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Research paper

Stability and stabilization of fractional-order singular interconnected delay systems

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ABSTRACT

An analytical approach based on fractional calculus and singular value theory to finite-time stability and stabilization of fractional-order singular interconnected delay systems is proposed. Particularly, we study fractional singular equations with interval time-varying delays. We first give new sufficient conditions for finite-time stability of such equations. Then, the feedback stabilizing controllers are designed via solving a tractable linear matrix inequality (LMI) and Mittag-Leffler function. Finally, numerical examples with simulations are given to illustrate the feasibility and effectiveness of the proposed results.

1. Introduction

Consider the following fractional-order singular interconnected systems (FSISs) with time-varying delays

$$\begin{cases} E_i D^{\alpha_i} x_i(t) = & A_i x_i(t) + \sum_{j\neq i, j=1}^{K} A_{ij} x_j(t-\beta_{ij}(t)) + B_i u_i(t), \quad t \geq 0, \\ x_i(t) = & \xi_i(t), \quad t \in [-\beta, 0], \end{cases} \tag{1}$$

where $i = 1, 2, \ldots, K$, $D^{\alpha_i} x$ is the Caputo derivative of $x, \alpha_i \in (0, 1)$, $x_i \in R^{n_i}$ is the state, $u_i(t) \in R^{m_i}$ is the control, E_i is singular: rank $E_i = r_i \leq n_i$, A_i, A_{ij}, B_i are constant matrices, $\xi_i(.) \in C([-\beta, 0], R^{n_i})$, $\beta_{ij}(.)$ is continuous satisfying

$$0 < \beta_1 \leq \beta_{ij}(t) \leq \beta, \quad t \geq 0.$$

Over the past decades, fractional dynamical equations (FDEs), which offer more advantages than integer-order ones, have gained considerable importance due to their various applications in widespread fields of applied sciences and engineering [1–4]. At the same time, many practical systems are of large-scale interconnected equations, which are characterized by a large number of variables representing the system, a strong interaction between the system variables and a complex structure [5]. So far numerous results on control and stability have been reported for interconnected time-delay equations (see [6–8] and the references therein). It is worth noting that most of the existing results have concerned with asymptotic stability defined over an infinite-time interval. Nevertheless, in the framework of practical applications, system trajectories are required to lie within a prescribed time interval and this concept is known as finite-time stability (FTS) [9,10]. Some early results on FTS were obtained for interconnected equations [11–13] as well as for fractional-order equations [14,15].

On the other side, singular equations (or descriptor, implicit, differential–algebraic equations) have come to play an important role in many practical fields, such as power systems, chemical processes, internet systems, etc. [16–18]. During the last two decades,

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the stability and control problem for singular equations with delay has become more and more complicated, since singular equations have complex structures of modes as finite dynamic, nondynamic modes and impulse modes, we cannot directly use algebraic tools for the equations due to lack of characteristic polynomial with a rational power-multivalued function. One way to study stability problem of FDEs is to Lyapunov function method. It is noted that most of the existing results used Lyapunov function method, where the key issue is to find positive definite Lyapunov functionals to apply the fractional Lyapunov stability theorem [19,20]. However, the main difficulty is how to find Lyapunov functionals for satisfying the fractional stability theorem [21]. In some existing papers, the authors attempted to find appropriate Lyapunov functionals, but there was a gap in the proof due to incorrect use of the fractional stability theorem. This paper aims at studying the FSISs (1), which are more general class of FDEs having just caused a little research up to now. The complexity introduced by both the singularity and the large-scale dimension makes it difficult to study the stability and control of the singular FDEs. Comparing with the existing research on the topic, we are facing some difficulties:
(i) The system (1) under consideration consists of interconnected subequations, where the delays are interacted between the recipient fractional singular subequations. Such systems contain heterogeneous time delays and hence the result is a complicated fractional singular delay equation.
(ii) Because it requires to deal with not only stability, but also regularity and absence of impulses at the same time, and the latter two need not be considered not only in the regular models, but also in the fractional models.
(iii) Due to the coupling between the differential and algebraic subequations with interacted delays in the decomposed singular model, the solution estimation of such equations is still hard to find and estimate. In addition, since the interacted delays are of high dimension and it needs extensive computations.
(iv) One more difficulty is how to find a suitable Lyapunov–Krasovskii functional in order to apply the fractional stability theorem.

In [17,18] the authors studied stability and control for integer-order singular interconnected equations without delays. Stability analysis for fractional interconnected equations with or without constant delays was studied in [13,15], however the singularity was not considered therein, i.e. $E_i = I$. For singular FDEs with constant delays, the problem is considered in [3,22–24], however the large-scale structure was not considered therein, i.e. $K = 1$. As far as we know, the problem of stability and control for the FSISs (1) with time-variable delays has not been fully studied, which is very challenging and of great importance. This motivates us to carry out this study.

In this paper, based on the fractional technique as calculating Caputo derivative, using the Laplace transform, Mittag-Leffler function and singular value theory, we study FTS and stabilization of Eq. (1) with time-varying delay. The present paper contributes to the available literature through the following.

• The delay function interacted in all subequations is interval time-varying, continuous and bounded.

• New delay-dependent sufficient conditions for the finite-time stability and designing stabilizing controllers are proposed.

• The proposed conditions are presented via solving a tractable LMI and Mittag-Leffler functions, which can be solved by LMI Tool Box [25].

• The effectiveness and applicability of theoretical results are verified by two numerical examples and simulations.

The layout of this article is as follows. Section 2 presents some basic of fractional calculus, stability and control problem and auxiliary technical results. The results on FTS and design of stabilizing controllers are presented in Section 3. Section 4 includes numerical examples and simulations.

Notations. By $\mathbb{C}$ we denote the complex spaces, $\mathbb{R}^n$ and $R^{n\times m}$ stands for the n-dimensional space and real $(n \times m)$ matrices, respectively. I_r stands for identity matrix of order r. For any matrix $S \in R^{n\times n}$, $S > 0$ $(S < 0)$ denotes the positive-definite (negative-definite) matrix. By $\lambda_{\max}(S)$ and $\lambda_{\min}(S)$ we denote the maximal and the minimal eigenvalue of $S \in \mathbb{R}^{n\times n}$. $L^1[0, h]$ denotes the space of integrable functions on $[0, h]$, $C([-h, 0])$ denotes the space of continuous functions on $[0, h]$. By $[d]$ we denote the integer part of a number d and we use $\bigtriangledown V(.)$ for the gradient of $V(.)$. The symmetric term in a matrix is denoted by $*$.

2. Problem formulation and preliminaries

We first recall from [1,2] some backgrounds of fractional calculus and related basic properties used in the text. Let $\alpha \in (0, 1), g(t) \in L^1[0, b]$. The Riemann fractional integral of order α of $g(t)$ is defined as

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds, \quad b > t > 0,$$

where $\Gamma(\alpha)$ is Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. The Riemann fractional derivative and the Caputo fractional derivative of $g(t)$ respectively are defined by

$$D_R^\alpha g(t) = \frac{d}{dt}[I^{1-\alpha} g(t)],$$
$$D^\alpha g(t) = D_R^\alpha [g(t) - g(0)].$$

The Mittag-Leffler function is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z \in \mathbb{C}, \ \alpha > 0, \ \beta > 0.$$

The Laplace transform of $g(t)$ is defined by $L[g(t)](s) = \int_0^\infty e^{-st} g(t)dt$.

Lemma 1. [2]. *Given integrable functions $u(.), v(.)$ and $\alpha \in (0,1), \eta > 0, h > 0$, the following relations hold.*

(a) $L[D^\alpha u(t)](s) = s^\alpha L[u(t)](s) - s^{\alpha-1}u(0)$.

(b) $n \in N, Re(s) > h^{1/\alpha}$,

$$L[t^{\alpha n+\eta-1}E^{(n)}_{\alpha,\eta}(ht^\alpha)](s) = \frac{n!s^{\alpha-\eta}}{(s^\alpha - h)^{n+1}}.$$

(c) $L[u * v(t)](s) = L[u(t)](s)L[v(t)](s)$, *where* $u * v(t) = \int_0^t u(t-\tau)v(\tau)d\tau$.

Consider the following unforced system of (1) ($u_i(t) = 0$.)

$$\begin{cases} E_i D^{\alpha_i} x_i(t) = & A_i x_i(t) + \sum\limits_{j\neq i, j=1}^{K} A_{ij} x_j(t - \beta_{ij}(t)), \quad t \geq 0, \\ x_i(t) = & \xi_i(t), \ t \in [-\beta, 0]. \end{cases} \tag{2}$$

Definition 1. System (2) is
(i) regular if the polynomial $\det(s^{\alpha_i}E_i - A_i), i = 1, 2, \dots, K, s \in \mathbb{C}$, is not identically zero,
(ii) impulse-free if $\deg \det(s^{\alpha_i}E_i - A_i) = \operatorname{rank} E_i, \quad i = 1, 2, \dots, K, s \in \mathbb{C}$.

Proposition 1. *If the system (2) is impulse-free and regular, then it has a unique solution on* $[0, +\infty)$.

Proof. Setting

$$\alpha = (\alpha_1, \dots, \alpha_K)^\top,\ x = (x_1, \dots, x_K)^\top,\ x(t - \beta_{ij}(t)) = (x_1(t - \beta_{ij}(t)), \dots, x_K(t - \beta_{ij}(t)))^\top,$$

$$D^\alpha x = (D^{\alpha_1}x_1, D^{\alpha_2}x_2, \dots, D^{\alpha_K}x_K)^\top,\ \xi(t) = (\xi_1(t), \dots, \xi_K(t))^\top,$$

$$\bar{A} = \operatorname{diag}\{A_1, \dots, A_K\}, \bar{E} = \operatorname{diag}\{E_1, \dots, E_K\},$$

$$\bar{A}_{ij} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \dots & \dots & \dots \\ \vdots & \dots & \dots & A_{ij} & \dots \\ \vdots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{n\times n} \quad \text{where } A_{ij} \text{ in the } i\text{th row, } j\text{th column,}$$

system (2) is reduced to a singular fractional differential equation with delays:

$$\begin{cases} \bar{E} D^\alpha \dot{x}(t) & = \bar{A}x(t) + \sum\limits_{i\neq j, j=1}^{K} \bar{A}_{ij} x(t - \beta_{ij}(t)), \\ x(t) & = \xi(t), \quad t \in [-\beta, 0]. \end{cases} \tag{3}$$

Since $\bar{E}$ and $\bar{A}$ are the diagonal matrix of E_i and A_i, if the system (2) is regular and impulse-free then Eq. (3) is also regular and impulse-free. Hence, using a result of [21,26] on the existence of solutions of the singular fractional differential system with delays, Eq. (2) has an unique solution on $[0, +\infty)$. □

Note that rank $E_i < n_i$, we can find two invertible matrices H_i, Q_i satisfying

$$H_i E_i Q_i = \begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix}.$$

In the sequel, we denote for simplification $\beta_{ij} := \beta_{ij}(t)$ and

$$H_i A_i Q_i = \begin{bmatrix} A_i(11) & A_i(12) \\ A_i(21) & A_i(22) \end{bmatrix};\ H_i A_{ij} Q_j = \begin{bmatrix} A_{ij}(11) & A_{ij}(12) \\ A_{ij}(21) & A_{ij}(22) \end{bmatrix}.$$

Under the state transformation $y_i = Q_i^{-1}x_i := (y_{i1}, y_{i2})^\top$, system (2) takes the following form

$$\begin{cases} D^{\alpha_i} y_{i1}(t) = & A_i(11)y_{i1}(t) + A_i(12)y_{i2}(t) + \sum\limits_{j=1, j\neq i}^{K} \Big[A_{ij}(11)y_{j1}(t - \beta_{ij}) + A_{ij}(12)y_{j2}(t - \beta_{ij})\Big], \\ 0 = & A_i(21)y_{i1}(t) + A_i(22)y_{i2}(t) + \sum\limits_{j\neq i, j=1}^{K} \Big[A_{ij}(21)y_{j1}(t - \beta_{ij}) + A_{ij}(22)y_{j2}(t - \beta_{ij})\Big], \\ y_i(t) = & Q_i^{-1}\xi_i(t), \quad t \in [-\beta, 0]. \end{cases} \tag{4}$$

Lemma 2 (*[16,26]*). *System (4) is regular and impulse-free if matrix* $A_i(22)$ *is invertible, i.e., det* $A_i(22) \neq 0$.

Lemma 3 ([27]). *Assume that $f(t) \in C([0,+\infty],\mathbb{R}^n)$ and $V(.) : \mathbb{R}^n \to \mathbb{R}^+$ is a convex and differentiable function, $V(0)=0$. We have*

$$D^{\alpha}V(f(t)) \le \langle \bigtriangledown V(f(t)), D^{\alpha}f(t)\rangle, \quad t \ge 0.$$

Definition 2. Let c_1, c_2, T be given positive numbers. Eq. (2) is said to be FTS w.r.t. (c_1, c_2, T) if

$$\|\xi\|^2 \le c_1 \Rightarrow \|x(t)\|^2 \le c_2, \quad \forall t \in [0,T],$$

where $\xi = (\xi_1,\dots,\xi_K)^{\top}$, $x = (x_1,\dots, x_K)^{\top}$, $\|\xi\| = \sqrt{\sum_{i=1}^K \|\xi_i\|^2}$, and $\|x(t)\| = \sqrt{\sum_{i=1}^K \|x_i(t)\|^2}$.

Definition 3. Let c_1, c_2, T be given positive numbers. System (1) is finite-time stabilizable w.r.t. (c_1, c_2, T) if there are controllers $u_i(t) = K_i x_i(t)$ such that the closed-loop equation

$$E_i D^{\alpha_i} x_i(t) = [A_i + B_i K_i]x_i(t) + \sum_{j\neq i, j=1}^{K} A_{ij}x_j(t-\beta_{ij}(t)), \quad t \ge 0, \tag{5}$$

is regular, impulse-free and FTS w.r.t. (c_1, c_2, T).

Proposition 2. *Assume that $S(t) : [-\beta,\ b] \to \mathbb{R}^+$ is increasing and satisfies*

$$S(t) \le a_1 S(0) + a_2 S(t-\beta), \quad t \in [0,b],$$

where $b>0, \beta>0,\ a_1 \ge 1,\ a_2 \ge 0$. Then

$$S(t) \le S(0)a_1 \sum_{i=0}^{[b/\beta]+1} a_2^i, \quad t \in [0,\ b].$$

Proof. We first note that for $t \in [0,b]$, we can find $n \in N$ such that $n\beta \le t < (n+1)\beta$. Using mathematical induction we get

$$S(t) \le S(0)\sum_{j=0}^{n} a_1 a_2^j + a_2^{n+1} S(t-(n+1)\beta),$$

for $n \ge 1$, and $S(t) \le a_1 S(0) + a_2 S(t-(n+1)\beta)$, for $n = 0$. Since $S(t)$ is increasing on $-\beta \le t-(n+1)\beta < 0$, $S(t-(n+1)\beta) \le S(0)$ and $a_1 \ge 1$, we get

$$S(t) \le \begin{cases} \left[a_1 + a_2 a_1 + \cdots + a_2^n a_1 + a_2^{n+1} a_1\right] S(0), & for \quad n \ge 1,\\ (a_1 + a_2 a_1)S(0), & for \quad n = 0, \end{cases}$$
$$= a_1 \sum_{j=0}^{n+1} a_2^j S(0).$$

Moreover, since $t \le b$, we get $n \le [b/\beta]$, then

$$S(t) \le a_1 \sum_{j=0}^{[b/\beta]+1} a_2^j S(0). \quad \square$$

3. Main results

In this section, on the basis of singular value decomposition approach combining fractional calculus and LMI technique, we first provide delay-dependent criteria for finite time stability, then propose a design of stabilizing controllers for the stabilization problem of FSISs with time-varying delay. The following notations will be used in the sequel.

$$\beta_{ij} := \beta_{ij}(t),\ [Q_i]^{\top} P_i E_i Q_i = \begin{bmatrix} P_i(11) & 0\\ 0 & 0\end{bmatrix},\ N_i = \begin{bmatrix} 0 & 0\\ 0 & I_{n_i-r_i}\end{bmatrix} H_i,$$

$$a_0 = \max_{i,j}\Big\{ |[A_i(22)]^{-1}A_{ij}(22)|,\ |[A_i(22)]^{-1}A_i(21)|,\ |[A_i(22)]^{-1}A_{ij}(21)| \Big\};$$

$$E_{\alpha}(z) := E_{\alpha,\ 1}(z), a_1 = \max_i E_{\alpha_i}(\beta T^{\alpha_i}),\ a_2 = (K-1)(a_1-1),\ q = a_1 \sum_{k=0}^{[T/\beta_1]+1} a_2^k;$$

$$\lambda_{max}(PE) = \max_i \lambda_{max}(P_i E_i),\ \lambda_{min}(P(11)) = \min_i \lambda_{min}(P_i(11));$$

$$q_1 = q\frac{\lambda_{max}(PE)}{\lambda_{min}(P(11))}, q_2 = q_1 + \max_i \lambda_{max}([Q_i^{-1}]^{\top} Q_i^{-1}), q_3 = a_0 K\sqrt{q_2},$$

$$q_4 = \max_{k\in 0,[T/\beta_1]+1}\Big(q_3\sum_{j=0}^{k}[a_0(K-1)]^j + [a_0(K-1)]^{k+1}\sqrt{q_2}\Big), \bar{\beta} = \max_i \lambda_{max}([Q_i]^{\top}Q_i).$$

Theorem 4. *Given $c_1 > 0, c_2 > 0, T > 0$, the system (2) is finite-time stable w.r.t (c_1, c_2, T) if there are invertible matrices P_i satisfying $P_iE_i = [P_iE_i]^\top \geq 0$, matrices Z_{ij}, $L_i, R_i, i, j = 1, 2, \ldots, K$ such that*

$$W(i) < 0, \quad i = 1, 2, \ldots, K, \tag{6}$$

$$\bar{\beta}(q_1 + Nq_4^2)c_1 \leq c_2, \tag{7}$$

where

$$W(i) = \begin{bmatrix} M(i)_{1,1} & M(i)_{1,2} & \ldots & M(i)_{1,K+1} \\ * & M(i)_{2,2} & \ldots & M(i)_{2,K+1} \\ . & . & \ldots & . \\ * & * & \ldots & M(i)_{K+1,K+1} \end{bmatrix},$$

$$M(i)_{ii} = P_iA_i + [P_iA_i]^\top - \beta P_iE_i + R_iL_iA_i + [R_iL_iA_i]^\top,$$

$$M(i)_{jj} = -\beta P_jE_j + Z_{ij}A_{ij} + [Z_{ij}A_{ij}]^\top,$$

$$M(i)_{ij} = P_iA_{ij} + [Z_{ij}A_i]^\top + R_iM_iA_{ij}, \quad M(i)_{ji} = [M(i)_{ij}]^\top,$$

$$M(i)_{K+1,K+1} = -L_i - L_i^\top, \ M(i)_{K+1,i} = L_iA_i, \ M(i)_{i,K+1} = [L_iA_i]^\top,$$

$$M(i)_{K+1,j} = L_iA_{ij} - [Z_{ij}]^\top, \ M(i)_{j,K+1} = [M(i)_{K+1,j}]^\top, \ j \neq i, \ j = 1, 2, \ldots, K,$$

the other terms of the $W(i)$ are zero.

Proof. Let us set

$$Q_i^\top = \begin{bmatrix} Q_i(11) & Q_i(12) \\ Q_i(21) & Q_i(22) \end{bmatrix}, \ Q_i^\top P_iH_i^{-1} = \begin{bmatrix} P_i(11) & P_i(12) \\ P_i(21) & P_i(22) \end{bmatrix}.$$

From

$$Q_i^\top P_iE_iQ_i = Q_i^\top P_iH_i^{-1}H_iE_iQ_i = Q_i^\top P_iH_i^{-1}\begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_i(11) & 0 \\ P_i(21) & 0 \end{bmatrix} \geq 0,$$

$$Q_i^\top E_i^\top P_i^\top Q_i = \begin{bmatrix} P_i(11)^\top & P_i(21)^\top \\ 0 & 0 \end{bmatrix} \geq 0,$$

and the condition $P_iE_i = [P_iE_i]^\top \geq 0$ it follows that

$$P_i(21) = 0, \ P_i(11) = P_i(11)^\top \geq 0, \ Q_i^\top P_iE_iQ_i = \begin{bmatrix} P_i(11) & 0 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

Since P_i is invertible, we have $Q_i^\top P_iH_i^{-1} = \begin{bmatrix} P_i(11) & P_i(12) \\ 0 & P_i(22) \end{bmatrix}$. From (8) it follows that $\det(P_i(11)) \neq 0, \ P_i(11) > 0$. Besides, LMI (6) leads to

$$M(i)_{i,i} = P_iA_i + [P_iA_i]^\top - \beta P_iE_i < 0, \quad i = 1, 2, \ldots, K,$$

and hence

$$Q_i^\top[P_iA_i + [P_iA_i]^\top - \beta P_iE_i]Q_i < 0. \tag{9}$$

Moreover, we have

$$Q_i^\top P_iA_iQ_i = Q_i^\top P_iH_i^{-1}H_iA_iQ_i = \begin{bmatrix} P_i(11) & P_i(12) \\ 0 & P_i(22) \end{bmatrix}\begin{bmatrix} A_i(11) & A_i(12) \\ A_i(21) & A_i(22) \end{bmatrix}$$
$$= \begin{bmatrix} P_i(11)A_i(11) + P_i(12)A_i(21) & P_i(11)A_i(12) + P_i(12)A_i(22) \\ P_i(22)A_i(21) & P_i(22)A_i(22) \end{bmatrix},$$

$$Q_i^\top P_iE_iQ_i = \begin{bmatrix} P_i(11) & 0 \\ 0 & 0 \end{bmatrix},$$

and using (9) we get

$$P_i(22)A_i(22) + A_i(22)^\top P_i(22)^\top < 0,$$

which derives $\det(A_i(22)) \neq 0$. Using Lemma 2, the equation is impulse-free and regular.

Next, we show the FTS of system (2). For this, we take the non-negative convex and differentiable function

$$V_i(x_i(t)) = x_i(t)^\top P_iE_ix_i(t), \quad i = 1, 2, \ldots, K.$$

Taking the Caputo derivative of $V_i(x_i(t))$ with respect to t, using Lemma 3 and the following equalities

$$0 = 2E_i D^{\alpha_i} x_i(t) L_i \Big[-E_i D^{\alpha_i} x_i(t) + A_i x_i(t) + \sum_{j=1,j\neq i}^{K} A_{ij} x_j(t-\beta_{ij})\Big],$$

$$0 = 2x_j^\top(t-h_{ij}) Z_{ij} \Big[-E_i D^{\alpha_i} x_i(t) + A_i x_i(t) + \sum_{j=1,j\neq i}^{K} A_{ij} x_j(t-\beta_{ij})\Big],$$

$$0 = 2x_i^\top(t) R_i N_i \Big[A_i x_i(t) + \sum_{j=1,j\neq i}^{K} A_{ij} x_j(t-\beta_{ij})\Big],$$

we obtain that

$$\begin{aligned} D^{\alpha_i} V_i(x_i(t)) &\le 2x_i(t)^\top P_i E_i D^{\alpha_i} x_i(t) = 2x_i(t)^\top P_i \Big[A_i x_i(t) + \sum_{j\neq i,j=1}^{K} A_{ij} x_j(t-\beta_{ij})\Big] \\ &= 2x_i(t)^\top P_i \Big[A_i x_i(t) + \sum_{j\neq i,j=1}^{K} A_{ij} x_j(t-\beta_{ij})\Big] \\ &\quad - \beta x_i(t)^\top P_i E_i x_i(t) - \sum_{j=1,j\neq i}^{K} \beta x_j(t-\beta_{ij})^\top P_j E_j x_j(t-\beta_{ij}) \\ &\quad + \beta V_i(x_i(t)) + \sum_{j\neq i,j=1}^{N} \beta V_j(x_j(t-\beta_{ij})) \\ &= v_i(t)^\top W(i) v_i(t) + \beta V_i(x_i(t)) + \sum_{j\neq i,j=1}^{K} \beta V_j(x_j(t-\beta_{ij})), \end{aligned}$$

where $v_i(t) = [v_{i1}(t), \dots, v_{i(K+1)}(t)]^\top$, and

$$v_{ii}(t) = x_i(t),\ v_{ij}(t) = x_j(t-\beta_{ij}),\ v_{i(K+1)}(t) = E_i D^{\alpha_i} x_i(t),\ j = 1,\dots,K, j \neq i.$$

applying the condition (6) gives

$$D^{\alpha_i} V_i(x_i(t)) \le \beta V_i(x_i(t)) + \sum_{j=1,j\neq i}^{K} \beta V_j(x_j(t-\beta_{ij})). \tag{10}$$

Let us set $U_i(t) = D^{\alpha_i} V_i(x_i(t)) - \beta V_i(x_i(t))$. Applying the Laplace transform and using Lemma 1, we have

$$s^\alpha \bar{V}_i(s) - V_i(x(0)) s^{\alpha-1} = \beta \bar{V}_i(s) + \bar{U}_i(s),$$

with $\bar{V}_i(s) = L[V_i(x(t))](s),\ \bar{U}_i(s) = L[U_i(t)](s)$, we get

$$\bar{V}_i(s) = (s^\alpha - \beta)^{-1}(V_i(x(0)) s^{\alpha-1} + \bar{U}_i(s)), \tag{11}$$

and then

$$\sup_{s\in[0,t]} U_i(s) \le \sup_{s\in[0,t]} \sum_{j=1,j\neq i}^{K} \beta V_j(x_j(s-\beta_{ij})) \le \beta \sum_{j\neq i,j=1}^{K} \Big(\sup_{\eta\in[-\beta,t-\beta_1]} V_j(x_j(\eta))\Big).$$

Therefore,

$$\begin{aligned} V_i(x_i(t)) =& V_i(x_i(0)) E_{\alpha_i}(\beta t^{\alpha_i}) + \int_0^t U_i(s)(t-s)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(\beta(t-s)^{\alpha_i}) ds \\ \le& V_i(x_i(0)) E_{\alpha_i}(\beta t^{\alpha_i}) + \sup_{s\in[0,t]} U_i(s) \int_0^t (t-s)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(\beta(t-s)^{\alpha_i}) ds \\ =& V_i(x_i(0)) E_{\alpha_i}(\beta t^{\alpha_i}) + \sup_{s\in[0,t]} U_i(s)[E_{\alpha_i}(\beta t^{\alpha_i}) - 1]/\beta \\ \le& V_i(x_i(0)) E_{\alpha_i}(\beta t^{\alpha_i}) + \sum_{j=1,j\neq i}^{K} \Big(\sup_{\eta\in[-\beta,t-\beta_1]} V_j(x_j(\eta))\Big)[E_{\alpha_i}(\beta t^{\alpha_i}) - 1]. \end{aligned}$$

Moreover, note that the functions $E_{\alpha_i}(\beta t^{\alpha_i})$ is increasing, we have for all $t \in [0,T]$:

$$\begin{aligned} \sup_{\eta\in[-\beta,t]} V_i(x_i(\eta)) \le& \sup_{\eta\in[-\beta,0]} V_i(x_i(\eta)) E_{\alpha_i}(\beta T^{\alpha_i}) + \sum_{j\neq i,j=1}^{K} \Big(\sup_{\eta\in[-\beta,t-h_1]} V_j(x_j(\eta))\Big)[E_{\alpha_i}(\beta T^{\alpha_i}) - 1] \\ \le& \sup_{\eta\in[-\beta,0]} V_i(x_i(\eta)) a_1 + \sum_{j=1,j\neq i}^{K} \Big(\sup_{\eta\in[-\beta,t-\beta_1]} V_j(x_j(\eta))\Big)(a_1 - 1), \end{aligned}$$

where $a_1 \geq 1$. Setting $S(t) = \sum_{i=1}^{K} \sup_{\eta\in[-\beta,t]} V_i(x_i(\eta))$, we have

$$S(t) \leq S(0)a_1 + a_2 S(t-\beta_1), \quad \forall t \in [0,T].$$

Since $S(t)$ is non-decreasing, by Proposition 2 we have

$$S(t) \leq S(0)q. \tag{12}$$

Let us denote

$$\mathcal{Y}_1(t) = \sqrt{\sum_{j=1}^{K} \|y_{j1}(t)\|^2}, \quad \mathcal{Y}_2(t) = \sqrt{\sum_{j=1}^{K} \|y_{j2}(t)\|^2}.$$

Using the following inequalities

$$\begin{aligned}
S(0) &= \sum_{i=1}^{K} \sup_{\eta\in[-\beta,0]} V_i(x_i(\eta)) = \sum_{i=1}^{K} \sup_{\eta\in[-\beta,0]} x_i^\top(\eta) P_i E_i x_i(\eta)\\
&\leq \sum_{i=1}^{K} \lambda_{max}(P_i E_i) \sup_{\eta\in[-\beta,0]} x_i^\top(\eta) x_i(\eta)\\
&\leq \lambda_{max}(PE) \sum_{i=1}^{K} \sup_{\eta\in[-\beta,0]} \xi_i^\top(\eta)\xi_i(\eta) = \lambda_{max}(PE)\sum_{i=1}^{K}\|\xi_i\|^2 = \lambda_{max}(PE)\|\xi\|^2,\\
S(t) &= \sum_{i=1}^{K} \sup_{\eta\in[-\beta,t]} V_i(x_i(\eta)) = \sum_{i=1}^{K} \sup_{\eta\in[-\beta,t]} x_i^\top(\eta) P_i E_i x_i(\eta)\\
&\geq \sum_{i=1}^{K} x_i^\top(t) P_i E_i x_i^\top(t) = \sum_{i=1}^{K} x_i^\top(t)[Q_i^{-1}]^\top [Q_i^\top P_i E_i Q_i] Q_i^{-1} x_i^\top(t)\\
&\geq \sum_{i=1}^{K} \lambda_{min}(P_i(11)) y_{i1}^\top(t) y_{i1}(t) \geq \lambda_{min}(P(11))\mathcal{Y}_1(t)^2,
\end{aligned}$$

and the condition (12), we have

$$\lambda_{min}(P(11))\mathcal{Y}_1(t)^2 \leq S(t) \leq qS(0) = q\lambda_{max}(PE)\|\xi\|^2, \ \forall t \in [0,T].$$

Consequently, if $\|\xi\|^2 \leq c_1$, then for all $t \in [0,T]$,

$$\mathcal{Y}_1(t)^2 \leq q\frac{\lambda_{max}(PE)}{\lambda_{min}(P(11))}\|\xi\|^2 \leq q_1 c_1, \ \forall t \in [0,T]. \tag{13}$$

Besides, we see that for $\theta \in [-\beta, 0]$, the following derivations hold

$$\begin{aligned}
\mathcal{Y}_1(\theta)^2 &= \sum_{i=1}^{K} \|y_{i1}(\theta)\|^2 \leq \sum_{i=1}^{K} \|y_i(\theta)\|^2 = \sum_{i=1}^{K} x_i(\theta)^\top [Q_i^{-1}]^\top Q_i^{-1} x_i(\theta)\\
&\leq \max_i \Big(\lambda_{max}([Q_i^{-1}]^\top Q_i^{-1})\Big) \sum_{i=1}^{K} x_i(\theta)^\top x_i(\theta)\\
&\leq \max_i \Big(\lambda_{max}([Q_i^{-1}]^\top Q_i^{-1})\Big) \sum_{i=1}^{K} \|\xi_i\|^2 = \max_i \Big(\lambda_{max}([Q_i^{-1}]^\top Q_i^{-1})\Big)\|\xi\|^2\\
&\leq \max_i \Big(\lambda_{max}([Q_i^{-1}]^\top Q_i^{-1})\Big) c_1.
\end{aligned}$$

Therefore, we have

$$\mathcal{Y}_1(t)^2 \leq q_2 c_1, \quad t \in [-\beta, T]. \tag{14}$$

To estimate $\mathcal{Y}_2(t)$, we consider two cases:

• Case $t \in [-\beta, 0]$:

$$\mathcal{Y}_2(t)^2 = \sum_{j=1}^{K} \|y_{j2}(t)\|^2 \leq \sum_{j=1}^{K} \|y_j(t)\|^2 \leq \max_i \lambda_{max}([Q_j^{-1}]^\top Q_j^{-1}) c_1 \leq q_2 c_1, \tag{15}$$

• Case $t \in [0,T]$: For $i = 1,2,\dots,K$, from the second equation of (4) it follows that

$$\begin{aligned}\|y_{i2}(t)\| \le& |[A_{22}(i)]^{-1}A_{21}(i)||y_{i1}(t)| \\ &+ \sum_{j\neq i,j=1}^{K}\Big[|[A_{22}(i)]^{-1}A_{21}(ij)\|y_{j1}(t-\beta_{ij})| + |[A_{22}(i)]^{-1}A_{22}(ij)\|y_{j2}(t-\beta_{ij})|\Big] \\ &\le a_0\Big(|y_{i1}(t)| + \sum_{j\neq i,j=1}^{K}|y_{j1}(t-\beta_{ij})|\Big) + a_0\sum_{j=1,j\neq i}^{K}|y_{j2}(t-\beta_{ij})| \\ &\le a_0\Big(\mathcal{Y}_1(t) + \sum_{j\neq i,j=1}^{K}\mathcal{Y}_1(t-\beta_{ij})\Big) + a_0\sum_{j=1,j\neq i}^{K}|y_{j2}(t-\beta_{ij})| \\ &\le q_3\sqrt{c_1} + a_0\sum_{j\neq i,j=1}^{K}|y_{j2}(t-\beta_{ij})|,\end{aligned}$$

hence for $t \in [0,\beta_1]$, we get

$$\|y_{i2}(t)\| \le q_3\sqrt{c_1} + a_0\sum_{j=1,j\neq i}^{K}\mathcal{Y}_2(t-\beta_{ij}) \le \Big(q_3 + a_0(K-1)\sqrt{q_2}\Big)\sqrt{c_1}.$$

Similarly, for $t \in [\beta_1, 2\beta_1]$, we get

$$\begin{aligned}\|y_{i2}(t)\| &\le q_3\sqrt{c_1} + a_0\sum_{j\neq i,j=1}^{K}|y_{j2}(t-\beta_{ij})| \\ &\le q_3\sqrt{c_1} + (K-1)a_0\Big(q_3 + a_0(K-1)\sqrt{q_2}\Big)\sqrt{c_1} \\ &= \Big(q_3 + (K-1)a_0q_3 + [a_0(K-1)]^2\sqrt{q_2}\Big)\sqrt{c_1}.\end{aligned}$$

By induction, for $t \in [k\beta_1, k\beta_1+\beta_1],\ k\beta_1 \le T$, we have

$$\|y_{i2}(t)\| \le \Big(q_3\sum_{j=1}^{k}[a_0(K-1)]^j + [a_0(K-1)]^{k+1}\sqrt{q_2}\Big)\sqrt{c_1} \le q_4\sqrt{c_1}. \tag{16}$$

Taking the estimation (15), (16) into account, we have

$$\mathcal{Y}_2(t)^2 \le Kq_4^2c_1, \quad t \in [0,T]. \tag{17}$$

Consequently, using (13) and (17) for all $t \in [0,T]$, gives

$$\begin{aligned}\|x(t)\|^2 = \sum_{i=1}^{K}\|x_i(t)\|^2 = \sum_{i=1}^{K}y_i(t)^\top[Q_i]^\top Q_iy_i(t) &\le \max_i \lambda_{max}([Q_i]^\top Q_i)\sum_{i=1}^{K}y_i(t)^\top y_i(t) \\ &= \bar{\beta}\sum_{i=1}^{K}y_{i1}(t)^\top y_{i1}(t) + \bar{\beta}\sum_{i=1}^{K}y_{i2}(t)^\top y_{i2}(t) \\ &= \bar{\beta}\mathcal{Y}_1(t)^2 + \bar{\beta}\mathcal{Y}_2(t)^2 \le \bar{\beta}(q_1 + Kq_4^2)c_1 \le c_2. \quad \square\end{aligned}$$

In the sequel, we present sufficient conditions for designing stabilizing controllers $u_i(t) = K_ix_i(t)$ for the stabilization problem. For this, the following notations are used:

$$\bar{A}_i = A_i + B_iY_iP_i^{-1},\ H_i\bar{A}_iQ_i = \begin{bmatrix}\bar{A}_i(11) & \bar{A}_i(12)\\ \bar{A}_i(21) & \bar{A}_i(22)\end{bmatrix},$$

$$[Q_i]^\top P_i^{-1}E_iQ_i = \begin{bmatrix}\bar{P}_i(11) & 0\\ 0 & 0\end{bmatrix},\ i = 1,2,\dots,K,\ N_i = \begin{bmatrix}0 & 0\\ 0 & I_{n_i-r_i}\end{bmatrix}H_i,$$

$$\bar{\gamma} = \max_{i,j}\{|[\bar{A}_i(22)]^{-1}A_{ij}(22)|,\ |[\bar{A}_i(22)]^{-1}\bar{A}_i(21)|,\ |[\bar{A}_i(22)]^{-1}A_{ij}(21)|\};$$

$$\lambda_{max}(P^{-1}E) = \max_i \lambda_{max}(P_i^{-1}E_i),$$

$$\lambda_{min}(\bar{P}(11)) = \min_i \lambda_{min}(\bar{P}_i(11));$$

$$\bar{q}_1 = q\frac{\lambda_{max}(P^{-1}E)}{\lambda_{min}(\bar{P}(11))},\ \bar{q}_2 = \bar{q}_1 + \max_i \lambda_{max}([Q_i^{-1}]^\top Q_i^{-1}),\ \bar{q}_3 = \bar{\gamma}K\sqrt{\bar{q}_2};$$

$$\bar{q}_4 = \max_{k\in 0,[T/\beta_1]+1}\Big(\bar{q}_3\sum_{j=0}^{k}[\bar{\gamma}(K-1)]^j + [\bar{\gamma}(K-1)]^{k+1}\sqrt{\bar{q}_2}\Big),\ \bar{\beta} = \max_i \lambda_{max}([Q_i]^\top Q_i).$$

Theorem 5. *Given numbers $c_1 > 0, c_2 > 0, T > 0$, the system (1) is finite-time stabilizable w.r.t (c_1, c_2, T), if there exist invertible matrices P_i satisfying $E_i P_i^\top = [E_i P_i^\top]^\top \geq 0$ and matrices Y_i such that*

$$\Sigma(i) < 0, \quad i = 1, 2, \dots, K, \tag{18}$$

$$\bar{\beta}(\bar{q}_1 + K\bar{q}_4^2)c_1 \leq c_2. \tag{19}$$

Moreover, the stabilizing controllers are defined as

$$u_i(t) = K_i x_i(t) = Y_i P_i^{-1} x_i(t), \quad i = 1, 2, \dots, K,$$

where

$$\Sigma(i) = \begin{bmatrix} \Omega(i)_{1,1} & \Omega(i)_{1,2} & \dots & \Omega(i)_{1,K+1} \\ * & \Omega(i)_{22} & \dots & \Omega(i)_{2,K+1} \\ . & . & \dots & . \\ * & * & \dots & \Omega(i)_{K+1,K+1} \end{bmatrix},$$

$$\Omega(i)_{ii} = A_i P_i^\top + [A_i P_i^\top]^\top + B_i Y_i + [B_i Y_i]^\top - \beta E_i P_i^\top + L_i A_i P_i^\top + [L_i A_i P_i^\top]^\top + L_i B_i Y_i + [L_i B_i Y_i]^\top,$$

$$\Omega(i)_{jj} = -\beta E_j P_j^\top - [A_{ij}]^\top A_{ij} P_j^\top - P_j [A_{ij}]^\top A_{ij},$$

$$\Omega(i)_{ij} = A_{ij} P_j^\top - P_i [A_i]^\top A_{ij} - [B_i Y_i]^\top A_{ij} + L_i A_{ij} P_j^\top, \quad \Omega(i)_{ji} = [\Omega(i)_{ij}]^\top,$$

$$\Omega(i)_{K+1,K+1} = -P_i - P_i^\top, \ \Omega(i)_{K+1,i} = A_i P_i + B_i Y_i, \ \Omega(i)_{i,K+1} = [\Omega(i)_{K+1,i}]^\top,$$

$$\Omega(i)_{K+1,j} = A_{ij} P_j^\top + P_i A_{ij}, \quad j \neq i, \ j = 1, 2, \dots, K,$$

the other terms of the $\Sigma(i)$ are zero.

Proof. Let $\bar{A}_i = A_i + B_i K_i$. Taking $V_i(x_i) = x_i^\top P_i^{-1} E_i x_i$ and

$$L_i = P_i^{-1}, \ K_i = Y_i P_i^{-1}, \ Z_{ij} = -P_j^{-1}[A_{ij}]^\top, \ R_i = P_i^{-1}, \quad i = 1, 2, \dots, K.$$

by the similar way of the proof of Theorem 4, the closed-loop equation

$$E_i D^{\alpha_i} x_i(t) = \bar{A}_i x_i(t) + \sum_{j=1, j\neq i}^{K} A_{ij} x_j(t - \beta_{ij}(t)), \quad t \geq 0,$$

is FTS w.r.t (c_1, c_2, T) if there exist invertible matrices P_i such that

$$W(i) < 0, \quad i = 1, 2, \dots, K, \tag{20}$$

$$\bar{\beta}(\bar{q}_1 + K\bar{q}_4^2)c_1 \leq c_2, \tag{21}$$

where

$$W(i)_{i,i} = P_i^{-1}\bar{A}_i + [P_i^{-1}\bar{A}_i]^\top - \beta P_i^{-1} E_i + R_i L_i \bar{A}_i + [R_i L_i \bar{A}_i]^\top,$$

$$W(i)_{j,j} = -\beta P_j^{-1} E_j + Z_{ij} A_{ij} + [Z_{ij} A_{ij}]^\top, \ W(i)_{i,j} = P_i^{-1} A_{ij} + [Z_{ij}\bar{A}_i]^\top + R_i L_i A_{ij},$$

$$W(i)_{K+1,K+1} = -L_i - [L_i]^\top, \ \bar{W}(i)_{K+1,i} = L_i \bar{A}_i, \ \bar{W}(i)_{i,K+1} = [L_i \bar{A}_i]^\top,$$

$$W(i)_{K+1,j} = L_i A_{ij} - [Z_{ij}]^\top, \ j \neq i, \ j = 1, 2, \dots, K,$$

the other elements of the matrix $W(i)$ are zero.

Indeed, the LMI condition (20) leads to

$$W(i)_{i,i} = P_i^{-1}\bar{A}_i + [P_i^{-1}\bar{A}_i]^\top - \beta P_i^{-1} E_i < 0, \quad i = 1, 2, \dots, K.$$

and the regularity and impulse-free are derived by the same augments of the proof in Theorem 4. Besides, the condition $P_i^{-1} E_i = [P_i^{-1} E_i]^\top$ leads to

$$E_i P_i^\top = P_i\Big(P_i^{-1} E_i\Big) P_i^\top = P_i\Big([P_i^{-1} E_i]^\top\Big) P_i^\top = P_i E_i^\top.$$

Moreover, from the relations

$$\Sigma(i) = \mathrm{diag}\{P_1, P_2, \dots, P_K, P_i\} W(i)\, \mathrm{diag}\{P_1^\top, P_2^\top, \dots, P_K^\top, P_i^\top\},$$

$$E_iP_i^\top = P_i\left(P_i^{-1}E_i\right)P_i^\top,$$

and from the invertibility of P_i, it follows that the condition (20) and (21) are equivalent to the condition (18) and (19), respectively. □

Remark 1. Theorem 4 and Theorem 5 provide FTS criteria and stabilization conditions via solving tractable LMIs (18), (20), which are solved by using LMI Control Toolbox [25]. Note that the condition $P_iE_i = [P_iE_i]^\top \geq 0$ is not an LMI, but it can be reduced into a single strict LMI. Moreover, since the parameters c_1, c_2, do not involve in the LMI (6) (or (18)), we first determine matrix solutions P_i, Z_i, L_i, R_i from the LMI and then we can easily check the conditions (7), (19).

Remark 2. Note that the derived condition (18), (19) for designing the stabilizing controllers involve some parameters c_1, c_2, K, T. In addition, for given K, T if $c_1 > 0$, the parameter $c_2 > 0$ can be defined as an optimization parameter problem for finding the minimal value of c_2.

Remark 3. Compared with previous works, the problem is studied in [13,15] for fractional interconnected systems, however the singularity was not considered therein. For singular fractional systems, the problem is considered in [3,22–24], however, the large-scale structure was not involved therein. To the author's best knowledge, it is for the first time that the sufficient conditions are proposed for the FTS and stabilization of the FSISs with interval time-variable delays.

Remark 4. It is notable that in this paper, we have considered the system (1) with the fractional derivative $0 < \alpha_i < 1$. For the large-scale classical systems ($\alpha_i = 1$), our results are still valid since the technical lemmas (Lemma 1, Lemma 3) used in the proofs are applicable. The problem was studied in [12,18,28,29] for the singular large-scale classical systems, however, the system considered therein subjected to no delays.

4. Examples and simulations

Example 1. Consider unforced FSISs (2) ($K = 3$) described by the following equations

$$\begin{cases} E_1D^{\alpha_1}x_1(t) &= A_1x_1(t) + A_{12}x_2(t-\beta_{12}) + A_{13}x_3(t-\beta_{13}),\\ x_1(t) &= \xi_1(t), \quad t\in[-\beta,0],\end{cases}$$
$$\begin{cases} E_2D^{\alpha_2}x_2(t) &= A_2x_2(t) + A_{21}x_1(t-\beta_{21}) + A_{23}x_3(t-\beta_{23}),\\ x_2(t) &= \xi_2(t), \quad t\in[-\beta,0],\end{cases}$$
$$\begin{cases} E_3D^{\alpha_3}x_3(t) &= A_3x_3(t) + A_{31}x_1(t-\beta_{31}) + A_{32}x_2(t-\beta_{32}),\\ x_3(t) &= \xi_3(t), \quad t\in[-\beta,0],\end{cases}$$

where $\alpha_1 = 0.1, \alpha_2 = 0.15, \alpha_3 = 0.2$, and

$$E_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad E_2 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad E_3 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$
$$A_1 = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}, A_2 = \begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}, A_3 = \begin{bmatrix}-1.5 & 0.5\\ 0 & -1\end{bmatrix},$$
$$A_{12} = \begin{bmatrix}0.015 & 0.01\\ 0.01 & 0.015\end{bmatrix},\quad A_{13} = \begin{bmatrix}0.01 & 0.015\\ 0.015 & 0.01\end{bmatrix}, A_{21} = \begin{bmatrix}0.02 & 0.01\\ 0.01 & 0.02\end{bmatrix},$$
$$A_{23} = \begin{bmatrix}0.01 & 0.02\\ 0.02 & 0.01\end{bmatrix}, A_{31} = \begin{bmatrix}0.015 & 0.015\\ 0.01 & 0.01\end{bmatrix},\quad A_{32} = \begin{bmatrix}0.01 & 0.01\\ 0.015 & 0.015\end{bmatrix},$$
$$\xi_1(t) = [0.4; 0.3]^T,\ \xi_2(t) = [0.4; 0.3]^T,\ \xi_3(t) = [0.4; 0.3]^T,$$
$$\beta_{12}(t) = \frac{1}{10} + \frac{5\sin^2(t)}{100},\ \beta_{13}(t) = \frac{1}{10} + \frac{\cos^2(t)}{100}, \beta_{21}(t) = \frac{1}{10} + \frac{2\sin^2(t)}{100},$$
$$\beta_{23}(t) = \frac{1}{10} + \frac{3\cos^2(t)}{100}, \beta_{31}(t) = \frac{1}{10} + \frac{3\sin^2(t)}{100},\ \beta_{32}(t) = \frac{1}{10} + \frac{2\cos^2(t)}{100},$$
$$\beta_1 = 0.1, \quad \beta = 0.15.$$

In the case we have

$$H_1 = H_2 = H_3 = Q_1 = Q_2 = Q_3 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

By using LMI Toolbox [25], the LMIs (6) have the following solutions:

$$P_1 = \begin{bmatrix}3.6400 & 0\\ 0 & 3.0323\end{bmatrix},\ P_2 = \begin{bmatrix}2.2297 & 0\\ 0 & 3.0323\end{bmatrix},\ P_3 = \begin{bmatrix}2.8608 & 0\\ 0 & 3.0323\end{bmatrix},$$

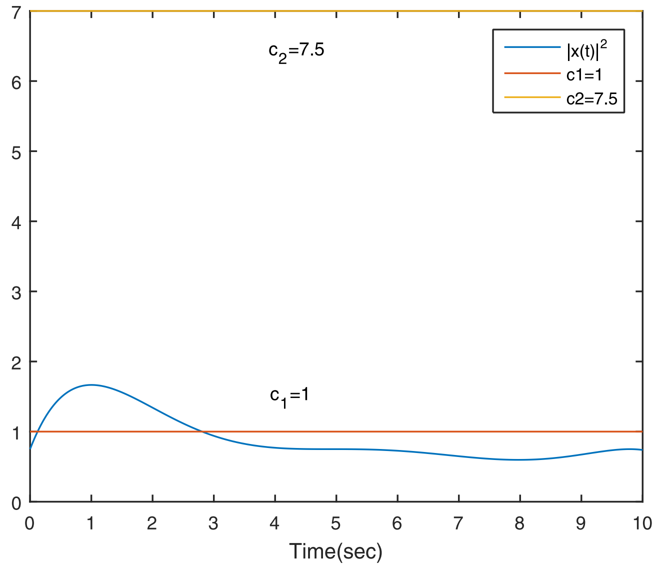

Fig. 1. The behaviour of $\|x(t)\|^2$ in Example 1.

$$Z_{12}=\begin{bmatrix}-0.0484 & 0.0120\\ -0.1263 & -0.0010\end{bmatrix},\ Z_{13}=\begin{bmatrix}-0.0781 & 0.0210\\ -0.2432 & 0.0059\end{bmatrix},$$

$$Z_{21}=\begin{bmatrix}-0.1024 & 0.0102\\ -0.2642 & 0.0068\end{bmatrix},\ Z_{23}=\begin{bmatrix}-0.0607 & 0.0217\\ -0.4226 & -0.0200\end{bmatrix},$$

$$Z_{31}=\begin{bmatrix}-0.0599 & 0.0230\\ -0.3315 & -0.0025\end{bmatrix},\ Z_{32}=\begin{bmatrix}-0.0275 & 0.0210\\ -0.1196 & -0.0051\end{bmatrix},$$

$$L_1=\begin{bmatrix}1.6481 & 0.2413\\ 0.2413 & 1.7949\end{bmatrix},\ L_2=\begin{bmatrix}1.4735 & 0.2030\\ 0.2030 & 1.7053\end{bmatrix},\ L_3=\begin{bmatrix}1.6443 & 0.1115\\ 0.1115 & 1.7598\end{bmatrix},$$

$$R_1=\begin{bmatrix}0 & 2.9413\\ 0 & -0.7712\end{bmatrix},\ R_2=\begin{bmatrix}0 & 0.7131\\ 0 & -0.8478\end{bmatrix},\ R_2=\begin{bmatrix}0 & 0.8703\\ 0 & -1.0104\end{bmatrix}.$$

Moreover, we check the condition (7) as follows.

$$P_1E_1=E_1^\top P_1^\top=\begin{bmatrix}3.6400 & 0\\ 0 & 0\end{bmatrix}\ge 0,\ P_2E_2=E_2^\top P_2^\top=\begin{bmatrix}2.2297 & 0\\ 0 & 0\end{bmatrix}\ge 0,$$

$$P_3E_3=E_3^\top P_3^\top=\begin{bmatrix}2.8608 & 0\\ 0 & 0\end{bmatrix}\ge 0,$$

and

$$\gamma=0.02,\ a_1=1.3421,\ a_2=0.6841,\ \bar\beta=1,$$

$$q=4.2485,\ q_1=6.9356,\ q_2=7.9356,\ q_3=0.1690,\ q_4=0.2817.$$

For $c_1=1;\ c_2=7.5,\ T=10$, the condition (7) holds:

$$\bar\beta(q_1+Kq_4^2)c_1=7.1737\le c_2=7.5.$$

Applying Theorem 4, the finite-time stability w.r.t. $(1,7.5,10)$ of the equation is derived.

Fig. 1 demonstrates the behaviour of $\|x(t)\|^2$ of Example 1.

Example 2. Consider control FSISs (1), where $K=3$, $\alpha_1=0.1$, $\alpha_2=0.15$, $\alpha_3=0.2$,

$$E_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad E_2=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad E_3=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

$$A_1=\begin{bmatrix}0.5 & 0\\ 0.1 & -1\end{bmatrix},\quad A_2=\begin{bmatrix}1 & 0.1\\ 0.1 & -1\end{bmatrix},\quad A_3=\begin{bmatrix}1.5 & 0.2\\ 0.2 & -1\end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0.01 & 0.02 \\ 0.01 & 0.01 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 0.02 & 0.03 \\ 0.01 & 0.001 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.02 & 0.01 \\ 0.01 & 0.01 \end{bmatrix},$$

$$A_{23} = \begin{bmatrix} 0.02 & 0.01 \\ 0.01 & 0.001 \end{bmatrix}, \quad A_{31} = \begin{bmatrix} 0.03 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.001 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\xi_1(t) = [0.4; 0.3]^T, \; \xi_2(t) = [0.4; 0.3]^T, \; \xi_3(t) = [0.4; 0.3]^T,$$

$$\beta_{12}(t) = \frac{1.2}{10} + \frac{3\sin^2(t)}{100}, \; \beta_{13}(t) = \frac{1.3}{10} + \frac{2\cos^2(t)}{100}, \beta_{21}(t) = \frac{1}{10} + \frac{2\sin^2(t)}{100},$$

$$\beta_{23}(t) = \frac{1.2}{10} + \frac{3\cos^2(t)}{100}, h_{31}(t) = \frac{1}{10} + \frac{3\sin^2(t)}{100}, \; \beta_{32}(t) = \frac{1.2}{10} + \frac{2\cos^2(t)}{100},$$

$$\beta_1 = 0.1, \; \beta = 0.15, \; H_1 = H_2 = H_3 = Q_1 = Q_2 = Q_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The LMI conditions (18) are feasible with

$$P_1 = 10^5 \begin{bmatrix} 1.7500 & 0 \\ 0 & 0.1175 \end{bmatrix}, \quad P_2 = 10^5 \begin{bmatrix} 1.1778 & 0 \\ 0 & 0.0601 \end{bmatrix},$$

$$P_3 = 10^5 \begin{bmatrix} 1.2764 & 0 \\ 0 & 0.1262 \end{bmatrix}, \quad Y_1 = 10^4 \begin{bmatrix} -7.9977 & -0.5398 \\ -0.5398 & -1.3703 \end{bmatrix},$$

$$Y_2 = 10^5 \begin{bmatrix} -1.1537 & -0.1249 \\ -0.1249 & -0.2117 \end{bmatrix}, \quad Y_3 = 10^5 \begin{bmatrix} -1.8713 & -0.1397 \\ -0.1397 & -0.4325 \end{bmatrix}.$$

Also, we check the condition (19) as follows.

$$E_1 P_1^\top = P_1 E_1^\top = 10^5 \begin{bmatrix} 1.7500 & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

$$E_2 P_2^\top = P_2 E_2^\top = 10^5 \begin{bmatrix} 1.1778 & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

$$E_3 P_3^\top = P_3 E_3^\top = 10^5 \begin{bmatrix} 1.2764 & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

and we can find the positive scalars

$$\bar{\gamma} = 0.0319, \; a_1 = 1.3421, \; a_2 = 0.6841, \; \bar{\beta} = 1,$$

$$\bar{q}_1 = 6.3124, \; \bar{q}_2 = 7.3124, \; \bar{q}_3 = 0.2590, \; \bar{q}_4 = 0.4317$$

such that for $c_1 = 1; \; c_2 = 7, \; T = 10$, the condition (19) is satisfied:

$$\bar{\beta}(\bar{q}_1 + K\bar{q}_4^2)c_1 = 6.8715 \leq c_2 = 7.$$

Applying Theorem 5, the finite-time stabilizability w.r.t. $(1, 7, 10)$ of the equation is derived. The stabilizing controllers are defined by

$$u_1(t) = K_1 x_1(t) = \begin{bmatrix} -0.4570 & -0.4593 \\ -0.0308 & -1.1658 \end{bmatrix} x_1(t),$$

$$u_2(t) = K_2 x_2(t) = \begin{bmatrix} -0.9795 & -2.0776 \\ -0.1060 & -3.5225 \end{bmatrix} x_2(t).$$

$$u_3(t) = K_3 x_3(t) = \begin{bmatrix} -1.4661 & -1.1072 \\ -0.1094 & -3.4277 \end{bmatrix} x_3(t).$$

Fig. 2 illustrates the behaviour of $\|x(t)\|^2$ of the system (1) without control, which shows that the system (1) without control ($u_i(t) = 0$) is not FTS. However, Fig. 3 shows that the control system (1) under the above defined controllers $u_i(t) = K_i x_i(t), i = 1, 2, 3,$ is FTS w.r.t $(1, 7, 10)$.

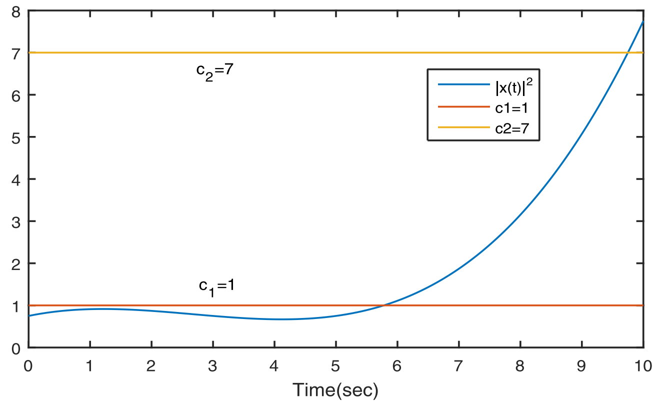

Fig. 2. The behaviour of $\|x(t)\|^2$ in Example 2 without controllers.

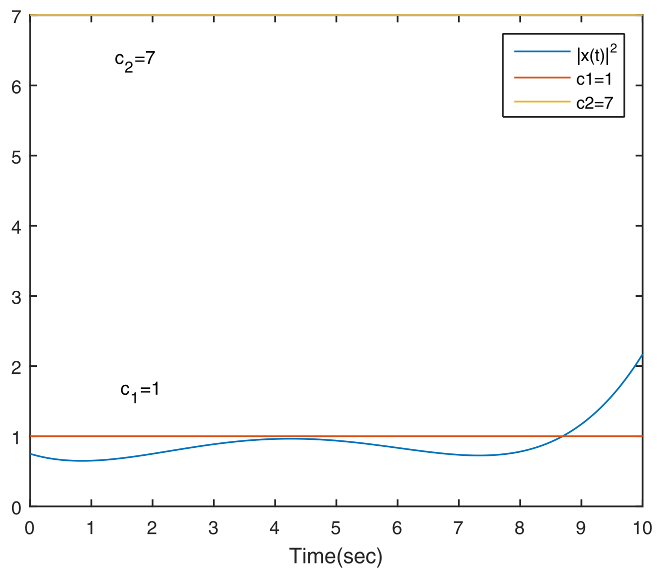

Fig. 3. The behaviour of $\|x(t)\|^2$ in Example 2 with controllers.

5. Conclusion

In this paper, an analytical approach based on fractional calculus and singular value theory is proposed to study problem of stability and stabilization for FSISs with time-varying delay. We have first shown the finite-time stability conditions and then, proposed a control design of feedback stabilizing controllers in the terms of the Laplace transform, the Mittag-Leffler functions and a tractable LMI. Finally, two numerical examples with simulations are included to demonstrate the validity and effectiveness of the obtained results.

CRediT authorship contribution statement

Nguyen T. Thanh: Writing – review & editing, Software, Investigation. **Vu N. Phat:** Supervision, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Podlubny I. Fractional differential equations. San Diego: Academic Press; 1999.
[2] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam: Elsevier Science; 2006.
[3] Liu S, Zhou XF, Li X, Jiang W. Asymptotical stability of Riemann–Liouville fractional singular systems with multiple time-varying delays. Appl Math Lett 2017;65:32–9.
[4] Camacho NA, Duarte-Mermoud MA, Gallegos JA. Lyapunov functions for fractional order systems. Commun Nonl Sci Numer Simul 2014;19:2951–7.
[5] Mahmoud M, Hassen M, Darwish M. Large-scale control systems: theories and techniques. New York: Marcel-Dekker; 1985.
[6] Kwon OM, Park JH. Decentralized guaranteed cost control for uncertain large-scale systems using delayed feedback: LMI optimization approach. J Optim Theory Appl 2006;129:391–414.
[7] Yin W, Cao J. On stability of large-scale G-SDEs: A decomposition approach. Appl Math Comput 2021;388:125466.
[8] Lee K, Bhattacharya R. Stability analysis of large-scale distributed networked control systems with random communication delays: A switched system approach. Systems Control Lett 2015;85:77–83.
[9] Dorato P. Short time stability in linear time-varying systems. In: Proceedings of IRE int. convention record. Vol. 4, 1961, p. 83–7.
[10] Amato F, Ambrosino R, Ariola M, Cosentino C. Finite-time stability and control lecture notes in control and information sciences. New York: Springer; 2014.
[11] La-inchua T, Niamsup P, Liu X. Finite-time stability of large-scale systems with interval time-varying delay in interconnection. Complexity 2017;1972748:1–11.
[12] Tharanidharan V, Sakthivel R, Kaviarasan B, Faris A, Anthoni S. Finite-time boundedness of large-scale systems with actuator faults and gain fluctuations. Int J Robust Nonlinear Control 2019;29:3042–62.
[13] Zhang D, Xu H, Wang H, Chen X. Stability analysis for interconnected system with quantized sampled-data control. Appl Math Model 2022;102:313–30.
[14] Thanh NT, Phat VN, Niamsup P. New finite-time stability analysis of singular fractional differential equations with time-varying delay. Fract Calc Appl Anal 2020;23:504–17.
[15] Boukal Y, Darouach M, Zasadzinski M, Radhy NE. Large-scale fractional-order systems: stability analysis and their decentralised functional observers. IET Control Theory Appl 2018;12:359–67.
[16] Dai L. Singular control systems. Berlin: Springer; 1981.
[17] Ma Y, Jia X, Liu D. Finite-time dissipative control for singular discrete-time Markovian jump systems with actuator saturation and partly unknown transition rates. Appl Math Model 2018;53:49–70.
[18] Wo S, Han X. Finite-time robust decentralized control for uncertain singular large-scale systems with exogenous disturbances. IMA J Math Control Inform 2019;36:1133–48.
[19] Liu S, Yang R, Zhou XF, Jiang W, Li X, Zhao XW. Stability analysis of fractional delayed equations and its applications on consensus of multi-agent systems Author links open overlay panel. Commun Nonl Sci Numer Simul 2019;73:351–62.
[20] Wen Y, Zhou XF, Zhang Z, Liu S. Lyapunov method for nonlinear fractional differential systems with delay. Nonlinear Dynam 2015;82:1015–25.
[21] Zhang Z, Jiang W. Some results of the degenerate fractional differential system with delay. Comput Math Appl 2011;62:1284–91.
[22] Mathiyalagan K, Balachandran K. Finite-time stability of fractional- order stochastic singular systems with time delay and white noise. Complexity 2016;21:370–9.
[23] Yin C, Zhong SM, Huang X, Cheng Y. Robust stability analysis of fractional-order uncertain singular nonlinear system with external disturbance. Appl Math Comput 2015;269:351–62.
[24] Pang D, Jiang W. Finite-time stability analysis of fractional singular time-delay systems. Adv Differ Equ 2014;259:1–11.
[25] Gahinet P, Nemirovskii A, Laub AJ, Chilali M. LMI control toolbox for use with matlab. Massachusetts: The MathWorks Inc; 1985.
[26] Zhang H, Wu D, Cao J, Zhang H. Stability analysis for fractional-order linear singular delay differential systems. Discrete Dyn Nat Soc 2014;850279:1–8.
[27] Trinh H, Tuan HT. Stability of fractional-order nonlinear systems by Lyapunov direct method. IET Control Theory Appl 2018;12:2417–22.
[28] Labibi B, Lohmann B, Sedigh AK, Maralani PJ. Decentralized stabilization of large-scale systems via state-feedback and using descriptor systems. IEEE Trans Syst Man Cyber A: Syst Hum 2023;33:771–6.
[29] Zamani I, Shafiee M, Shafieirad M, Zeinali M. Optimal control of large-scale singular linear systems via hierarchical strategy. Trans Inst Measur Control 2018;41:2250–67.