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Linear Singular Continuous Time-varying Delay Equations: Stability and Filtering via LMI Approach

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Abstract
In this paper, we propose an LMI-based approach to study stability and H_∞ filtering for linear singular continuous equations with time-varying delay. Particularly, the delay pattern is quite general and includes non-differentiable time-varying delay. First, new delay-dependent sufficient conditions for the admissibility of the equation are extended to the time-varying delay case. Then, we propose a design of H_∞ filters via feasibility problem involving linear matrix inequalities, which can be solved by the standard numerical algorithm. The proposed result is demonstrated through an example and simulations.

Keywords Stability · Singularity · Filters · Time-varying delay · Linear matrix inequalities

Mathematics Subject Classification (2010) 34D10 · 93D20 · 49M7

1 Introduction

Consider the following linear singular differential equations (LSDEs) with time-varying delay

$$\begin{cases} E\dot{y}(t) = Ay(t) + A_\tau y(t-\tau(t)), \quad t \ge 0, \\ y(t) = \xi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1}$$

where $y(t) \in R^n, E \in R^{n\times n}$ is singular: rank $E = r < n$; $A, A_\tau \in R^{n\times n}$, $\xi(t) \in C([-\tau, 0], R^n)$, $\tau(t)$ is continuous and satisfies $0 \le \tau(t) \le \tau,\ t \ge 0$.

Over the past decades, considerable attention has been devoted to state estimation problem such as Kalman and H_∞ filtering due to its various applications in systems and control area [3, 15]. The Kalman filtering gives an optimal estimation of the state error variables,

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however, a main disadvantage of the Kalman filtering is that the statistical information of the external disturbance noise on the system must be known. To overcome this disadvantage, an estimation technique based on H_∞ filtering approach has been used in [8, 10]. It is notable that an advantage of the H_∞ filtering is that one does not need to exactly know the statistical features of the external disturbance noise, we only require the boundedness of the noise. The H_∞ filtering problem considered in this paper is to design a filter guaranteeing stability of the filtering error singular system with a maximum H_∞ performance. In the last few decades, numerous mathematical and control approaches, including polynomial equation and interpolation approaches, Lyapunov function and LMI approaches have been proposed to solve the H_∞ filtering problem [2, 17, 18, 22].

With the growing complexity of dynamic systems, singular (or descriptor, implicit, differential-algebraic) equations have become popular research topics and widely studied, since the singular equations have many interesting applications in control and engineering field [5, 19]. Especially, study of singular delay equations (SDEs) becomes more and more difficult, because SDEs are coupled with delay differential and algebraic equations. In order to guarantee the existence of solutions, the proposed conditions should guarantee the equations not only to be stable but also to be regular and impulse free. There are two approaches have been used to investigate the stability of SDEs. The first approach is to decompose the system into differential and algebraic subequations, and the stability of the differential subequation is proved by using Lyapunov-Krasovskii function method [16, 19]. The second approach consists of constructing Lyapunov-Krasovskii functionals that corresponds directly to the descriptor form of the equation [7, 8]. In [8, 13, 20], using the first approach, the authors propose a delay-dependent H_∞ filtering design for system (1) with constant delays $\tau(t) = \tau$. The results on the H_∞ filtering were extended in [4, 21, 23] to linear singular equations (LSEs) with time-varying delay by using the second approach. However, the time-varying delay $\tau(t)$ considered in the aforementioned papers is assumed to be differentiable, which limit the scope of applications of the derived conditions. Moreover, from the existing results, we may conclude that to study stability of LSEs with time-varying delay $\tau(t)$, one needs to find appropriate Lyapunov-Krasovskii functionals, which are possible to apply the Lyapunov stability theorem. However, most of the existing results on this topic tackled only the case of constant delay ($\tau(t) = \tau$) or of the bounded differentiable delay ($\dot{\tau}(t) \le \delta$). In this paper, we show that by constructing properly augmented Lyapunov-Krasovskii functionals, we can obtain less conservative conditions for system (1) with more general time-varying delay. Namely, the system with non-differentiable, continuous and bounded delay ($0 \le \tau(t) \le \tau$). As far as we know, the H_∞ filtering problem of system (1) with non-differentiable time-varying delay has not been fully studied, which is very challenging and of great importance.

Based on the above discussion, we study stability and H_∞ filtering problem for LSEs with time-varying delay. This paper is our first attempt at exploring an LMI approach to the design of H_∞ filters for LSEs with time-varying delay. The novelty and contributions of this work are the following.

- Different from the existing results in the literature, the delay function was required to be differentiable or even its time derivative was assumed to be smaller than one. In our paper the time-varying delay appeared in both the observation and the disturbance inputs is only assumed to be continuous and bounded.
- Newly proposed technical results (Lemma 1, Lemma 2, Lemma 4, Lemma 5) are presented to develop and to extend the stability results for LSEs with time-varying delay.
- Novel criteria for H_∞ filtering design are proposed via solving tractable LMIs [6].
- Numerical examples and its simulations show the effectiveness of the theoretical results.

The remainder of this paper is arranged as follows. In Section 2, we introduce the problem to be treated and some auxiliary technical lemmas needed for the proof of the main results. In Section 3, the stability conditions and the H_∞ filter design are provided with an illustrated numerical example.

Notations. By $\mathbb{R}$ we denote the set of real numbers; $\mathbb{C}$ we denote the set of complex numbers; by R^+ and Z^+ we denote the set of nonnegative numbers and nonnegative integers, respectively; by R^n we denote the $n-$dimensional Euclidean space. $R^{n\times m}$ stands for the space of $n \times m$ matrices. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ stand for the maximal and minimal eigenvalues sets of A, respectively. $C([-\tau, 0], R^n)$ is the space of R^n- valued continuous functions on $[-\tau, 0]$. $\|x_t\|$ is the norm of $x(\cdot)$ on $[t-\tau, t]$ defined by $\|x_t\| = \sup_{s\in[-\tau,0]} \|x(t+s)\|$. $[M_{ij}]_{k\times k}$ is a $(k \times k)-$dimension symmetric matrix of elements $M_{ij}, i, j = 1, 2, \ldots, k$.

2 Preliminaries

In this section, we present some mathematical basic of singular systems and auxiliary technical lemmas to be used in the next section.

Definition 1 System (1) is said to be
(i) Regular if $\det(\alpha E - A)$, $\alpha \in \mathbb{C}$, is not identically zero,
(ii) Impulse-free if $\deg(\det(\alpha E - A)) = \operatorname{rank} E,\ \alpha \in \mathbb{C}$,
(iii) Asymptotically stable if it is stable and $\lim_{t\to\infty} \|y(t)\| = 0$,
(iv) Admissible if it is regular, impulse-free and asymptotically stable.

It is well known that the LSEs (1) may have an impulsive solution, however, if the equation is regular and impulse-free then its solution exists and is unique on $[0, \infty)$, which is shown in ([7, 9]).

The following lemma is slightly modified from [12, Lemma 3.4].

Lemma 1 *Let* $x \in C([-\tau, \infty), R^+)$ *and* $x(t) \le \beta\|x_t\| + N,\ t \ge c$, *where* $N > 0,\ 0 < \beta < 1,\ c \ge 0$. *Then*

$$x(t) < \beta\|x_c\| + \frac{N}{1-\beta}, \quad t \ge c.$$

Proof We have

$$x(c) \le \beta\|x_c\| + N < \gamma := \beta\|x_c\| + \frac{N}{1-\beta}.$$

Next, we will prove that $x(t) < \beta\|x_c\| + \frac{N}{1-\beta},\ \forall t \ge c$. Contrarily, if there is a real number $t^* \ge c$ such that

$$x(t^*) = \gamma,\ x(t) < \gamma,\ \forall t \in [c, t^*),$$

which implies that $\sup_{s\in[c,t^*]} x(s) = \gamma$.

From $t^* + \theta \in [c-\tau, c] \cup [c, t^*],\ \forall\theta \in [-\tau, 0]$, we have

$$\begin{aligned}\|x_{t^*}\| = \sup_{\theta\in[-\tau,0]} x(t^*+\theta) &\le \max\left\{\sup_{s\in[c-\tau,c]} x(s) \text{ and } \sup_{s\in[c,t^*]} x(s)\right\}\\ &\le \max\{\|x_c\| \text{ and } \gamma\}.\end{aligned}$$

Using the assumption again, we obtain

$$\gamma = x(t^*) \le \beta\|x_{t*}\| + N \le \beta\max\{\|x_c\| \text{ and } \gamma\} + N,$$

it follows that

$$\gamma \le \begin{cases} \beta\|x_c\| + N & \text{if } \|x_c\| \ge \gamma \\ \beta\gamma + N & \text{if } \|x_c\| \le \gamma \end{cases} < \gamma,$$

because $\beta\|x_c\| + N < \gamma$ and $\beta\gamma + N < \gamma$. This yields a contradiction. Hence,

$$x(t) < \beta\|x_c\| + \frac{N}{1-\beta}, \quad t \ge c.$$

The lemma is proved. □

Lemma 2 *Let* $a(.) \in C([-\tau, +\infty), R^+)$ *and* $b(.) : R^+ \to R^+$ *is a continuous and bounded function satisfying* $a(t) \le \alpha\|a_t\| + b(t), t \ge 0,$ *where* $\alpha \in (0, 1)$. *If* $\lim_{t\to\infty} b(t) = 0$, *then* $\lim_{t\to\infty} a(t) = 0$.

Proof From the assumption we have

$$a(t) \le \alpha\|a_t\| + \sup_{t\ge c} b(t), \quad t \ge c.$$

Using Lemma 1 we get

$$a(t) \le \alpha\|a_c\| + \frac{1}{1-\alpha}\sup_{t\ge c} b(t), \quad t \ge c.$$

Since the nonnegative function $a(t)$ is bounded, there is a sequence $\{t_k\}$

$$0 = t_0 < t_1 < t_2 < \cdots, \text{ and } t_{k+1} - t_k > \tau, \ \forall k = 1, 2, \ldots$$

and $\delta \ge 0$ such that $\limsup_{t\to\infty} \|a_t\| = \lim_{k\to\infty} \|a_{t_k}\| = \delta \ge 0$ and

$$\|a(t)\| \le \alpha\|a_{t_k}\| + \frac{1}{1-\alpha}\sup_{t\ge t_k} b(t), \quad t \ge t_k, \quad k = 1, 2, \ldots.$$

Since $t_{k+1} - t_k > \tau$, we have $t_{k+1} + s > t_k$, $s \in [-\tau, 0]$, and hence

$$\|a(t_{k+1} + s)\| \le \alpha\|a_{t_k}\| + \frac{1}{1-\alpha}\sup_{t\ge t_k} b(t), \quad s \in [-\tau, 0].$$

Consequently,

$$\|a_{t_{k+1}}\| \le \alpha\|a_{t_k}\| + \frac{1}{1-\alpha}\sup_{t\ge t_k} b(t), \quad k = 1, 2, \ldots.$$

Giving $k \to \infty$, $\lim_{k\to\infty} \sup_{t\ge t_k} b(t) = 0$, we have $\delta \le \alpha\delta$, such that $\delta = 0$ due to $\alpha < 1$. Thus, $\lim_{t\to\infty} a(t) = 0$. The lemma is proved. □

The following Barbalat's Lemma stated in [1] will be used.

Lemma 3 (Barbalat lemma [1]) *If* $f : R^+ \to \mathbb{R}$ *is uniformly continuous and* $\int_0^\infty f(s)ds < \infty$, *then* $\lim_{t\to\infty} f(t) = 0$.

3 Stability

In this section, we provide sufficient conditions for regularity, impulse-free property and asymptotical stability of system (1).

From matrix theory, we can find two invertible matrices H_1, H_2 satisfying $\mathbb{E} = H_1 E H_2 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ such that the system (1) under transformation $u(t) = H_2^{-1} y(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$, $u_1(t) \in R^r, u_2(t) \in R^{n-r}$ is formulated in the form

$$\mathbb{E}\dot{u}(t) = \mathbb{A}u(t) + \mathbb{A}_\tau u(t - \tau(t)), \tag{2}$$

where

$$\mathbb{A} = H_1 A H_2 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \ \mathbb{A}_\tau = H_1 A_\tau H_2 = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

System (2) is reduced to the following differential-algebraic equations written by

$$\begin{cases} \dot{u}_1(t) & = A_{11}u_1(t) + A_{12}u_2(t) + D_{11}u_1(t - \tau(t)) + D_{12}u_2(t - \tau(t)), \\ 0 & = A_{21}u_1(t) + A_{22}u_2(t) + D_{21}u_1(t - \tau(t)) + D_{22}u_2(t - \tau(t)), \end{cases} \tag{3}$$

with the initial conditions $u(t) = H_2^{-1}\xi(t) := \phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}$, $t \in [-\tau, 0]$.

Lemma 4 below extends a result of [19] to the time-varying delay case.

Lemma 4 *System (1) is regular, impulse-free if there exist a nonsingular matrix P satisfying $E^\top P^\top = PE \geq 0$, a symmetric matrix $Q > 0$ and a matrix R such that the following LMI holds*

$$\begin{pmatrix} A^\top P^\top + PA + Q + RE + (RE)^\top & PA_\tau \\ * & -Q \end{pmatrix} < 0. \tag{4}$$

Moreover, $\|A_{22}^{-1} D_{22}\| < 1$, where A_{22}, D_{22} are defined in the algebraic equation of (3).

Proof Let

$$\hat{P} = H_2^T P H_1^{-1} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \ \hat{Q} = H_2^T Q H_2 = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

Note that, from the assumption it follows that $\hat{P}\mathbb{E} = \mathbb{E}^T \hat{P}^T \geq 0$, $P_{21} = 0$, $P_{11} > 0$, and hence $\hat{P} = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$. Moreover, since $H_2^T(PA + A^T P^T)H_2 = \hat{P}\mathbb{A} + \mathbb{A}^T \hat{P}^T$, left and right-multiplying LMI (4) by $\mathrm{diag}(H_2, H_2)^T$ and $\mathrm{diag}(H_2, H_2)$, respectively gives

$$\begin{pmatrix} \hat{P}\mathbb{A} + \mathbb{A}^T \hat{P}^T + \hat{Q} + H_2^T REH_2 + [H_2^T REH_2]^T & \hat{P}\mathbb{A}_\tau \\ * & -\hat{Q} \end{pmatrix} < 0. \tag{5}$$

Since

$$H_2^T REH_2 = H_2^T RH_1^{-1}\mathbb{E} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix},$$

$$H_2^T PA_\tau H_2 = H_2 P H_1^{-1} H_1 A_\tau H_2 = \hat{P}\mathbb{A}_\tau = \begin{pmatrix} * & * \\ * & P_{22}D_{22} \end{pmatrix},$$

$$H_2^T PAH_2 = H_2 P H_1^{-1} H_1 A H_2 = \hat{P}\mathbb{A} = \begin{pmatrix} * & * \\ * & P_{22}A_{22} \end{pmatrix},$$

where the terms $*$ are not relevant and can be ignored. Left and right-multiplying LMI (5) by $\begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$ and its transpose gives

$$\begin{pmatrix} P_{22}A_{22} + A_{22}^T P_{22}^T + Q_{22} & P_{22}D_{22} \\ * & -Q_{22} \end{pmatrix} < 0, \tag{6}$$

which gives $P_{22}A_{22}+A_{22}^TP_{22}^T<0$, because of $Q_{22}>0$. We obtain matrix A_{22} is invertible, which shows the regularity and impulse-free (see, e.g., [5, 19]). Now left and right-multiplying LMI (6) by $[(-A_{22}^{-1}D_{22})^T,\ I]$ and its transpose, we have

$$
\begin{aligned}
0 >&[(-A_{22}^{-1}D_{22})^T,\ I]\begin{pmatrix} P_{22}A_{22}+A_{22}^TP_{22}^T+Q_{22} & P_{22}D_{22}\\ [P_{22}D_{22}]^T & -Q_{22}\end{pmatrix}\begin{bmatrix}(-A_{22}^{-1}D_{22})\\ I\end{bmatrix}\\
&=(-A_{22}^{-1}D_{22})^T\Big(P_{22}A_{22}+A_{22}^TP_{22}^T+Q_{22}\Big)(-A_{22}^{-1}D_{22})+(-A_{22}^{-1}D_{22})^TP_{22}D_{22}\\
&\quad+[P_{22}D_{22}]^T(-A_{22}^{-1}D_{22})-Q_{22}\\
&=(-A_{22}^{-1}D_{22})^TQ_{22}(-A_{22}^{-1}D_{22})-Q_{22},
\end{aligned}
$$

which gives $\rho(A_{22}^{-1}D_{22})<1$, and hence

$$\|A_{22}^{-1}D_{22}\|<1. \tag{7}$$

The lemma is proved. □

For a function $\mathcal{V}(.): C([-\tau,0],R^n)\to R^+$ we define the derivative of $\mathcal{V}(.)$ (see, e.g., [7, 11]) by

$$\dot{\mathcal{V}}(\phi)=\lim\sup_{h\to 0^+}\frac{1}{h}[\mathcal{V}(x_{t+h}(t,\phi))-\mathcal{V}(\phi)].$$

The following lemma extends [7, Lemma 1] to the time-varying delay case.

Lemma 5 *Let* (1) *be regular, impulse-free and the condition* (7) *holds. Equation* (1) *is asymptotically stable if there are numbers* $\alpha_1>0,\alpha_2>0,\alpha_3>0$, *an absolutely continuous function* $\mathcal{V}(.): C([-\tau,0],R^n)\to R^+$ *such that*
(i) $\alpha_1|\phi_1(0)|^2\le\mathcal{V}(\phi)\le\alpha_2|\phi|^2$,
(ii) $\dot{\mathcal{V}}(\phi)\le-\alpha_3|\phi(0)|^2$.

Proof Using (i) and $\mathcal{V}(u_t)\le\mathcal{V}(u_0)$, where $u_0: C[-\tau,0]\to R^n$, $u_0(s)=\phi(s)$, $s\in[-\tau,\ 0]$, and

$$\|u_1(s)\|\le\|u(s)\|\le\|u_0\|=\sup_{s\in[-\tau,0]}\|u(s)\|,$$

we have

$$\alpha_1|u_1(t)|^2=\alpha_1|(u_t)_1(0)|^2\le\mathcal{V}(u_t)\le\mathcal{V}(u_0)\le\alpha_2|u_0|^2,\ t\ge 0.$$

Hence

$$\exists\beta_1>0:\quad \|u_1(t)\|\le\beta_1\|u_0\|,\quad t\in[-\tau,\infty). \tag{8}$$

Moreover, from the second equation of (3) it follows that

$$u_2(t)=-A_{22}^{-1}[A_{21}u_1(t)+D_{21}u_1(t-\tau(t))]-A_{22}^{-1}D_{22}u_2(t-\tau(t))$$

and hence

$$\|u_2(t)\|\le\|A_{22}^{-1}\|\|[A_{21}u_1(t)+D_{21}u_1(t-\tau(t))]\|+\|A_{22}^{-1}D_{22}\|\|u_2(t-\tau(t))\|.$$

Applying (8), there exists $\beta_2>0$ such that

$$\|A_{22}^{-1}\|\|[A_{21}u_1(t)+D_{21}u_1(t-\tau(t))]\|\le\beta_2\|u_0\|,\quad t\ge 0,$$

hence

$$\|u_2(t)\|\le\beta_2\|u_0\|+\eta\|u_2(t-\tau(t))\|,$$

where $\eta = \|A_{22}^{-1}D_{22}\| < 1$. Setting $x(t) = \|u_2(t)\|$, we have

$$x(t) \le \eta \|x_t(\cdot)\| + \beta_2\|u_0\|, \quad t \ge 0,$$

and using Lemma 1, we get

$$x(t) \le \eta\|x_0\| + \frac{\beta_2\|u_0\|}{1-\eta}, \quad t \ge 0,$$

consequently,

$$\|u_2(t)\| \le \eta\|u_0\| + \frac{\beta_2\|u_0\|}{1-\eta} \le \beta_3\|u_0\|, \quad t \ge 0, \tag{9}$$

where $\beta_3 = \eta + \frac{\beta_2}{1-\eta}$. From (8) and (9) it follows that

$$\|y(t)\| \le \|H_2\|\|u(t)\| \le \|H_2\|(\beta_1+\beta_3)\|u_0\|, \quad t \ge 0,$$

hence

$$\exists N > 0: \quad \|y(t)\| \le N\|y_0\|, \quad t \ge 0,$$

which shows that $y(t)$ is stable. To show asymptotic stability, i.e., $\lim_{t\to\infty} y(t) = 0$, using the condition (ii) and integrating $\dot{\mathcal{V}}(.)$,

$$\mathcal{V}(u_t) - \mathcal{V}(u_0) = \int_0^t \dot{\mathcal{V}}(u_s)ds \le -\int_0^t \alpha_3|u_s(0)|^2 ds = -\int_0^t \alpha_3|u(s)|^2 ds,$$

which gives

$$\int_0^t \alpha_3|u(s)|^2 ds \le \mathcal{V}(u_0) - \mathcal{V}(u_t) \le \mathcal{V}(u_0) \le \alpha_2|u_0|^2,$$

due to $V(u_t) \ge 0$ and (i). Letting $t \to +\infty$, we obtain that

$$\exists \alpha_4 > 0: \quad \int_0^\infty \|u(t)\|^2 dt \le \alpha_4\|u_0\|^2,$$

which implies $u(t) \in L_2[0,+\infty)$, and hence $y(t) = H_2u(t) \in L_2[0,+\infty)$. Setting $f(t) = \|u_1(t)\|^2$, we have $\int_0^\infty f(t) < +\infty$. Using the first equation of (3) gives $\dot{u}_1(t)$ is bounded on $[0,+\infty)$, then $\dot{f}(t) = 2u_1(t)^T\dot{u}_1(t)$ is bounded, which gives $f(t)$ is uniformly continuous on $[0,+\infty)$. Applying the Barbalat's Lemma (Lemma 3), we get $\lim_{t\to\infty} f(t)dt = 0$, which gives $\lim_{t\to\infty} u_1(t) = 0$. On the other hand, using the second equations of (3) gives

$$\|u_2(t)\| \le \|A_{22}^{-1}\|\|[A_{21}u_1(t) + D_{21}u_1(t-\tau(t))]\| + \|A_{22}^{-1}D_{22}\|\|u_2(t-\tau(t))\|,$$

then

$$\exists \alpha_5 > 0: \quad \|u_2(t)\| \le \eta \sup_{s\in[-\tau,0]}\|u_2(t+s)\| + \alpha_5 \sup_{s\in[-\tau,0]}\|u_1(t+s)\|, \quad t \ge 0,$$

where $\eta = \|A_{22}^{-1}D_{22}\| < 1$. Applying Lemma 2, where $a(t) = \|u_2(t)\|$, $b(t) = \alpha_5 \sup_{s\in[-\tau,0]}\|u_1(t+s)\|$, we get $\lim_{t\to\infty} u_2(t) = 0$. Therefore, $\lim_{t\to\infty} y(t) = 0$. The lemma is proved. □

4 H_∞ Filtering

In this section, we propose an LMI-based design of the H_∞ filters for LSEs (1). Consider the observer-based LSEs with time-varying delay defined by

$$\begin{cases} E\dot{y}(t) = Ay(t) + A_\tau y(t-\tau(t)) + Bw(t), \quad t \geq 0, \\ o(t) = Cy(t) + C_\tau y(t-\tau(t)), \\ z(t) = Dy(t) + D_\tau y(t-\tau(t)), \\ y(t) = \xi(t), \quad t \in [-\tau, 0], \end{cases} \tag{10}$$

where $o(t)$ is the observation vector, $z(t)$ is the measured vector, $w(t)$ is the disturbance vector; B, C, C_τ, D, D_τ are given constant matrices. Consider the following filtering system

$$\begin{cases} \mathcal{E}\dot{\bar{y}}(t) = \mathcal{A}\bar{y}(t) + \mathcal{B}o(t), \\ \bar{z}(t) = \mathcal{C}\bar{x}(t) + \mathcal{G}o(t), \end{cases} \tag{11}$$

where $\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{G}$ are the filters to be designed. Setting $r(t) = (y(t), \bar{y}(t))^\top$, $e(t) = z(t) - \bar{z}(t)$, the error system for (10) is

$$\begin{cases} \bar{E}\dot{r}(t) = \bar{A}r(t) + \bar{A}_\tau r(t-\tau(t)) + \bar{B}w(t), \\ e(t) = \bar{C}r(t) + \bar{C}_\tau r(t-\tau(t)), \\ r(t) = [\xi(t), 0], \quad t \in [-\tau, 0], \end{cases} \tag{12}$$

where

$$\bar{C} = [D - \mathcal{G}C, -\mathcal{C}], \ \bar{C}_\tau = [D_\tau - \mathcal{G}C_\tau, 0],$$

$$\bar{E} = \begin{pmatrix} E & 0 \\ 0 & \mathcal{E} \end{pmatrix}, \ \bar{A} = \begin{pmatrix} A & 0 \\ \mathcal{B}C & \mathcal{A} \end{pmatrix}, \ \bar{A}_\tau = \begin{pmatrix} A_\tau \\ \mathcal{B}C_\tau \end{pmatrix}[I, \ 0], \ \bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

For given $\gamma > 0$, the H_∞ filtering problem of system (10) has a solution if there are the filters (11) such that (12) is admissible and for all zero initial conditions and non-zero $w \in L_2[0, +\infty)$ the following condition holds

$$\int_0^\infty \|e(t)\|^2 dt \leq \gamma \int_0^\infty \|w(t)\|^2 dt. \tag{13}$$

Theorem 1 *The H_∞ filtering for (10) has a solution if there exist invertible matrices P_1, P_2 satisfying $\mathbb{P}\bar{E} = \bar{E}^\top\mathbb{P}^\top \geq 0$, $\bar{K}_i, i = 1, 2, \ldots, 5$, $\mathbb{K} = \mathbb{K}^\top > 0$, and X, Y, Z, V_1, V_2 such that*

$$\begin{pmatrix} \mathbb{N}_{11} & \mathbb{P}\bar{A}_\tau \\ * & -\bar{K}_5 \end{pmatrix} < 0, \tag{14}$$

$$[\mathbb{R}_{ij}]_{10\times 10} < 0. \tag{15}$$

The filters are defined by

$$\mathcal{E} = E, \ \mathcal{A} = P_2^{-1}X, \ \mathcal{B} = P_2^{-1}Y, \ \mathcal{C} = V_1, \ \mathcal{G} = V_2,$$

where

$$\mathbb{P} = \mathrm{diag}\{P_1, P_2\},\ \mathbb{S} = [Z, 0],\ \mathbb{K} = \begin{pmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \\ * & \bar{K}_4 & \bar{K}_5 \\ * & * & \bar{K}_5 \end{pmatrix},$$

$$\mathbb{N}_{11} = \mathbb{P}\bar{A} + \bar{A}^\top\mathbb{P}^\top + \bar{K}_3\bar{E} + \bar{E}^\top\bar{K}_3^\top + \bar{K}_5,\ \mathbb{R}_{11} = \tau\bar{K}_1 - \bar{K}_5 + \mathbb{P}\bar{A} + \bar{A}^\top\mathbb{P}^\top,$$

$$\mathbb{R}_{13} = \bar{A}^\top\mathbb{P}^\top - \mathbb{P}, \mathbb{R}_{12} = \tau\bar{K}_2 - \bar{K}_3\bar{E} + \bar{E}^\top\bar{K}_5^\top + \mathbb{P}\bar{A}_\tau + \bar{A}^\top\mathbb{P}^\top,\ \mathbb{R}_{1j} = 0,\ j = 7, 8, 10,$$

$$\mathbb{R}_{14} = \bar{A}^\top\mathbb{S}^\top, \mathbb{R}_{15} = \mathbb{R}_{16} = \mathbb{P}\bar{B},\ \mathbb{R}_{19} = [D - V_2C, -V_1]^\top,\ \mathbb{R}_{2j} = 0,\ j = 5, 6, 8, 9,$$

$$\mathbb{R}_{22} = \tau\bar{K}_4 - \bar{K}_5\bar{E} - \bar{E}^\top\bar{K}_5^\top + \mathbb{P}\bar{A}_\tau + \bar{A}_\tau^\top\mathbb{P}^\top + \bar{K}_5,\ \mathbb{R}_{23} = \bar{A}_\tau^\top\mathbb{P}^\top - \mathbb{P},\ \mathbb{R}_{24} = \bar{A}_\tau^\top\mathbb{S}^\top,$$

$$\mathbb{R}_{33} = \tau\bar{K}_5 - \mathbb{P} - \mathbb{P}^\top,\ \mathbb{R}_{27} = \mathbb{P}\bar{B},\ \ \mathbb{R}_{2,10} = [D_\tau - V_2C_\tau, 0]^\top,\ \ \mathbb{R}_{34} = \mathbb{S}^\top,\ \mathbb{R}_{38} = \mathbb{P}\bar{B},$$

$$\mathbb{R}_{3j} = 0,\ j = 5, 6, 7, 9, 10,\ \mathbb{R}_{44} = \mathbb{S}\bar{B} + \bar{B}^\top\mathbb{S}^\top,\ \mathbb{R}_{4j} = 0,\ j = 5, 6, \ldots, 10,$$

$$\mathbb{R}_{5j} = 0,\ j = 6, 7, \ldots, 10,\ \ \mathbb{R}_{6j} = 0,\ j = 7, 8, \ldots, 10,\ \ \mathbb{R}_{7j} = 0,\ j = 8, 9, 10,$$

$$\mathbb{R}_{9,10} = 0,\ \ \mathbb{R}_{ii} = -\frac{\gamma}{4}I,\ i = 5, \ldots, 10.$$

Proof *Step 1. Singularity and absence impulse of* (12). Employing Lemma 4, we will show that there exist matrices $\bar{Q} > 0,\ \bar{R}$ satisfying $\mathbb{P}\bar{E} = \bar{E}^\top\mathbb{P}^\top \geq 0$ such that

$$\begin{pmatrix} \mathbb{P}\bar{A} + \bar{A}^\top\mathbb{P}^\top + \bar{Q} + \bar{R}\bar{E} + \bar{E}^\top\bar{R}^\top & \mathbb{P}\bar{A}_\tau \\ * & -\bar{Q} \end{pmatrix} < 0. \tag{16}$$

It is seen that LMI (16) is equivalent to (14) by taking $\bar{R} = \bar{K}_3, \bar{Q} = \bar{K}_5$, which derives the regularity and absence of impulse. In addition, we get $\|\bar{A}_{22}^{-1}\bar{D}_{22}\| < 1$, where $\bar{A}_{22}^{-1}, \bar{D}_{22}$ are the block matrices of the differential-algebraic equations of (12), similar to (3) of (2), defined by

$$\bar{\mathbb{A}} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix},\quad \bar{\mathbb{A}}_\tau = \begin{pmatrix} \bar{D}_{11} & \bar{D}_{12} \\ \bar{D}_{21} & \bar{D}_{22} \end{pmatrix}.$$

Step 2. Asymptotical stability. Consider the Lyapunov function $\mathbb{V}(r_t) = \sum_{i=1}^{3} \mathbb{V}_i(r_t)$, where

$$\mathbb{V}_1(r_t) = r^\top(t)\mathbb{P}\bar{E}r(t),$$

$$\mathbb{V}_2(r_t) = \int_{-\tau}^{0}\int_{t+s}^{t} \dot{r}^\top(\theta)\bar{E}^\top\bar{K}_5\bar{E}\dot{r}(\theta)d\theta ds,$$

$$\mathbb{V}_3(v_t) = \int_{0}^{t}\int_{\theta-\tau(\theta)}^{\theta} e^\top(s,\theta)\mathbb{K}e(s,\theta)dsd\theta,$$

where $e^\top(s,\theta) = [r(\theta)^\top, r(\theta - \tau(\theta))^\top, (\bar{E}\dot{r}(s))^\top]$.

Let $\hat{\mathbb{P}} = \bar{H}_2^\top\mathbb{P}\bar{H}_1^{-1} = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix}$, where matrices $\bar{H}_1, \bar{H}_2$ are invertible such that $\hat{\bar{E}} = \bar{H}_1\bar{E}\bar{H}_2 = \begin{pmatrix} I_{2r} & 0 \\ 0 & 0 \end{pmatrix}$. From $\mathbb{P}\bar{E} = \bar{E}^\top\mathbb{P}^\top \geq 0$, it follows that $\hat{\mathbb{P}}\hat{\bar{E}} = \hat{\bar{E}}^\top\hat{\mathbb{P}}^\top$. Since $\hat{\mathbb{P}}$ is invertible, we have $\bar{P}_{21} = 0,\ \bar{P}_{11}^\top > 0$, and then $\hat{\mathbb{P}}\hat{\bar{E}} = \begin{pmatrix} \bar{P}_{11} & 0 \\ 0 & 0 \end{pmatrix}$. We will prove that there exist $\alpha_1 > 0, \alpha_2 > 0$ such that

$$\alpha_1\|\bar{u}_1(t)\|^2 \leq \mathbb{V}(r_t) \leq \alpha_2\|r_t\|^2,\quad t \geq 0, \tag{17}$$

where $\bar{u}(t) = \bar{H}_2^{-1} r(t) = [\bar{u}_1(t), \bar{u}_2(t)]^\top, \bar{u}_1(t) \in R^{2r},\ \bar{u}_2(t) \in R^{2n-2r}$. For this, we first estimate $\mathbb{V}_1(r_t)$ as follows. From

$$\mathbb{P}\bar{E} = [\bar{H}_2^{-1}]^\top \hat{\mathbb{P}} \hat{\bar{E}} [\bar{H}_2^{-1}] = [\bar{H}_2^{-1}]^\top \begin{pmatrix} \bar{P}_{11} & 0 \\ 0 & 0 \end{pmatrix} [\bar{H}_2^{-1}],$$

it follows that

$$\begin{aligned} \mathbb{V}_1(r_t) &= r^\top(t)\mathbb{P}\bar{E}r(t) = r^\top(t)[\bar{H}_2^{-1}]^\top \begin{pmatrix} \bar{P}_{11} & 0 \\ 0 & 0 \end{pmatrix} [\bar{H}_2^{-1}] r(t) \\ &= [\bar{u}_1(t)]^\top(t)\bar{P}_{11}\bar{u}_1(t), \end{aligned}$$

and hence

$$\begin{aligned} \lambda_{\min}(\bar{P}_{11})\|\bar{u}_1(t)\|^2 \le \mathbb{V}_1(r_t) &\le \lambda_{\max}(\bar{P}_{11})\|\bar{u}_1(t)\|^2 \\ &\le \lambda_{\max}(\bar{P}_{11})\|\bar{u}(t)\|^2 \le \lambda_{\max}(\bar{P}_{11})\|[\bar{H}_2^{-1}]\|^2.\|r(t)\|^2 \\ &\le \lambda_{\max}(\bar{P}_{11})\|[\bar{H}_2^{-1}]\|^2.\|r_t\|^2. \end{aligned}$$

Next, upon some similar calculations, we can estimate $\mathbb{V}_2(r_t)$, $\mathbb{V}_3(r_t)$, by using $\|r_t\| \ge \max\{r(t),\ r(t-\tau(t))\}$ such that

$$\exists a > 0: \quad \mathbb{V}_2(r_t) \le a\|r_t\|^2,\ \mathbb{V}_3(r_t) \le a\|r_t\|^2,$$

which shows the condition (17). Taking the derivative of $\mathbb{V}(.)$, we have

$$\begin{aligned} \dot{\mathbb{V}}_1(r_t) =& 2r^\top(t)\mathbb{P}\bar{E}\dot{r}(t) \\ =& \eta(t)^\top \begin{pmatrix} \mathbb{P}\bar{A} + \bar{A}^\top\mathbb{P}^\top & \mathbb{P}\bar{A}_\tau \\ \bar{A}_\tau^\top\mathbb{P}^\top & 0 \end{pmatrix} \eta(t) + 2r^\top(t)\mathbb{P}\bar{B}w(t), \\ \dot{\mathbb{V}}_2(r_t) =& \tau\dot{r}^\top(t)\bar{E}^\top\bar{K}_5\bar{E}\dot{r}(t) - \int_{t-\tau}^t \dot{r}^\top(s)\bar{E}^\top\bar{K}_5\bar{E}\dot{r}(s)ds, \\ \dot{\mathbb{V}}_3(r_t) =& \int_{t-\tau(t)}^t e^\top(s,t)\mathbb{K}e(s,t)ds \\ =& \tau(t)\eta^\top(t)\hat{X}\eta(t) + 2\eta^\top(t)\begin{pmatrix} \bar{K}_3 \\ \bar{K}_5 \end{pmatrix}[\bar{E}r(t) - \bar{E}r(t-\tau(t))] \\ &+ \int_{t-\tau(t)}^t \dot{r}^\top(s)\bar{E}^\top\bar{K}_5\bar{E}\dot{r}(s)ds \\ =& \tau\eta^\top(t)\hat{X}\eta(t) + 2[r(t)^\top\bar{K}_3 + r(t-\tau(t))^\top\bar{K}_5][\bar{E}r(t) - \bar{E}r(t-\tau(t))] \\ &+ \int_{t-\tau}^t \dot{r}^\top(s)\bar{E}^\top\bar{K}_5\bar{E}\dot{r}(s)ds, \end{aligned}$$

where $\eta(t) = [r(t),\ r(t-\tau(t))]$ and $\hat{X} = \begin{pmatrix} \bar{K}_1 & \bar{K}_2 \\ * & \bar{K}_4 \end{pmatrix}$. Therefore, we have

$$\begin{aligned} \dot{\mathbb{V}}(r_t) \le& \eta(t)^\top \begin{pmatrix} \mathbb{P}\bar{A} + \bar{A}^\top\mathbb{P}^\top & \mathbb{P}\bar{A}_\tau \\ \bar{A}_\tau^\top\mathbb{P}^\top & 0 \end{pmatrix} \eta(t) \\ &+ \tau\dot{r}^\top(t)\bar{E}^\top\bar{K}_5\bar{E}\dot{r}(t) + 2r^\top(t)\mathbb{P}\bar{B}w(t) + \tau\eta^\top(t)\hat{X}\eta(t) \\ &+ 2\Big[r(t)^\top\bar{K}_3 + r(t-\tau(t))^\top\bar{K}_5\Big]\Big[\bar{E}r(t) - \bar{E}r(t-\tau(t))\Big]. \end{aligned}$$

Multiplying two sides of (12) by $-2\dot{r}^\top(t)\bar{E}^\top\mathbb{P}$, $-2r^\top(t)\mathbb{P}$, $-2r^\top(t-\tau(t))\mathbb{P}$, $-2w^\top(t)\mathbb{S}$, adding the zero terms and using the following inequation

$$0 \le -\|e(t)\|^2 + 2r(t)^\top\bar{C}^\top\bar{C}r(t) + 2r(t-\tau(t))^\top\bar{C}_\tau^\top\bar{C}_\tau r(t-\tau(t)),$$

where $\bar{C} = [D - V_2C, -V_1]$, $\bar{C}_\tau = [D_\tau - V_2C_\tau, 0]$, we have

$$\dot{\mathbb{V}}(r_t) \le \eta^\top(t)\mathcal{W}_1\eta(t) + \mu^\top(t)\mathcal{W}_2\mu(t) + \gamma\|w(t)\|^2 - \|e(t)\|^2, \tag{18}$$

where $\mu(t)^\top = [r(t)^\top, r(t-\tau(t))^\top, (\bar{E}\dot{r}(t))^\top,\ w(t)^\top]$,

$$\mathcal{W}_1 = \begin{pmatrix} \mathbb{N}_{11} & \mathbb{P}\bar{A}_\tau \\ * & -\bar{K}_5 \end{pmatrix},\ \mathcal{W}_2 = [N_{ij}]_{4\times 4},$$

$$\mathbb{N}_{11} = \mathbb{P}\bar{A} + \bar{A}^\top\mathbb{P}^\top + \bar{U}_3\bar{E} + \bar{E}^\top\bar{K}_3^\top + \bar{K}_5,$$

$$N_{11} = \tau\bar{K}_1 - \bar{K}_5 + \mathbb{P}\bar{A} + \bar{A}^\top\mathbb{P}^\top + \frac{4}{\gamma}\mathbb{P}\bar{B}\bar{B}^\top\mathbb{P}^\top + \frac{4}{\gamma}\mathbb{P}\bar{B}\bar{B}^\top\mathbb{P}^\top + 2\bar{C}^\top\bar{C},$$

$$N_{12} = \tau\bar{K}_2 - \bar{K}_3\bar{E} + \bar{E}^\top\bar{K}_5^\top + \mathbb{P}\bar{A}_\tau + \bar{A}^\top\mathbb{P}^\top,\ N_{13} = \bar{A}^\top\mathbb{P}^\top - \mathbb{P},$$

$$N_{14} = \bar{A}^\top\mathbb{S}^\top,\ N_{23} = \bar{A}_\tau^\top\mathbb{P}^\top - \mathbb{P},\ N_{24} = \bar{A}_\tau^\top\mathbb{S}^\top,$$

$$N_{22} = \tau\bar{K}_4 - \bar{K}_5\bar{E} - \bar{E}^\top\bar{K}_5^\top + \mathbb{P}\bar{A}_\tau + \bar{A}_\tau^\top\mathbb{P}^\top + \bar{K}_5 + \frac{4}{\gamma}\mathbb{P}\bar{B}\bar{B}^\top\mathbb{P}^\top + 2\bar{C}_\tau^\top\bar{C}_\tau,$$

$$N_{44} = \mathbb{S}\bar{B} + \bar{B}^\top\mathbb{S}^\top,\ N_{33} = \tau\bar{K}_5 - \mathbb{P} - \mathbb{P}^\top + \frac{4}{\gamma}\mathbb{P}\bar{B}\bar{B}^\top\mathbb{P}^\top, N_{34} = \mathbb{S}^\top.$$

Using (14), (15) and the Schur complement lemma, we obtain $\mathcal{W}_i < 0$, which gives

$$\exists\lambda_3 > 0 : \quad \dot{\mathbb{V}}(r_t) \le \eta^\top(t)\mathcal{W}_1\eta(t) + \mu^\top(t)\mathcal{W}_2\mu(t) < -\lambda_3\|r(t)\|^2 \tag{19}$$

for $w(t) \equiv 0$. Finally, applying Lemma 5 and the conditions (17), (19), we have proved the asymptotical stability of the system.

Step 3. H_∞ performance. To show the condition (13), we use the derived inequality (18) and $\mathcal{W}_i < 0, i = 1, 2$, such that

$$\int_0^t [\|e(s)\|^2 - \gamma\|w(s)\|^2]ds \le -\int_0^t \dot{\mathbb{V}}(r_s)ds = \mathbb{V}(r_0) - \mathbb{V}(r_t) \le \mathbb{V}(r_0).$$

Letting the initial condition $r_0 = 0$ and $t \to \infty$, we have

$$\int_0^\infty \|e(s)\|^2 ds \le \gamma \int_0^\infty \|w(s)\|^2 ds,$$

which implies the condition (13). The theorem is proved. □

Remark 1 It is notable that in Theorem 1, the conditions (14), (15) are LMIs if we set $\mathcal{A} = P_2^{-1}X$, $\mathcal{B} = P_2^{-1}Y$, we have

$$\mathbb{P}\bar{A} = \begin{bmatrix} P_1A & 0 \\ YC & X \end{bmatrix},\ \mathbb{P}\bar{A}_\tau = \begin{bmatrix} P_1A_\tau \\ YC_\tau \end{bmatrix}H,\ \mathbb{P}\bar{B} = \begin{bmatrix} P_1B \\ 0 \end{bmatrix},$$

$$\mathbb{S}\bar{A} = \begin{bmatrix} ZA & 0 \end{bmatrix},\ \mathbb{S}\bar{A}_\tau = \begin{bmatrix} ZA_\tau & 0 \end{bmatrix},\ \mathbb{S}\bar{B} = ZB.$$

Remark 2 In the proof of Theorem 1, we construct improved Lyapunov-Krasovsii functionals $\mathbb{V}_i(.)$, $i = 1, 2, 3$ and when we take their derivatives we do not need the smooth assumption on $\tau(t)$. Therefore, the method used in the existing works [4, 21–23], where the differentiability of the delay function is required, cannot be applicable.

Example 1 We consider system (10) described by an economical Leontief model [14], which is a quantitative technique representing the interdependency between production of different commodities. Using description of (10), y_i represents production of ith commodity, A represents the rate of production of commodities, A_τ gives the influence of the past production, B corresponds to the known supply uncertainties, and the disturbance $w(t)$ presents the supply uncertainty, $z(t)$ corresponds to the productions of commodities available for evaluation, $e(t)$ is the error of such an evaluation, where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} -5 & 1 \\ 0 & -5 \end{bmatrix}, \ A_\tau = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}, C_\tau = \begin{bmatrix} -1 & 0.1 \\ 1 & -0.1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.01 & 0.1 \\ 0.01 & 0.01 \end{bmatrix}, \ D_\tau = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$
$$\tau(t) = 1/10 + 2/5|\sin(t)|, \ \gamma = 0.01, \ \tau = 1/2.$$

The LMIs (14), (15) are feasibly solved by the LMI Control Toolbox [6] as

$$P_1 = \begin{bmatrix} 0.0031 & 0 \\ 0 & 0.0027 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0755 & 0 \\ 0 & 0.0227 \end{bmatrix},$$
$$X = \begin{bmatrix} -0.1165 & 0 \\ 0 & -0.0332 \end{bmatrix}, \ Y = \begin{bmatrix} -0.1165 & -0.0004 \\ -0.0004 & 0.0014 \end{bmatrix},$$
$$Z = 10^{-3} \begin{bmatrix} -0.2953 & -0.0750 \\ -0.0750 & -0.0321 \end{bmatrix}, \ V_1 = \begin{bmatrix} 0.0001 & 0 \\ -0.0012 & 0.0005 \end{bmatrix},$$
$$V_2 = \begin{bmatrix} 0.0023 & 0.0996 \\ -0.0357 & 0.0244 \end{bmatrix}, \ \bar{K}_1 = \begin{bmatrix} 0.0501 & -0.0087 & 0.0036 & -0.0001 \\ -0.0087 & -0.0054 & -0.0001 & -0.0001 \\ 0.0036 & -0.0001 & 0.0671 & 0.0000 \\ -0.0001 & -0.0001 & 0.0000 & 0.0467 \end{bmatrix},$$
$$\bar{K}_2 = \begin{bmatrix} -0.0392 & -0.0048 & -0.0039 & 0.0006 \\ 0.0014 & 0.0191 & -0.0004 & -0.0014 \\ -0.0060 & 0.0005 & 0.0047 & -0.0000 \\ -0.0008 & 0.0000 & 0.0001 & 0.0214 \end{bmatrix},$$
$$\bar{K}_3 = \begin{bmatrix} -0.0112 & -0.0024 & -0.0011 & 0.0001 \\ 0.0032 & 0.0062 & -0.0004 & 0.0002 \\ -0.0018 & -0.0001 & 0.0410 & 0.0000 \\ -0.0003 & 0.0004 & 0.0001 & 0.0008 \end{bmatrix},$$
$$\bar{K}_4 = \begin{bmatrix} 0.0579 & -0.0005 & 0.0038 & -0.0015 \\ -0.0005 & 0.0137 & -0.0005 & -0.0006 \\ 0.0038 & -0.0005 & 0.0586 & -0.0000 \\ -0.0015 & -0.0006 & -0.0000 & 0.0088 \end{bmatrix},$$
$$\bar{K}_5 = \begin{bmatrix} 0.0240 & 0.0024 & -0.0010 & -0.0002 \\ 0.0024 & 0.0101 & 0.0001 & 0.0004 \\ -0.0010 & 0.0001 & 0.0582 & 0.0000 \\ -0.0002 & 0.0004 & 0.0000 & 0.0011 \end{bmatrix}.$$

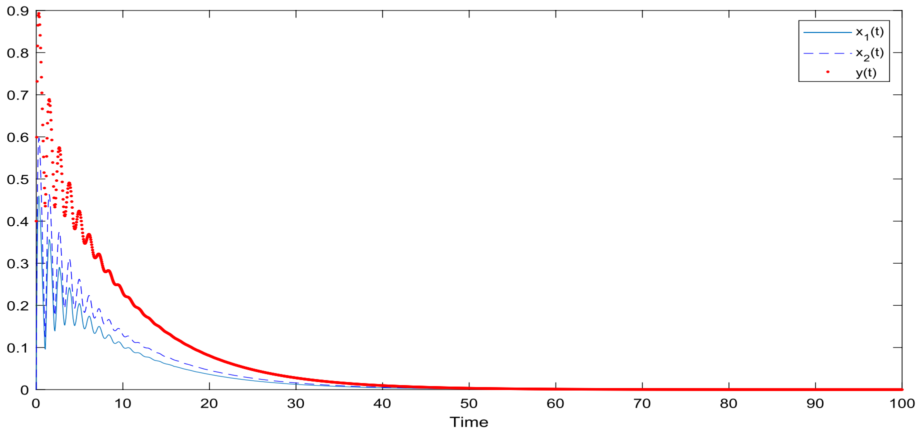

Fig. 1 The state y_1 and $\hat{y}_1$

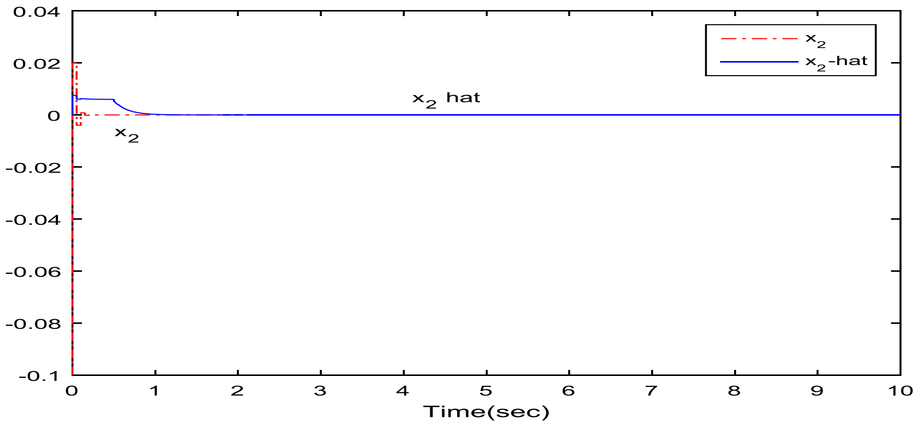

Fig. 2 The state y_2 and $\hat{y}_2$

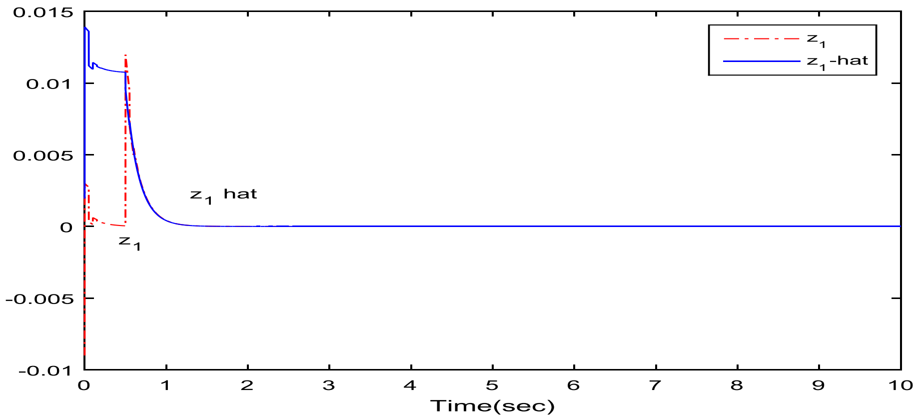

Fig. 3 The measures z_1 and $\hat{z}_1$

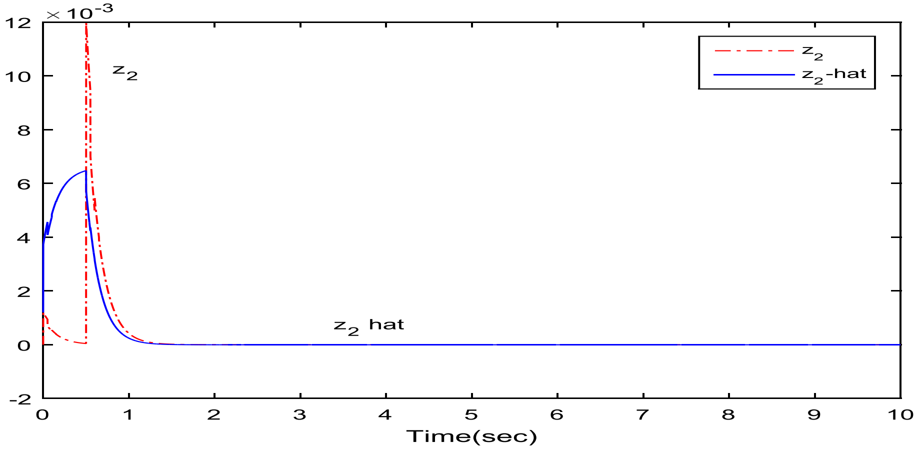

Fig. 4 The measures z_2 and $\hat{z}_2$

The H_∞ filtering problem, by Theorem 1, has a solution and the filters are given as

$$\mathcal{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} -0.01291 & 0 \\ 0 & -0.0037 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0.0011 & 0 \\ 0 & 0.0002 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 0.0001 & 0 \\ -0.0012 & 0.0005 \end{bmatrix}, \mathcal{G} = \begin{bmatrix} 0.0023 & 0.0996 \\ -0.0357 & 0.0244 \end{bmatrix}.$$

Figures 1–4 show the response states $y = [y_1, y_2]^\top$, $\bar{y} = [\hat{y}_1, \hat{y}_2]^\top$, $z = [z_1, z_2]$ and estimate signal $\bar{z} = [\bar{z}_1, z_2]^\top$ with $\xi(t) = [0.1, -0.1]^\top$.

5 Conclusions

The LMI-based conditions for stability and filtering of LSEs with time-varying delay have been presented. By newly proposed delay estimation techniques and improved Lyapunov-Krasovskii functionals, we have converted the filtering design into the problem of finding some parameters of the stability and H_∞ filtering, which could be certainly obtained by solving tractable LMIs. A numerical example is given to demonstrate the validity of the proposed results.

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