

VANISHING AND NON-NEGATIVITY OF THE FIRST NORMAL HILBERT COEFFICIENT

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Dedicated to Professor Ngo Viet Trung on the occasion of his 70th birthday

ABSTRACT. Let (R, \mathfrak{m}) be a Noetherian local ring such that \widehat{R} is reduced. We prove that, when \widehat{R} is S_2 , if there exists a parameter ideal $Q \subseteq R$ such that $\bar{e}_1(Q) = 0$, then R is regular and $\nu(\mathfrak{m}/Q) \leq 1$. This leads to an affirmative answer to a problem raised by Goto-Hong-Mandal [10]. We also give an alternative proof (in fact a strengthening) of their main result. In particular, we show that if \widehat{R} is equidimensional, then $\bar{e}_1(Q) \geq 0$ for all parameter ideals $Q \subseteq R$, and in characteristic $p > 0$, we actually have $e_1^*(Q) \geq 0$. Our proofs rely on the existence of big Cohen-Macaulay algebras.

1. INTRODUCTION

Let (R, \mathfrak{m}) be a Noetherian local ring of dimension d such that \widehat{R} is reduced and let $I \subseteq R$ be an \mathfrak{m} -primary ideal. Then for $n \gg 0$, $\ell(R/\overline{I^{n+1}})$ agrees with a polynomial in n of degree d , and we have integers $\bar{e}_0(I), \dots, \bar{e}_d(I)$ such that

$$\ell(R/\overline{I^{n+1}}) = \bar{e}_0(I) \binom{n+d}{d} - \bar{e}_1(I) \binom{n+d-1}{d-1} + \dots + (-1)^d \bar{e}_d(I).$$

These integers $\bar{e}_i(I)$ are called the normal Hilbert coefficients of I .

It is well-known that $\bar{e}_0(I)$ is the Hilbert-Samuel multiplicity of I , which is always a positive integer. In this paper, we are interested in the first coefficient $\bar{e}_1(I)$. It was proved by Goto-Hong-Mandal [10] that when \widehat{R} is unmixed, $\bar{e}_1(I) \geq 0$ for all \mathfrak{m} -primary ideals $I \subseteq R$ (which answers a question posed by Vasconcelos [30]). They proposed a further problem in [10, Section 3] regarding the vanishing of $\bar{e}_1(I)$ and the regularity of the normalization of R . Since any \mathfrak{m} -primary ideal I is integral over a parameter ideal when the residue field is infinite, to study $\bar{e}_1(I)$ we may assume that $I = Q$ is a parameter ideal (i.e., it is generated by a system of parameters). In this paper, we prove the following main result which will lead to an affirmative answer to the question proposed in [10]. This theorem is also a generalization of the main result of [27].

Theorem 1.1 (Theorem 3.7). *Let (R, \mathfrak{m}) be a Noetherian local ring such that \widehat{R} is reduced and S_2 . If $\bar{e}_1(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is regular and $\nu(\mathfrak{m}/Q) \leq 1$.*

In [7], it was shown that when R has characteristic $p > 0$, for $n \gg 0$, $\ell(R/(I^{n+1})^*)$ also agrees with a polynomial of degree d and one can define the tight Hilbert coefficients $e_0^*(I), \dots, e_d^*(I)$ in

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a similar way (see Section 2 for more details). It is easy to see that $\bar{e}_1(I) \geq e_1^*(I)$. We strengthen the main result of [10] in characteristic $p > 0$ by showing that $e_1^*(Q) \geq 0$ for any parameter ideal $Q \subseteq R$ under mild assumptions.

Theorem 1.2 (Corollary 3.3). *Let (R, \mathfrak{m}) be an excellent local ring of characteristic $p > 0$ such that \widehat{R} is reduced and equidimensional. Then we have $e_1^*(Q) \geq 0$ for all parameter ideals $Q \subseteq R$.*

Our proofs of both theorems rely on the existence of big Cohen-Macaulay algebras. In fact, we show that the tight Hilbert coefficients $e_i^*(I)$ is a special case of what we call the BCM Hilbert coefficients $e_i^B(I)$ associated to a big Cohen-Macaulay algebra B , and the latter can be defined in arbitrary characteristic. In this context, we will show in Theorem 3.1 that $\bar{e}_1(Q) \geq e_1^B(Q) \geq 0$ for all parameter ideals $Q \subseteq R$ when B satisfies some mild assumptions. This recovers and extends the main result of [10] in arbitrary characteristic.

Throughout this article, all rings are commutative with multiplicative identity 1. We will use (R, \mathfrak{m}) to denote a Noetherian local ring with unique maximal ideal \mathfrak{m} . We refer the reader to [4, Chapter 1-4] for basic notions such as Cohen-Macaulay rings, regular sequence, Euler characteristic, integral closure, and the Hilbert-Samuel multiplicity. We refer the reader to [29, Section 07QS] for the definition and basic properties of excellent rings. The paper is organized as follows. In Section 2 we collect the definitions and some basic results on big Cohen-Macaulay algebras and variants of Hilbert coefficients. In Section 3 we prove our main results and we propose some further questions.

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2. PRELIMINARIES

Recall that an element x in a ring R is integral over an ideal $I \subseteq R$ if it satisfies an equation of the form $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, where $a_k \in I^k$. The set of all elements integral over I forms an ideal and is denoted by \bar{I} , called the integral closure of I . An ideal $I \subseteq R$ is called integrally closed if $I = \bar{I}$. It is well-known that an element $x \in R$ is integral over I if and only if the image of x in R/\mathfrak{p} is integral over $I(R/\mathfrak{p})$ for all minimal primes \mathfrak{p} , see [21, Proposition 1.1.5].

Suppose R is a Noetherian ring of prime characteristic $p > 0$. The tight closure of an ideal $I \subseteq R$, introduced by Hochster–Huneke, is defined as follows:

$$I^* := \{x \in R \mid \text{there exists } c \in R - \cup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p} \text{ such that } cx^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0\}.$$

An ideal $I \subseteq R$ is called tightly closed if $I = I^*$. In general, tight closure is always contained in the integral closure, that is, $I^* \subseteq \bar{I}$ (see [15, Proposition on Page 58]). Similar to integral closure, an element $x \in R$ is in the tight closure of I if and only if the image of x in R/\mathfrak{p} is in the tight closure of $I(R/\mathfrak{p})$ for all minimal primes \mathfrak{p} , see [15, Theorem on page 49].

Let R be a Noetherian complete local domain and let $I \subseteq R$ be an ideal. The solid closure of I , denoted by I^\star , consists of those element $x \in R$ such that there exists an R -algebra S such that $\text{Hom}_R(S, R) \neq 0$ and such that $x \in IS$. One can define solid closure of ideals in more general rings, see [16, Definition 1.2], but we will only need this notion for complete local domains. It was shown in [16, Theorem 5.10] that solid closure is contained in the integral closure, i.e., $I^\star \subseteq \bar{I}$. If R has prime characteristic $p > 0$, then solid closure agrees with tight closure $I^\star = I^*$, see [16, Theorem 8.6].

2.1. Big Cohen-Macaulay algebras. Let (R, \mathfrak{m}) be a Noetherian local ring. An R -algebra B , not necessarily Noetherian, is called balanced big Cohen-Macaulay over R if every system of parameters of R is a regular sequence on B and $\mathfrak{m}B \neq B$. Balanced big Cohen-Macaulay algebras exist, in equal characteristic, this is due to Hochster-Huneke [18], and in mixed characteristic, this is proved by André [1] (see also [12, 2, 3]). In this article, we need to compare the closure operation induced by a balanced big Cohen-Macaulay algebra with integral closure. We begin with the following result.

In what follows, when $R \rightarrow S$ is a (not necessarily injective) homomorphism of rings, $IS \cap R$ should be interpreted as the contraction of IS to R . That is, those elements of R whose image in S are contained in IS .

Lemma 2.1. *Let (R, \mathfrak{m}) be a Noetherian local ring. Then the following conditions are equivalent:*

- (1) \widehat{R} is equidimensional.
- (2) There exists a balanced big Cohen-Macaulay R -algebra B such that

$$I^B := IB \cap R \subseteq \bar{I} \text{ for all } \mathfrak{m}\text{-primary ideals } I \subseteq R. \quad (\dagger)$$

- (3) There exists a balanced big Cohen-Macaulay R -algebra B such that $I^B \subseteq \bar{I}$ for all $I \subseteq R$.

Proof. Since (3) \Rightarrow (2) is obvious, we only need to show (1) \Rightarrow (3) and (2) \Rightarrow (1). Suppose \widehat{R} is equidimensional and let P_1, \dots, P_n be the minimal primes of \widehat{R} . Let B_i be any balanced big Cohen-Macaulay algebra over \widehat{R}/P_i . Since \widehat{R} is equidimensional, each system of parameters of \widehat{R} is also a system of parameters of \widehat{R}/P_i and thus B_i is a balanced big Cohen-Macaulay algebra over \widehat{R} . It follows that $B := \prod_{i=1}^n B_i$ is a balanced big Cohen-Macaulay algebra over \widehat{R} .

Claim 2.2. $(I\widehat{R})^B = IB \cap \widehat{R} \subseteq \overline{I\widehat{R}}$.

Proof of Claim. Since integral closure can be checked after modulo each minimal prime, it suffices to show that $(I\widehat{R})^B \cdot (\widehat{R}/P_i) \subseteq \overline{I(\widehat{R}/P_i)}$. It is easy to see (by our construction of B) that

$$(I\widehat{R})^B \cdot (\widehat{R}/P_i) = (I(\widehat{R}/P_i))^{B_i}.$$

Since B_i is a solid algebra over the complete local domain \widehat{R}/P_i by [16, Corollary 2.4], we have

$$(I(\widehat{R}/P_i))^{B_i} \subseteq (I(\widehat{R}/P_i))^\star \subseteq \overline{I(\widehat{R}/P_i)},$$

where the second inclusion follows from [16, Theorem 5.10]. \square

By the claim above, we have

$$I^B \subseteq (I\widehat{R})^B \cap R \subseteq \overline{I\widehat{R}} \cap R = \overline{I},$$

where the last equality follows from [21, Proposition 1.6.2].

We next assume there exists a balanced big Cohen-Macaulay R -algebra B that satisfies (\dagger) . We first note that \widehat{B} (the \mathfrak{m} -adic completion of B) is still a balanced big Cohen-Macaulay algebra over \widehat{R} by [4, Corollary 8.5.3]. If I is an \mathfrak{m} -primary ideal, then we have $R/I \cong \widehat{R}/I\widehat{R}$ and $B/IB \cong \widehat{B}/I\widehat{B}$ (see [29, Tag 05GG]). It follows that $(I\widehat{R})^{\widehat{B}} = (I^B)\widehat{R} \subseteq \overline{I\widehat{R}} = \overline{I\widehat{R}}$ (where the last equality follows from [21, Lemma 9.1.1]). Thus without loss of generality, we may replace R by \widehat{R} and B by \widehat{B} to assume R is complete. Suppose R is not equidimensional. Let P_1, \dots, P_n be all the minimal primes of R such that $\dim(R/P_i) = \dim(R)$, and Q_1, \dots, Q_m be all the minimal primes of R such that $\dim(R/Q_j) < d$. We pick $y \in Q_1 \cap \dots \cap Q_m \setminus P_1 \cup \dots \cup P_n$. Then y is a parameter element in R , and thus y is a nonzerodivisor on B , since B is balanced big Cohen-Macaulay. Since $y \cdot (P_1 \cap \dots \cap P_n) \subseteq \sqrt{0}$, there exists t such that $y^t \cdot (P_1 \cap \dots \cap P_n)^t = 0$. It follows that $(P_1 \cap \dots \cap P_n)^t B = 0$. Hence

$$(P_1 \cap \dots \cap P_n)^t \subseteq \mathfrak{m}^k B \cap R \subseteq \overline{\mathfrak{m}^k}$$

for all k by (\dagger) . Thus $(P_1 \cap \dots \cap P_n)^t \subseteq \cap_k \overline{\mathfrak{m}^k} = \sqrt{0}$ by [21, Exercise 5.14], which is a contradiction. \square

Remark 2.3. In the proof of Lemma 2.1, we have proved the fact that when (R, \mathfrak{m}) is a Noetherian complete local domain, then every balanced big Cohen-Macaulay algebra B satisfies (\dagger) . We suspect that when (R, \mathfrak{m}) is Noetherian, complete, reduced and equidimensional, then every balanced big Cohen-Macaulay algebra B such that $\text{Supp}(\widehat{B}) = \text{Spec}(R)$ satisfies (\dagger) .

2.2. Hilbert coefficients. Let (R, \mathfrak{m}) be a Noetherian local ring of dimension d and let $I \subseteq R$ be an \mathfrak{m} -primary ideal. Then for all $n \gg 0$ we have

$$\ell(R/I^{n+1}) = e_0(I) \binom{n+d}{d} - e_1(I) \binom{n+d-1}{d-1} + \dots + (-1)^d e_d(I),$$

where $e_0(I), \dots, e_d(I)$ are all integers, and are called the Hilbert coefficients of I .

Now suppose $R \oplus \overline{I}t \oplus \overline{I}^2 t^2 \oplus \dots$ is module-finite over the Rees algebra $R[It]$. For instance, by a famous result of Rees (see [21, Corollary 9.2.1]), this is the case when \widehat{R} is reduced. Then one can show that for all $n \gg 0$, $\ell(R/\overline{I}^{n+1})$ agrees with a polynomial in n and one can write

$$\ell(R/\overline{I}^{n+1}) = \bar{e}_0(I) \binom{n+d}{d} - \bar{e}_1(I) \binom{n+d-1}{d-1} + \dots + (-1)^d \bar{e}_d(I),$$

where the integers $\bar{e}_0(Q), \dots, \bar{e}_d(Q)$ are called the normal Hilbert coefficients. It is well-known that $e_0(I) = \bar{e}_0(I)$ agrees with the Hilbert-Samuel multiplicity $e(I, R)$ of I .

We also recall the tight Hilbert coefficients studied in [7]. Again, we suppose that \widehat{R} is reduced and R has characteristic $p > 0$. Then we have

$$\ell(R/(I^{n+1})^*) = e_0^*(I) \binom{n+d}{d} - e_1^*(I) \binom{n+d-1}{d-1} + \dots + (-1)^d e_d^*(I),$$

for all $n \gg 0$, and the integers $e_0^*(I), \dots, e_d^*(I)$ are called the tight Hilbert coefficients, see [7] for more details.

Now if B is a balanced big Cohen-Macaulay R -algebra that satisfies (\dagger) , then we know that $R \oplus I^B t \oplus (I^2)^B t^2 \oplus \dots$ is an R -algebra that is also module-finite over $R[It]$: the fact that it is an R -algebra follows from the fact that $(I^a)^B (I^b)^B \subseteq (I^{a+b})^B$ for all a, b (i.e., $\{(I^n)^B\}_n$ form a graded family of ideals), and that it is module-finite over $R[It]$ follows because by (\dagger) , it is an $R[It]$ -submodule of $R \oplus \bar{I}t \oplus \bar{I}^2 t^2 \oplus \dots$, and the latter is module-finite over $R[It]$ (note that $R[It]$ is Noetherian). Based on the discussion above, one can show that for all $n \gg 0$, $\ell(R/(I^{n+1})^B)$ also agrees with a polynomial in n , and we write

$$\ell(R/(I^{n+1})^B) = e_0^B(I) \binom{n+d}{d} - e_1^B(I) \binom{n+d-1}{d-1} + \dots + (-1)^d e_d^B(I),$$

for all $n \gg 0$ (see [19] for more general results). We call the integers $e_0^B(I), \dots, e_d^B(I)$ the BCM Hilbert coefficients with respect to B . It is easy to see that $e_0^B(I) = e(I, R)$ is still the Hilbert-Samuel multiplicity of I , and that we always have $\bar{e}_1(I) \geq e_1^B(I) \geq e_1(I)$ by comparing the coefficients of n^{d-1} and noting that $I^n \subseteq (I^n)^B \subseteq \bar{I}^n$ for all n by (\dagger) .

Remark 2.4. We point out that when (R, \mathfrak{m}) is excellent and \widehat{R} is reduced and equidimensional of characteristic $p > 0$, the tight Hilbert coefficient is a particular case of BCM Hilbert coefficient. This follows from the fact that under these assumptions, there exists a balanced big Cohen-Macaulay algebra B such that $I^* = I^B$ for all $I \subseteq R$ (and any such B will satisfy (\dagger) , since tight closure is contained in the integral closure [20, Theorem 1.3]). When R is a complete local domain this is proved in [15, Theorem on page 250]. In general, one can take such a B_i for each complete local domain \widehat{R}/P_i , where P_i is a minimal prime of \widehat{R} , and let $B = \prod B_i$. Since R is excellent, $I^* \widehat{R} = (I \widehat{R})^*$ (see [20, Proposition 1.5]) and as tight closure can be checked after modulo each minimal prime, it follows that $I^* \widehat{R} = (I \widehat{R})^B$ and thus $I^* = I^B$.

Throughout the rest of this article, we will be mainly working with parameter ideals, i.e., ideals generated by a system of parameters. As we mentioned in the introduction, this will not affect the study of $\bar{e}_1(I)$, since we can often enlarge the residue field and replace I by its minimal reduction.

3. THE MAIN RESULTS

In this section we prove our main results that $e_1^B(Q)$ (and hence $\bar{e}_1(Q)$) is always nonnegative for a parameter ideal Q , and that $\bar{e}_1(Q) = 0$ for some parameter ideal Q implies R is regular.

3.1. Non-negativity of $\bar{e}_1(Q)$ and $e_1^B(Q)$.

Theorem 3.1. *Let (R, \mathfrak{m}) be a Noetherian local ring such that \widehat{R} is reduced and equidimensional. Let B be any balanced big Cohen-Macaulay R -algebra that satisfies (\dagger) . Then for all parameter ideals $Q \subseteq R$ we have*

$$\bar{e}_1(Q) \geq e_1^B(Q) \geq 0 \geq e_1(Q).$$

Remark 3.2. $\bar{e}_1(Q) \geq 0$ was the main theorem of [10, Theorem 1.1], and $0 \geq e_1(Q)$ was first proved in full generality in [26, Theorem 3.6]. Our method gives alternative proofs, and is inspired by some work of Goto [9] (in fact the proof that $e_1(Q) \leq 0$ via this method is due to Goto [9], see also [13, Theorem 1.1] for a generalization).

Corollary 3.3. *Let (R, \mathfrak{m}) be an excellent local ring of characteristic $p > 0$ such that \widehat{R} is reduced and equidimensional. Then we have $e_1^*(Q) \geq 0$ for all parameter ideals $Q \subseteq R$.*

Proof. This follows from Theorem 3.1 and Remark 2.4. \square

Proof of Theorem 3.1. Let $Q = (x_1, \dots, x_d) \subseteq R$. Set $S = R[[y_1, \dots, y_d]]$ and $\mathfrak{q} = (y_1 - x_1, \dots, y_d - x_d) \subseteq S$. For all $n \geq 0$ we have y_1, \dots, y_d is a system of parameters on S/\mathfrak{q}^{n+1} , and that

$$R/Q^{n+1} = S/(\mathfrak{q}^{n+1} + (y_1, \dots, y_d)).$$

We next note that

$$\begin{aligned} e_0(Q) = e(Q, R) &= \chi(x_1, \dots, x_d; R) \\ &= \chi(x_1, \dots, x_d, y_1, \dots, y_d; S) \\ &= \chi(y_1, \dots, y_d, y_1 - x_1, \dots, y_d - x_d; S) \\ &= \chi(y_1, \dots, y_d; S/\mathfrak{q}) \\ &= e(y_1, \dots, y_d; S/\mathfrak{q}), \end{aligned}$$

where the equalities on the second and the fourth line follow from the fact that y_1, \dots, y_d and $y_1 - x_1, \dots, y_d - x_d$ are both regular sequences on S . Now since S/\mathfrak{q}^{n+1} has a filtration by $\binom{n+d}{d}$ copies of S/\mathfrak{q} , by the additivity formula for multiplicity (see [21, Theorem 11.2.3]) we have

$$e(y_1, \dots, y_d; S/\mathfrak{q}^{n+1}) = \binom{n+d}{d} e(y_1, \dots, y_d; S/\mathfrak{q}).$$

Putting these together, we have

$$\binom{n+d}{d} e_0(Q) = \binom{n+d}{d} e(y_1, \dots, y_d; S/\mathfrak{q}) = e(y_1, \dots, y_d; S/\mathfrak{q}^{n+1}),$$

and

$$\ell(R/Q^{n+1}) = \ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \dots, y_d)S/\mathfrak{q}^{n+1}}\right).$$

Since y_1, \dots, y_d is a system of parameters of S/\mathfrak{q}^{n+1} , we have

$$\ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \dots, y_d)S/\mathfrak{q}^{n+1}}\right) \geq e(y_1, \dots, y_d; S/\mathfrak{q}^{n+1}).$$

It follows that

$$\ell(R/Q^{n+1}) \geq \binom{n+d}{d} e_0(Q),$$

and thus $e_1(Q) \leq 0$ (note that this does not require any assumption on \widehat{R}).

It remains to show that $e_1^B(Q) \geq 0$ (since $\bar{e}_1(Q) \geq e_1^B(Q)$ always holds, see the discussion in Section 2.2). Since $e_0^B(Q) = e_0(Q)$, it is enough to show that

$$\ell(R/(Q^{n+1})^B) \leq \binom{n+d}{d} e_0(Q) \quad (1)$$

for any balanced big Cohen-Macaulay algebra B . Below we will prove a slightly stronger result. Recall that for a parameter ideal (z_1, \dots, z_d) of R , the limit closure is defined as $(z_1, \dots, z_d)^{\text{lim}_R} := \bigcup_t (z_1^{t+1}, \dots, z_d^{t+1}) : (z_1 z_2 \cdots z_d)^t$. The limit closure does not depend on the choice of the elements z_1, \dots, z_d (i.e., it only depends on the ideal (z_1, \dots, z_d)). This is because $(z_1, \dots, z_d)^{\text{lim}_R} / (z_1, \dots, z_d)$ is the kernel of the natural map $R/(z_1, \dots, z_d) \rightarrow H_m^d(R)$.

Claim 3.4. *Set $\Lambda_{n+1} = \{(\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d \mid \alpha_i \geq 1 \text{ and } \sum_{i=1}^d \alpha_i = 1 + n\}$ and for each $\alpha = (\alpha_1, \dots, \alpha_d) \in \Lambda_{n+1}$, set $Q(\alpha) = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$. Then we have*

$$\ell \left(R / \left(\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\text{lim}_R} \right) \right) \leq \binom{n+d}{d} e_0(Q).$$

Proof of Claim. Recall that we have already proved that

$$\binom{n+d}{d} e_0(Q) = e(y_1, \dots, y_d; S/\mathfrak{q}^{n+1}).$$

Moreover, we always have (for example, see [23, Theorem 9])

$$e(y_1, \dots, y_d; S/\mathfrak{q}^{n+1}) \geq \ell \left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \dots, y_d)^{\text{lim}_{S/\mathfrak{q}^{n+1}}}} \right).$$

Therefore it is enough to prove that

$$\ell \left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \dots, y_d)^{\text{lim}_{S/\mathfrak{q}^{n+1}}}} \right) \geq \ell \left(R / \left(\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\text{lim}_R} \right) \right). \quad (2)$$

Consider $z \in S$ whose image in S/\mathfrak{q}^{n+1} is contained in $(y_1, \dots, y_d)^{\text{lim}_{S/\mathfrak{q}^{n+1}}}$. This means there exists some $t \geq 1$ such that

$$\begin{aligned} (y_1 y_2 \cdots y_d)^t z &\in (y_1^{t+1}, \dots, y_d^{t+1}, (y_1 - x_1, \dots, y_d - x_d)^{n+1}) \\ &\subseteq (y_1^{t+1}, \dots, y_d^{t+1}, (y_1 - x_1)^{\alpha_1}, \dots, (y_d - x_d)^{\alpha_d}) \end{aligned}$$

for each $\alpha = (\alpha_1, \dots, \alpha_d) \in \Lambda_{n+1}$. This implies

$$z \in (y_1, \dots, y_d, (y_1 - x_1)^{\alpha_1}, \dots, (y_d - x_d)^{\alpha_d})^{\text{lim}_S} = (y_1, \dots, y_d, x_1^{\alpha_1}, \dots, x_d^{\alpha_d})^{\text{lim}_S}.$$

But since $S = R[[y_1, \dots, y_d]]$, it is straightforward to check that

$$(y_1, \dots, y_d, x_1^{\alpha_1}, \dots, x_d^{\alpha_d})^{\text{lim}_S} = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})^{\text{lim}_R} S + (y_1, \dots, y_d) S.$$

Thus if the image of z is contained in $(y_1, \dots, y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}$, then after modulo $(y_1, \dots, y_d)S$, $\bar{z} \in (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})^{\lim_R}$ for each $(\alpha_1, \dots, \alpha_d) \in \Lambda_{n+1}$, i.e., $\bar{z} \in \bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R}$. It follows that the natural surjection

$$S/\mathfrak{q}^{n+1} \xrightarrow{\text{mod } (y_1, \dots, y_d)S} R/Q^{n+1}$$

induces a surjection

$$\frac{S/\mathfrak{q}^{n+1}}{(y_1, \dots, y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}} \twoheadrightarrow \frac{R}{\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R}}.$$

This clearly establishes (2) and completes the proof of claim. \square

Finally, since x_1, \dots, x_d is a regular sequence on B , we have $Q(\alpha)^{\lim_R} \subseteq Q(\alpha)^B$ for each α . It follows that $\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R} \subseteq \bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^B$. Now if $x \in \bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^B$, then we have $x \in (\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)B) \cap R$. But since x_1, \dots, x_d is a regular sequence on B , it is not hard to check that $\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)B = Q^{n+1}B$ (see [28, Remark 3.3] or [11]) and thus $x \in Q^{n+1}B \cap R = (Q^{n+1})^B$. Therefore we have $\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^B = (Q^{n+1})^B$. Putting these together, we have

$$\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R} \subseteq \bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^B = (Q^{n+1})^B.$$

Therefore by Claim 3.4, we have

$$\ell(R/(Q^{n+1})^B) \leq \binom{n+d}{d} e_0(Q)$$

as wanted. \square

Remark 3.5. With notation as in Theorem 3.1, we do not know whether we have

$$\ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \dots, y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}}\right) = \ell\left(\frac{R}{\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R}}\right).$$

Remark 3.6. With notation as in Claim 3.4, fix a generating set (x_1, \dots, x_d) of Q , one may try to define $(Q^n)^{\lim} := \bigcap_{\alpha \in \Lambda_n} Q(\alpha)^{\lim}$ and call this the limit closure of Q^n . However, it is not clear to us whether this is independent of the choice of the generators x_1, \dots, x_d . It is also not clear to us (even when fixing the generators (x_1, \dots, x_d) of Q) whether $\{(Q^n)^{\lim}\}_n$ form a graded family of ideals, i.e., we do not know whether $(Q^a)^{\lim}(Q^b)^{\lim} \subseteq (Q^{a+b})^{\lim}$ for all a, b .

3.2. Vanishing of $\bar{e}_1(Q)$. In this subsection we prove our main result. Recall that for a finitely generated R -module M , we use the notation $\nu(M)$ to denote its minimal number of generators.

Theorem 3.7. *Let (R, \mathfrak{m}) be a Noetherian local ring such that \widehat{R} is reduced and S_2 . If $\bar{e}_1(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is regular and $\nu(\mathfrak{m}/Q) \leq 1$.*

Proof. We first note that if R is Cohen-Macaulay, then by [19, Corollary 4.9], Q is integrally closed.¹ But then by the main result of [8], R is regular and $\nu(\mathfrak{m}/Q) \leq 1$.

¹Using the language of [19], $\bar{e}_1(Q) = 0$ in a Cohen-Macaulay ring implies that the reduction number of the filtration $\{\overline{Q^n}\}_n$ is 0, i.e., a minimal reduction of \overline{Q} is equal to \overline{Q} , this is saying that Q is integrally closed.

We may assume that R is complete. We use induction on $d := \dim(R)$. If $d \leq 2$, then R is Cohen-Macaulay and we are done by the previous paragraph. Now suppose $d \geq 3$ and we have established the theorem in dimension $< d$. Let $Q = (x_1, \dots, x_d)$, $R' = R[t_1, \dots, t_d]_{\mathfrak{m}_{R[t_1, \dots, t_d]}}$, and $x = t_1x_1 + \dots + t_dx_d$.

Claim 3.8. *We have $R'' := \widehat{R'}/x\widehat{R'}$ is reduced, equidimensional, and S_2 on the punctured spectrum. Moreover, we have $\bar{e}_1(QR'') = 0$.*

Proof. This is essentially contained in [10, Proof of Theorem 1.1] under the assumption that R is (complete and) normal. The key ingredient is [22, Theorem 2.1]. Since [22] does not require the normal assumption, the same proof as in [10] works in our setup. For the ease of the reader (and also because the S_2 on the punctured spectrum conclusion is not stated in [10]), we give a complete and self-contained argument here.

First of all, since R' is S_2 and R_0 , we know that R'/xR' is S_1 and R_0 (see [25, Lemma 10]), so R'/xR' and thus R'' is reduced (as R'/xR' is excellent). R'' is clearly equidimensional since $\widehat{R'}$ is so and x is a parameter in $\widehat{R'}$. To see R'' is S_2 on the punctured spectrum, it is enough to show R'/xR' is S_2 on the punctured spectrum (as R'/xR' is excellent). Now we use a similar argument as in [25, Lemma 10] (the idea follows from [14]): every non-maximal $P' \in \text{Spec}(R'/xR')$ corresponds to a prime ideal of R' that contracts to a non-maximal $P \in \text{Spec}(R)$, thus $(R'/xR')_{P'}$ is a localization of $R_P[t_1, \dots, t_d]/(t_1x_1 + \dots + t_dx_d)$, but at least one x_i is invertible in R_P (say x_1 is invertible) so the latter is isomorphic to $R_P[t_2, \dots, t_d]$, which is S_2 as R_P is S_2 , thus R'/xR' is S_2 on the punctured spectrum as wanted.

It remains to show that $\bar{e}_1(QR'') = 0$. By [21, Corollary 6.8.13], we have a short exact sequence

$$0 \rightarrow R'/\overline{Q^n} \xrightarrow{x} R'/\overline{Q^{n+1}} \rightarrow R'/(x, \overline{Q^{n+1}}) \rightarrow 0.$$

Since $\bar{e}_1(Q) = 0$, for $n \gg 0$ we have

$$\begin{aligned} \ell(R'/\overline{Q^{n+1}}) &= \bar{e}_0(Q) \cdot \binom{n+d}{d} + \bar{e}_2(Q) \cdot \binom{n+d-2}{d-2} + o(n^{d-2}), \\ \ell(R'/\overline{Q^n}) &= \bar{e}_0(Q) \cdot \binom{n+d-1}{d} + \bar{e}_2(Q) \cdot \binom{n+d-3}{d-2} + o(n^{d-2}). \end{aligned}$$

It follows that

$$\ell(R'/(x, \overline{Q^{n+1}})) = \bar{e}_0(Q) \cdot \binom{n+d-1}{d-1} + o(n^{d-2}). \quad (3)$$

We next show that for all $n \gg 0$, $\overline{Q^n}(R'/xR') = \overline{Q^n}(R'/xR')$. Once this is proved, we will have $\overline{Q^n}R'' = \overline{Q^n}R''$ for all $n \gg 0$ by [21, Lemma 9.1.1] and thus (3) will tell us that

$$\ell(R''/\overline{Q^{n+1}}R'') = \bar{e}_0(Q) \binom{n+d-1}{d-1} + o(n^{d-2}).$$

Since x is a general element of Q , we have $\bar{e}_0(Q) = e(Q, R') = e(QR'', R'') = \bar{e}_0(QR'')$ and so the above equation implies that $\bar{e}_1(QR'') = 0$ as wanted.

To show $\overline{Q^n}(R'/xR') = \overline{Q^n(R'/xR')}$ for $n \gg 0$, let \mathcal{R}' denote the integral closure of $R'[Qt, t^{-1}]$ inside $R'[t, t^{-1}]$. Concretely, \mathcal{R}' is the \mathbb{Z} -graded ring such that $\mathcal{R}'_n = \overline{Q^n t^n}$ for $n > 0$ and $\mathcal{R}'_n = R' t^n$ for $n \leq 0$. Consider the map

$$\mathcal{R}'/(xt)\mathcal{R}' \rightarrow R'[t, t^{-1}]/(xt)R'[t, t^{-1}].$$

If we localize at any prime ideal \mathcal{P} of $R'[Qt, t^{-1}]$ that does not contain (Qt, t^{-1}) , then we note that $(\mathcal{R}'/(xt)\mathcal{R}')_{\mathcal{P}}$ is integrally closed inside $(R'[t, t^{-1}]/(xt)R'[t, t^{-1}])_{\mathcal{P}}$. To see this, one can “unlocalize” the ring R' , and consider the integral closure of $R[t_1, \dots, t_d][Qt, t^{-1}]$ inside $R[t_1, \dots, t_d][t, t^{-1}]$, call this ring \mathcal{R} . If one localizes the map $\mathcal{R}/(xt)\mathcal{R} \rightarrow R[t_1, \dots, t_d][t, t^{-1}]/(xt)R[t_1, \dots, t_d][t, t^{-1}]$ at any prime ideal that does not contain (Qt, t^{-1}) (say it does not contain $x_1 t$), then the resulting map is a localization of $R[t_2, \dots, t_d][\overline{Q^n t^n}, t^{-1}]_{[\frac{1}{x_1 t}]} \rightarrow R[t_2, \dots, t_d][t, t^{-1}]_{[\frac{1}{x_1 t}]}$, and the former is already integrally closed in the latter.

Since the radical of (Qt, t^{-1}) is the unique homogeneous maximal ideal of $R[Qt, t^{-1}]$, it follows that $\mathcal{R}'/(xt)\mathcal{R}'$ and the integral closure of $R'[Qt, t^{-1}]/(xt)R'[Qt, t^{-1}]$ inside $R'[t, t^{-1}]/(xt)R'[t, t^{-1}]$ agree in large degree. But note that for $n > 0$,

$$[\mathcal{R}'/(xt)\mathcal{R}']_n \cong \frac{\overline{Q^n}}{x\overline{Q^{n-1}}} \cdot t^n \cong \frac{\overline{Q^n}}{x(\overline{Q^n : x})} \cdot t^n \cong \frac{\overline{Q^n}}{(xR') \cap \overline{Q^n}} \cdot t^n \cong \overline{Q^n}(R'/xR') \cdot t^n,$$

where we have used [21, Corollary 6.8.13] again, while the degree n part of the integral closure of $R'[Qt, t^{-1}]/(xt)R'[Qt, t^{-1}]$ inside $R'[t, t^{-1}]/(xt)R'[t, t^{-1}]$ is $\overline{Q^n(R'/xR')} \cdot t^n$. Thus the fact that they agree in degree $n \gg 0$ is precisely saying that $\overline{Q^n}(R'/xR') = \overline{Q^n(R'/xR')}$ for $n \gg 0$. \square

Now we come back to the proof of the theorem. Let S be the S_2 -ification of R'' . We have a short exact sequence

$$0 \rightarrow R'' \rightarrow S \rightarrow S/R'' \rightarrow 0$$

such that S/R'' has finite length (since R'' is S_2 on the punctured spectrum). Also note that (S, \mathfrak{n}) is (complete) local by [17, Proposition (3.9)] and that S is reduced (since S is a subring of the total quotient ring of R''). Since $R'' \rightarrow S$ is an integral extension, we have $\overline{IS} \cap R'' = \overline{I}$ for every ideal $I \subseteq R''$ by [21, Proposition 1.6.1]. It follows that $\ell_{R''}(S/\overline{Q^n S}) \geq \ell_{R''}(R''/\overline{Q^n R''})$ for all $n \geq 0$. Thus for $n \gg 0$ we have

$$\begin{aligned} & \bar{e}_0(QR'') \binom{n+d}{d} - \bar{e}_1(QR'') \binom{n+d-1}{d-1} + o(n^{d-1}) \\ &= \ell_{R''}(R''/\overline{Q^{n+1}R''}) \\ &\leq \ell_{R''}(S/\overline{Q^{n+1}S}) \\ &= [S/\mathfrak{n} : R/\mathfrak{m}] \cdot \ell_S(S/\overline{Q^{n+1}S}) \\ &= [S/\mathfrak{n} : R/\mathfrak{m}] \cdot \left(\bar{e}_0(QS) \binom{n+d}{d} - \bar{e}_1(QS) \binom{n+d-1}{d-1} + o(n^{d-1}) \right). \end{aligned}$$

Since S is a rank one module over R'' , we also know that

$$\bar{e}_0(QR'') = e(QR'', R'') = [S/\mathfrak{n} : R/\mathfrak{m}] \cdot e(QS, S) = [S/\mathfrak{n} : R/\mathfrak{m}] \cdot \bar{e}_0(QS),$$

where the second equality is the projection formula for Hilbert-Samuel multiplicity (which can be seen by combining [21, Theorem 11.2.4 and Theorem 11.2.7]). Putting these together we have

$$[S/\mathfrak{n} : R/\mathfrak{m}] \cdot \bar{e}_1(QS) \leq \bar{e}_1(QR'') = 0.$$

But since $\bar{e}_1(QS) \geq 0$ by [10, Theorem 1.1] (see Theorem 3.1), we must have $\bar{e}_1(QS) = 0$. Now (S, \mathfrak{n}) is a reduced complete local ring that is S_2 and $\dim(S) = d - 1$, such that $\bar{e}_1(QS) = 0$. By our inductive hypothesis, we know that S is regular. But since S/R'' has finite length, by the long exact sequence of local cohomology induced by $0 \rightarrow R'' \rightarrow S \rightarrow S/R'' \rightarrow 0$, we obtain that

$$H_{\mathfrak{m}}^i(R'') = 0 \text{ for all } i < \dim(R'') \text{ and } i \neq 1, \text{ and } H_{\mathfrak{m}}^1(R'') \cong S/R''.$$

At this point, we consider the long exact sequence of local cohomology induced by $0 \rightarrow \widehat{R}' \xrightarrow{x} \widehat{R}' \rightarrow R'' \rightarrow 0$, we get

$$0 = H_{\mathfrak{m}}^1(R') \rightarrow H_{\mathfrak{m}}^1(R'') \rightarrow H_{\mathfrak{m}}^2(R') \xrightarrow{x} H_{\mathfrak{m}}^2(R') \rightarrow H_{\mathfrak{m}}^2(R'') \rightarrow \cdots.$$

If $d \geq 4$, then $\dim(R'') \geq 3$ and thus $H_{\mathfrak{m}}^2(R'') = 0$. Since \widehat{R}' is S_2 , $H_{\mathfrak{m}}^2(R')$ has finite length and the above exact sequence tells us that $H_{\mathfrak{m}}^2(R') = 0$ by Nakayama's lemma. But then by the above exact sequence again, we have $H_{\mathfrak{m}}^1(R'') = 0$ and hence $S/R'' = 0$. Thus $R'' \cong S$ is regular. But then \widehat{R}' and hence R is regular as wanted.

Finally, suppose $d = 3$. Let B be a balanced big Cohen-Macaulay algebra of \widehat{R}' that is \mathfrak{m} -adic complete, then B/xB is a balanced big Cohen-Macaulay algebra of R'' . It follows that the canonical map $R'' \rightarrow B/xB$ factors through S .

Claim 3.9. *B/xB is a balanced big Cohen-Macaulay algebra over S .*

Proof of Claim. It is clear that some system of parameters of S (namely those coming from R'') are regular sequences on B/xB . To see that every system of parameters of S is a regular sequence on B/xB , we first note that B/xB is \mathfrak{m} -adically complete: since B is \mathfrak{m} -adic complete, B/xB is derived \mathfrak{m} -complete by [29, Tag 091U], take (y, z) that is a system of parameters of R'' , then as y, z is a regular sequence on B/xB , the derived completion with respect to (y, z) , which is B/xB itself, agrees with the usual completion with respect to (y, z) by [29, Tag 0920] (equivalently, with respect to \mathfrak{m} as $\sqrt{(y, z)} = \mathfrak{m}$). Hence by [4, Corollary 8.5.3], every system of parameters of S is a regular sequence on $\widehat{B/xB} \cong B/xB$. \square

Note that $\dim(R'') = \dim(S) = 2$ and S is regular, thus the long exact sequence of local cohomology induced by $0 \rightarrow R'' \rightarrow S \rightarrow S/R'' \rightarrow 0$ implies that $H_{\mathfrak{m}}^2(R'') \cong H_{\mathfrak{m}}^2(S)$. Hence we have

the following commutative algebra:

$$\begin{array}{ccccccccc}
& & & & H_{\mathfrak{m}}^2(S) & & & & \\
& & & & \parallel & & & & \\
H_{\mathfrak{m}}^2(R') & \xrightarrow{\cdot x} & H_{\mathfrak{m}}^2(R') & \longrightarrow & H_{\mathfrak{m}}^2(R'') & \longrightarrow & H_{\mathfrak{m}}^3(R') & \longrightarrow & H_{\mathfrak{m}}^3(R') & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \phi & & \downarrow & & \downarrow & & \\
& & 0 = H_{\mathfrak{m}}^2(B) & \longrightarrow & H_{\mathfrak{m}}^2(B/xB) & \longrightarrow & H_{\mathfrak{m}}^3(B) & \longrightarrow & H_{\mathfrak{m}}^3(B) & \longrightarrow & 0
\end{array}$$

where the injectivity of ϕ follows from the fact that B/xB is a balanced big Cohen-Macaulay algebra over S and thus faithfully flat over S (as S is regular). Chasing this diagram we find that the map $H_{\mathfrak{m}}^2(R') \xrightarrow{\cdot x} H_{\mathfrak{m}}^2(R')$ is surjective. But since $\widehat{R'}$ is S_2 , $H_{\mathfrak{m}}^2(R')$ has finite length, thus $H_{\mathfrak{m}}^2(R') = 0$ by Nakayama's lemma. Hence $\widehat{R'}$ is Cohen-Macaulay and thus R'' is also Cohen-Macaulay. But then $R'' \cong S$ and so R'' is regular and thus $\widehat{R'}$ is regular. Thus R is regular as wanted.

Now we have established that R is regular, we can repeat the argument in the first paragraph of the proof to show that $\nu(\mathfrak{m}/Q) \leq 1$ (essentially, this follows from the main result of [8]). \square

As a consequence, we answer the problem raised in [10, Section 3] for excellent rings.

Corollary 3.10. *Let R be an excellent local ring such that \widehat{R} is reduced and equidimensional. Suppose $I \subseteq R$ is an \mathfrak{m} -primary ideal such that $\bar{e}_1(I) = 0$. Then $R^{\mathbb{N}}$, the normalization of R , is regular and $IR^{\mathbb{N}}$ is normal (i.e., all powers of $IR^{\mathbb{N}}$ are integrally closed in $R^{\mathbb{N}}$).*

Proof. Replacing R by $R[t]_{\mathfrak{m}R[t]}$, we may assume that the residue field of R is infinite (we leave it to the readers to check that the hypotheses and conclusions are stable under such a base change). Let S be the S_2 -ification of R . We will show that the \mathfrak{m} -adic completion of \widehat{S} is regular. Since R is excellent, \widehat{S} agrees with the S_2 -ification of \widehat{R} by [17, Proposition 3.8]. Thus \widehat{S} is semilocal, reduced, and S_2 . Since $\overline{J\widehat{S}} \cap \widehat{R} = \overline{J}$ for every \mathfrak{m} -primary ideal $J \subseteq \widehat{R}$ by [21, Proposition 1.6.1], we have $\ell_{\widehat{R}}(\widehat{R}/\overline{J}) \leq \ell_{\widehat{R}}(\widehat{S}/\overline{J\widehat{S}})$.

Let $\mathfrak{n}_1, \dots, \mathfrak{n}_s$ be the maximal ideals of \widehat{S} and let $S_i := (\widehat{S})_{\mathfrak{n}_i}$ (in fact, since \widehat{S} is complete, we have $\widehat{S} \cong \prod_{i=1}^s S_i$, and each S_i is complete local, reduced, and S_2). Then we have

$$\begin{aligned}
& \bar{e}_0(I) \binom{n+d}{d} - \bar{e}_1(I) \binom{n+d-1}{d-1} + o(n^{d-1}) \\
&= \ell_R(R/\overline{I^{n+1}}) = \ell_{\widehat{R}}(\widehat{R}/\overline{I^{n+1}\widehat{R}}) \\
&\leq \ell_{\widehat{R}}(\widehat{S}/\overline{I^{n+1}\widehat{S}}) \\
&= \sum_{i=1}^s [S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \ell_{S_i}(S_i/\overline{I^{n+1}S_i}) \\
&= \sum_{i=1}^s [S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \left(\bar{e}_0(IS_i) \binom{n+d}{d} - \bar{e}_1(IS_i) \binom{n+d-1}{d-1} + o(n^{d-1}) \right),
\end{aligned}$$

where we have used [21, Lemma 9.1.1] for the equality in the second line. Since \widehat{S} is a rank one module over \widehat{R} , we also know that

$$\bar{e}_0(I) = e(I\widehat{R}, \widehat{R}) = \sum_{i=1}^s [S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot e(IS_i, S_i) = \sum_{i=1}^s [S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \bar{e}_0(IS_i),$$

where we have used the projection formula for the Hilbert-Samuel multiplicity (see [21, Theorem 11.2.4 and Theorem 11.2.7]). The above inequality implies that

$$\sum_{i=1}^s [S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \bar{e}_1(IS_i) \leq \bar{e}_1(I) = 0.$$

But since $\bar{e}_1(IS_i) \geq 0$ by [10, Theorem 1.1], we must have $\bar{e}_1(IS_i) = 0$ for all i . Let Q be a minimal reduction of I (note that Q is a parameter ideal of R , since we have reduced to the case that R has an infinite residue field). It follows that $\bar{e}_1(QS_i) = 0$ and thus by Theorem 3.7, S_i is regular and $\nu(\mathfrak{n}_i/Q) \leq 1$. But then QS_i is normal in S_i . It follows that $\widehat{S} \cong \prod_{i=1}^s S_i$ is regular, $Q\widehat{S}$ is normal in \widehat{S} and in particular, $Q\widehat{S} = I\widehat{S}$.

Since $S \rightarrow \widehat{S} \cong \widehat{R} \otimes_R S$ is faithfully flat with geometrically regular fibers (as R is excellent). We have S is regular and $QS = IS$ is normal in S by [21, Theorem 19.2.1]. Finally, since S is regular, S agrees with the normalization R^N of R . \square

Remark 3.11. The condition \widehat{R} is S_2 cannot be dropped in Theorem 3.7. This was already observed in [10, Section 3]. We give a different example that is a complete local domain. Let $R = k[[x, xy, y^2, y^3]]$ where k is a field. Then the S_2 -ification of R is $S = k[[x, y]]$ and we have $0 \rightarrow R \rightarrow S \rightarrow S/R \cong k \cdot \bar{y} \rightarrow 0$. Let $Q = (x, y^2) \subseteq R$ and we claim that $\bar{e}_1(Q) = 0$. To see this, note that $QS = (x, y^2) \subseteq S$ is normal and $\ell(S/Q^{n+1}S) = 2 \cdot \binom{n+2}{2}$. It follows from the short exact sequence

$$0 \rightarrow R/\overline{Q^{n+1}} \rightarrow S/Q^{n+1}S \rightarrow k \rightarrow 0$$

that $\ell(R/\overline{Q^{n+1}}) = 2 \cdot \binom{n+2}{2} - 1$. In particular, $\bar{e}_1(Q) = 0$.

Recall that a Noetherian local ring (R, \mathfrak{m}) of prime characteristic $p > 0$ is called F -rational if every ideal generated by a system of parameters is tightly closed. It was mentioned in [5] that Huneke asked that when \widehat{R} is reduced and equidimensional of prime characteristic $p > 0$, whether $e_1^*(Q) = 0$ for some system of parameters $Q \subseteq R$ implies R is F -rational. In general, counter-examples to the question were constructed in [5, Example 5.4 and 5.5] (in fact, the example in Remark 3.11 is a counter-example that is a complete local domain). However, all these examples do not satisfy Serre's S_2 condition.

Let (R, \mathfrak{m}) be a Noetherian local ring and let B be a big Cohen-Macaulay R -algebra. Recall that R is called BCM_B -rational if R is Cohen-Macaulay and the natural map $H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(B)$ is injective, where $d = \dim(R)$. If R is an excellent local ring of prime characteristic $p > 0$, then R is F -rational if and only if R is BCM_B -rational for all big Cohen-Macaulay algebra B , see [24, Proposition 3.5].

We propose the following conjecture relating the vanishing of $e_1^B(Q)$ and BCM_B -rational singularities, which modifies Huneke's question and makes sense in all characteristics.

Conjecture 3.12. *Let (R, \mathfrak{m}) be a Noetherian local ring such that \widehat{R} is reduced and S_2 . Let B be a balanced big Cohen-Macaulay R -algebra that satisfies (\dagger) . If $e_1^B(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is BCM_B -rational.*

In particular, if R is excellent and has characteristic $p > 0$ (such that \widehat{R} is reduced and S_2), and $e_1^(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is F -rational.*

We have the following partial result towards the Conjecture 3.12, which is an analog of the main result of [27].

Proposition 3.13. *Let (R, \mathfrak{m}) be a Noetherian local ring such that \widehat{R} is reduced and equidimensional. Let B be a balanced big Cohen-Macaulay R -algebra that satisfies (\dagger) . If $e_1^B(Q) = e_1(Q)$ for some parameter ideal $Q \subseteq R$, then R is BCM_B -rational.*

In particular, if R is excellent and has characteristic $p > 0$, and $e_1^(Q) = e_1(Q)$ for some parameter ideal $Q \subseteq R$, then R is F -rational.*

Proof. By Theorem 3.1, we know that $e_1^B(Q) = e_1(Q) = 0$. By the main result of [6], $e_1(Q) = 0$ implies that R is Cohen-Macaulay. By [19, Corollary 4.9], we have $Q^B = Q$. Now we consider the commutative diagram:

$$\begin{array}{ccc} R/Q & \hookrightarrow & B/QB \\ \downarrow & & \downarrow \\ H_{\mathfrak{m}}^d(R) & \longrightarrow & H_{\mathfrak{m}}^d(B) \end{array}$$

where the injectivity of the top row follows from $Q^B = Q$, the injectivity of the left column is because R is Cohen-Macaulay, and the injectivity of the right column is because B is balanced big Cohen-Macaulay. Since R is Cohen-Macaulay, we know that $\text{Soc}(R/Q) \cong \text{Soc}(H_{\mathfrak{m}}^d(R))$. Chasing the commutative diagram we find that $H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(B)$ injective. Therefore R is BCM_B -rational. \square

Remark 3.14. It is clear from the proof of Proposition 3.13 that Conjecture 3.12 holds when R is Cohen-Macaulay, and this essentially follows from [19, Corollary 4.9].

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