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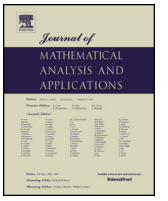

Regular Articles

Defect relations for holomorphic curves of finite lower order intersecting hypersurfaces ☆

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ABSTRACT

In this paper, we will give a better upper bound for the defect relation for a class of holomorphic maps, this result is generalized to hypersurfaces from the hyperplane case in [9]. More precisely, let $D_1, D_2, ..., D_q$ be hypersurfaces in general position, and let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be holomorphic map of lower order μ, such that $f(\mathbb{C}) \not\subset D_i$ for all $i = 1, 2, ..., q$. If $0 < \mu \leq 1/2$ then

$$\sum_{i=1}^{q} \delta_f(D_i) \leq n.$$

1. Introduction

Let f be a holomorphic map from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$ and D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$. We use the definitions of the Nevanlinna-Cartan functions $m_f(r, D)$, $N_f(r, D)$, $T_f(r)$ as usual in [8] and [11].

The deficiency of f associated with D is defined as follows:

$$\delta_f(D) = 1 - \limsup_{r\to\infty} \frac{N_f(r, D)}{T_f(r)}.$$

If f is a non-constant meromorphic function, then Nevanlinna [8] (respectively, Yamanoi [16]) proved the sharp defect relation a are taken in all $\mathbb{C}$ (respectively, a are taken on all of small functions) that

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$$\sum_a \delta_f(a) \leq 2. \tag{1.1}$$

Many mathematicians have tried to obtain defect relations for higher dimensional spaces. For example, Cartan [3] considered f a non-degenerate holomorphic map to $\mathbb{P}^n(\mathbb{C})$, and $H_j\,(j = 1, 2, \ldots, q)$ hyperplanes in general position. He showed that

$$\sum_{j=1}^{q} \delta_f(H_j) \leq n + 1.$$

Min Ru in [12] and [13] considered $f : \mathbb{C} \to X$ an algebraic non-degenerate holomorphic curve to an algebraic variety X of degree n and $D_j\,(j = 1, \ldots, q)$ hypersurfaces in general position and he proved

$$\sum_{j=1}^{q} \delta_f(D_j) \leq n + 1. \tag{1.2}$$

Until now, the above results gave the lowest upper bounds for the sum of deficiencies of a holomorphic curve. One question is raised: can we reduce the bound if we only consider a special set of curves. If the lower order μ of f is considered, then the bound on the right side of those defect relations can be reduced. For example, for a meromorphic function f of lower order μ, it was a problem of long standing to replace the 2 on the right-hand side in (1.1) by a best possible constant. If $\mu = 0$, then Valiron [15] proved that a meromorphic function of order zero cannot have more than one deficient value, Edrei and Fuchs [6] showed that the upper bound in (1.1) is replaced by 1. When $0 < \mu \leq 1$, Baernstein ([1]) and Edrei ([4] and [5]) obtained the sharp upper bound $2 - \sin \pi\mu$. When $\mu > 1$, Drasin and Weitsman stated a conjecture for a better upper bound in (1.1) and they proved that the upper bound in the conjecture, if true, can not be improved. For holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, Krytov [7] and Niino [9] gave lower bound of the inequality (1.2) if D_i are hyperplanes in general position. The proofs of the results in [7] and [9] are based on the spread relation of the hyperplanes.

In this paper, we will extend the results in [1] and [9] for holomorphic curves of lower order $\mu \leq 1/2$, intersecting hypersurfaces in general position.

From now, we always assume that f is a holomorphic map from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$. Assume that $\tilde{f} = (f_0, f_1, \ldots, f_n)$ is a reduced representative of f, where $f_0, f_1, \ldots, f_n$ are entire functions without common zeros. Let D_j be hypersurfaces in general position which are defined by homogeneous polynomials $Q_j \in \mathbb{C}[z_0, ..., z_n]$, $j = 1, 2, ..., q$. Here, the hypersurfaces $D_1, \ldots, D_q$, $q > n$, are *in general position* in $\mathbb{P}^n(\mathbb{C})$ if

$$\bigcap_{j=1}^{n+1} \operatorname{supp}(D_{i_j}) = \emptyset,$$

for any subset $\{i_1, i_2, \ldots, i_{n+1}\} \subset \{1, \ldots, q\}$. The lower order μ of f is defined by

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

Our results are stated as following:

Theorem 1. *Let f be a holomorphic curve in $\mathbb{P}^n(\mathbb{C})$ of lower order μ and $D_1, \ldots, D_q$ be hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$. Suppose $Q_j(\tilde{f})(z) \not\equiv 0$ for all $j = 1, 2, \ldots, q$.*

(a) *Assume* $\mu = 0$ *and* f *satisfies*

$$\lim_{r\to\infty} \frac{T_f(r)}{\log r} = \infty.$$

Then, there exist at most n *deficient hypersurfaces in the set* $\{D_1, \dots, D_q\}$.

(b) *Assume* $0 < \mu \leq 1/2$.

(i) *If there are* n *hypersurfaces* D_i, $i \in \{1, 2, ..., q\}$ *such that* $\delta_f(D_i) \geq 1 - \cos\pi\mu$, *then all the other deficiencies are equal to zero.*

(ii) *If there are* p, $(p < n)$, *hypersurfaces* D_i, $i \in \{1, 2, ..., q\}$ *such that* $\delta_f(D_i) > 1 - \cos\pi\mu$, *then*

$$\sum_{\substack{i\in\{1,2,\dots,q\}\\ \delta_f(D_i)\leq 1-\cos\pi\mu}} \delta_f(D_i) \leq (n-p)(1-\cos\pi\mu),$$

where the equality holds if and only if there are $n - p$ *hypersurfaces* D_i, $i \in \{1, 2, ..., q\}$ *such that* $\delta_f(D_i) = 1 - \cos\pi\mu$, *and* $\delta_f(D) = 0$ *otherwise.*

As an immediately consequence of (a) and (b), we obtain the following defect relation

Corollary 2. *Assume that* f *and* D_j $j = 1, 2, \dots, q$ *are given as in Theorem 1. If* $0 \leq \mu \leq 1/2$, *then*

$$\sum_{i=1}^{q} \delta_f(D_i) \leq n.$$

A part of our proof uses ideas from Edrei and Fuchs [6], Edrei [5], and Niino [9].

2. Spread relation of hypersurface

Assume that f is a holomorphic map and D is a hypersurface of degree d, which is defined by the homogeneous polynomial Q. The following definitions can be found in [10].

Definition 1. An increasing, positive, unbounded sequence $\{r_m\}$ is called a *sequence of Pólya peaks* of order μ of $T_f(r)$ if there are positive sequences $\{r'_m\}, \{r''_m\}$, and $\{\epsilon_m\}$ satisfying

$$r'_m \to \infty, \frac{r_m}{r'_m} \to \infty, \frac{r''_m}{r_m} \to \infty, \epsilon_m \to 0, \quad \text{as } m \to \infty,$$

and

$$\frac{T_f(t)}{T_f(r_m)} \leq \left(\frac{t}{r_m}\right)^{\mu} (1 + \epsilon_m) \quad (r'_m < t < r''_m). \tag{2.1}$$

Let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T_f(r)$ and $\Lambda(r)$ a positive function satisfying

$$\Lambda(r) = o(T_f(r)), \quad r \to \infty. \tag{2.2}$$

A set of arguments is defined by

$$E_\Lambda(r, D) = \{\theta : \log||f(re^{i\theta}), D|| < -\Lambda(r)\} \subset (-\pi, \pi],$$

where

$$||f(z), D|| = \frac{|Q(\tilde{f})(z)|}{||\tilde{f}(z)||^d}.$$

Let

$$\sigma_\Lambda(D) = \liminf_{m\to\infty} \text{meas} E_\Lambda(r_m, D).$$

Then, we define *the spread* of D with respect to f by

$$\sigma(D) = \inf_\Lambda \sigma_\Lambda(D),$$

where the infimum is taken over all functions $\Lambda(r)$ satisfying (2.2).

Using ideas from Baernstein [1], we generalize the spread relation in Baernstein [1] and Krytov [7] for hypersurfaces.

Theorem 3 *(Spread relation). Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve of lower order μ $(0 < \mu < +\infty)$. Then*

$$\sigma(D) \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D)}{2}}\right\}$$

for every hypersurface D in $\mathbb{P}^n(\mathbb{C})$ satisfying $Q(\tilde{f})(z) \not\equiv 0$.

Proof. If $\delta_f(D) = 0$ there is nothing to prove, so we may assume $\delta_f(D) > 0$.

(i) First, we assume that

$$0 < \frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D)}{2}} < 2\pi. \tag{2.3}$$

Without loss of generality, assume that $||\tilde{f}(0)|| = 1$.

We can take $c \neq 0$ and an integer $k \geq 0$ such that

$$h(z) = cz^{-k}Q(\tilde{f})(z)$$

is an entire function and $h(0) = 1$. Hence

$$N_f(r, D) = N\big(r, \frac{1}{Q(\tilde{f})}\big) = N\big(r, \frac{1}{h}\big) + k\log r, \tag{2.4}$$

which implies

$$\delta_f(D) = 1 - \limsup_{r\to\infty} \frac{N\big(r, \frac{1}{h}\big)}{T_f(r)}. \tag{2.5}$$

Put

$$F(z) = d\log||\tilde{f}(z)|| - \log|h(z)|,$$
$$\Lambda_1(r) = \Lambda(r) + k\log r - \log|c|.$$

Then, we deduce that

$$E_{\Lambda_1}(r,F)=\{\theta: F(re^{i\theta})>\Lambda_1(r)\}=E_{\Lambda}(r,D), \tag{2.6}$$

$$\Lambda_1(r)=o(T_f(r)), \quad r\to\infty. \tag{2.7}$$

Thus, to finish a proof of Theorem 3, we have to prove the following claim

Claim:

$$\liminf_{r\to\infty}\text{meas}E_{\Lambda_1}(r,F)\geq\min\left\{2\pi,\frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D)}{2}}\right\}. \tag{2.8}$$

Fix $z=re^{i\theta}$, where $0\leq\theta\leq\pi$ and $0<r<+\infty$. Following the ideas of Baernstein in [1] and [2, Theorem A'], we define a modification function $T^*(z)$ by

$$T^*(z)=\frac{1}{d}\left\{\sup_E\frac{1}{2\pi}\int_E F(re^{i\varphi})d\varphi+N(r,0,h)\right\}$$

here the supremum is considered to all sets $E\subset(-\pi,\pi]$ of measurement 2θ. Then, $T^*(z)$ is continuous on $\{z:\text{Im}(z)\geq 0\}$, and subharmonic in $\{z:\text{Im}(z)>0\}$ except at the origin. By putting $T^*(0)=0$ and the hypothesis $h(0)=1$, we can remove this discontinuity. We have

$$T^*(re^{i\pi})=T^*(-r)=T_f(r), \tag{2.9}$$

$$T^*(r)=\frac{1}{d}N\big(r,0,h\big). \tag{2.10}$$

Since $|Q(\tilde{f})(z)|\leq c_1||\tilde{f}(z)||^d$, where $c_1\geq 0$ depends only on the coefficients of Q, it follows

$$|h(z)|\leq A|z|^{-k}||\tilde{f}(z)||^d$$

for all z, where $A=|c|c_1>0$. Then, $F(z)\geq k\log|z|-\log A$ for all z. Hence, we deduce that

$$\begin{aligned}
T^*(re^{i\theta})&=T^*(re^{i\pi})-\frac{1}{2\pi d}\inf_E\int_{(-\pi,\pi]\setminus E}F(re^{i\varphi})d\varphi\\
&\leq T^*(re^{i\pi})+\frac{\log A-k\log r}{2\pi d}\inf_E(\text{meas}(-\pi,\pi]\setminus E)\\
&=T^*(re^{i\pi})+\frac{\pi-\theta}{\pi d}(\log A-k\log r)\\
&\leq T^*(re^{i\pi})+\log A\\
&\leq T_f(r)+K,
\end{aligned} \tag{2.11}$$

with $K>\log^+ A$.

Put

$$\gamma=\frac{2}{\pi\mu}\arcsin\sqrt{\frac{\delta_f(D)}{2}}$$

and

$$v(z) = \begin{cases} 0, & z = 0, \\ T^*(z^\gamma), & z = re^{i\theta},\ 0 < r < \infty,\ 0 \le \theta \le \pi. \end{cases}$$

Then

$$0 < \gamma < 1, \quad 0 < \gamma\mu \le \frac{1}{2} \text{ and } 1 - \delta_f(D) = \cos \pi\gamma\mu, \tag{2.12}$$

and $v(z)$ is continuous on $\{z : \mathrm{Im}(z) \ge 0\}$ and subharmonic in $\{z : \mathrm{Im}(z) > 0\}$.

Combining (2.9), (2.10) and (2.11), for $t > 0$ and $0 \le \varphi \le \pi$, we have

$$v(t) = T^*(t^\gamma) = \frac{1}{d} N\big(t^\gamma, \frac{1}{h}\big), \tag{2.13}$$

$$v(-t) = v(te^{i\pi}) = T^*(t^\gamma e^{i\gamma\pi}) \le T_f(t^\gamma) + K, \tag{2.14}$$

$$v(Re^{i\varphi}) = T^*(R^\gamma e^{i\gamma\varphi}) \le T_f(R^\gamma) + K. \tag{2.15}$$

We have

$$\begin{aligned} 0 < A(t, r, \theta, R) &:= \frac{r \sin\theta}{\pi} \Big(\frac{1}{t^2 - 2tr\cos\theta + r^2} - \frac{1}{|R - \frac{rt}{R} e^{i\theta}|} \Big) \\ &\le \frac{r \sin\theta}{\pi} \Big(\frac{1}{t^2 - 2tr\cos\theta + r^2} \Big) \\ &:= P(t, r, \pi - \theta). \end{aligned} \tag{2.16}$$

Also,

$$0 < A(-t, r, \theta, R) \le P(t, r, \theta). \tag{2.17}$$

In addition, by simple computation, we have

$$\int_0^\infty P(t, r, \theta) d\theta = \frac{\theta}{\pi} < 1 \quad (0 < \theta < \pi, r > 0). \tag{2.18}$$

Similarly,

$$B(\varphi, r, \theta, R) := \frac{2Rr(R^2 - r^2)\sin\varphi\, \sin\theta}{\pi |R^2 e^{ei\varphi} - 2rRe^{i\varphi}\cos\theta + r^2|^2}.$$

Since

$$|R^2 e^{ei\varphi} - 2rRe^{i\varphi}\cos\theta + r^2|^2 \ge (R - r)^4,$$

we have, for $0 < \theta < \pi, 0 < \varphi < \pi, 0 < r < \frac{R}{2}$,

$$0 < B(\varphi, r, \theta, R) \le \frac{32r}{\pi R}. \tag{2.19}$$

On the other hand, take $z = re^{i\theta} \in D_R$, we have from (2.13) - (2.18) for $0 < \theta < \pi, 0 < r < \frac{R}{2}$ that

$$v(re^{i\theta}) \le \int_{-R}^{R} A(t, r, \theta, R) v(t) dt + \int_0^\pi B(\varphi, r, \theta, R) v(Re^{i\varphi}) d\varphi$$

$$\leq \frac{1}{d}\int_0^R P(t,r,\pi-\theta)N\big(t^\gamma,\frac{1}{h}\big)dt + \int_0^R P(t,r,\theta)T_f(t^\gamma)dt + \frac{32r}{R}(T_f(R^\gamma)+K)+K. \tag{2.20}$$

Assume that $\{r_m\}$ is a sequence of Pólya peaks of order μ of $T_r(f)$, and $\{r'_m\}, \{r''_m\}$ and $\{\epsilon_m\}$ are its associated sequences. Put

$$s'_m = (r'_m)^{1/\gamma},\ s_m = r_m^{1/\gamma},\ s''_m = (r''_m)^{1/\gamma}.$$

We have

$$T_f(t^\gamma) \leq (1+\epsilon_m)\big(\frac{t}{s_m}\big)^{\gamma\mu}T_f(r_m) \quad (s'_m < t < s''_m). \tag{2.21}$$

Choose m_0 so that $m \geq m_0$ implies $2s'_m < s_m < \frac{s_m}{2}$. Then

$$P(t,s_m,\theta) = \frac{1}{\pi}\mathrm{Im}\left(\frac{1}{t+s_m e^{-i\theta}}\right) \leq \frac{1}{\pi}\left|\frac{1}{t+s_m e^{-i\theta}}\right| < \frac{2}{\pi s_m} < \frac{1}{s_m} \tag{2.22}$$

for $0 < t < s'_m, m \geq m_0$.

From (2.21) and (2.22), we find that for $0 < \theta < \pi$ and $m \geq m_0$,

$$\int_0^{s''_m} P(t,s_m,\theta)T_f(t^\gamma)dt = \int_0^{s'_m} P(t,s_m,\theta)T_f(t^\gamma)dt + \int_{s'_m}^{s''_m} P(t,s_m,\theta)T_f(t^\gamma)dt$$
$$\leq T_f((s'_m)^\gamma)\frac{s'_m}{s_m} + (1+\epsilon_m)T_f(r_m)\int_0^\infty P(t,s_m,\theta)\left(\frac{t}{s_m}\right)^{\gamma\mu}dt.$$

Since $0 < \gamma\mu \leq \frac{1}{2}$, by a change of variable and the residue theorem, we obtain

$$\int_0^\infty \big(\frac{t}{s_m}\big)^{\gamma\mu}P(t,s_m,\theta)dt = \int_0^\infty t^{\gamma\mu}P(t,1,\theta)dt = \frac{\sin\theta\gamma\mu}{\sin\pi\gamma\mu}.$$

It is obvious that

$$\lim_{m\to\infty}\frac{s'_m}{s_m} = \lim_{m\to\infty}\left(\frac{r'_m}{r_m}\right)^{1/\gamma} = 0.$$

Hence, we get

$$\int_0^{s''_m} T_f(t^\gamma)P(t,s_m,\theta)dt \leq T_f(r_m)\left[\frac{\sin\theta\gamma\mu}{\sin\pi\gamma\mu}+o(1)\right] \quad (m\to\infty) \tag{2.23}$$

uniformly in θ for $0 < \theta < \pi$.

From (2.5) we deduce that

$$N\big(t^\gamma,\frac{1}{h}\big) \leq (1-\delta_f(D)+o(1))dT_f(t^\gamma) \quad (s'_m < t < s''_m). \tag{2.24}$$

By the First main theorem, we have

$$N\big(t^\gamma, \frac{1}{h}\big) \le N_f(t^\gamma, D) \le dT_f(t^\gamma) + O(1). \tag{2.25}$$

From (2.24), (2.25) and a similar argument as above, we obtain

$$\int_0^{s''_m} P(t, s_m, \pi-\theta) N\big(t^\gamma, \frac{1}{h}\big) dt \le (1-\delta_f(D)) dT_f(r_m) \left[\frac{\sin(\pi-\theta)\gamma\mu}{\sin \pi\gamma\mu} + o(1)\right] \tag{2.26}$$

when $m \to \infty$ and for all $0 < \theta < \pi$.

We have

$$\begin{aligned}
\frac{s_m}{s''_m} T_f((s''_m)^\gamma) &= \left(\frac{r_m}{r''_m}\right)^{\frac{1}{\gamma}} T_f(r''_m) \\
&\le \left(\frac{r_m}{r''_m}\right)^{\frac{1}{\gamma}} (1+\epsilon_m) \left(\frac{s''_m}{s_m}\right)^{\gamma\mu} T_f(r_m) \\
&\le (1+\epsilon_m) \left(\frac{r_m}{r''_m}\right)^{\frac{1}{\gamma}-\mu} T_f(r_m).
\end{aligned}$$

Hence

$$\frac{32 s_m}{s''_m} (T_f((s''_m)^\gamma) + K) + K = o(T(r_m)) \quad (m \to \infty). \tag{2.27}$$

In (2.20), we consider for $r = s_m, R = s''_m$. From (2.23), (2.26) and (2.27), together with (2.12) that $1 - \delta_f(D) = \cos \pi\gamma\mu$, we obtain

$$\begin{aligned}
v(s_m e^{i\theta}) &\le T_f(r_m) \left[\frac{(1-\delta_f(D)) \sin(\pi-\theta)\gamma\mu + \sin\theta\gamma\mu}{\sin\pi\gamma\mu} + o(1)\right] \\
&= T_f(r_m) \left[\frac{\cos\pi\gamma\mu \sin(\pi-\theta)\gamma\mu + \sin\theta\gamma\mu}{\sin\pi\gamma\mu} + o(1)\right] \\
&= T_f(r_m)(\cos(\pi-\theta)\gamma\mu + o(1))
\end{aligned}$$

when $m \to \infty$ and $0 < \theta < \pi$, where $o(1)$ is independent of θ. Hence, there is a sequence $\{\alpha_m\} \to 0$ such that for $m = 1, 2, \ldots, 0 < \theta < \pi$ we have

$$v(s_m e^{i\theta}) \le T_f(r_m)(\cos(\pi-\theta)\gamma\mu + \alpha_m). \tag{2.28}$$

Let

$$\sigma_m = \text{meas} E_{\Lambda_1}(r_m, F).$$

Then (2.8) is equivalent to the inequality

$$\liminf_{m\to\infty} \sigma_m \ge 2\pi\gamma. \tag{2.29}$$

Since $h(0) = 1$, we have

$$\begin{aligned} T_f(r_m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log||\tilde{f}(r_m e^{i\varphi})||d\varphi \\ &= \frac{1}{2\pi d}\int_{-\pi}^{\pi} (F(r_m e^{i\varphi}) + \log|h(r_m e^{i\varphi})|)d\varphi \\ &= \frac{1}{d}\left\{\frac{1}{2\pi}\int\limits_{E_{\Lambda_1}(r_m,F)} F(r_m e^{i\varphi})d\varphi + N\big(r_m, \frac{1}{h}\big)\right\} \\ &\quad + \frac{1}{2\pi d}\int\limits_{(-\pi,\pi]\setminus E_{\Lambda_1}(r_m,F)} F(r_m e^{i\varphi})d\varphi \\ &\leq T^*(r_m e^{i\sigma_m/2}) + \frac{1}{d}\Lambda_1(r_m). \end{aligned}$$

From this and (2.11), we obtain

$$\lim_{m\to\infty} \frac{T^*(r_m e^{i\sigma_m/2})}{T_f(r_m)} = 1. \tag{2.30}$$

Let

$$M = \{m : \sigma_m < 2\pi\gamma\}.$$

Clearly, (2.29) holds when M is a finite set. Hence we now consider that M is infinite. We have $m \in M$ if and only if the point $(r_m e^{i\sigma_m/2})^{1/\gamma} = s_m e^{i\sigma_m/2\gamma}$ belongs to the domain of $v(z)$, which means the upper half-plane. In this case, we have

$$T^*(r_m e^{i\sigma_m/2}) = v(s_m e^{i\sigma_m/2\gamma}) \quad (m \in M).$$

Using this in (2.30), comparing with (2.28) and combining with $0 < \gamma\mu \leq \frac{1}{2}$, we deduce that

$$\lim_{\substack{m\to\infty \\ m\in M}} \sigma_m = 2\pi\gamma,$$

which implies (2.29) is satisfied.

(ii) Second, we assume that

$$\frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D)}{2}} \geq 2\pi.$$

Then, we take a number κ with $0 < \kappa < \delta_f(D)$ such that

$$0 < \frac{4}{\mu}\arcsin\sqrt{\frac{\kappa}{2}} < 2\pi.$$

Put

$$\gamma = \frac{2}{\pi\mu}\arcsin\sqrt{\frac{\kappa}{2}}.$$

It follows from the above arguments that

$$\sigma(D) \geq \frac{4}{\mu} \arcsin \sqrt{\frac{\kappa}{2}}.$$

In this inequality, let κ tend to $2\sin^2(\mu\pi/2)$, to obtain $\sigma(D) \geq 2\pi$. $\square$

Lemma 4. *Let f be a holomorphic curve in $\mathbb{P}^n(\mathbb{C})$ and $D_1, \ldots, D_{n+1}$ be hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$ such that $Im(f) \not\subset D_j$ for all $j = 1, 2, \ldots, n+1$. Let $\Lambda(r)$ be a function satisfying (2.2). Then*

$$\text{meas}\left\{ \bigcap_{j=1}^{n+1} E_\Lambda(r, D_j) \right\} \leq \frac{2\pi K}{\Lambda(r)},$$

where $K > 0$ is independent of Λ.

Proof. Denote $d = \gcd(d_1, \ldots, d_{n+1})$.

Since $\cap_{i=1}^{n+1} D_i = \emptyset$, we can apply Hilbert's Nullstellensatz [17] to get that for any $k, (0 \leq k \leq n)$, there are homogeneous polynomials $b_{jk} \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ of degree $m_k - d_j, 1 \leq j \leq n+1$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{jk} Q_j(x_0, \ldots, x_n),$$

where m_k is an integer. So

$$||\tilde{f}(z)||^d \leq c_1 \sum_{j=1}^{n+1} \{||\tilde{f}(z)||^{d-d_j} |Q_j(\tilde{f})(z)|\},$$

where $c_1 > 0$ depends only on the coefficients of Q_j. Hence,

$$d \log ||\tilde{f}(z)|| \leq \log \left(\sum_{j=1}^{n+1} ||\tilde{f}(z)||^{d-d_j} |Q_j(\tilde{f})(z)| \right) + \log c_1. \tag{2.31}$$

If $z = re^{i\theta}$, such that $\theta \in \bigcap_{j=1}^{n+1} E_\Lambda(r, D_j)$, then

$$\frac{|Q_j(\tilde{f})(z)|}{||\tilde{f}(z)||^{d_j}} < \exp(-\Lambda(r)), \quad j = 1, 2, \ldots, n+1.$$

Hence, we get

$$\frac{1}{||\tilde{f}(z)||^d} \left(\sum_{j=1}^{n+1} ||\tilde{f}(z)||^{d-d_j} |Q_j(\tilde{f})(z)| \right) < (n+1) \exp(-\Lambda(r)),$$

so

$$\log \left(\sum_{j=1}^{n+1} ||\tilde{f}(z)||^{d-d_j} |Q_j(\tilde{f})(z)| \right) < d \log ||\tilde{f}(z)|| - \Lambda(r) + \log(n+1). \tag{2.32}$$

Therefore, (2.31) and (2.32) imply

$$\Lambda(r) < \log c_1 (n+1), \tag{2.33}$$

for $z = re^{i\theta}, \theta \in \bigcap_{j=1}^{n+1} E_\Lambda(r, D_j)$. Put $E(r) = \bigcap_{j=1}^{n+1} E_\Lambda(r, D_j)$. It follows from (2.33) that

$$\begin{aligned}\frac{1}{2\pi}\Lambda(r)\text{meas}E(r) &\leq \frac{1}{2\pi}\int\limits_{E(r)} \log c_1(n+1)d\theta \\ &\leq \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \log c_1(n+1)d\theta \\ &= \log c_1(n+1).\end{aligned}$$

Take $K \geq \log^+ c_1(n+1)$. Then, we have

$$\text{meas}E(r) \leq \frac{2\pi K}{\Lambda(r)},$$

which proves Lemma 4. □

We define $I_\Lambda(r, D_k)$ by

$$I_\Lambda(r, D_k) = E_\Lambda(r, D_k) - \bigcup_{\{i_1,\ldots,i_n\}} \left(E_\Lambda(r, D_k) \cap \big(\bigcap_{j=1}^{n} E_\Lambda(r, D_{i_j})\big)\right), \tag{2.34}$$

where the union is taken over all subsets of $\{1, \ldots, k-1, k+1, \ldots, q\}$. Let

$$\sigma'_\Lambda(D_k) = \liminf_{m\to\infty} \text{meas}I_\Lambda(r_m, D_k). \tag{2.35}$$

From the definition of $I_\Lambda(r, D_k)$, it is obvious that we have

Lemma 5. *If $\{i_1, i_2 \ldots, i_{n+1}\} \subset \{1, 2, \ldots, q\}$, then*

$$\bigcap_{j=1}^{n+1} I_\Lambda(r, D_{i_j}) = \emptyset.$$

Lemma 6. *Let f and $D_1, D_2 \ldots, D_q (q \geq n+1)$ be as in Lemma 4. Then, for any $1 \leq k \leq q$, there exists a sequence of positive functions $\Lambda_\nu(r)$ satisfying (2.2) such that*

$$\liminf_{\nu\to\infty} \sigma'_{\Lambda_\nu}(D_k) \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D_k)}{2}}\right\}.$$

Proof. By Lemma 4, there is a constant K satisfying

$$\text{meas}\left\{\bigcap_{j=1}^{n+1} E_\Lambda(r, D_{i_j})\right\} \leq \frac{2\pi K}{\Lambda(r)},$$

for all subsets $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, q\}$. Hence,

$$\begin{aligned}&\sum_{\{i_1,\ldots,i_n\}} \text{meas}\left\{E_\Lambda(r, D_k) - \bigcup_{\{i_1,\ldots,i_n\}} \left(E_\Lambda(r, D_k) \cap \big(\bigcap_{j=1}^{n} E_\Lambda(r, D_{i_j})\big)\right)\right\} \\ &\leq \sum_{\{i_1,\ldots,i_n\}} \frac{2\pi K}{\Lambda(r)} = \frac{2\pi K}{\Lambda(r)}\binom{q-1}{n}.\end{aligned} \tag{2.36}$$

Let $\epsilon_\nu \to 0$ as $\nu \to \infty$ where ϵ_ν be positive numbers. Put

$$\Lambda_\nu(r) = \frac{2\pi K}{\epsilon_\nu}\binom{q-1}{n}. \tag{2.37}$$

Fix ν. We have $\Lambda_\nu(r)$ satisfying (2.2). Assume that $\{r_m\}$ is a sequence of Polya peaks which has order μ of $T_f(r)$. By (2.34) - (2.37), we have

$$\begin{aligned}\mathrm{meas}I_{\Lambda_\nu}(r_m, D_k) &\geq \mathrm{meas}E_{\Lambda_\nu}(r_m, D_k) - \sum_{\{i_1,\ldots,i_n\}} \mathrm{meas}\Big\{E_\Lambda(r, D_k)\\ &\quad - \bigcup_{\{i_1,\ldots,i_n\}}\Big(E_\Lambda(r, D_k) \cap \big(\bigcap_{j=1}^{n} E_\Lambda(r, D_{i_j})\big)\Big)\Big\}\\ &\geq \mathrm{meas}E_{\Lambda_\nu}(r_m, D_k) - \epsilon_\nu,\end{aligned}$$

which implies

$$\sigma'_{\Lambda_\nu}(D_k) \geq \sigma_{\Lambda_\nu}(D_k) - \epsilon_\nu.$$

By Theorem 3 we get

$$\sigma'_{\Lambda_\nu}(D_k) \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D_k)}{2}}\right\} - \epsilon_\nu.$$

The lemma is proved when $\nu \to \infty$. □

Lemma 7. *Let f and $D_1, D_2, \ldots, D_q, (q \geq n+1)$, be as in Lemma 4. Then*

$$\sum_{k=1}^{q} \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D_k)}{2}}\right\} \leq 2n\pi,$$

where $\mu, (0 < \mu < +\infty)$, is the lower order of f.

Proof. Let $s(\theta)$ be the largest number of sets $I_\Lambda(r, D_k)$ which have a common point θ in the family $\{I_\Lambda(r, D_k)\}(1 \leq k \leq q)$. By Lemma 5 we deduce that $s(\theta) \leq n$.

Put

$$I = \bigcup_{k=1}^{q} I_\Lambda(r, D_k).$$

By the definition of $I_\Lambda(r, D_k)$, we have $\mathrm{meas}I \leq 2\pi$. Let $\chi_k : I_\Lambda(r, D_k) \to \mathbb{R}$ be a characteristic function of $I_\Lambda(r, D_k)$ for $k = 1, \ldots, q$. Then

$$\chi_k(\theta) = \begin{cases}1, & \theta \in I_\Lambda(r, D_k),\\ 0, & \theta \notin I_\Lambda(r, D_k).\end{cases}$$

We have

$$s(\theta) = \sum_{k=1}^{q} \chi_k(\theta). \tag{2.38}$$

Hence, it follows from (2.38) and the definition and the basic properties of the Lebesgue Integral that

$$\begin{aligned}\sum_{k=1}^{q}\text{meas}I_{\Lambda}(r,D_k) &= \sum_{k=1}^{q}\int\limits_{I_{\Lambda}(r,D_k)}\chi_k(\theta)d\mu\\ &\leq \sum_{k=1}^{q}\int\limits_{I}\chi_k(\theta)d\mu = \int\limits_{I}s(\theta)d\mu\\ &\leq n\text{meas}I \leq 2n\pi. \end{aligned} \tag{2.39}$$

From (2.35) we deduce that

$$\sum_{k=1}^{q}\sigma'_{\Lambda_\nu}(D_k) \leq 2n\pi.$$

Hence, it follows from Lemma 6 that

$$\sum_{k=1}^{q}\min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta_f(D_k)}{2}}\right\} \leq 2n\pi,$$

which proves Lemma 7. □

3. Proof of Theorem 1

Theorem 1(a) is a consequence of the following theorem.

Theorem 8. *Let f be a holomorphic curve of lower order μ in $\mathbb{P}^n(\mathbb{C})$ satisfying*

$$\lim_{r\to\infty}\frac{T_f(r)}{\log r} = \infty,$$

let D_j be hypersurfaces in general position of degree $d_j, 1 \leq j \leq n+1$. Denote $d = \gcd(d_1,\ldots,d_{n+1})$. Suppose $Im(f) \not\subset D_j$ for all $j = 1,2,\ldots,n+1$. If

$$\gamma = \max_{1\leq j\leq n+1}\{1-\delta_f(D_j)\} < 1,$$

then

$$\mu \geq \begin{cases}\frac{\log\left(\frac{1}{\gamma(2-\gamma)}\right)}{d\log\left(1+\frac{4}{\gamma(1-\gamma)}\right)} & (\gamma \neq 0),\\ \frac{1}{d} & (\gamma = 0).\end{cases}$$

Let $x = \varphi(s)\,(s \in [0,1])$ be a real continuous function which satisfies:

(i) $\varphi(1) = 1, \varphi(0) = 0$.

(ii) For any $s \in [0,1]$, the first and the second derivative of $\varphi(s)$ exist and they are strictly positive and continuous in $[0,1]$.

Denote $\psi(x) = \varphi^{-1}(x)\,(0 \leq x \leq 1)$.

We will recall the results:

Lemma 9. *[5, Lemma 1] Let the quantities x_j be subject to the constraints*

$$0 \le x_j \le 1 (1 \le j \le k), \quad \sum_{j=1}^{k} \psi(x_j) \le H < +\infty,$$

where $k\,(k \ge 2)$ may be $+\infty$.

a) If $k \le H$, then $\sum_{j=1}^{k} x_j \le k$.

b) If $H < k < +\infty$, then

$$\sum_{j=1}^{k} x_j \le [H] + \varphi(H - [H]). \tag{3.1}$$

Equality is possible in (3.1) if and only if

(i) exactly $[H]$ of the x_j are equal to 1;

(ii) one x_j is $\varphi(H - [H])$;

(iii) all other x_j, if they exist, are equal to 0.

c) If $k = +\infty$ and $x_j > 0$ for all $j \ge 1$, then

$$\sum_{j=1}^{k} x_j < [H] + \varphi(H - [H]).$$

Lemma 10. *[14, Lemma 4] Let f be a holomorphic curve in $\mathbb{P}^n(\mathbb{C})$ and $\tilde{f} = (f_0, \dots, f_n)$ be a reduced representative of f. If $\varrho > 1$ and $r > 0$, then*

$$T_f(r) \le \frac{4}{\varrho - 1} T_f(\varrho r) + \max_{0 \le j \le n} N\big(\varrho r, \frac{1}{f_j}\big) + O(\log r) \quad (r \to \infty).$$

Proof of Theorem 8. For $1 \le j \le n+1$, we denote by $\widetilde{D}_j$ the hypersurfaces given by the homogeneous polynomials $Q_j^{d/d_j}(x_0, \dots, x_n) = 0$. If $z \in \mathbb{C}$ satisfies $Q_j \circ \tilde{f}(z) = 0$ with multiplicity p, then $Q_j^{d/d_j} \circ \tilde{f}(z) = 0$ with multiplicity $p\frac{d}{d_j}$. Hence

$$N_f(r, \widetilde{D}_j) = \frac{d}{d_j} N_f(r, D_j).$$

Therefore, replacing the hypersurfaces D_j by the hypersurfaces $\widetilde{D}_j$ if necessary, we may assume that $D_1, \dots, D_{n+1}$ have the same degree d.

By Hilbert's Nullstellensatz theorem [17], with any $k \in \{0, ..., n\}$, there is an integer m_k such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{jk} Q_j(x_0, x_1, \dots, x_n),$$

where $b_{jk} \in \mathbb{C}[x_0, \dots, x_n]$, $1 \le j \le n+1$, are the homogeneous polynomials of degree $m_k - d$. Therefore,

$$||\tilde{f}(z)||^d \le K \max\{|Q_1(\tilde{f})(z)|, Q_2(\tilde{f})(z)|, \dots, |Q_{n+1}(\tilde{f})(z)|\}|,$$

for a constant $K > 0$ depending only on the coefficients of $Q_j, 1 \le j \le n+1$.

We put $g_j = Q_j \circ \tilde{f}$ for $j = 1, \dots, n+1$. Then $g_1, \dots, g_{n+1}$ are entire functions without common zeros. Consider $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ given by $\widetilde{g} = (g_1, \dots, g_{n+1}) : \mathbb{C} \to \mathbb{C}^{n+1} \setminus \{0\}$. Then g is a holomorphic curve from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$. It follows that

$$dT_f(r) \leq T_g(r) + O(1),$$

where $O(1)$ is independent of r. Hence, we have

$$1 - \delta_f(D_j) = \limsup_{r\to\infty} \frac{N_f(r, D_j)}{dT_f(r)} \geq \limsup_{r\to\infty} \frac{N\big(r, \frac{1}{g_j}\big)}{T_g(r)}, \tag{3.2}$$

and

$$\lim_{r\to\infty} \frac{T_g(r)}{\log r} = \lim_{r\to\infty} \frac{T_f(r)}{\log r} = \infty. \tag{3.3}$$

Applying Lemma 10 to g, we have

$$T_g(r) \leq \frac{4}{\varrho - 1} T_g(\varrho r) + \max_{0\leq j\leq n} N\big(\varrho r, \frac{1}{g_j}\big) + O(\log r), \tag{3.4}$$

for $\varrho > 1$ and $r > 0$.

Take c and c' satisfying $\gamma < c' < c < 1$. Then, it follows from (3.2) that

$$N\big(r, \frac{1}{g_j}\big) < c' T_g(r) \quad (j = 1, \ldots, n+1) \tag{3.5}$$

for all r big enough.

Choose

$$\varrho = 1 + \frac{4}{c(1-c)}.$$

Then, we deduce from (3.3), (3.4) and (3.5) that

$$T_g(r) < c(2-c) T_g(\varrho r)$$

for all r big enough. By using a similar calculation of Edrei and Fuchs [6], we have

$$\liminf_{r\to\infty} \frac{\log T_g(r)}{\log r} \geq \frac{\log\big(\frac{1}{c(2-c)}\big)}{\log \varrho}. \tag{3.6}$$

Clearly, there exists a constant $A > 0$ satisfying

$$|Q_j(\widetilde{f})| \leq A||\widetilde{f}||^d$$

for every $1 \leq j \leq n+1$. Hence,

$$||\widetilde{g}|| \leq A||\widetilde{f}||^d.$$

It implies that

$$T_g(r) \leq dT_f(r) + O(1).$$

Therefore, we obtain

$$\mu = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} \geq \frac{1}{d} \liminf_{r\to\infty} \frac{\log T_g(r)}{\log r}.$$

Together with (3.6), we have

$$\mu = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} \geq \frac{1}{d} \frac{\log\left(\frac{1}{c(2-c)}\right)}{\log \varrho}.$$

Theorem 8 is obtained by letting $c \to \gamma$. □

Proof of Theorem 1(b). Assume that

$$\delta_f(D_k) \geq 1 - \cos \pi\mu,\ k = 1, \ldots, n.$$

Hence, we have

$$\min\left\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta_f(D_k)}{2}}\right\} = 2\pi, \quad k = 1, \ldots, n.$$

We deduce from Lemma 7 that

$$\sum_{k=n+1}^{q} \min\left\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta_f(D_k)}{2}}\right\} = 0.$$

This implies

$$\delta_f(D_k) = 0, \quad k = n+1, \ldots, q.$$

Now, we assume that $\delta_f(D_k) > 1 - \cos \pi\mu$ for $k = 1, \ldots, p$; $0 \leq p < n$ and $\delta_f(D_k) \leq 1 - \cos \pi\mu$ for $k = p+1, \ldots, q$. Then, from Lemma 7, we have

$$\sum_{k=p+1}^{q} \frac{4}{\mu} \arcsin \sqrt{\frac{\delta_f(D_k)}{2}} \leq 2(n-p)\pi.$$

Hence, by the same arguments as in Section 3 of Edrei [5], we get

$$\sum_{k=p+1}^{q} \delta_f(D_k) \leq (n-p)(1 - \cos \pi\mu). \tag{3.7}$$

By Lemma 9, the equality holds in (3.7) if and only if there are exactly $n-p$ of $\{\delta_f(D_k)\}_{k=p+1}^{q} = 1 - \cos \pi\mu$ and all other $\delta_f(D_k) = 0$. Theorem 1(b) is proved.

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