T. Cuong introduced the notion of p-standard system of parameter (s.o.p for short). Note that if $x_1, \ldots, x_d$ is a p-standard s.o.p of $M$ then it is a d-sequence on $M$ and there exist non-negative integers $\lambda_0, \ldots, \lambda_d$ such that

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i$$

Key words and phrases: Almost p-standard system of parameters; idealization; Macaulayfication; Hilbert coefficient.

2020 Subject Classification: 13E05, 13A30, 13D40, 13H10.
for all \( n_1, \ldots, n_d \geq 1 \) (see [6, Theorem 2.6]). In generalized Cohen-Macaulay modules, every p-standard s.o.p is a standard s.o.p in the sense of [22], and in general, the notion of p-standard s.o.p plays a key role in the study of the singularity of Cohen-Macaulay type of Noetherian rings and modules (see [16, 17, 10]).

Let \( x_1, \ldots, x_d \) be a s.o.p of \( M \). If there exists non-negative integers \( \lambda_0, \ldots, \lambda_d \) such that

\[
\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i
\]

for all \( n_1, \ldots, n_d \geq 1 \), then \( x_1^{n_1}, \ldots, x_d^{n_d} \) is a p-standard s.o.p for all \( n_i \geq i \), for \( i = 1, \ldots, d \) (see [7, Corollary 3.9]), however \( x_1, \ldots, x_d \) is not necessary a p-standard s.o.p (see [8, Example 3.11]). This fact leads to the following notion (see [4, Definition 2.1]).

**Definition 1.1.** A s.o.p \( x_1, \ldots, x_d \) of \( M \) is called **almost p-standard** if there exist non-negative integers \( \lambda_0, \ldots, \lambda_d \) such that

\[
\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i
\]

for all \( n_1, \ldots, n_d \geq 1 \).

Following [10, Theorem 1.2], \( R \) admits an almost p-standard s.o.p if and only if \( R \) is a quotient of a Cohen-Macaulay local ring, if and only if every finitely generated \( R \)-module admits an almost p-standard s.o.p. Note that every almost p-standard s.o.p is a d-sequence, this fact helps to compute several numerical invariants, the Hilbert coefficients, the partial Euler-Poincaré characteristics of the Koszul complex with respect to an almost p-standard s.o.p of \( M \), see [4]. The notion of almost p-standard s.o.p makes an important role in the study of sequentially Cohen-Macaulay modules and sequentially generalized Cohen-Macaulay modules [7, 9].

The notion of the idealization was introduced by M. Nagata [20]. We provide a multiplication on the additive group \( R \oplus M \)

\[
(a, x), (b, y) = (ab, ay + bx)
\]

for all \( (a, x), (b, y) \in R \oplus M \), then \( R \oplus M \) forms a Noetherian local ring with the unique maximal ideal \( \mathfrak{m} \times M \). This local ring is called the idealization of \( M \) over \( R \) and denoted by \( R \ltimes M \). Note that \( \dim(R \ltimes M) = \dim(R) \). The structure of the idealization and its applications have attracted the interest of mathematicians (see [2, 20, 13]).

The aim of this paper is to construct almost p-standard s.o.p of \( R \ltimes M \). As applications, we build Cohen-Macaulay Rees algebras for \( R \ltimes M \), Cohen-Macaulay Rees modules for unmixed module \( M \), and find a tight relation between Macaulayfications of \( R \) and \( R \ltimes M \) in several particular cases. Then we give precise formulas computing Hilbert coefficients of \( R \ltimes M \) with respect to certain almost p-standard s.o.p.

The following theorem is the first main result of this paper.

**Theorem 1.2.** Let \( x_1, \ldots, x_r \) be elements in \( \mathfrak{m} \). Set \( u_i = (x_i, 0) \) for \( i = 1, \ldots, r \) and \( u = u_1, \ldots, u_r \). The following statements are equivalent:
(i) $u$ is an almost p-standard s.o.p of $R \times M$.

(ii) $x_1, \ldots, x_d$ is an almost p-standard s.o.p of $M$ and $x_1, \ldots, x_r$ is an almost p-standard s.o.p of $R$ and $x_{d+1}, \ldots, x_r \in \text{Ann}_R(M)$.

As a consequence, we give a characterization for $R \times M$ being a quotient of a Cohen-Macaulay local ring (Corollary 2.6).

Denote by $\widehat{R}$ and $\widehat{M}$ the $m$-adic completion of $R$ and $M$, respectively. Following M. Nagata [20], $M$ is said to be unmixed if $\dim(\widehat{R}/\mathfrak{P}) = \dim(\widehat{M})$ for any $\mathfrak{P} \in \text{Ass}_R(\widehat{M})$. Note that $R \times M$ is unmixed if and only if $\dim(R) = \dim(M) = r$ and $R, M$ are unmixed. The first application of Theorem 1.2 is to clarify certain Hilbert coefficients of the idealization.

**Theorem 1.3.** Suppose that $R$ is a quotient of a Cohen-Macaulay local ring, $R$ and $M$ are unmixed, and $\dim(R) = \dim(M) = r > 1$. Let $x_1, \ldots, x_r$ be an almost p-standard s.o.p of both $R$ and $M$ (such a s.o.p exists). For $i = 1, \ldots, r$, put $u_i = (x_i, 0)$, $P_i = (u_1, \ldots, u_r)$ and $P = P_1P_2 \ldots P_{r-2}$. Then the Rees algebra $\mathfrak{R}(R \times M, P)$ is Cohen-Macaulay.

From an almost p-standard s.o.p of $M$, we can construct subquotient modules $U^{i,j}_M, \overline{U}^{i,j}_M$ which are independent of the choice of almost p-standard s.o.p (see [4, Proposition 2.2]). The second application of Theorem 1.2 is to clarify certain Hilbert coefficients of the idealization.

**Theorem 1.4.** Let $x_1, \ldots, x_r$ be an almost p-standard s.o.p of $R$ such that $x_1, \ldots, x_d$ is an almost p-standard s.o.p of $M$ and $x_{d+1}, \ldots, x_r \in \text{Ann}_R(M)$. Set $Q = (u_1, \ldots, u_r)$, where $u_i = (x_i, 0)$ for $i = 1, \ldots, r$. Put $I = (x_1, \ldots, x_d)$ and $J = (x_1, \ldots, x_r)$. Then

$$\ell((R \times M)/Q^{n+1}) = e_0(Q, R \times M) \binom{n+r}{r} + e_1(Q, R \times M) \binom{n+r-1}{r-1} + \ldots + e_r(Q, R \times M)$$

for all $n \geq 0$, where for $d = r$,

$$e_{r-i}(Q, R \times M) = \begin{cases} \sum_{t=0}^{i} e(x_1, \ldots, x_t; U^{t,i+1}_R), & \text{if } 0 \leq i < r, \\ e_0(J; R) + e_0(J, M), & \text{if } i = r; \end{cases}$$

and for $d < r$,

$$e_{r-i}(Q, R \times M) = \begin{cases} e_0(J; R), & \text{if } i = r, \\ \sum_{t=0}^{i} e(x_1, \ldots, x_t; U^{t,i+1}_R), & \text{if } d < i < r, \\ \sum_{t=0}^{d} e(x_1, \ldots, x_t; \overline{U}^{t,d+1}_R) + e_0(I, M), & \text{if } i = d, \\ \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_R) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_M), & \text{if } 0 \leq i < d. \end{cases}$$

We also describe the Hilbert coefficients of $R \times M$ in case where $R$ and $M$ are sequentially generalized Cohen-Macaulay (Corollary 4.4).

In the next section, after giving some preliminaries on almost p-standard systems of parameters, we prove Theorem 1.2. In Section 3 and Section 4, we present the proofs of Theorem 1.3 and Theorem 1.4, respectively.
2 Almost p-standard system of parameters and idealization

We first recall some properties of almost p-standard s.o.p that will be used in the sequel, see [7, Corollaries 3.5, 3.6], [4, Lemma 2.9].

**Lemma 2.1.** Let $x_1, \ldots, x_d$ be an almost p-standard s.o.p of $M$. For $i = 0, \ldots, d$, put $\lambda_i = e(x_1, \ldots, x_i; (0 : x_{i+1})_M/(x_{i+2}, \ldots, x_d)_M)$. Then

(i) $\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})_M) = \sum_{i=0}^d \lambda_i n_1 \ldots n_i$ for all $n_1, \ldots, n_d \geq 1$.

(ii) $N \cap (x_1, \ldots, x_d)_M = 0$ for any submodule $N$ of $M$ and any integer $i > \dim_R(N)$.

Let $y = x_1, \ldots, x_d$ be a s.o.p of $M$ and $n_1, \ldots, n_d \geq 1$ be positive integers. We set $y^n = x_1^{n_1}, \ldots, x_d^{n_d}$. The following function in $n_1, \ldots, n_d$ is very helpful in the study of almost p-standard s.o.p

$$\bar{I}_{M,y^n} := \ell(M/y^n_M) - e(y^n; M) - \sum_{i=0}^{d-1} n_1 \ldots n_i e(x_1, \ldots, x_i; (0 : x_{i+1})_M/(x_{i+2}, \ldots, x_d)_M).$$

From Lemma 2.1 and [4, Proposition 2.6], we have the following properties of $\bar{I}_{M,y^n}$.

**Lemma 2.2.** Let $y = x_1, \ldots, x_d$ be a s.o.p of $M$. Then

(i) $\bar{I}_{M,y^n}$ is a non-decreasing function and $\bar{I}_{M,y^n} \geq 0$ for all $n_1, \ldots, n_d \geq 1$.

(ii) $y$ is almost p-standard if and only if $\bar{I}_{M,y^n} = 0$ for all $n_1, \ldots, n_d \geq 1$.

**Lemma 2.3.** Let $x_1, \ldots, x_r$ be elements in $m$. For $i = 1, \ldots, r$, put $u_i = (x_i, 0)$. Then

$$(0 : u_{i+1})(R \times M)/_{(u_{2r}+1), \ldots, u_j)(R \times M) \simeq (0 : x_{i+1})_R/(x_{i+2}, \ldots, x_j)_R \times (0 : x_{i+1})_M/(x_{i+2}, \ldots, x_d)_M,$$

for all $0 \leq i < j \leq r$.

**Proof.** For all $0 \leq i < j \leq r$, we have

$$(0 : u_{i+1})(R \times M)/_{(u_{i+2}, \ldots, u_j)(R \times M)} = [(u_{i+2}, \ldots, u_j)(R \times M) :_{R \times M} u_{i+1}]/(u_{i+2}, \ldots, u_j)(R \times M);$$

$$(u_{i+2}, \ldots, u_j)(R \times M) = (x_{i+2}, \ldots, x_j)_R \times (x_{i+2}, \ldots, x_j)_M.$$

We claim that

$$[(u_{i+2}, \ldots, u_j)(R \times M) :_{R \times M} u_{i+1}] = [(x_{i+2}, \ldots, x_j)_R :_{R} x_{i+1}] \times [(x_{i+2}, \ldots, x_j)_M :_{M} x_{i+1}].$$

Indeed, take an element $(a, m) \in (u_{i+2}, \ldots, u_j)(R \times M) :_{R \times M} u_{i+1}$, then

$$(a, m)(x_{i+1}, 0) = (ax_{i+1}, x_{i+1}m) \in (u_{i+2}, \ldots, u_j)(R \times M).$$

Hence $a \in (x_{i+2}, \ldots, x_j)_R :_{R} x_{i+1}$ and $m \in (x_{i+2}, \ldots, x_j)_M :_{M} x_{i+1}$. Conversely, let

$$(a, m) \in (x_{i+2}, \ldots, x_j)_R :_{R} x_{i+1} \times (x_{i+2}, \ldots, x_j)_M :_{M} x_{i+1}.$$
Then \( ax_{i+1} \in (x_{i+2}, \ldots, x_j)R \) and \( x_{i+1}m \in (x_{i+2}, \ldots, x_j)M \). Hence
\[
(a, m)(x_{i+1}, 0) = (ax_{i+1}, x_{i+1}m) \\
\in (x_{i+2}, \ldots, x_j)R \times (x_{i+2}, \ldots, x_j)M = (u_{i+2}, \ldots, u_j)(R \ltimes M),
\]
therefore, \((a, m) \in (u_{i+2}, \ldots, u_j)(R \ltimes M) : R \ltimes M u_{i+1}, \) the claim is proved. Now, the result is clear by the claim. \( \square \)

\textbf{Lemma 2.4.} Let \( \mathbf{x} = x_1, \ldots, x_r \) be a s.o.p of \( R \). Set \( \mathbf{u} = u_1, \ldots, u_r \), where \( u_i = (x_i, 0) \) for \( i = 1, \ldots, r \). Then \( \mathbf{u} \) is a s.o.p of \( R \ltimes M \). Moreover, if \( x_1, \ldots, x_d \) is a s.o.p of \( M \) and \( (x_{d+1}, \ldots, x_r)M = 0 \), then for any \( n_1, \ldots, n_r \geq 1 \) we have
\[
\tilde{I}_{R \ltimes M, \mathbf{u}}(\mathbf{n}) = \tilde{I}_{R, \mathbf{x}}(\mathbf{n}) + I_{M, x_1, \ldots, x_d}(\mathbf{n}).
\]

\textit{Proof.} For a tuple of positive integers \( \mathbf{n} = n_1, \ldots, n_r \), set \( \mathbf{u}(\mathbf{n}) = u_1^{n_1}, \ldots, u_r^{n_r} \) and \( \mathbf{x}(\mathbf{n}) = x_1^{n_1}, \ldots, x_r^{n_r} \). We have
\[
(u_1^{n_1}, \ldots, u_r^{n_r})(R \ltimes M) \simeq (x_1^{n_1}, \ldots, x_r^{n_r})R \times (x_1^{n_1}, \ldots, x_r^{n_r})M.
\]
Thus \( \mathbf{u} \) is a s.o.p of \( R \ltimes M \) and
\[

\ell((R \ltimes M)/\mathbf{u}(\mathbf{n})(R \ltimes M)) = \ell((R/\mathbf{x}(\mathbf{n})R) + \ell(M/\mathbf{x}(\mathbf{n})M)).
\]
It is clear that \( e(\mathbf{u}; R \ltimes M) = e(\mathbf{x}; R) + e(\mathbf{x}; M) \), where \( e(\mathbf{x}; M) = 0 \) whenever \( d < r \). So, by Lemma 2.3 we obtain
\[
\tilde{I}_{R \ltimes M, \mathbf{u}}(\mathbf{n}) = \ell((R \ltimes M)/\mathbf{u}(\mathbf{n})(R \ltimes M)) - n_1 \ldots n_re(\mathbf{u}; R \ltimes M)
\]
\[
- \sum_{i=0}^{r-1} n_1 \ldots n_i e(u_1, \ldots, u_i; (0 : u_{i+1})(R \ltimes M)/(u_{i+2}, \ldots, u_r)(R \ltimes M))
\]
\[
= \tilde{I}_{R, \mathbf{x}}(\mathbf{n}) + \ell(M/\mathbf{x}(\mathbf{n})M) - n_1 \ldots n_re(\mathbf{x}; M)
\]
\[
- \sum_{i=0}^{r-1} n_1 \ldots n_i e(x_1, \ldots, x_i; (0 : x_{i+1})M/(x_{i+2}, \ldots, x_r)M).
\]
If \( d = r \), then \( \mathbf{x} \) is a s.o.p of \( M \) and the above equality gives
\[
\tilde{I}_{R \ltimes M, \mathbf{u}}(\mathbf{n}) = \tilde{I}_{R, \mathbf{x}}(\mathbf{n}) + I_{M, \mathbf{x}}(\mathbf{n}),
\]
for all \( n_1, \ldots, n_r \geq 1 \). Let \( d < r \). As \( x_{d+1}, \ldots, x_r \in \text{Ann}_R(M) \), we get \( e(\mathbf{x}; M) = 0 \) and
\[
e(x_1, \ldots, x_i; (0 : x_{i+1})M/(x_{i+2}, \ldots, x_r)M) = 0
\]
for \( d < i < r \). Moreover,
\[
e(x_1, \ldots, x_d; (0 : x_{d+1})M/(x_{d+2}, \ldots, x_r)M) = e(x_1, \ldots, x_d; M);
\]
\[
e(x_1, \ldots, x_i; (0 : x_{i+1})M/(x_{i+2}, \ldots, x_r)M) = e(x_1, \ldots, x_i; (0 : x_{i+1})M/(x_{i+2}, \ldots, x_d)M)
\]
for \( i < d \). From the above computations we have
\[
\tilde{I}_{R \ltimes M, \mathbf{u}}(\mathbf{n}) = \tilde{I}_{R, \mathbf{x}}(\mathbf{n}) + \tilde{I}_{M, x_1, \ldots, x_d}(\mathbf{n})
\]
for all \( n_1, \ldots, n_r \geq 1 \). \( \square \)
Now we are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) $\Rightarrow$ (ii). Since $\underline{y}$ is a s.o.p of $R \ltimes M$, it follows that $\underline{x}$ is a s.o.p of $R$ and $\underline{x}$ is a multiplicity system of $M$ (i.e. $\ell(M/(x_1,\ldots,x_r)M) < \infty$).

If $d = r$, then $\underline{x}$ is a s.o.p of $M$. Using the assumption (i) together with Lemma 2.2(ii) and Lemma 2.4, we have

$$0 = \tilde{I}_{R \ltimes M, \underline{y}}(n) = \tilde{I}_{R, \underline{x}}(n) + \tilde{I}_{M, \underline{x}}(n)$$

for all $n_1,\ldots,n_r \geq 1$. By Lemma 2.2(i), each term on the right hand side is non-negative. Therefore, $\tilde{I}_{R, \underline{x}}(n) = \tilde{I}_{M, \underline{x}}(n) = 0$ for all $n_1,\ldots,n_r \geq 1$. By Lemma 2.2(ii), $\underline{x}$ is an almost p-standard s.o.p of both $M$ and $R$.

Suppose $d < r$. Via the canonical inclusion $\varepsilon : M \rightarrow R \ltimes M$ defined by $\varepsilon(x) = (0,x)$, each $R$-submodule of $M$ can be identified with an $R \ltimes M$-submodule of $R \ltimes M$. Consider the submodule $\varepsilon(M) = 0 \times M$ of $R \ltimes M$. We have $\dim_{R \ltimes M}(0 \times M) = d < r$. Since $\underline{y}$ is an almost p-standard s.o.p of $R \ltimes M$, we get by Lemma 2.1(ii) that

$$0 = \tilde{I}_{R \ltimes M, \underline{y}}(n) = \tilde{I}_{R, \underline{x}}(n) + \tilde{I}_{M, \underline{x}}(n)$$

for all $n_1,\ldots,n_r \geq 1$. By Lemma 2.2(ii), $\underline{x}$ is an almost p-standard s.o.p. So from the assumption (i) together with Lemma 2.2(ii) and Lemma 2.4, we obtain

$$0 = \tilde{I}_{R \ltimes M, \underline{y}}(n) = \tilde{I}_{R, \underline{x}}(n) + \tilde{I}_{M, \underline{x}}(n)$$

for all $n_1,\ldots,n_r \geq 1$. By Lemma 2.2(ii), $\tilde{I}_{R, \underline{x}}(n) = \tilde{I}_{M, \underline{x}}(n) = 0$ for all $n_1,\ldots,n_r \geq 1$. By Lemma 2.2(ii), $\underline{x}$ is an almost p-standard s.o.p of $R$ and $x_1,\ldots,x_d$ is an almost p-standard s.o.p of $M$.

(ii) $\Rightarrow$ (i). Since $\underline{x}$ is an almost p-standard s.o.p of $R$ and $\underline{y} = x_1,\ldots,x_d$ is an almost p-standard s.o.p of $M$, we get by Lemma 2.2(ii) that

$$\tilde{I}_{R, \underline{x}}(n) = \tilde{I}_{M, \underline{y}}(n) = 0$$

for all $n_1,\ldots,n_r \geq 1$. Therefore, we have by assumption (ii) and Lemma 2.4 that

$$\tilde{I}_{R \ltimes M, \underline{y}}(n) = \tilde{I}_{R, \underline{x}}(n) + \tilde{I}_{M, \underline{x}}(n) = 0.$$

By Lemma 2.2(ii), $\underline{y}$ is an almost p-standard s.o.p of $R \ltimes M$.

Theorem 1.2 leads to the following consequence for the existence of almost p-standard s.o.p of idealization.

Corollary 2.5. The following statements are equivalent:

(i) $R$ admits an almost p-standard s.o.p;

(ii) $R \ltimes M$ admits an almost p-standard s.o.p;

(iii) $R \ltimes M$ admits an almost p-standard s.o.p of the form $(x_1,0),\ldots,(x_r,0)$, where $x_1,\ldots,x_r$ is an almost p-standard s.o.p of $R$ and $x_1,\ldots,x_d$ is an almost p-standard s.o.p of $M$.  

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Proof. (iii) ⇒ (ii) is clear.

(ii) ⇒ (i). By assumption (ii), we get by [10, Theorem 1.2] that \( R \lhd M \) is a quotient of a Cohen-Macaulay local ring. Note that \( R \) is a quotient of \( R \lhd M \). Therefore, \( R \) is a quotient of a Cohen-Macaulay local ring. Now, the result follows by [10, Theorem 1.2].

(i) ⇒ (iii). By assumption (i), we get by [10, Theorem 1.2] that \( R \lhd M \) is a quotient of a Cohen-Macaulay local ring. Therefore, \( \dim(R/a(N)) < \dim(R) \) for any finitely generated \( R \)-module \( N \), where \( a(N) = a_0(N) a_1(N) \ldots a_{\dim(R)(N) - 1}(N) \) and \( a_i(N) = \text{Ann}_R(H^i_{\mathfrak{m}}(N)) \) for \( i = 0, \ldots, \dim(R)(N) - 1 \). Therefore, by Prime Avoidance, there exists a \( p \)-standard s.o.p \( x_1, \ldots, x_r \) of \( R \) such that \( x_{d+1}, \ldots, x_r \in \text{Ann}_R(M) \) and \( x_1, \ldots, x_d \) is a \( p \)-standard s.o.p of \( M \) (see the definition of \( p \)-standard s.o.p in [6]). Hence \( x_1, \ldots, x_r \) is an almost \( p \)-standard s.o.p of \( R \lhd M \), where \( u_i = (x_i, 0) \) for all \( i = 1, \ldots, r \). \( \square \)

From Corollary 2.5 and [10, Theorem 1.2], we get immediately the following consequence.

**Corollary 2.6.** A Noetherian local ring is a quotient of a Cohen-Macaulay local ring if and only if so is one of its idealization, if and only if so are all of its idealizations by finitely generated modules.

### 3 Macaulayfication of idealization

In this section, we discuss an application of Theorem 1.2 to construct Cohen-Macaulay Rees algebras of idealization and then to prove the existence of Cohen-Macaulay Rees modules of unmixed modules.

Let \( I \) be an ideal of \( R \) and \( T \) be a variable over \( R \). The **Rees algebra of \( R \) with respect to \( I \)** defined by

\[
\mathfrak{R}(R, I) = R[T] = \left\{ \sum_{i=0}^{\infty} a_i T^i \mid n \in \mathbb{N}, a_i \in I^i \right\} = \bigoplus_{n \geq 0} I^n T^n,
\]

where \( I^0 = R \). Similarly, the **Rees module of \( M \) with respect to \( I \)** defined by

\[
\mathfrak{R}(M, I) = \left\{ \sum_{i=0}^{\infty} a_i x_i T^i \mid n \in \mathbb{N}, a_i \in I^i, x_i \in M \right\} = \bigoplus_{n \geq 0} I^n MT^n,
\]

where \( I^0 M = M \). A Rees algebra \( \mathfrak{R}(R, I) \) is called an **arithmetic Macaulayfication of \( R \)** if it is Cohen-Macaulay and \( I \) is of positive height. If \( \mathfrak{R}(R, I) \) is an arithmetic Macaulayfication of \( R \), then the canonical algebra homomorphism \( R \rightarrow \mathfrak{R}(R, I) \) induces a morphism of Noetherian schemes \( \text{Proj}(\mathfrak{R}(R, I)) \rightarrow \text{Spec}(R) \) which is called a **projective Macaulayfication**. More generally, a Macaulayfication of \( \text{Spec}(R) \) is a birational and proper morphism \( X \rightarrow \text{Spec}(R) \) where \( X \) is a Cohen-Macaulay locally Noetherian scheme.

The existence of arithmetic Macaulayfication and of Macaulayfication have been established by several authors. Kawasaki [17, Theorem 1.1] showed that a Noetherian local ring has an arithmetic Macaulayfication if and only if it is unmixed and all its formal fibers are

**Definition 3.1.** A Noetherian scheme $X$ is **CM-quasi-excellent** if

(a) Every formal fiber of local rings of $X$ is Cohen-Macaulay, and

(b) Any integral subscheme of $X$ has an open Cohen-Macaulay locus.

A Noetherian ring is **CM-quasi-excellent** if its prime spectrum is a CM-quasi-excellent affine scheme. In [3, Theorem 1.6], Česnavičius showed that if $R$ is CM-quasi-excellent then $\text{Spec}(R)$ admits a Macaulayfication.

Arithmetic Macaulayfication has been studied from other perspective by Kurano [19], Aberbach-Huneke-Smith [1], Cutkosky-Tai [12], Tai-Trung [21]. In [10], N.T. Cuong and D.T. Cuong extended Kawasaki’s theorem for modules. They showed that there is an ideal $I$ such that the Rees module $\mathfrak{R}(M,I)$ is Cohen-Macaulay if and only if $M$ is unmixed and $R/\text{Ann}_R(M)$ is a quotient of a Cohen-Macaulay ring.

Note that the idealization $R \ltimes M$ is a finite $R$-algebra (see, for example, [2, Proposition 2.2]). By Corollary 2.6, if $R$ is a quotient of a Cohen-Macaulay ring, then so is $R \ltimes M$, therefore we get by [17, Theorem 1.1] that if $R$ admits an arithmetic Macaulayfication and the idealization $R \ltimes M$ is unmixed then $R \ltimes M$ also admits an arithmetic Macaulayfication. Similarly, if $R$ is CM-quasi-excellent then so is $R \ltimes M$ (see [3, Remark 1.5]). Česnavičius’s theorem implies that in that case both $\text{Spec}(R)$ and $\text{Spec}(R \ltimes M)$ admit Macaulayfications.

We now investigate further relations between arithmetic Macaulayfications and Macaulayfications respectively on $R$ and $R \ltimes M$. We first prove Theorem 1.3.

**Proof of Theorem 1.3.** Since $R, M$ are unmixed of the same dimension $r$, we get by [2, Theorem 4.11, 3.2] that the idealization $R \ltimes M$ is unmixed of dimension $r$. Since $R$ is a quotient of a Cohen-Macaulay, $R$-module $R \oplus M$ admits an almost p-standard s.o.p $\underline{x} = x_1, \ldots, x_r$. By Lemma 2.2(ii),

$$0 = \tilde{I}_{R \oplus M, \underline{x}}(n) = \tilde{I}_{R, \underline{x}}(n) + \tilde{I}_{M, \underline{x}}(n).$$

By Lemma 2.2(i), we get $\tilde{I}_{R, \underline{x}}(n) = \tilde{I}_{M, \underline{x}}(n) = 0$. Hence $x_1, \ldots, x_r$ is an almost p-standard s.o.p of both $R$ and $M$ by Lemma 2.2(ii). By Theorem 1.2, $(x_1, 0), \ldots, (x_r, 0)$ is an almost p-standard s.o.p of $R \ltimes M$. Therefore, Theorem 1.3 is then implied from [18, Proposition 8.2].

Theorem 1.3 has an interesting application in constructing Cohen-Macaulay Rees module.

Let $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathfrak{m}$ and put $u_i = (x_i, 0), v_j = (y_j, 0) \in R \ltimes M$, for $i = 1, \ldots, n, j = 1, \ldots, m$. Denote $I = (x_1, \ldots, x_n)$, $J = (y_1, \ldots, y_m)$, and $P = (u_1, \ldots, u_n)$, $Q = (v_1, \ldots, v_m)$. The following properties are obvious

$$P + Q = (I + J) \times (I + J)M,$$
$$PQ = ((x_iy_j, 0))_{i,j} = IJ \times IJM,$$
$$P^t = I^t \times I^t M,$$

for all $t > 0$. They lead to the following lemma.
Lemma 3.2. We have an algebra isomorphism
\[ \mathfrak{R}(R \ltimes M, P) \simeq \mathfrak{R}(R, I) \ltimes \mathfrak{R}(M, I). \]

Consequently, the Rees algebra \( \mathfrak{R}(R \ltimes M, P) \) is Cohen-Macaulay if and only if \( \mathfrak{R}(R, I) \) and \( \mathfrak{R}(M, I) \) are Cohen-Macaulay of the same dimension.

Using Theorem 1.3 and Kawasaki’s theorem on arithmetic Macaulayfication, we obtain another proof for the construction of Cohen-Macaulay Rees module in [10, Theorem 4.4].

Corollary 3.3. Let \( R \) be a quotient of a Cohen-Macaulay local ring. Suppose that \( M \) is unmixed and of dimension \( d > 1 \). Then there is an ideal \( I \) such that the Rees module \( \mathfrak{R}(M, I) \) is Cohen-Macaulay.

Proof. Replace \( R \) by \( R/\text{Ann}_R(M) \), we may assume that \( R \) is unmixed of the same dimension with \( M \). Since \( R \) is a quotient of a Cohen-Macaulay local ring, \( R \) admits an almost p-standard s.o.p. By Corollary 2.5 and Theorem 1.2, \( R \ltimes M \) admits an almost p-standard s.o.p \( u_1, \ldots, u_d \), where \( u_i = (x_i, 0) \) for \( i = 1, \ldots, d \) such that \( x_1, \ldots, x_d \) is an almost p-standard s.o.p of both \( R \) and \( M \). Put \( I_i = (x_i, \ldots, x_d) \) for \( i = 1, \ldots, d \), and \( I = I_1 \cdots I_{d-2} \). Also we denote \( u_i = (x_i, 0) \), \( P_i = (u_i, \ldots, u_d) \) for \( i = 1, \ldots, d \), and \( P = P_1 \cdots P_{d-2} \). Then \( \mathfrak{R}(R, I) \) and \( \mathfrak{R}(R \ltimes M, P) \) are Cohen-Macaulay. The Rees module \( \mathfrak{R}(M, I) \) has the same dimension with \( \mathfrak{R}(R, I) \) and \( \mathfrak{R}(R \ltimes M, P) \). So the short exact sequence
\[ 0 \to \mathfrak{R}(M, I) \to \mathfrak{R}(R \ltimes M, P) \to \mathfrak{R}(R, I) \to 0, \]
implies that \( \mathfrak{R}(M, I) \) is Cohen-Macaulay.

Conversely, using [10, Theorem 4.4] we are able to give the second proof for Theorem 1.3 as following: Denote \( I_i = (x_i, \ldots, x_r) \) and
\[ I := I_1 \cdots I_{r-3}I_{r-2}. \]
Following [18, Proposition 8.2] and [10, Theorem 4.4], \( \mathfrak{R}(R, I) \) and \( \mathfrak{R}(M, I) \) are Cohen-Macaulay. By Lemma 3.2, \( \mathfrak{R}(R \ltimes M, P) \simeq \mathfrak{R}(R, I) \ltimes \mathfrak{R}(M, I) \) which is thus Cohen-Macaulay, hence Theorem 1.3 is proved.

Another consequence of Theorem 1.3 is the following characterization for the existence of arithmetic Macaulayfication for idealizations.

Corollary 3.4. The idealization \( R \ltimes M \) has an arithmetic Macaulayfication if and only if \( R \) has an arithmetic Macaulayfication and \( M \) is unmixed with \( \dim(R) = \dim_R(M) \).

Proof. Suppose \( R \ltimes M \) has an arithmetic Macaulayfication. By [10, Corollary 5.4], \( R \ltimes M \) is unmixed and is a quotient of a Cohen-Macaulay ring. Then \( R \) and \( M \) are unmixed of the same dimension and \( R \) is also a quotient of a Cohen-Macaulay ring. Using again [10, Corollary 5.4], \( R \) admits an arithmetic Macaulayfication.

Conversely, suppose that \( R \) has an arithmetic Macaulayfication and \( M \) is unmixed with \( \dim_R(M) = \dim(R) \). Then \( R \) is a quotient of a Cohen-Macaulay local ring, Theorem 1.3 then implies that the idealization \( R \ltimes M \) admits an arithmetic Macaulayfication. \( \square \)
For Macaulayfication, we find a tight relation between certain Macaulayfications of $R$ and $R \ltimes M$ in several particular cases.

First, suppose $R$ and $M$ are unmixed of the same dimension. If $R$ is a quotient of a Cohen-Macaulay ring then by Theorem 1.3, there are arithmetic Macaulayfications of $R$, $M$ and $R \ltimes M$ with relation
\[ \mathfrak{R}(R \ltimes M, P) \simeq \mathfrak{R}(R, I) \ltimes \mathfrak{R}(M, I). \]

On the other hand, the canonical morphism $\mathfrak{R}(R, I) \to \mathfrak{R}(R \ltimes M, P)$ induces a morphism of $R$-schemes $\text{Proj}(\mathfrak{R}(R \ltimes M, P)) \to \text{Proj}(\mathfrak{R}(R, I))$ which is actually an isomorphism. Note that $\text{Proj}(\mathfrak{R}(R \ltimes M, P))$ and $\text{Proj}(\mathfrak{R}(R, I))$ are Cohen-Macaulay which are Macaulayfications of $\text{Spec}(R \ltimes M)$ and $\text{Spec}(R)$ respectively. Therefore in this case, the Macaulayfication of $R$ and the idealization are isomorphic.

Now suppose that $R$ is quasi-CM-excellent. The canonical map $R \ltimes M \to R$ induces a bijective morphism of affine schemes $\rho : \text{Spec}(R) \to \text{Spec}(R \ltimes M)$ (see [2, Theorem 3.2(b)]). Let $p$ be a minimal prime ideal of $R$, then $\rho(p) = p \ltimes M$ is the corresponding prime ideal of the idealization. By [2, Theorem 4.1], we have
\[ (R \ltimes M)_{p \ltimes M} \simeq R_p \ltimes M_p. \]

In particular, if $p$ does not belong to in the support of $M$ then
\[ (R \ltimes M)_{p \ltimes M} \simeq R_p. \]

This proves the following proposition.

**Proposition 3.5.** Assume that no associated prime ideals of $M$ are minimal prime ideals of $R$. Then the morphism $\rho : \text{Spec}(R) \to \text{Spec}(R \ltimes M)$ is a birational morphism. Consequently, if $\varphi : X \to \text{Spec}(R)$ is a Macaulayfication then $\varphi \circ \rho : X \to \text{Spec}(R \ltimes M)$ is a Macaulayfication.

**Proof.** Let $p$ be a minimal prime ideal of $R$. Then $(R \ltimes M)_{p \ltimes M} \simeq R_p$. Since the morphism $\rho$ is bijective, then it is clearly birational. Furthermore, $\rho$ is obviously proper. So $\varphi \circ \rho$ is proper and birational, which is therefore a Macaulayfication of $\text{Spec}(R \ltimes M)$.

## 4 Hilbert function of idealization

Firstly, we recall the following property (see [4, Proposition 3.2, Corollary 3.5]).

**Lemma 4.1.** Let $\mathbf{x} = x_1, \ldots, x_d$ be an almost $p$-standard s.o.p of $M$. Let $i, j$ be integers such that $0 \leq i < j \leq d$. The following statements are true.

(i) The subquotient module $U^{i,j}_{M} := (0 :_{M/(x_i^{n_i+2}, \ldots, x_j^{n_j})M} x_{i+1})$ is independent of the choice of the s.o.p $\mathbf{x}$ and of the exponents $n_{i+2}, \ldots, n_j \geq 2$.

(ii) If $j > i + 1$, then there is an injective homomorphism $\varphi_{i,j} : U^{i,j-1}_{M} \to U^{i,j}_{M}$ such that $\text{Im}(\varphi_{i,j})$ is a direct summand of $U^{i,j}_{M}$. In particular, set $U^{i,j}_{M} = \text{Coker}(\varphi_{i,j})$, then
\[ U^{i,j}_{M} \simeq U^{i,j}_{M} \oplus U^{i,j-1}_{M} \oplus \cdots \oplus U^{i,i+2}_{M} \oplus U^{i,i+1}_{M}. \]
For an integer $0 \leq i < d$, set $U^{i,i+1}_{i,M} := U^{i+1}_M$. Note that $U^{d-1,d}_M$ is the largest submodule of $M$ of dimension less than $d$, and $U^{0,1}_M = H^0_0(M)$. The subquotient modules $U^{i,j}_M, U^{i,j}_{i,M}$ give a lot of information on structure of $M$. For example, $M$ is Cohen-Macaulay if and only if $U^{i,j}_M = 0$ for all $i < j$, and if and only if $U^{i,j}_{i,M} = 0$ for all $i < j$. Moreover, $M$ is generalized Cohen-Macaulay if and only if $\ell(U^{i,j}_M) < \infty$ for all $i < j$, if and only if $\ell(U^{i,j}_{i,M}) < \infty$ for all $i < j$, see [4, Proposition 3.9].

From now on, we assume that $R$ is a quotient of a Cohen-Macaulay local ring. Before proving Theorem 1.4, we compute the subquotient modules $U^{i,j}_{R \ltimes M}$ and $U^{i,j}_{R \ltimes M}$ of the idealization.

**Lemma 4.2.** The following statements are true.

(i) If $d = r$, then $U^{i,j}_{R \ltimes M} \simeq U^{i,j}_R \times U^{i,j}_M$ for all $0 \leq i < j \leq r$.

(ii) If $d < r$, then

$$
U^{i,j}_{R \ltimes M} \simeq \begin{cases} 
U^{i,j}_R \times U^{i,j}_M & \text{if } 0 \leq i < j < d, \\
U^{i,j}_R \times U^{i,j}_M & \text{if } 0 \leq i < d \leq j \leq r, \\
U^{i,j}_R \times M & \text{if } d \leq i < j \leq r.
\end{cases}
$$

**Proof.** Since $R$ is a quotient of a Cohen-Macaulay local ring, $R$ admits an almost p-standard s.o.p. By Corollary 2.5 and Theorem 1.2, $R \ltimes M$ admits an almost p-standard s.o.p $u_1, \ldots, u_r$, where $u_i = (x_i, 0)$ for $i = 1, \ldots, r$ such that $x_1, \ldots, x_r$ is an almost p-standard s.o.p of $R$, $x_1, \ldots, x_d$ is an almost p-standard of $M$ and $x_{d+1}, \ldots, x_r \in \text{Ann}_R(M)$.

For integers $0 \leq i < j \leq r$, by Lemma 2.3 we have

$$
U^{i,j}_{R \ltimes M} := (0 : u_{i+1})(R \ltimes M)/(u^2_{i+1}, \ldots, u^2_j(R \ltimes M))
\simeq (0 : x_{i+1})_R/(x_{i+1}^2, \ldots, x_j^2) \times (0 : x_{i+1})_M/(x_{i+1}^2, \ldots, x_j^2)
\simeq U^{i,j}_R \times (0 : x_{i+1})_M/(x_{i+1}^2, \ldots, x_j^2).
$$

(i) If $d = r$, then $(0 : x_{i+1})_M/(x_{i+1}^2, \ldots, x_j^2)M \simeq U^{i,j}_M$ for $0 \leq i < j \leq r$, so $U^{i,j}_{R \ltimes M} \simeq U^{i,j}_R \times U^{i,j}_M$.

(ii) Suppose that $d < r$. If $0 \leq i < j < d$ then $(0 : x_{i+1})_M/(x_{i+1}^2, \ldots, x_j^2)M \simeq U^{i,j}_M$. Let $0 \leq i < d \leq j \leq r$. Since $x_{d+1}, \ldots, x_r \in \text{Ann}_R(M)$, we have

$$
(0 : x_{i+1})_M/(x_{i+1}^2, \ldots, x_j^2)M = (0 : x_{i+1})_M/(x_{i+1}^2, \ldots, x_j^2)M \simeq U^{i,j}_M.
$$

It is clear that $(0 : x_{i+1})_M/(x_{i+1}^2, \ldots, x_j^2)M \simeq M$ for all $d \leq i < j \leq r$, the statement follows.  

For the subquotients $U^{i,j}_{R \ltimes M}$ we have the following lemma.

**Lemma 4.3.** The following statements are true.

(i) If $d = r$, then $U^{i,j}_{R \ltimes M} \simeq U^{i,j}_R \times U^{i,j}_M$ for all $0 \leq i < j \leq r$. 


(ii) If \( d < r \), then

\[
\mathcal{U}_{R \times M}^{i,j} \simeq \begin{cases} 
R^i_j \times R^i_j & \text{if } 0 \leq i < j \leq d, \\
R^i_j & \text{if } 0 \leq i < d < j \leq r, \text{ or } d < i + 1 < j \leq r, \\
R^{i+1}_j \times M & \text{if } d < i + 1 = j \leq r.
\end{cases}
\]

Proof. (i) Suppose that \( d = r \) and \( 0 \leq i < j \leq r \). If \( j = i + 1 \), then we get by Lemma 4.2(i)

\[
\mathcal{U}_{R \times M}^{i+1,j} = U_{R \times M}^{i+1,j} \simeq U^{i+1,j} \times U^{i+1,j} = U^{i+1,j} \times U^{i+1,j}.
\]

Let \( j > i + 1 \). Then \( U_{R \times M}^{i,j-1} \simeq U^{i,j-1} \times U_{M}^{i,j-1} \) by Lemma 4.2(i), and hence

\[
U_{R \times M}^{i,j} \simeq U^{i,j-1} / U^{i,j-1} \times U_{M}^{i,j-1} / U_{M}^{i,j-1}.
\]

We get by Proposition 4.1(ii) that

\[
U_{R \times M}^{i,j} \simeq U_{R \times M}^{i,j} \oplus U_{M}^{i,j-1}, U_{R}^{i,j} \simeq U_{R}^{i,j} \oplus U_{R}^{i,j-1}, U_{M}^{i,j} \simeq U_{M}^{i,j} \oplus U_{M}^{i,j-1}.
\]

Therefore

\[
U_{R \times M}^{i,j} \simeq U_{R \times M}^{i,j} / U_{R \times M}^{i,j-1} \simeq U_{R}^{i,j} / U_{R}^{i,j-1} \times U_{M}^{i,j-1} / U_{M}^{i,j-1} \simeq U_{R}^{i,j} \times U_{M}^{i,j}.
\]

(ii) Suppose that \( d < r \) and \( 0 \leq i < j \leq r \). If \( j \leq d \), then by the same arguments as in the proof of (i), we have \( U_{R \times M}^{i,j} \simeq U_{R}^{i,j} \times U_{M}^{i,j} \).

Let \( j > d \). As in the proof of Lemma 4.2, there exists an almost p-standard s.o.p \( x_1, \ldots, x_r \) of \( R \) such that \( x_1, \ldots, x_d \) is an almost p-standard s.o.p of \( M, x_{d+1}, \ldots, x_r \in \text{Ann}_R(M) \) and

\[
U_{R \times M}^{i,j} \simeq U_{R}^{i,j} \times (0 : x_{i+1})_{M/(x_1^2, \ldots, x_j^2)M}.
\]

Note that \((0 : x_{i+1})_{M/(x_1^2, \ldots, x_j^2)M} = U_{M}^{i,d} \) for all \( i < d \) and \((0 : x_{i+1})_{M/(x_1^2, \ldots, x_j^2)M} = M \) for all \( i \geq d \). Therefore, if \( i < d \) then

\[
U_{R \times M}^{i,j} \simeq U_{R}^{i,j} \times U_{R}^{i,j-1} / U_{R}^{i,j-1} \times U_{M}^{i,d} / U_{M}^{i,d} \simeq U_{R}^{i,j}.
\]

If \( j > i + d \) then

\[
U_{R \times M}^{i,j} \simeq U_{R \times M}^{i,j} / U_{R \times M}^{i,j-1} \simeq U_{R}^{i,j} / U_{R}^{i,j-1} \times M / M \simeq U_{R}^{i,j}.
\]

If \( j = i + d \) then

\[
U_{R \times M}^{i+i+1} = U_{R \times M}^{i,i+1} \simeq U_{R}^{i,i+1} \times M = U_{R}^{i,i+1} \times M.
\]

\( \square \)

Proof of Theorem 1.4. Theorem 1.2 tells us that \( u = u_1, \ldots, u_r \) is an almost p-standard s.o.p of \( R \times M \). By [4, Theorem 4.7], we have

\[
l((R \times M) / Q^{n+1}) = e_0(Q, R \times M)\binom{n + r}{r} + e_1(Q, R \times M)\binom{n + r - 1}{r - 1} + \ldots + e_r(Q, R \times M)
\]

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for all $n \geq 0$, where $e_{r-i}(Q, R \times M) = \sum_{t=0}^{i} e(u_1, \ldots, u_t; \overline{U}_{R \times M}^{t,i+1})$ for all $0 \leq i \leq r - 1$.

- Let $d = r$. Then $J$ is a parameter ideal of $M$, therefore
  \[ e_0(Q, R \times M) = e_0(J, R) + e_0(J, M). \]

Since $\overline{U}_{R \times M}^{t,i+1} \cong \overline{U}_R^{t,i+1} \times \overline{U}_M^{t,i+1}$ by Lemma 4.3, we get
  \[ e(u_1, \ldots, u_t; \overline{U}_{R \times M}^{t,i+1}) = e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) + e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}) \]
  for all $0 \leq t \leq i < r$. Therefore, for all $0 \leq i < r$ we have
  \[ e_{r-i}(Q, R \times M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}). \]

- Let $d < r$. Then $e_0(Q, R \times M) = e_0(J, R)$. If $0 \leq i < d$, then $\overline{U}_{R \times M}^{t,i+1} \cong \overline{U}_R^{t,i+1} \times \overline{U}_M^{t,i+1}$ by Lemma 4.3 for all $t \leq i$, therefore,
  \[ e_{r-i}(Q, R \times M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}). \]

If $d \leq i < r$ then we get by Lemma 4.3 that
  \[ \overline{U}_{R \times M}^{t,i+1} \cong \begin{cases} \overline{U}_R^{t,i+1} & \text{if } 0 \leq t < i, \\ \overline{U}_R^{t,i+1} \times M & \text{if } t = i, \end{cases} \]
  therefore,
  \[ e_{r-d}(Q, R \times M) = \sum_{t=0}^{d} e(x_1, \ldots, x_t; \overline{U}_R^{t,d+1}) + e_0(I, M) \]
  and $e_{r-i}(Q, R \times M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1})$ for all $i > d$. \(\square\)

Let the notations and assumptions be as in Theorem 1.4. Consider the case where $R$ and $M$ are generalized Cohen-Macaulay. We use Theorem 1.4 and [5, Lemma 2.4] to compute Hilbert coefficients of $R \times M$. If $d = 0$ or $d = r$ then $R \times M$ is generalized Cohen-Macaulay. In this case, if $d = r$ then
  \[ e_{r-i}(Q, R \times M) = \begin{cases} \sum_{t=1}^{i} (i-1) \ell_R(H^t_m(R)) + \sum_{t=1}^{i} (i-1) \ell_R(H^t_m(M)) & \text{if } 0 \leq i < r, \\ e_0(J, R) + e_0(J, M) & \text{if } i = r. \end{cases} \]

and if $d = 0$ then
  \[ e_{r-i}(Q, R \times M) = \begin{cases} \ell_R(H^0_m(R)) + \ell_R(M) & \text{if } i = 0, \\ \sum_{t=1}^{i} (i-1) \ell_R(H^t_m(R)) & \text{if } 0 < i < r, \\ e_0(J, R) & \text{if } i = r. \end{cases} \]
If $0 < d < r$, then $R \times M$ is not generalized Cohen-Macaulay. In this case we have

$$e_{r-i}(Q, R \times M) = \begin{cases} e_0(J; R) & \text{if } i = r, \\ \sum_{t=1}^{i} (i-1) \ell_R(H_m^t(R)) & \text{if } d < i < r, \\ \sum_{t=1}^{d} (d-1) \ell_R(H_m^t(R)) + e_0(I; M) & \text{if } i = d, \\ \sum_{t=1}^{i} (i-1) \ell_R(H_m^t(R)) + \sum_{t=1}^{i} (i-1) \ell_R(H_m^t(M)) & \text{if } 0 \leq i < d. \end{cases}$$

Let $M_0 = H_m^0(M) \subseteq M_1 \subseteq \ldots \subseteq M_t = M$ be the dimension filtration of $M$, i.e. $M_i$ is the largest submodule of $M_{i+1}$ satisfying $\dim_R(M_i) < \dim_R(M_{i+1})$ for $i < t$. Following [11], $M$ is sequentially generalized Cohen-Macaulay if each quotient $M_{i+1}/M_i$ is generalized Cohen-Macaulay. Let $R_0 = H_m^0(R) \subseteq R_1 \subseteq \ldots \subseteq R_s = R$ be the dimension filtration of $R$. For $i = 0, \ldots, s$ and $j = 0, \ldots, t$, put $d_i = \dim_R(R_i)$ and $d_j' = \dim_R(M_j)$. Denote $\Delta_R = \{d_1, \ldots, d_s\}$ and $\Delta_M = \{d_1', \ldots, d_t'\}$ and set $\Delta := \Delta_R \cap \Delta_M$.

**Corollary 4.4.** Let the notations and assumptions be as in Theorem 1.4. For $0 < i \leq r$, set $\underline{x}_i = x_1, \ldots, x_i$. Suppose that $R$ and $M$ are sequentially generalized Cohen-Macaulay.

(i) If $d = r$ then for all $0 \leq i < r$ we have

$$e_{r-i}(Q, R \times M) = \begin{cases} \ell(U_R^{0,d_j+1}) + e(x_{d_j}; R_j) + e(x_{d_j}; M_j) + \ell(U_M^{0,d_j+1}) & \text{if } d_j \in \Delta, \\ \ell(U_R^{0,i+1}) + \ell(U_M^{0,i+1}) & \text{if } i \notin \Delta_R \cup \Delta_M, \\ \ell(U_R^{0,d_j+1}) + e(x_{d_j}; R_j) + \ell(U_M^{0,d_j+1}) & \text{if } d_j \in \Delta_R \setminus \Delta_M, \\ \ell(U_R^{0,d_j'+1}) + e(x_{d_j'}; M_j) + \ell(U_M^{0,d_j'+1}) & \text{if } d_j' \in \Delta_M \setminus \Delta_R. \end{cases}$$

(ii) If $d < r$ then for $d < i < r$, we have

$$e_{r-i}(Q, R \times M) = \begin{cases} \ell(U_R^{0,d_j+1}) + e(x_{d_j}; R_j) & \text{if } d_j \in \Delta_R, \\ \ell(U_R^{0,i+1}) & \text{if } i \notin \Delta_R; \end{cases}$$

and for all $0 \leq i < d < r$ we have

$$e_{r-i}(Q, R \times M) = \begin{cases} \ell(U_R^{0,d_j+1}) + e(x_{d_j}; R_j) + e(x_{d_j}; M_j) + \ell(U_M^{0,d_j+1}) & \text{if } d_j \in \Delta, \\ \ell(U_R^{0,i+1}) + \ell(U_M^{0,i+1}) & \text{if } i \notin \Delta_R \cup \Delta_M, \\ \ell(U_R^{0,d_j+1}) + e(x_{d_j}; R_j) + \ell(U_M^{0,d_j+1}) & \text{if } d_j \in \Delta_R \setminus \Delta_M, \\ \ell(U_R^{0,d_j'+1}) + e(x_{d_j'}; M_j) + \ell(U_M^{0,d_j'+1}) & \text{if } d_j' \in \Delta_M \setminus \Delta_R; \end{cases}$$

and finally for $i = d$ we have

$$e_{r-d}(Q, R \times M) = \begin{cases} \ell(U_R^{0,d+1}) + e(x_d; R_j) + e_0(I, M) & \text{if } d = d_j \in \Delta_R, \\ \ell(U_R^{0,d+1}) + e_0(I, M) & \text{if } d \notin \Delta_R. \end{cases}$$
Proof. We get by Lemma 4.1(ii) that
\[ U_R^{i,n} \simeq U_R^{i,n} \oplus U_R^{i,n-1} \oplus \cdots \oplus U_R^{i+2} \oplus U_R^{i+1} \text{ for all } 0 \leq i < n \leq r; \]
\[ U_M^{j,m} \simeq U_M^{j,m} \oplus U_M^{j,m-1} \oplus \cdots \oplus U_M^{j+2} \oplus U_M^{j+1} \text{ for all } 0 \leq j < m \leq d. \]

It follows by [8, Lemma 3.5] that \( M_j = U_M^{i+1,j} \) for any integers \( i, j \) such that \( d_j \leq i < d_{j+1} \), and \( R_j = U_R^{i+1,j} \) for any integers \( i, j \) such that \( d_j \leq i < d_{j+1} \). So, by [4, Proposition 2.9 (2)], \( U_M^{i,j} \oplus U_M^{i,j+1} \oplus \cdots \oplus U_M^{i+2,j} \) and \( U_R^{i,j} \oplus U_R^{i,j+1} \oplus \cdots \oplus U_R^{i+2,j} \) are of finite length. Hence

\[ e(x_1, \ldots, x_t; U_R^{i,n}) = \begin{cases} e(x_1, \ldots, x_{d_j}; R_j) & \text{if } n = i+1, i = d_j, \\ 0 & \text{otherwise.} \end{cases} \]

\[ e(x_1, \ldots, x_t; U_M^{j,m}) = \begin{cases} e(x_1, \ldots, x_{d'_j}; M_k) & \text{if } m = j+1, j = d'_k, \\ 0 & \text{otherwise.} \end{cases} \]

(i) Let \( d = r \). By Theorem 1.4, we have

\[ e_{r-i}(Q, R \ltimes M) = \sum_{t=0}^i e(x_1, \ldots, x_t; U_R^{i+1}) + \sum_{t=0}^i e(x_1, \ldots, x_t; U_M^{i+1}) \]

for all \( 0 \leq i < r \). We divide into four cases.

- If \( i = d_j \in \Delta \), then \( e(x_1, \ldots, x_t; U_R^{i+1}) = e(x_1, \ldots, x_t; U_M^{i+1}) = 0 \) for all \( t \notin \{0, d_j\} \). Hence

\[ e_{r-i}(Q, A) = \ell(U_R^{0,d_j+1}) + e(x_1, \ldots, x_{d_j}; R_j) + \ell(U_M^{0,d_j+1}) + e(x_1, \ldots, x_{d_j}; M_j). \]

- If \( i \notin \Delta_R \cup \Delta_M \), then \( e(x_1, \ldots, x_t; U_R^{i+1}) = e(x_1, \ldots, x_t; U_M^{i+1}) = 0 \) for all \( t \neq 0 \). Hence

\[ e_{r-i}(Q, R \ltimes M) = \ell(U_R^{0,d_j+1}) + \ell(U_M^{0,d_j+1}). \]

- If \( i = d_j \in \Delta_R \setminus \Delta_M \), then \( e(x_1, \ldots, x_t; U_R^{i+1}) = 0 \) for all \( t \notin \{0, d_j\} \). Moreover, \( e(x_1, \ldots, x_t; U_M^{i+1}) = 0 \) for all \( t \neq 0 \). Therefore,

\[ e_{r-i}(Q, R \ltimes M) = \ell(U_R^{0,d_j+1}) + e(x_1, \ldots, x_{d_j}; R_j) + \ell(U_M^{0,d_j+1}). \]

- If \( i = d'_j \in \Delta_M \setminus \Delta_R \), then \( e(x_1, \ldots, x_t; U_R^{i+1}) = 0 \) for all \( t \neq 0 \); \( e(x_1, \ldots, x_t; U_M^{i+1}) = 0 \) for all \( t \notin \{0, d'_j\} \). Therefore

\[ e_{r-i}(Q, A) = \ell(U_R^{0,d'_j+1}) + e(x_1, \ldots, x_{d'_j}; M_j) + \ell(U_M^{0,d'_j+1}). \]

(ii) Let \( d < r \). We divide into three cases.

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• Assume that \( d < i < r. \) By Theorem 1.4, \( e_{r-i}(Q, R \times M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_R) \). Note that if \( i = d_j \in \Delta_R \), then \( e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_R) = 0 \) for all \( t \notin \{0, d_j\} \). Moreover, if \( i \notin \Delta_R \), then \( e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_R) = 0 \) for all \( t \neq 0. \) Therefore,

\[
e_{r-i}(Q, R \times M) = \begin{cases} \ell(\overline{U}^{0,d_j+1}_R) + e(x_1, \ldots, x_{d_j}; R_j) & \text{if } i = d_j \in \Delta_R, \\ \ell(\overline{U}^{0,i+1}_R) & \text{if } i \notin \Delta_R. \end{cases}
\]

• Assume that \( 0 \leq i < d. \) Then by Theorem 1.4, we have

\[
e_{r-i}(Q, R \times M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_R) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_M),
\]

and the result follows by the same arguments as in the proof of (i).

• Assume that \( i = d. \) Then by Theorem 1.4, we have

\[
e_{r-d}(Q, R \times M) = \sum_{t=0}^{d} e(x_1, \ldots, x_t; \overline{U}^{t,d+1}_R) + e_0(I, M).
\]

We note that if \( d \notin \Delta_R \) then \( e(x_1, \ldots, x_t; \overline{U}^{t,d+1}_R) = 0 \) for all \( t \neq 0. \) Moreover, if \( d \in \Delta_R \) then \( e(x_1, \ldots, x_t; \overline{U}^{t,d+1}_R) = 0 \) for all \( t \notin \{0, d\}. \) Therefore, the result follows.

\[\square\]

**Remark 4.5.** Suppose that \( R, M \) are sequentially Cohen-Macaulay. Then \( \overline{U}^{0,1}_M = H^0_H^{0}(M), \overline{U}^{0,1}_R = H^0_H^{0}(R) \) and \( \overline{U}^{0,i}_M = 0, \overline{U}^{0,i}_R = 0 \) for all \( i \geq 2. \) Now, applying Corollary 4.4, we obtain a much better formula for Hilbert coefficients in this case.

We end this paper with an example of computing Hilbert coefficients of \( R \times M \) in case where \( R, M \) are sequentially generalized Cohen-Macaulay.

**Example 4.6.** Let \( S = k[[x_1, x_2, x_3, x_4, x_5]] \) be the formal power series ring over a field \( k, \) let \( a = (x_1, x_2) \cap (x_3, x_4, x_5) \) and \( b = (x_1, x_2, x_3) \cap (x_3, x_4, x_5). \) Let \( R = S/a, M = S/b. \) Then \( \dim(R) = 3 \) and the filtration of \( R \) is \( (0) = R_0 \subsetneq (x_1, x_2)R = R_1 \subsetneq R_2 = R; \) \( \dim(M) = 2 \) and the filtration of \( M \) is \( (0) = M_0 \subsetneq M_1 = M. \) Denote by \( K^i_R \) is the \( i \)-th deficiency of \( R. \) Since \( K^0_R = 0, K^1_R \) is of length 1 and \( K^2_R \) is Cohen-Macaulay of dimension 2, it follows by [11] that \( R \) is sequentially generalized Cohen-Macaulay, not sequentially Cohen-Macaulay. It is clear that \( M \) is generalized Cohen-Macaulay, not Cohen-Macaulay. Note that \( \overline{U}^{0,1}_R = 0 \) and \( \overline{U}^{0,1}_M = 0. \) We have \( \Delta_R = \{2, 3\} \) and \( \Delta_M = \{1\}. \) We choose \( a_1, a_2, a_3 \) respectively the image of \( x_1 + x_4, (x_2 + x_5)^2, x_3 \) in \( R. \) Then \( a_3 \in \text{Ann}_R(M) \) and

\[
\ell(R/(a_1^{n_1}, a_2^{n_2}, a_3^{n_3})R) = 2n_1n_2n_3 + 2n_1n_2 + 1,
\]

\[
\ell(M/(a_1^{n_1}, a_2^{n_2})M) = 4n_1n_2 + 1,
\]

for all \( n_1, n_2, n_3 \geq 1. \) Hence \( a_1, a_2, a_3 \) (resp. \( a_1, a_2 \)) is an almost p-standard s.o.p of \( R \) (resp. \( M \)). Moreover \( \ell(U^{0,3}_R) = \ell(U^{0,3}_R) + \ell(U^{0,2}_R) = 1 \) and \( \ell(U^{0,2}_M) = \ell(U^{0,2}_M) = 1, \) since \( \overline{U}^{0,1}_M = 0 \) and
Let $J = (a_1, a_2, a_3)$ and $I = (a_1, a_2)$. Then
\[
\ell(R/J^{n+1}) = 2\binom{n+3}{3} + 2\binom{n+2}{2} + \binom{n+1}{1},
\]
\[
\ell(M/I^{n+1}M) = 4\binom{n+2}{2} + \binom{n+1}{1},
\]
for all $n \geq 0$. Since $a_1, a_2, a_3$ is an almost p-standard s.o.p of $R$ and $\mathcal{U}_R^{2,3} = R_1$, we get
\[
e_1(J, R) = \ell(\mathcal{U}_R^{0,3}) + e(a_1, \mathcal{U}_R^{1,3}) + e(a_1, a_2; R_1) = 2,
\]
\[
e_2(J, R) = \ell(\mathcal{U}_R^{0,2}) + e(a_1; \mathcal{U}_R^{1,2}) = 1.
\]
Thus $\ell(\mathcal{U}_R^{0,3}) = e(a_1, \mathcal{U}_R^{1,3}) = 0$ and so $\ell(\mathcal{U}_R^{0,2}) = 1$. We set $Q = (u_1, u_2, u_3)$, where $u_i = (x_i, 0)$ for $i = 1, 2, 3$. By applying Corollary 4.4, we get $e_0(Q, R \ltimes M) = e_0(J, R) = 2$. Since $2 = \dim R(M) \in \Delta_R \cap \Delta_M$,
\[
e_1(Q, R \ltimes M) = \ell(\mathcal{U}_R^{0,2+1}) + e(a_1, a_2; R_1) + e_0(I, M) = 6.
\]
Since $1 \notin \Delta_R \cup \Delta_M$, we have $e_2(Q, R \ltimes M) = \ell(\mathcal{U}_R^{0,1+1}) = 2$. Since $0 \notin \Delta_R \cup \Delta_M$, we get $e_3(Q, R \ltimes M) = \ell(\mathcal{U}_R^{0,0+1}) = 0$.

Acknowledgement. The first and the second authors are supported by the Ministry of Education and Training of Vietnam under grant number B2021-TNA-03. The second author thanks the Institute of Mathematics, Vietnam Academy of Science and Technology, for hospitality and support through the Simons Foundation Targeted Grant (No. 558672) during his visit to the institute. The first author thanks the VIASM for hospitality and support during his visit to the institute.

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