PARTIAL DIFFERENTIAL EQUATIONS *AND* APPLICATIONS 2017

SCIENTIFIC WORKSHOP

PARTIAL DIFFERENTIAL EQUATIONS *AND* APPLICATIONS

On the occasion of
Professor Nguyễn Mạnh Hùng's 60th birthday

Hà Nội, Oct 2017

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AND

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Time & Venue

Time: Wednesday, October 4, 2017
Venue: C2-714, VIASM, 7th floor, Ta Quang Buu library

Objective

The aim of this workshop is to create a forum for Vietnamese experts in PDEs to exchange new ideas and results. There will be a special session of the Workshop devoted to Professor Nguyen Manh Hung's 60th birthday.

Invited Speakers

Dinh Nho Hao (Institute of Mathematics, VAST)

Nguyen Xuan Thao (Hanoi University of Science and Technology)

Cung The Anh (Hanoi National University of Education)

Ngo Quoc Anh (University of Science, VNU)

Phan Quoc Hung (Duy Tan University)

Vu Trong Luong (Tay Bac University)

Organizing Committee

Nguyen Huu Du (VIASM)

Cung The Anh (HNUE)

Tran Dinh Ke (HNUE)

Sponsors

Vietnam Institute for Advanced Study in Mathematics (VIASM)

Division of Mathematical Analysis, Department of Mathematics, Hanoi National University of Education (HNUE)

Contents

Workshop on Partial Differential Equations and Applications

VIASM, October 4, 2017

On the occasion of Professor Nguyen Manh Hung's 60th birthday

SCIENTIFIC PROGRAM

Morning Session

8h30-9h00: Registration in the 7th floor of Ta Quang Buu library

9h00-9h05: Opening

Chair: Trần Đình Kế

9h05-9h35: Đinh Nho Hào

Inverse problems with nonnegative and sparse solutions: Algorithms and application to the phase retrieve problem

9h35-10h05: **Ngô Quốc Anh**

Bất đẳng thức Hardy-Littlewood-Sobolev ngược trên R^n

10h05-10h30: Coffee Break

Chair: Lê Văn Hiện

10h30-11h00: Vũ Trọng Lưỡng

Nonlinear hyperbolic partial differential equations on nonsmooth domains

11h00-11h30: Phan Quốc Hưng

A Liouville-type theorem for a cooperative parabolic system

11h45-14h00: Lunch

Afternoon Session

Chair: Đinh Nho Hào

14h00-14h30: Nguyễn Xuân Thảo

Discrete-time Fourier sine integral transforms

14h30-15h00: Cung Thế Anh

Local exact controllability to trajectories of the magneto-micropolar fluid equations

15h00-15h30: Coffee Break

Chair: Cung Thế Anh

15h30-16h00: Trần Đình Kế

A brief on Professor Nguyen Manh Hung's scientific life

16h00-16h30:

Celebration of Professor Nguyen Manh Hung's 60th birthday

16h30-16h35: Closing

16h45-19h00: Workshop Banquet

A brief of Professor Nguyễn Mạnh Hùng

VÀI NÉT VỀ GIÁO SƯ NGUYỄN MẠNH HÙNG

NGƯT.GS.TSKH. Nguyễn Mạnh Hùng sinh ngày 6 tháng 10 năm 1957 tại Đan Phượng, Hà Nội. Năm 1978, ông tốt nghiệp loại giỏi Khoa Toán (nay là Khoa Toán-Tin), Trường Đại học Sư phạm Hà Nội, và được giữ lại Trường làm cán bộ giảng dạy. Năm 1980, ông tốt nghiệp Hệ sau đại học (nay là hệ Thạc sĩ) của Khoa. Từ năm 1978 đến năm 1990, ông giảng dạy tại Khoa Toán, trường Đại học Sư phạm Hà Nội.

Trong giai đoạn 1990-1999, ông là nghiên cứu sinh và sau đó là thực tập sinh sau tiến sĩ tại Khoa Toán Cơ, Đại học Tổng hợp Quốc gia Moskva mang tên Lomonosov (MGU) dưới sự hướng dẫn của GS nổi tiếng V.A. Kondratiev. Ông bảo vệ luận án TS năm 1994 và TSKH năm 1999 tại trường đại học danh tiếng này. Ông được phong PGS năm 2002 và GS năm 2011.

Sau khi về nước năm 1999, ông tiếp tục công tác tại Khoa Toán-Tin, Trường Đại học Sư phạm Hà Nội. Ông đã trải qua nhiều vị trí khác nhau. Từ 1999 đến 2004 là giảng viên, từ 2004 đến 2011 là Trưởng Bộ môn Giải tích, năm học 2006-2007 là Trưởng Ban thanh tra nhân dân Trường ĐHSP Hà Nội, từ năm 2007 đến 2012 là Trưởng phòng Tạp chí và Thông tin khoa học công nghệ, Trường ĐHSP Hà Nội. Trong giai đoạn là Trưởng Bộ môn, ông đã có công lớn trong việc xây dựng Bộ môn Giải tích trở thành một bộ môn mạnh của Trường, được tặng Giấy chứng nhận của Bộ Giáo dục và Đào tạo về điển hình tiên tiến giai đoạn 2006-2010 và Giải tập thể tiêu biểu về Khoa học Công nghệ của Trường ĐHSP Hà Nội năm 2012. Dưới sự lãnh đạo của ông, Bộ môn Giải tích gồm nhiều cán bộ trẻ có năng lực đã trở thành một trung tâm mạnh về nghiên cứu và đào tạo sau đại học chuyên ngành Phương trình vi phân và tích phân ở Việt Nam. Trong giai đoạn làm Trưởng phòng Tạp chí và Thông tin Khoa học công nghệ, ông là người có công lớn trong việc từng bước xây dựng Phòng theo hướng chuẩn hóa và đưa Tạp chí khoa học của Trường ĐHSP Hà Nội trở thành một trong bốn tạp chí khoa học hàng đầu của các trường đại học và đã được Bộ Giáo dục và Đào tạo chọn để đầu tư nâng cấp theo chuẩn tạp chí khoa học ISI.

Từ tháng 10/2012 đến nay, ông chuyển sang làm công tác quản lí tại Học viện Quản lí Giáo dục. Từ tháng 10/2012 đến tháng 12/2013 ông làm Phó Giám đốc Học viện, sau đó là Phó Giám đốc phụ trách Học viện từ tháng 1/2014 đến tháng 11/2016, và từ tháng 11/2016 đến nay, ông là Phó Giám đốc Học viện. Ông cũng là Bí thư Đảng ủy Học viện Quản lí Giáo dục từ tháng 1/2014 đến nay.

Về chuyên môn, GS.TSKH Nguyễn Mạnh Hùng là một chuyên gia có uy tín của Việt Nam về Phương trình đạo hàm riêng. Hướng nghiên cứu chính của Giáo sư là nghiên cứu một cách hệ thống các bài toán biên ban đầu đối với các hệ phương trình đạo hàm riêng tuyến tính không dừng, bao gồm hệ parabolic, hệ hyperbolic và hệ Schrodinger, trong các trụ hữu hạn hoặc vô hạn có đáy là miền không trơn. Các nghiên cứu tập trung vào sự tồn tại duy nhất nghiệm, tính trơn của nghiệm và công thức biểu diễn tiệm cận nghiệm trong lân cận các điểm kì dị.

Theo hướng nghiên cứu này, GS Nguyễn Mạnh Hùng đã công bố hơn 50 bài báo khoa học trên các tạp chí chuyên ngành quốc tế được tờ Mathematical Reviews của Hội Toán học Mỹ điểm danh, với hơn 30 bài báo trên các tạp chí trong danh mục ISI, trong đó có những tạp chí uy tín cao như J. Differential Equations, Nonlinear Anal., Sbornik Math., Russian Math. Surveys, Dokl. Akad. Nauk, Differential Equations,…

Giáo sư cũng là chủ nhiệm của 02 đề tài nghiên cứu cấp Nhà nước và 02 đề tài nghiên cứu cơ bản do Quỹ NAFOSTED tài trợ. Các đề tài này đều được hoàn thành đúng hạn với chất lượng tốt.

GS Nguyễn Mạnh Hùng là người có công lớn trong việc gây dựng và phát triển nhóm nghiên cứu về Phương trình đạo hàm riêng tại Khoa Toán-Tin, trường Đại học Sư phạm Hà Nội. Ngày nay, đây là một trong những trung tâm nghiên cứu và đào tạo sau đại học uy tín của Việt Nam về lĩnh vực này. Trong những năm gần đây, trung bình mỗi năm các thành viên của Bộ môn Giải tích, Khoa Toán-Tin, Trường ĐHSP Hà Nội, hướng dẫn 04 NCS bảo vệ thành công luận án tiến sĩ, hơn 20 học viên cao học bảo vệ thành công luận văn thạc sĩ, và công bố khoảng 20 bài báo khoa học trên các tạp chí quốc tế uy tín trong danh mục ISI.

GS Nguyễn Mạnh Hùng đã được mời tham gia Ban Chương trình và làm báo cáo mời tại nhiều hội nghị và hội thảo khoa học trong nước và quốc tế. Giáo sư cũng tham gia phản biện bài cho một số tạp chí chuyên ngành quốc tế.

Giáo sư cũng đã thiết lập được quan hệ quốc tế và hợp tác khoa học với một số giáo sư ở ĐHQG Sun Yet-sen (Đài Loan), ĐHQG Pusan (Hàn Quốc), ĐHTHQG Moskva mang tên Lomonosov, ĐHQG Voronezh (Nga), ĐH West Georgia (Mĩ); ĐH Bharathiar Bang Coimbatore (Ấn Độ), ĐHTH Gottingen (Đức).

GS Nguyễn Mạnh Hùng cũng là người rất thành công trong công tác đào tạo sau đại học. Giáo sư hướng dẫn và đồng hướng dẫn 10 NCS, trong đó 09 NCS đã bảo vệ thành công luận án tiến sĩ, và đã hướng dẫn hơn 30 học viên cao học bảo vệ thành công luận văn thạc sĩ. Các nghiên cứu sinh đã bảo vệ bao gồm: Cung Thế Anh (2006), Phạm Triều Dương (2006), Nguyễn Thành Anh (2010), Nguyễn Thị Kim Sơn (2010), Vũ Trọng Lưỡng (2011), Đỗ Văn Lợi (2011), Phùng Kim Chức (2012), Nguyễn Thị Liên (2016) và Nguyễn Thanh Tùng (2017). Giáo sư cũng là Chủ tịch hoặc Phản biện của nhiều Hội đồng đánh giá luận án tiến sĩ.

Giáo sư cũng rất tích cực trong việc viết sách phục vụ đào tạo. Giáo sư là tác giả của một số sách chuyên khảo, giáo trình đại học và giáo trình sau đại học về Phương trình đạo hàm riêng. Cuốn giáo trình Phương trình đạo hàm riêng của Giáo sư đã được sử dụng rộng rãi trong các trường đại học sư phạm.

Do những đóng góp nổi bật trong công tác quản lí, nghiên cứu khoa học, giảng dạy và đào tạo, GS.TSKH Nguyễn Mạnh Hùng đã được Nhà nước tặng thưởng nhiều danh hiệu và chức danh cao quý: Huân chương Lao động hạng Ba (2016), Nhà giáo ưu tú (2010), Giáo sư (2011).

Nhân dịp GS Nguyễn Mạnh Hùng tròn 60 tuổi, những người học trò cũ chúng tôi xin kính chúc Giáo sư mạnh khỏe và tiếp tục có những đóng góp trong sự nghiệp giáo dục và đào tạo.

Cung Thế Anh – Trần Đình Kế

Khoa Toán-Tin, Trường ĐHSP Hà Nội

ABSTRACTS

Local exact controllability to trajectories of the magneto-micropolar fluid equations

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We prove the exact controllability to trajectories of the magneto-micropolar fluid equations with distributed controls. We first establish new Carleman inequalities for the associated linearized system which lead to its null controllability. Then, combining the null controllability of the linearized system with an inverse mapping theorem, we deduce the local exact controllability to trajactories of the nonlinear problem.

This is joint work with Vu Manh Toi.

Bất đẳng thức Hardy-Littlewood-Sobolev ngược trên $\mathbb{R}^n$

Ngô Quốc Anh

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Bất đẳng thức Hardy-Littewood-Sobolev (HLS) trên $\mathbb{R}^n$ có lịch sử lâu đời bắt nguồn từ các kết quả về tích phân Riemann-Liouville của Hardy và Littewood những năm 20 thế kỷ trước. Năm 1938, để chứng minh kết quả về phép nhúng mà sau này mang tên ông, bằng cách sử dụng các kết quả nội suy Marcinckiewicz, Sobolev đã tổng quát kết quả của Hardy và Littewood cho trường hợp $\mathbb{R}^n$ và thu được bất đẳng thức tích chập với nhân kỳ dị dạng

$$||f * |x|^{-\alpha}||_p \lesssim ||f||_q,$$

trong đó $\alpha > 0$, $q>1$, và $p>1$ là các hằng số thích hợp.

Tuy nhiên phải mất gần 50 năm thì bài toán tìm hằng số tốt nhất và việc phân loại các hàm tối ưu để bất đẳng thức HLS xảy ra dấu bằng mới được giải quyết trong một công trình của E. Lieb năm 1983. Kể từ công trình của Lieb, bất đẳng thức HLS trở thành chủ đề nghiên cứu nóng bỏng thu hút rất nhiều nhà toán học bởi mối liên hệ giữa nó với các bất đẳng thức Sobolev, Moser-Trudinger-Onofri, v.v.

Năm 2015, khi nghiên cứu trường hợp kỳ dị của bài toán xác định metric bảo giác với độ cong vô hướng cho trước (prescribed scalar curvature problem), J. Dou và M. Zhu lần đầu tiên giới thiệu dạng ngược của bất đẳng thức HLS trên $\mathbb{R}^n$

$$||f * |x|^{\alpha}||_p \gtrsim ||f||_q,$$

trong đó $\alpha > 0$, $q>1$, và $p>1$. Để chứng minh bất đẳng thức và chỉ ra sự tồn tại của hàm tối ưu, Dou và Zhu đã đề xuất và sử dụng các kết quả nội suy Marcinckiewicz ngược.

Trong báo cáo này, tôi sẽ giới thiệu một cách tiếp cận mới sử dụng biểu diễn tích phân dưới dạng lớp (layer cake representation) để chứng minh bất đẳng thức HLS ngược. Việc chứng minh sự tồn tại của hàm tối ưu cũng như phân loại chúng cũng sẽ được đề cập trong báo cáo. Đây là kết quả cộng tác với Nguyễn Văn Hoàng (Đại học Paul Sabatier, Toulouse, CH Pháp).

Inverse problems with nonnegative and sparse solutions: Algorithms and application to the phase retrieve problem

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We study a gradient-type method and a semismooth Newton method for minimization problems in regularizing inverse problems with nonnegative and sparse solutions. We propose a special penalty functional forcing the minimizers of regularized minimization problems to be nonnegative and sparse and then apply the suggested algorithms for finding the solution to the problem. The strong convergence of the gradient-type method and the local superlinear convergence of the semismooth Newton method are proved. Then, we use these algorithms for the phase retrieval problem and illustrate their efficiency in numerical examples, particularly in the practical problem of optical imaging through scattering media where all the noises from experiment are presented.

This is joint work with Pham Quy Muoi and Dang Cuong.

A Liouville-type theorem for a cooperative parabolic system

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We prove the nonexistence of entire positive solutions to a cooperative parabolic system. By nontrivial modifications of the techniques of Gidas and Spruck and of Bidaut-Véron, we partially improve the results of Quittner in space dimensions $N \geq 3$. In particular, our result solves the important case of the parabolic Gross-Pitaevskii system in space dimension $N = 3$. We also give the results on universal singularity estimates, universal bounds for global solutions, and blow-up rate estimates for the corresponding initial value problem.

Nonlinear hyperbolic partial differential equations on nonsmooth domains

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In this report, we give a discussion on our results related to nonlinear hyperbolic partial differential equations on nonsmooth domains. The concrete results are local existence and regularity of solutions of certain semilinear hyperbolic partial differential equation on domains with an edge or cone with edges.

This is joint work with Nguyen Thanh Tung.

Discrete-time Fourier sine integral transforms

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Some integral transforms have attracted the attention of many mathematicians such as Fourier transform, fractional Fourier transform, discrete Fourier transform, Fourier transform on time scale and discrete-time Fourier transform. In this talk, we will construct and study the discrete-time Fourier sine transform

$$X_s(\omega) = \mathcal{F}_{sDT}\{x(n)\}(\omega) = 2\sum_{n=0}^{\infty} x(n)\sin(n\omega)$$

and its inverse

$$x(n) = \mathcal{F}_{sDT}^{-1}\{X_s(\omega)\}(n) = \frac{1}{\pi}\int_0^{\pi} X_s(\omega)\sin(n\omega)d\omega,$$

where $X_s(\omega)$ is a periodic function with period 2π.

We also study its operator properties, Parseval's identity, discrete-time Fourier cosine transform, Fourier sine generalized convolution theorems, and the Titchmarsh theorem. They are useful for solving infinite systems of linear algebraic equations.

This is joint work with Nguyen Anh Dai.

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LOCAL EXACT CONTROLLABILITY TO TRAJECTORIES OF THE MAGNETO-MICROPOLAR FLUID EQUATIONS

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Abstract. In this paper we prove the exact controllability to trajectories of the magneto-micropolar fluid equations with distributed controls. We first establish new Carleman inequalities for the associated linearized system which lead to its null controllability. Then, combining the null controllability of the linearized system with an inverse mapping theorem, we deduce the local exact controllability to trajectories of the nonlinear problem.

1. **Introduction and statement of main results.** Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d \in \{2,3\}$, whose boundary $\partial\Omega$ is regular enough. Let $T>0$ and we will use the notations $Q = \Omega \times (0,T), \Sigma = \partial\Omega \times (0,T)$, and we denote by $n(x)$ the outward unit normal to $\partial\Omega$ at the point $x \in \partial\Omega$.

We consider the controllability of the following magneto-micropolar fluid equations:

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y - (B\cdot\nabla)B + \nabla p + \nabla\Big(\dfrac{|B|^2}{2}\Big) = \mathrm{curl}\omega + u1_{\mathcal{O}} & \text{in } Q,\\ \omega_t - \Delta\omega - (d-2)\nabla(\nabla\cdot\omega) + (y\cdot\nabla)\omega + \omega = \mathrm{curl} y + w1_{\mathcal{O}} & \text{in } Q,\\ B_t - \Delta B + (y\cdot\nabla)B - (B\cdot\nabla)y = P(v1_{\mathcal{O}}) & \text{in } Q,\\ \nabla\cdot y = \nabla\cdot B = 0 & \text{in } Q,\\ y=0,\ \omega=0,\ B=0 & \text{on } \Sigma,\\ y(0)=y^0,\ \omega(0)=\omega^0,\ B(0)=B^0 & \text{in } \Omega, \end{cases} \tag{1}$$

where y and B respectively describe the flow velocity vector and the magnetic field vector,

$$\omega = \begin{cases} \text{scalar angular velocity if } d=2,\\ (\omega_1(x,t),\omega_2(x,t),\omega_3(x,t)) \text{ angular velocity vector if } d=3, \end{cases}$$

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p is a scalar pressure, while y^0, ω^0 and B^0 are the given initial velocity, initial angular velocity and initial magnetic field, and (u, w, v) stands for control functions acting on a small nonempty open subset $\mathcal{O}$ of Ω.

Here we have used the following notations:

In the case $d = 2$, we denote $\text{curl}a = \partial_{x_1} a_2 - \partial_{x_2} a_1$ for a vector function $a = (a_1, a_2)$, and $\text{curl}b = (\partial_{x_2} b, -\partial_{x_1} b)$ for a scalar function b.

In the case $d = 3$, we denote

$$\text{curl}a = (\partial_{x_2} a_3 - \partial_{x_3} a_2, \partial_{x_3} a_1 - \partial_{x_1} a_3, \partial_{x_1} a_2 - \partial_{x_2} a_1)$$

for a vector function $a = (a_1, a_2, a_3)$.

In this work, the control function acting on the equations satisfied by the magnetic B is assumed to have the form

$$P(v 1_{\mathcal{O}}) = v 1_{\mathcal{O}} + \nabla \chi, \text{ for some } \chi \in L^2(0, T; H^1(\Omega)). \tag{2}$$

This form of the control v has been also considered in recent works on the local exact controllability of the MHD system [4, 5, 18, 19]. There is only a recent result on the controllability of MHD system [3] in which the control acting on the magnetic field has support in an arbitrarily small open subset of the spatial domain, i.e., the control has the form $1_{\mathcal{O}} P_{\mathcal{O}}(v 1_{\mathcal{O}})$, where $P_{\mathcal{O}}$ is the classical Helmholtz projector related to $\mathcal{O}$ (i.e., the orthogonal projection operator from $L^2(\mathcal{O})^d$ onto the completion of the set $\{v \in C_0^\infty(\mathcal{O})^d \,|\, \nabla \cdot v = 0 \text{ in } \mathcal{O}\}$ in the norm of $L^2(\mathcal{O})^d$. However, since the boundary conditions on the magnetic field in our system is different from that in [3], so here we cannot use ideas in [3] to establish our Carleman estimate for the component C of the adjoint system respectively to the magnetic field. Hence, we are not able to get an estimate of the right-hand side of the component C having the form $\iint_{\mathcal{O}} e^{-2s\alpha} \xi^3 |P_{\mathcal{O}} C|^2 dxdt$ as in [3]. So we only obtain the controllability of (1) with the control function acting on the magneto field has the form (2). The controllability of (1) with the control function acting on the magneto field has the form $1_{\mathcal{O}} P_{\mathcal{O}}(v 1_{\mathcal{O}})$ remains an open question.

The magneto-micropolar fluid is a model of fluids in which micro-structures of the fluid and its electronic-magnetic properties are taken into account. In the past years, there have been a number of works devoted to studying mathematical questions related to the magneto-micropolar fluid equations. The existence and uniqueness of weak/strong solutions to (1) were studied in [8, 14, 25, 27, 28]. The regularity and blow-up criterion of solutions were studied in [13, 23, 33, 35]. Besides, the long-time behavior of solutions was investigated in [1, 6, 21, 22, 24, 29]. However, to the best of our knowledge, there is no work on the controllability of the magneto-micropolar fluid equations. This is the motivation of the present paper. Because here we focus on the controllability, we have omitted some physical constants in this model.

It is noticed that the magneto-micropolar fluid equations contain the micropolar equations (when $B = 0$), the MHD equations (when $\omega = 0$), the Navier-Stokes equations (when $B = 0$ and $\omega = 0$) as particular cases. The local exact controllability of the Navier-Stokes equations has been studied extensively in many works, see e.g. [10, 12, 26] and references therein. In recent years, the local exact controllability of the MHD system was also studied by a number of authors in [3, 4, 5, 18, 19], and that of the micropolar fluid equation was studied in [9, 17].

To study system (1), we use the following function spaces

$$H = \left\{ y \in L^2(\Omega)^d \,|\, \nabla \cdot y = 0 \text{ and } y \cdot n = 0 \text{ on } \partial\Omega \right\}$$

with the norm

$$\|y\|_H = \Big(\sum_{i=1}^{d}\int_\Omega |y_j|^2 dx\Big)^{1/2},$$

and

$$V = \big\{y \in H_0^1(\Omega)^d \mid \nabla\cdot y = 0 \text{ in } \Omega\big\}$$

with the norm

$$\|y\|_V = \Big(\sum_{i=1}^{d}\int_\Omega |\nabla y_j|^2 dx\Big)^{1/2}.$$

The main question considered in this paper is that whether (1) is locally exactly controllable to the trajectories.

Let us fix a regular trajectory $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ of the non-controlled system corresponding to (1), i.e.,

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} - (\overline{B}\cdot\nabla)\overline{B} + \nabla\overline{p} + \nabla\Big(\dfrac{|\overline{B}|^2}{2}\Big) = \text{curl}\overline{\omega} & \text{in } Q,\\ \overline{\omega}_t - \Delta\overline{\omega} - (d-2)\nabla(\nabla\cdot\overline{\omega}) + (\overline{y}\cdot\nabla)\overline{\omega} + \overline{\omega} = \text{curl}\overline{y} & \text{in } Q,\\ \overline{B}_t - \Delta\overline{B} + (\overline{y}\cdot\nabla)\overline{B} - (\overline{B}\cdot\nabla)\overline{y} = 0 & \text{in } Q,\\ \nabla\cdot\overline{y} = \nabla\cdot\overline{B} = 0 & \text{in } Q,\\ \overline{y} = 0,\ \overline{\omega} = 0,\ \overline{B} = 0 & \text{on } \Sigma,\\ \overline{y}(0) = \overline{y}^0,\ \overline{\omega}(0) = \overline{\omega}^0,\ \overline{B}(0) = \overline{B}^0 & \text{in } \Omega, \end{cases} \tag{3}$$

for some initial data $(\overline{y}^0, \overline{\omega}^0, \overline{B}^0)$.

We will assume that $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ satisfies

$$(\overline{y}, \overline{\omega}, \overline{B}) \in L^\infty(Q)^5 \text{ if } d = 2, \tag{4}$$

and

$$(\overline{y}, \overline{\omega}, \overline{B}) \in L^\infty(Q)^9 \text{ if } d = 3. \tag{5}$$

As long as the initial conditions are concerned, we will assume that

$$(y^0, \omega^0, B^0) \in E_0 := \begin{cases} H \times L^2(\Omega) \times H & \text{if } d = 2,\\ (H\cap L^4(\Omega)^3) \times L^4(\Omega)^3 \times (H\cap L^4(\Omega)^3) & \text{if } d = 3. \end{cases} \tag{6}$$

We are now ready to formulate the main results in the present paper. First, the result in the case of two dimensions is given in the following theorem.

Theorem 1.1. *Let $d = 2$. Assume that $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ satisfies (4). Then (1) is locally exactly controllable to $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ at any time $T > 0$, that is, there exists $\varepsilon > 0$ such that, for any initial data (y^0, ω^0, B^0) satisfying (6) and*

$$\|y^0 - \overline{y}^0\|_H + \|\omega^0 - \overline{\omega}^0\|_{L^2(\Omega)} + \|B^0 - \overline{B}^0\|_H < \varepsilon,$$

there exist controls $(u, w, v) \in L^2(\mathcal{O}\times(0,T))^5$ such that the solution (y, p, ω, B) of (1) satisfying

$$y(\cdot,T) = \overline{y}(\cdot,T), \omega(\cdot,T) = \overline{\omega}(\cdot,T) \text{ and } B(\cdot,T) = \overline{B}(\cdot,T) \text{ in } \Omega.$$

The following theorem is the result in the case of three dimensions.

Theorem 1.2. *Let* $d = 3$. *Assume that* $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ *satisfies* (5). *Then* (1) *is locally exactly controllable to* $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ *at any time* $T > 0$, *that is, there exists* $\varepsilon > 0$ *such that, for any initial data* (y^0, ω^0, B^0) *satisfying* (6) *and*

$$\|y^0 - \overline{y}^0\|_{H\cap L^4(\Omega)^3} + \|\omega^0 - \overline{\omega}^0\|_{L^4(\Omega)^3} + \|B^0 - \overline{B}^0\|_{H\cap L^4(\Omega)^3} < \varepsilon,$$

there exist controls $(u, w, v) \in L^2(\mathcal{O} \times (0,T))^9$ *such that the solution* (y, p, ω, B) *of* (1) *satisfying*

$$y(\cdot, T) = \overline{y}(\cdot, T),\ \omega(\cdot, T) = \overline{\omega}(\cdot, T) \text{ and } B(\cdot, T) = \overline{B}(\cdot, T) \text{ in } \Omega.$$

Remark 1. From the above theorems, by taking $\omega = 0$ and $B = 0$ we recover the local exact controllability result in [26] for Navier-Stokes equations, which improved the previous results in [10] and references therein. Moreover, by taking $B = 0$ only, we improved the previous result on local exact controllability to trajectories of the micropolar fluids in [9] in the sense that a weaker regularity of the given trajectory and initial data is required.

Our strategy is as follows: Let the trajectory $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ be given in (3) satisfying (4) or (5). Firstly, let us introduce the auxiliary nonlinear system:

$$\begin{cases} \widetilde{y}_t - \Delta\widetilde{y} + ((\widetilde{y} + \overline{y})\cdot\nabla)\widetilde{y} + (\widetilde{y}\cdot\nabla)\overline{y} - ((\widetilde{B} + \overline{B})\cdot\nabla)\widetilde{B} & \\ \quad -(\widetilde{B}\cdot\nabla)\overline{B} + \nabla\widetilde{p} + \dfrac{1}{2}\nabla((\widetilde{B} + \overline{B})\cdot\widetilde{B}) + \dfrac{1}{2}\nabla(\overline{B}\cdot\widetilde{B}) = \text{curl}\widetilde{\omega} + u1_{\mathcal{O}} & \text{in } Q, \\ \widetilde{\omega}_t - \Delta\widetilde{\omega} - (d-2)\nabla(\nabla\cdot\widetilde{\omega}) + ((\widetilde{y} + \overline{y})\cdot\nabla)\widetilde{\omega} & \\ \quad +(\widetilde{y}\cdot\nabla)\overline{\omega} + \widetilde{\omega} = \text{curl}\widetilde{y} + w1_{\mathcal{O}} & \text{in } Q, \\ \widetilde{B}_t - \Delta\widetilde{B} + ((\widetilde{y} + \overline{y})\cdot\nabla)\widetilde{B} + (\widetilde{y}\cdot\nabla)\overline{B} & \\ \quad -((\widetilde{B} + \overline{B})\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\overline{y} = P(v1_{\mathcal{O}}) & \text{in } Q, \\ \nabla\cdot\widetilde{y} = \nabla\cdot\widetilde{B} = 0 & \text{in } Q, \\ \widetilde{y} = 0,\ \widetilde{\omega} = 0,\ \widetilde{B} = 0 & \text{on } \Sigma, \\ \widetilde{y}(0) = \widetilde{y}^0,\ \widetilde{\omega}(0) = \widetilde{\omega}^0,\ \widetilde{B}(0) = \widetilde{B}^0 & \text{in } \Omega. \end{cases} \tag{7}$$

Setting $(y, p, \omega, B) = (\widetilde{y} + \overline{y}, \widetilde{p} + \overline{p}, \widetilde{\omega} + \overline{\omega}, \widetilde{B} + \overline{B})$, it is seen that to prove the main results, what we have to do is to prove the local null controllability of (7). In other words, we have to show that, for some $\varepsilon > 0$, whenever the initial datum in (7) satisfies

$$\|(\widetilde{y}^0, \widetilde{\omega}^0, \widetilde{B}^0)\|_{E_0} < \varepsilon,$$

we can find controls u, w and v such that the associated solution $(\widetilde{y}, \widetilde{p}, \widetilde{\omega}, \widetilde{B})$ of (7) satisfies

$$\widetilde{y}(\cdot, T) = 0,\ \widetilde{\omega}(\cdot, T) = 0 \text{ and } \widetilde{B}(\cdot, T) = 0 \text{ in } \Omega.$$

To do this, we will follow the strategy introduced by Fursikov and Imanuvilov [12] in the context of Navier-Stokes equations. Let us consider the linearized system around $(\overline{y}, \overline{\omega}, \overline{B})$:

$$\begin{cases} \widetilde{y}_t - \Delta\widetilde{y} + (\overline{y}\cdot\nabla)\widetilde{y} + (\widetilde{y}\cdot\nabla)\overline{y} - (\overline{B}\cdot\nabla)\widetilde{B} \\ \qquad -(\widetilde{B}\cdot\nabla)\overline{B} + \nabla\widetilde{p} + \nabla(\overline{B}\cdot\widetilde{B}) = f_1 + \mathrm{curl}\widetilde{\omega} + u1_{\mathcal{O}} & \text{in } Q, \\ \widetilde{\omega}_t - \Delta\widetilde{\omega} - (d-2)\nabla(\nabla\cdot\widehat{\omega}) + (\overline{y}\cdot\nabla)\widetilde{\omega} \\ \qquad +(\widetilde{y}\cdot\nabla)\overline{\omega} + \widetilde{\omega} = f_2 + \mathrm{curl}\widetilde{y} + w1_{\mathcal{O}} & \text{in } Q, \\ \widetilde{B}_t - \Delta\widetilde{B} + (\overline{y}\cdot\nabla)\widetilde{B} + (\widetilde{y}\cdot\nabla)\overline{B} - (\overline{B}\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\overline{y} \\ \qquad = f_3 + P(v1_{\mathcal{O}}) & \text{in } Q, \\ \nabla\cdot\widetilde{y} = \nabla\cdot\widetilde{B} = 0 & \text{in } Q, \\ \widetilde{y} = 0,\ \widetilde{\omega} = 0,\ \widetilde{B} = 0 & \text{on } \Sigma, \\ \widetilde{y}(0) = \widetilde{y}^0,\ \widetilde{\omega}(0) = \widetilde{\omega}^0,\ \widetilde{B}(0) = \widetilde{B}^0 & \text{in } \Omega, \end{cases} \tag{8}$$

where f_1, f_2 and f_3 are functions that decay exponentially to zero as $t \to T^-$.

We will prove that, under appropriate assumptions for f_1, f_2 and f_3, these above linear system (8) is null controllable. After that, combining the null controllability of (8) with an inverse mapping theorem, it will lead to the local null exact controllability of (7).

A basic tool for proving the null controllability of (8) is a global Carleman inequality for solutions to the following associated adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi - (D^s\varphi)\overline{y} + (D^aC)\overline{B} + \nabla\pi = \mathrm{curl}\psi + ({}^t\nabla\psi)\overline{\omega} + g_1 & \text{in } Q, \\ -\psi_t - \Delta\psi - (d-2)\nabla(\nabla\cdot\psi) + (\overline{y}\cdot\nabla)\psi + \psi = \mathrm{curl}\varphi + g_2 & \text{in } Q, \\ -C_t - \Delta C + (D^s\varphi)\overline{B} - (D^aC)\overline{y} + \nabla r = g_3 & \text{in } Q, \\ \nabla\cdot\varphi = \nabla\cdot C = 0 & \text{in } Q, \\ \varphi = 0,\ \psi = 0,\ C = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T,\ \psi(T) = \psi^T,\ C(T) = C^T & \text{in } \Omega. \end{cases} \tag{9}$$

Here we have used the notations $D^s := \nabla + {}^t\nabla$ and $D^a := \nabla - {}^t\nabla$. In (9), the pressure functions are π, r.

To obtain the above main results, which particularly improve some recent related results, we have to establish new necessary Carleman inequalities. This is in fact the main contribution of our paper.

Let us explain the method used to construct our Carleman inequality. Firstly, using the Carleman estimate in [20, Theorem 4.1] (see also in [26, Theorem 3.4]) for the Stokes system with suitable f, we get the global integral estimates for the component φ in both cases $d = 2$ and $d = 3$. Since the magneto field has the homogeneous Dirichlet condition and the equation satisfying the magneto field has an addition pressure, then the global integral estimates for the component C can be established as same as the estimates for the component φ. The global integral estimate for the component ψ is obtained separately in two cases $d = 2$ and $d = 3$. In the case $d = 2$, we can use the Carleman inequality directly for the heat equation to the component ψ to get the estimate for ψ. However, in the case $d = 3$, we cannot use the Carleman inequality directly for the heat equations to the component ψ since the equation satisfying by ψ has the term $\nabla(\nabla\cdot\psi)$. To overcome this difficulty, we exploit some ideas in [17] by using the Carleman inequality [20, Theorem 2.2] for the nonhomogeneous heat equations with suitable powers of the weight functions. Then, we can establish our new Carleman estimates with slightly weaker requirement of the regularity of the trajectory as that in the case of micropolar fluid equations [9, Propositon 4].

The paper is organized as follows. In Section 2, we establish new Carleman inequalities for the solutions to the adjoint linearized system. Section 3 is devoted to proving Theorem 1.1 and Theorem 1.2. We first use the new Carleman inequality to prove the null controllability of the linearized system, then the conclusion of the proof of the main results is obtained by combining the null controllability of the linearized system and an inverse mapping theorem. In the Appendix we recall some well-known Carleman inequalities which are used in the proof.

2. **Carleman inequalities.**

2.1. **Statement of Carleman inequalities.** In this subsection, we will formulate a suitable Carleman estimate for the adjoint system (9). To do this, we introduce some weight functions. Let $\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$ and $\eta^0 \in C^2(\overline{\Omega})$ satisfy

$$\eta^0 > 0 \text{ in } \Omega, \ \eta^0 \equiv 0 \text{ on } \partial\Omega \text{ and } |\nabla \eta^0| > 0 \text{ in } \overline{\Omega} \setminus \widetilde{\mathcal{O}}. \tag{10}$$

The existence of such a function η^0 was given in [11, Lemma 1.1]. Let $\ell \in C^\infty([0,T])$ be a function such that

$$\begin{cases} \ell(t) > 0 & \text{for all } t \in [0,T], \\ \ell(t) = t & \text{for all } t \in [0,T/4], \\ \ell(t) = T - t & \text{for all } t \in [3T/4,T]. \end{cases}$$

We now consider the following weight functions

$$\begin{aligned} \alpha(x,t) &= \frac{e^{\lambda(\|\eta^0\|_\infty + m_2)} - e^{\lambda(\eta^0(x)+m_1)}}{\ell(t)^4}, \quad \xi(x,t) = \frac{e^{\lambda(\eta^0(x)+m_1)}}{\ell(t)^4}, \\ \alpha^*(t) &= \max_{x\in\overline{\Omega}} \alpha(x,t) = \alpha|_{\partial\Omega}(t) = \frac{e^{\lambda(\|\psi\|_\infty + m_2)} - e^{\lambda m_1}}{\ell(t)^4}, \\ \xi^*(t) &= \min_{x\in\overline{\Omega}} \xi(x,t) = \xi|_{\partial\Omega}(t) = \frac{e^{\lambda m_1}}{\ell(t)^4}, \end{aligned} \tag{11}$$

where $\lambda \geq 1$ and m_1, m_2 are two constants chosen for the moment such that $m_1 \leq m_2$ and $\exists \mathcal{C} > 0$ (independent of λ) such that $\forall \lambda \geq 1$,

$$|\partial_t \alpha| \leq \mathcal{C}\xi^{5/4}, \text{ and } |\partial^2_{tt}\alpha| \leq \mathcal{C}\xi^{3/2}.$$

For example, we can choose with $m_0 \geq 0$,

$$m_1 = (4 + m_0)\|\eta^0\|_\infty, \quad m_2 = (4 + m_0 + \frac{m_0}{4})\|\eta^0\|_\infty.$$

Theorem 2.1. *Let $d = 2$. Assume that the trajectory $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ satisfies (4), $(g_1, g_2, g_3) \in L^2(Q)^5$. Then there exist some positive constants $\widehat{\mathcal{C}}, \hat{s}_0$ and $\hat{\lambda}_0$, only depending on Ω and $\mathcal{O}$, such that the solution (φ, ψ, C) of (9) satisfies*

$$\begin{aligned} &s^{-1} \iint\limits_Q e^{-2s\alpha}\xi^{-1}\left(|\psi_t|^2 + |\Delta\psi|^2\right) dxdt + s^3\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2 dxdt \\ &+ s\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi|\nabla\psi|^2 dxdt + s^{-1} \iint\limits_Q e^{-2s\alpha}\xi^{-1}(|\nabla \text{curl}\varphi|^2 + |\nabla \text{curl} C|^2) dxdt \end{aligned}$$

$$+ s\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi(|\text{curl}\varphi| + |\text{curl}C|^2)dxdt + \lambda^2 \iint\limits_Q e^{-2s\alpha}(|\nabla\varphi|^2 + |\nabla C|^2)dxdt$$
$$+ s^2\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^2(|\varphi|^2 + |C|^2)dxdt$$
$$\leq \mathcal{C}\Big(s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(|\varphi|^2 + |\psi|^2 + |C|^2)dxdt$$
$$+ \iint\limits_Q e^{-2s\alpha}(|g_1|^2 + |g_2|^2 + |g_3|^2)dxdt\Big), \tag{12}$$

for $s \geq \hat{s}_0(T^3 + T^4)$ *and* $\lambda \geq \hat{\lambda}_0(1 + \|\overline{y}\|_\infty + \|\overline{B}\|_\infty + \|\overline{\omega}\|_\infty)$.

Theorem 2.2. *Let* $d = 3$. *Assume that the trajectory* $(\overline{y}, \overline{p}, \overline{\omega}, \overline{B})$ *satisfies* (5), $(g_1, g_2, g_3) \in L^2(Q)^9$. *Then there exist some positive constants* $\widehat{\mathcal{C}}, \hat{s}_0$ *and* $\hat{\lambda}_0$, *only depending on* Ω *and* $\mathcal{O}$, *such that the solution* (φ, ψ, C) *of* (9) *satisfies*

$$s^{-1} \iint\limits_Q e^{-2s\alpha}\xi^{-1}\left(|\psi_t|^2 + |\Delta\psi|^2\right)dxdt + s^3\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2dxdt$$
$$+ s\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi|\nabla\psi|^2dxdt + \iint\limits_Q e^{-2s\alpha}|\nabla(\nabla\cdot\psi)|^2dxdt$$
$$+ s^2\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt + s^{-1} \iint\limits_Q e^{-2s\alpha}\xi^{-1}\left(|\nabla\text{curl}\varphi|^2 + |\nabla\text{curl}C|^2\right)dxdt$$
$$+ s\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi\left(|\text{curl}\varphi|^2 + |\text{curl}C|^2\right)dxdt$$
$$+ \lambda^2 \iint\limits_Q e^{-2s\alpha}(|\nabla\varphi|^2 + |\nabla C|^2)dxdt + s^2\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^2(|\varphi|^2 + |C|^2)dxdt$$
$$\leq \mathcal{C}\Big(s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(|\varphi|^2 + |\psi|^2 + |C|^2)dxdt$$
$$+ \iint\limits_Q e^{-2s\alpha}(|g_1|^2 + s\xi|g_2|^2 + |g_3|^2)dxdt\Big) \tag{13}$$

for $s \geq \hat{s}_0(T^3 + T^4)$ *and* $\lambda \geq \hat{\lambda}_0(1 + \|\overline{y}\|_\infty + \|\overline{B}\|_\infty + \|\overline{\omega}\|_\infty)$.

Remark 2. By taking $C = 0$ and $\psi = 0$, we recover the improved versions of Carleman estimates for the Navier-Stokes equations, which were recently obtained in [26] (see also in [20]).

2.2. **Proof of Carleman inequalities.** We will prove Theorem 2.1 and Theorem 2.2 in several steps.

Step 1. Estimation of global terms φ **and** C**:** Notice that the system for the components φ (and C) in the adjoint system (9) can be viewed as the Stokes system (43) in the Appendix with t replaced by $T - t$ and $f = (D^s\varphi)\overline{y} - (D^aC)\overline{B} + \text{curl}\psi + ({}^t\nabla\psi)\overline{\omega} + g_1$ (and $f = -(D^s\varphi)\overline{B} + (D^aC)\overline{y} + g_3$). So, applying Lemma 4.3

in the Appendix to components φ (and C) in (9) we get some positive constants $s_0 \geq 1, \lambda_0 \geq 1$ and $\mathcal{C} > 0$ such that

$$
\begin{aligned}
& s^{-1} \iint\limits_Q e^{-2s\alpha}\xi^{-1}(|\nabla \mathrm{curl}\varphi|^2 + |\nabla \mathrm{curl} C|^2)dxdt \\
& + s\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi(|\mathrm{curl}\varphi|^2 + |\mathrm{curl}C|^2)dxdt \\
& + \lambda^2 \iint\limits_Q e^{-2s\alpha}(|\nabla\varphi|^2 + |\nabla C|^2)dxdt + s^2\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^2(|\varphi|^2 + |C|^2)dxdt \\
\leq\ & \mathcal{C}\Big((\|\overline{y}\|_\infty^2 + \|\overline{B}\|_\infty^2) \iint\limits_Q e^{-2s\alpha}(|\nabla\varphi|^2 + |\nabla C|^2)dxdt \\
& + (1 + \|\overline{\omega}\|_\infty^2) \iint\limits_Q e^{-2s\alpha}|\nabla\psi|^2 dxdt \\
& + \iint\limits_Q e^{-2s\alpha}(|g_1|^2 + |g_3|^2)dxdt + s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(|\varphi|^2 + |C|^2)dxdt\Big)
\end{aligned}
\tag{14}
$$

for any $s \geq s_0$ and $\lambda \geq \lambda_0$, where we have used the fact that $|\mathrm{curl}\varphi|^2 \leq \mathcal{C}|\nabla\varphi|^2$ and $|\mathrm{curl}\psi|^2 \leq \mathcal{C}|\nabla\psi|^2$.

Therefore, taking $\lambda \geq \max\{\lambda_0, \mathcal{C}(\|\overline{y}\|_\infty + \|\overline{B}\|_\infty)\}$, we have from (14) that

$$
\begin{aligned}
& s^{-1} \iint\limits_Q e^{-2s\alpha}\xi^{-1}(|\nabla \mathrm{curl}\varphi|^2 + |\nabla \mathrm{curl} C|^2)dxdt \\
& + s\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi(|\mathrm{curl}\varphi|^2 + |\mathrm{curl}C|^2)dxdt \\
& + \lambda^2 \iint\limits_Q e^{-2s\alpha}(|\nabla\varphi|^2 + |\nabla C|^2)dxdt + s^2\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^2(|\varphi|^2 + |C|^2)dxdt \\
\leq\ & \mathcal{C}\Big((1 + \|\overline{\omega}\|_\infty^2) \iint\limits_Q e^{-2s\alpha}|\nabla\psi|^2 dxdt \\
& + \iint\limits_Q e^{-2s\alpha}(|g_1|^2 + |g_3|^2)dxdt + s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(|\varphi|^2 + |C|^2)dxdt\Big).
\end{aligned}
\tag{15}
$$

Step 2. Estimation of global term ψ: We will consider two cases:

Case $d = 2$. Using the Carleman estimate (40) in the Appendix for ψ in (9) with $d = 2$, we deduce that

$$
\begin{aligned}
& s^{-1} \iint\limits_Q e^{-2s\alpha}\xi^{-1}\left(|\psi_t|^2 + |\Delta\psi|^2\right)dxdt + s^3\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2 dxdt \\
& + s\lambda^2 \iint\limits_Q e^{-2s\alpha}\xi|\nabla\psi|^2 dxdt \leq \mathcal{C}\Big(s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2
\end{aligned}
$$

$$+\iint_Q e^{-2s\alpha}(|\psi|^2+|\mathrm{curl}\varphi|^2+|g_2|^2)dxdt+\|\overline{y}\|_\infty^2\iint_Q e^{-2s\alpha}|\nabla\psi|^2dxdt\Big), \tag{16}$$

for $s\geq \mathcal{C}(T^3+T^4)$ and $\lambda\geq\mathcal{C}$.

Case $d=3$. We apply the divergence operator to the equation satisfied by ψ in (9) with $d=3$ to deduce that

$$-\partial_t(\nabla\cdot\psi)-2\Delta(\nabla\cdot\psi)=\nabla\cdot(\psi+(\overline{y}\cdot\nabla)\psi+g_2). \tag{17}$$

Thus, we apply the Carleman estimate (42) in the Appendix for the equation (17) with different powers of ξ. More precisely, we apply that Carleman inequality to $s^{1/2}\xi^{1/2}\nabla\cdot\psi$ and we get that

$$\begin{aligned}&\iint_Q e^{-2s\alpha}|\nabla(\nabla\cdot\psi)|^2dxdt+s^2\lambda^2\iint_Q e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt\\&\leq\mathcal{C}\Big(s^2\lambda^2\iint_{\widehat{\mathcal{O}}\times(0,T)}e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt+s^{1/2}\left\|e^{-s\alpha}\xi^{1/4}\nabla\cdot\psi\right\|^2_{H^{\frac12,\frac14}(\Sigma)}\\&\quad+s\iint_Q e^{-2s\alpha}\xi|\psi|^2dxdt+s\|\overline{y}\|_\infty^2\iint_Q e^{-2s\alpha}\xi|\nabla\psi|^2dxdt+s\iint_Q e^{-2s\alpha}\xi|g_2|^2dxdt\Big)\end{aligned} \tag{18}$$

for $s\geq s_0$ and $\lambda\geq\lambda_0$, where $\widetilde{\mathcal{O}}\subset\subset\widehat{\mathcal{O}}\subset\subset\mathcal{O}$.

On the other hand, since ψ satisfies the system

$$\begin{cases}-\psi_t-\Delta\psi=\nabla(\nabla\cdot\psi)-(\overline{y}\cdot\nabla)\psi-\psi+\mathrm{curl}\varphi+g_2 & \text{in } Q,\\ \psi=0 & \text{on } \Sigma,\end{cases} \tag{19}$$

then using the Carleman (40) in the Appendix for ψ in (19), we deduce that

$$\begin{aligned}&s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}\left(|\psi_t|^2+|\Delta\psi|^2\right)dxdt\\&+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\psi|^2dxdt+s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla\psi|^2dxdt\\&\leq\mathcal{C}\Big(s^3\lambda^4\iint_{\widehat{\mathcal{O}}\times(0,T)}e^{-2s\alpha}\xi^3|\psi|^2+\iint_Q e^{-2s\alpha}|\nabla(\nabla\cdot\psi)|^2dxdt\\&+\iint_Q e^{-2s\alpha}(|\psi|^2+|\mathrm{curl}\varphi|^2+|g_2|^2)dxdt+\|\overline{y}\|_\infty^2\iint_Q e^{-2s\alpha}|\nabla\psi|^2dxdt\Big),\end{aligned} \tag{20}$$

for $s\geq\mathcal{C}(T^3+T^4)$ and $\lambda\geq\mathcal{C}$. Combining (18) and (20) yields the estimate

$$\begin{aligned}&s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}\left(|\psi_t|^2+|\Delta\psi|^2\right)dxdt\\&+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\psi|^2dxdt+s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla\psi|^2dxdt\end{aligned}$$

$$+\iint\limits_Q e^{-2s\alpha}|\nabla(\nabla\cdot\psi)|^2dxdt+s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt$$
$$\le \mathcal{C}\Big(s^3\lambda^4\iint\limits_{\widehat{\mathcal{O}}\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2dxdt+s^2\lambda^2\iint\limits_{\widehat{\mathcal{O}}\times(0,T)} e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt$$
$$+s^{1/2}\left\|e^{-s\alpha}\xi^{1/4}\nabla\cdot\psi\right\|^2_{H^{\frac12,\frac14}(\Sigma)}$$
$$+s\iint\limits_Q e^{-2s\alpha}\xi|g_2|^2dxdt+\iint\limits_Q e^{-2s\alpha}|\mathrm{curl}\varphi|^2dxdt\Big),\tag{21}$$

for $s\ge\max\{s_0,\mathcal{C}(T^3+T^4)\}$ and $\lambda\ge\max\{\lambda_0,\mathcal{C}(1+\|\overline{y}\|_\infty)\}$.

Furthermore, integrating by parts and using the Cauchy inequality, we get

$$s^2\lambda^2\iint\limits_{\widehat{\mathcal{O}}\times(0,T)} e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt\le\varepsilon\iint\limits_Q e^{-2s\alpha}|\nabla(\nabla\cdot\psi)|^2dxdt$$
$$+\mathcal{C}\varepsilon^{-1}s^2\lambda^2\iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^2|\psi|^2dxdt$$

for any $\varepsilon>0$. Hence, choosing ε sufficiently small, one infers from (21) that

$$s^{-1}\iint\limits_Q e^{-2s\alpha}\xi^{-1}\left(|\psi_t|^2+|\Delta\psi|^2\right)dxdt$$
$$+s^3\lambda^4\iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2dxdt+s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi|\nabla\psi|^2dxdt$$
$$+\iint\limits_Q e^{-2s\alpha}|\nabla(\nabla\cdot\psi)|^2dxdt+s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt$$
$$\le\mathcal{C}\Big(s^3\lambda^4\iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2dxdt+s^{1/2}\left\|e^{-s\alpha}\xi^{1/4}\nabla\cdot\psi\right\|^2_{H^{\frac12,\frac14}(\Sigma)}$$
$$+s\iint\limits_Q e^{-2s\alpha}\xi|g_2|^2dxdt+\iint\limits_Q e^{-2s\alpha}|\mathrm{curl}\varphi|^2dxdt\Big).\tag{22}$$

We now estimate the trace terms. From the definition of $\|\cdot\|_{H^{\frac12,\frac14}(\Sigma)}$, we have

$$s^{1/2}\left\|e^{-s\alpha}\xi^{1/4}\nabla\cdot\psi\right\|^2_{H^{\frac12,\frac14}(\Sigma)}\le\mathcal{C}\left(\|\sigma_1\nabla\cdot\psi\|^2_{L^2(0,T;H^1(\Omega))}+\|\sigma_1\psi\|^2_{H^1(0,T;L^2(\Omega)^3)}\right),\tag{23}$$

where $\sigma_1:=s^{1/4}(\xi^*)^{1/4}e^{-s\alpha^*}$.

We see that $\sigma_1\psi$ satisfies

$$\begin{cases}-\partial_t(\sigma_1\psi)-\Delta(\sigma_1\psi)-\nabla(\nabla\cdot(\sigma_1\psi))=-\sigma_1(\overline{y}\cdot\nabla)\psi-\sigma_1\psi & \\ \qquad\qquad -\sigma_1'\psi+\sigma_1\mathrm{curl}\varphi+\sigma_1 g_2 & \text{in } Q,\\ \sigma_1\psi=0 & \text{on } \Sigma,\\ (\sigma_1\psi)(T)=0 & \text{in } \Omega.\end{cases}$$

Hence, using a similar classical energy estimate for the heat equation, we get

$$\|\sigma_1\psi\|^2_{L^2(0,T;H^2(\Omega)^3)} + \|\sigma_1\psi\|^2_{H^1(0,T;L^2(\Omega)^3)} \le \mathcal{C}\iint\limits_Q \sigma_1^2(|\mathrm{curl}\varphi|^2 + |g_2|^2)dxdt$$
$$+ \|\overline{y}\|^2_\infty \iint\limits_Q \sigma_1^2|\nabla\psi|^2dxdt + \iint\limits_Q ((\sigma_1')^2 + \sigma_1^2)|\psi|^2dxdt\Big). \quad (24)$$

Since $|\sigma_1'| \le \mathcal{C}s^{5/4}(\xi^*)^{11/8}e^{-s\alpha^*}$, one deduces from (24) and (23) that

$$s^{1/2}\left\|e^{-s\alpha}\xi^{1/4}\nabla\cdot\psi\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}$$
$$\le \mathcal{C}\Big(s^{1/2}\iint\limits_Q e^{-2s\alpha}\xi^{1/2}|\mathrm{curl}\varphi|^2dxdt + s^{1/2}\iint\limits_Q e^{-2s\alpha}\xi^{1/2}|g_2|^2dxdt$$
$$+ \|\overline{y}\|^2_\infty s^{1/2}\iint\limits_Q e^{-2s\alpha}\xi^{1/2}|\nabla\psi|^2dxdt + s^{5/2}\iint\limits_Q e^{-2s\alpha}\xi^{11/4}|\psi|^2dxdt\Big).$$

Combining (22) and (24), we get

$$s^{-1}\iint\limits_Q e^{-2s\alpha}\xi^{-1}\left(|\psi_t|^2 + |\Delta\psi|^2\right)dxdt + s^3\lambda^4\iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2dxdt$$
$$+ s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi|\nabla\psi|^2dxdt + \iint\limits_Q e^{-2s\alpha}|\nabla(\nabla\cdot\psi)|^2dxdt$$
$$+ s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|\nabla\cdot\psi|^2dxdt \le \mathcal{C}\Big(s^3\lambda^4\iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2dxdt$$
$$+ s^{1/2}\iint\limits_Q e^{-2s\alpha}\xi^{1/2}|\mathrm{curl}\varphi|^2dxdt + s\iint\limits_Q e^{-2s\alpha}\xi|g_2|^2dxdt\Big), \quad (25)$$

for $s \ge \max\{s_0, \mathcal{C}(T^3+T^4)\}$ and $\lambda \ge \max\{\lambda_0, \mathcal{C}(1+\|\overline{y}\|_\infty)\}$. Here we have used the fact that $s^{1/2}\xi^{1/2} \ge \mathcal{C}$ for $s \ge \mathcal{C}T^4$.

Step 3. Conclusion.

Conclusion of Theorem 2.1. Combining (15) and (16) with note that $|\mathrm{curl}\varphi|^2 \le \mathcal{C}|\nabla\varphi|^2$, we get (12) for $s \ge \hat{s}_0(T^3+T^4)$ and $\lambda \ge \hat{\lambda}_0(1+\|\overline{y}\|_\infty + \|\overline{B}\|_\infty + \|\overline{\omega}\|_\infty)$, where $\hat{\lambda}_0 = \max\{\lambda_0, \mathcal{C}\}$ and $\hat{s}_0 = \max\{s_0, \mathcal{C}\}$. This completes the proof of Theorem 2.1.

Conclusion of Theorem 2.2. Combining (15) and (25), we get (13) for $\lambda \ge \hat{\lambda}_0(1+\|\overline{y}\|_\infty + \|\overline{B}\|_\infty + \|\overline{\omega}\|_\infty)$ and for any $s \ge \hat{s}_0(T^3+T^4)$. This completes the proof of Theorem 2.2.

3. **Proof of the main results.** In this section, we will give the proof of Theorem 1.2, i.e. the result in the case of three dimensions. The proof of Theorem 1.1 (the result in the case $d=2$) is very similar to that in the case $d=3$, so it is omitted here.

3.1. **Null controllability for the linear system (8).** We now prove the null controllability for the system (8) and this will be crucial when proving the local controllability of (1) in the next subsection.

We can rewrite problem (8) as follows

$$\begin{cases} L(\widetilde{y}, \widetilde{\omega}, \widetilde{B}) + (\nabla p, 0, 0) = (f_1 + u1_{\mathcal{O}}, f_2 + w1_{\mathcal{O}}, f_3 + P(v1_{\mathcal{O}})) & \text{in } Q, \\ \nabla \cdot \widetilde{y} = \nabla \cdot \widetilde{B} = 0 & \text{in } Q, \\ \widetilde{y} = 0, \widetilde{\omega} = 0, \widetilde{B} = 0 & \text{on } \Sigma, \\ \widetilde{y}(0) = \widetilde{y}^0, \widetilde{\omega}(0) = \widetilde{\omega}^0, \widetilde{B}(0) = \widetilde{B}^0 & \text{in } \Omega, \end{cases} \tag{26}$$

where

$$L(\widetilde{y}, \widetilde{\omega}, \widetilde{B}) = (L_1(\widetilde{y}, \widetilde{\omega}, \widetilde{B}), L_2(\widetilde{y}, \widetilde{\omega}), L_3(\widetilde{y}, \widetilde{B}))$$

with

$$\begin{aligned} L_1(\widetilde{y}, \widetilde{\omega}, \widetilde{B}) &:= \widetilde{y}_t - \Delta\widetilde{y} + (\overline{y} \cdot \nabla)\widetilde{y} + (\widetilde{y} \cdot \nabla)\overline{y} - (\overline{B} \cdot \nabla)\widetilde{B} - (\widetilde{B} \cdot \nabla)\overline{B} - \mathrm{curl}\widetilde{\omega}, \\ L_2(\widetilde{y}, \widetilde{\omega}) &:= \widetilde{\omega}_t - \Delta\widetilde{\omega} - \nabla(\nabla \cdot \widetilde{\omega}) + (\overline{y} \cdot \nabla)\widetilde{\omega} + \widetilde{\omega} + (\widetilde{y} \cdot \nabla)\overline{\omega} - \mathrm{curl}\widetilde{y}, \\ L_3(\widetilde{y}, \widetilde{B}) &:= \widetilde{B}_t - \Delta\widetilde{B} + (\overline{y} \cdot \nabla)\widetilde{B} - (\widetilde{B} \cdot \nabla)\overline{y} + (\widetilde{y} \cdot \nabla)\overline{B} - (\overline{B} \cdot \nabla)\widetilde{y}. \end{aligned}$$

We would like to find the controls (u, w, v) such that the solution $(\widetilde{y}, \widetilde{\omega}, \widetilde{B})$ to (26) satisfies

$$\widetilde{y}(T) = 0, \ \widetilde{\omega}(T) = 0, \ \widetilde{B}(T) = 0 \text{ in } \Omega. \tag{27}$$

We first deduce the Carleman inequality with weight functions that do not vanish at $t = 0$. More precisely, let us consider the function

$$\widetilde{\ell}(t) = \begin{cases} \ell(T/2) & \text{if } 0 \le t \le T/2, \\ \ell(t) & \text{if } T/2 \le t \le T, \end{cases}$$

and we define new weight functions

$$\begin{aligned} \beta(x, t) &= \frac{e^{\lambda(\|\eta^0\|_\infty + m_2)} - e^{\lambda(\eta^0(x) + m_1)}}{\widetilde{\ell}(t)^4}, \\ \gamma(x, t) &= \frac{e^{\lambda(\eta^0(x) + m_1)}}{\widetilde{\ell}(t)^4}, \\ \beta^*(t) &= \max_{x \in \overline{\Omega}} \beta(x, t), \ \gamma^*(t) = \min_{x \in \overline{\Omega}} \gamma(x, t). \end{aligned}$$

We will prove the following lemma.

Lemma 3.1. *Let s and λ be like in Theorem 2.2. Then there exists a positive constant $\widehat{\mathcal{C}}_0$ depending on T, s and λ, such that every solution (φ, ψ, C) of (9) satisfies*

$$\begin{aligned} &\|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi(0)\|^2_{L^2(\Omega)^3} + \|C(0)\|^2_{L^2(\Omega)^3} \\ &+ \iint_Q e^{-2s\beta}\gamma^2(|\varphi|^2 + |C|^2)dxdt + \iint_Q e^{-2s\beta}\gamma^3|\psi|^2 dxdt \\ &+ \iint_Q e^{-2s\beta^*}\gamma^*(|\nabla\varphi|^2 + |\nabla\psi|^2 + |\nabla C|^2)dxdt \\ \le \mathcal{C}\Big(&\iint_Q e^{-2s\beta}(|g_1|^2 + |g_3|^2)dxdt + \iint_Q e^{-2s\beta}\gamma|g_2|^2 dxdt \\ &+ \iint_{\mathcal{O}\times(0,T)} e^{-2s\beta}\gamma^3(|\varphi|^2 + |\psi|^2 + |C|^2)dxdt\Big). \end{aligned} \tag{28}$$

Proof. The proof of this lemma is similar to those in some recent works on the controllability of the fluid models (see for instance [16]). More precisely, this lemma is a consequence of (13) and energy estimates satisfied by solutions of (9). In what follows, we only give the sketch of the proof.

We introduce a function $\vartheta \in C^1([0,T])$ such that

$$\vartheta \equiv 1 \text{ in } [0,T/2],\ \ \vartheta \equiv 0 \text{ in } [3T/4,T].$$

Then $(\vartheta\varphi, \vartheta\psi, \vartheta C)$ satisfies

$$\begin{cases} -(\vartheta\varphi)_t - \Delta(\vartheta\varphi) - (D^s\vartheta\varphi)\overline{y} + (D^a\vartheta C)\overline{B} + \nabla(\vartheta\pi) = \mathrm{curl}(\vartheta\psi) & \\ \qquad\qquad\qquad + ({}^t\nabla(\vartheta\psi))\overline{\omega} + \vartheta g_1 - \vartheta'\varphi & \text{in } Q, \\ -(\vartheta\psi)_t - \Delta(\vartheta\psi) - \nabla(\nabla\cdot(\vartheta\psi)) - (\overline{y}\cdot\nabla)(\vartheta\psi) + \vartheta\psi = \mathrm{curl}(\vartheta\varphi) + \vartheta g_2 & \text{in } Q, \\ -(\vartheta C)_t - \Delta(\vartheta C) + (D^s\vartheta\varphi)\overline{B} - (D^a\vartheta C)\overline{y} + \nabla(\vartheta r) = \vartheta g_3 - \vartheta' C & \text{in } Q, \\ \nabla\cdot(\vartheta\varphi) = \nabla\cdot(\vartheta C) = 0 & \text{in } Q, \\ \vartheta\varphi = 0, \vartheta\psi = 0,\ \vartheta C = 0 & \text{on } \Sigma, \\ (\vartheta\varphi)(T) = 0,\ (\vartheta\psi)(T) = 0, (\vartheta C)(T) = 0 & \text{in } \Omega. \end{cases} \tag{29}$$

Multiplying $(29)_1$ by $\vartheta\varphi$, $(29)_2$ by $\vartheta\psi$, $(29)_3$ by ϑC, then integrating over Ω and using the Cauchy inequality, there exists a positive constant $\mathcal{C}$ depending on $\|\overline{y}\|_\infty, \|\overline{\omega}\|_\infty$, $\|\overline{B}\|_\infty$ such that

$$\begin{aligned} &-\frac{d}{dt}\int_\Omega \left(|\vartheta\varphi|^2 + |\vartheta\xi|^2 + |\vartheta C|^2\right)dx + \int_\Omega \left(|\nabla(\vartheta\varphi)|^2 + |\nabla(\vartheta\psi)|^2 + |\nabla(\vartheta C)|^2\right)dx \\ &\le \mathcal{C}\Big(\int_\Omega \left(|\vartheta\varphi|^2 + |\vartheta\psi|^2 + |\vartheta C|^2\right)dx + \int_\Omega \left(|\vartheta g_1|^2 + |\vartheta g_2|^2 + |\vartheta g_3|^2\right)dx \\ &\quad + \int_\Omega |\vartheta'|^2(|\varphi|^2 + |\psi|^2 + |C|^2)dx\Big). \end{aligned} \tag{30}$$

So, from inequality (30) we get the energy estimate

$$\begin{aligned} &\|\vartheta\varphi\|^2_{L^\infty(0,T;H)} + \|\vartheta\varphi\|^2_{L^2(0,T;V)} + \|\vartheta\psi\|^2_{L^\infty(0,T;L^2(\Omega)^3)} + \|\vartheta\psi\|^2_{L^2(0,T;H^1_0(\Omega)^3)} \\ &+ \|\vartheta C\|^2_{L^\infty(0,T;H)} + \|\vartheta C\|^2_{L^2(0,T;V)} \\ &\le \mathcal{C}(T)\Big(\|\vartheta'\varphi\|^2_{L^2(Q)^3} + \|\vartheta'\psi\|^2_{L^2(Q)^3} + \|\vartheta' C\|^2_{L^2(Q)^3} + \|\vartheta(g_1,g_2,g_3)\|^2_{L^2(Q)^9}\Big). \end{aligned}$$

This implies that

$$\begin{aligned} &\|(\varphi(0),\psi(0),C(0)\|^2_{L^2(\Omega)^9} + \|\varphi\|^2_{L^2(0,T/2;H)} + \|\psi\|^2_{L^2(0,T/2;L^2(\Omega)^3)} + \|C\|^2_{L^2(0,T/2;H)} \\ &+ \|\varphi\|^2_{L^2(0,T/2;V)} + \|\psi\|^2_{L^2(0,T/2;H^1_0(\Omega)^3)} + \|C\|^2_{L^2(0,T/2;V)} \\ &\le \mathcal{C}(T)\Big(\|(\varphi,\psi,C)\|^2_{L^2(T/2,3T/4;L^2(\Omega)^9)} + \|(g_1,g_2,g_3)\|^2_{L^2(0,3T/4;L^2(\Omega)^9)}\Big). \end{aligned}$$

From the last inequality and the fact that

$$0 < e^{-2s\beta}\gamma^3,\ e^{-2s\beta}\gamma^2,\ e^{-2s\beta^*}\gamma^* \le \mathcal{C}, \forall t \in [0,T/2];\ \ e^{-2s\beta} \ge \mathcal{C},\ \forall t \in [0,3T/4],$$

we have

$$\|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi(0)\|^2_{L^2(\Omega)^3} + \|C(0)\|^2_{L^2(\Omega)^3}$$

$$
\begin{aligned}
&+\int_0^{T/2}\int_\Omega e^{-2s\beta}\gamma^2(|\varphi|^2+|C|^2)dxdt+\int_0^{T/2}\int_\Omega e^{-2s\beta}\gamma^3|\psi|^2dxdt\\
&+\int_0^{T/2}\int_\Omega e^{-2s\beta^*}\gamma^*(|\nabla\varphi|^2+|\nabla\psi|^2+|\nabla C|^2)dxdt\\
\le\ &\mathcal{C}\Big(\int_{T/2}^{3T/4}\int_\Omega e^{-2s\beta}\gamma^2(|\varphi|^2+|C|^2)dxdt+\int_{T/2}^{3T/4}\int_\Omega e^{-2s\beta}\gamma^3|\psi|^2dxdt\\
&+\int_0^{3T/4}\int_\Omega e^{-2s\beta}(|g_1|^2+|g_3|^2)dxdt+\int_0^{3T/4}\int_\Omega e^{-2s\beta}\gamma|g_2|^2dxdt\Big). \qquad (31)
\end{aligned}
$$

Note that, since $\beta=\alpha$ in $\Omega\times(T/2,T)$, we have

$$
\begin{aligned}
&\int_{T/2}^{T}\int_\Omega e^{-2s\beta}\gamma^2(|\varphi|^2+|C|^2)dxdt+\int_{T/2}^{T}\int_\Omega e^{-2s\beta}\gamma^3|\psi|^2dxdt\\
&+\int_{T/2}^{T}\int_\Omega e^{-2s\beta^*}\gamma^*(|\nabla\varphi|^2+|\nabla\psi|^2+|\nabla C|^2)dxdt\\
\le\ &\iint_Q e^{-2s\alpha}\xi^2(|\varphi|^2+|C|^2)dxdt+\iint_Q e^{-2s\alpha}\xi^3|\psi|^2dxdt\\
&+\iint_Q e^{-2s\alpha^*}\xi^*(|\nabla\varphi|^2+|\nabla\psi|^2+|\nabla C|^2)dxdt\\
\le\ &\mathcal{C}\Big(\iint_Q e^{-2s\alpha}(|g_1|^2+|g_3|^2)dxdt+\iint_Q e^{-2s\alpha}\xi|g_2|^2dxdt\\
&+\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(|\varphi|^2+|\psi|^2+|C|^2)dxdt\Big), \qquad (32)
\end{aligned}
$$

for some positive constant $\mathcal{C}$ depending on s_0,λ_0. Here, we have used the Carleman inequality (13) with note that

$$
\iint_Q e^{-2s\alpha^*}\xi^*|\nabla\varphi|^2dxdt\le\mathcal{C}\iint_Q e^{-2s\alpha^*}\xi^*|\mathrm{curl}\varphi|^2dxdt
$$

since $\varphi=0$ on Σ and $\nabla\cdot\varphi=0$ in Ω.

Now, since

$$
e^{-2s\beta},\ e^{-2s\beta}\gamma,\ e^{-2s\beta}\gamma^3\ge\mathcal{C}>0\ \ \forall t\in[0,T/2],
$$

we conclude from (32) that

$$
\begin{aligned}
&\int_{T/2}^{T}\int_\Omega e^{-2s\beta}\gamma^2(|\varphi|^2+|C|^2)dxdt+\int_{T/2}^{T}\int_\Omega e^{-2s\beta}\gamma^3|\psi|^2dxdt\\
&+\int_{T/2}^{T}\int_\Omega e^{-2s\beta^*}\gamma^*(|\nabla\varphi|^2+|\nabla\psi|^2+|\nabla C|^2)dxdt\\
&\le\mathcal{C}\Big(\iint_Q e^{-2s\beta}(|g_1|^2+|g_3|^2)dxdt+\iint_Q e^{-2s\beta}\gamma|g_2|^2dxdt\\
&\quad+\iint_{\mathcal{O}\times(0,T)} e^{-2s\beta}\gamma^3(|\varphi|^2+|\psi|^2+|C|^2)dxdt\Big). \qquad (33)
\end{aligned}
$$

Combining (33) and (31) we get (28). □

Now, we proceed to define the spaces where (26)-(27) will be solved. The main space will be

$$
\begin{aligned}
E = \Big\{ & (\widetilde{y}, \widetilde{p}, \widetilde{\omega}, \widetilde{B}, u, w, v) \,:\, e^{s\beta}\widetilde{y}, e^{s\beta}\gamma^{-1/2}\widetilde{\omega}, e^{s\beta}\widetilde{B} \in L^2(Q)^3, \\
& e^{s\beta}\gamma^{-3/2}(u1_{\mathcal{O}}, w1_{\mathcal{O}}, P(v1_{\mathcal{O}})) \in L^2(Q)^9, \\
& e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{y} \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap L^4(0,T;L^{12}(\Omega)^3), \\
& e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{\omega} \in L^2(0,T;H^1(\Omega)^3) \cap L^\infty(0,T;L^2(\Omega)^3) \cap L^4(0,T;L^{12}(\Omega)^3), \\
& e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{B} \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap L^4(0,T;L^{12}(\Omega)^3), \\
& e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_1(\widetilde{y}, \widetilde{\omega}, \widetilde{B}) + \nabla\widetilde{p} - u1_{\mathcal{O}}\big) \in L^2(0,T;W^{-1,6}(\Omega)^3), \\
& e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_2(\widetilde{y}, \widetilde{\omega}) - w1_{\mathcal{O}}\big) \in L^2(0,T;W^{-1,6}(\Omega)^3), \\
& e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_3(\widetilde{y}, \widetilde{B}) - P(v1_{\mathcal{O}})\big) \in L^2(0,T;W^{-1,6}(\Omega)^3)\Big\}.
\end{aligned}
$$

Observe that E is a Banach space with the norm

$$
\begin{aligned}
& \|(\widetilde{y}, \widetilde{p}, \widetilde{\omega}, \widetilde{B}, u, w, v)\|_E^2 \\
= & \|(e^{s\beta}\widetilde{y}, e^{s\beta}\gamma^{-1/2}\widetilde{\omega}, e^{s\beta}\widetilde{B})\|_{L^2(Q)^9}^2 + \|e^{s\beta}\gamma^{-3/2}(u1_{\mathcal{O}}, w1_{\mathcal{O}}, P(v1_{\mathcal{O}}))\|_{L^2(Q)^9}^2 \\
& + \|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{y}\|_{L^2(0,T;V)\cap L^\infty(0,T;H)\cap L^4(0,T;L^{12}(\Omega)^3)}^2 \\
& + \|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{\omega}\|_{L^2(0,T;H^1(\Omega)^3)\cap L^\infty(0,T;L^2(\Omega)^3)\cap L^4(0,T;L^{12}(\Omega)^3)}^2 \\
& + \|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{B}\|_{L^2(0,T;V)\cap L^\infty(0,T;H)\cap L^4(0,T;L^{12}(\Omega)^3)}^2 \\
& + \|e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_1(\widetilde{y}, \widetilde{\omega}, \widetilde{B}) + \nabla\widetilde{p} - u1_{\mathcal{O}}\big)\|_{L^2(0,T;W^{-1,6}(\Omega)^3)}^2 \\
& + \|e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_2(\widetilde{y}, \widetilde{\omega}) - w1_{\mathcal{O}}\big)\|_{L^2(0,T;W^{-1,6}(\Omega)^3)}^2 \\
& + \|e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_3(\widetilde{y}, \widetilde{B}) - P(v1_{\mathcal{O}})\big)\|_{L^2(0,T;W^{-1,6}(\Omega)^3)}^2.
\end{aligned}
$$

Remark 3. We can see that if $(\widetilde{y}, \widetilde{p}, \widetilde{\omega}, \widetilde{B}, u, w, v) \in E$ then $\widetilde{y}(.,T) = 0, \widetilde{\omega}(.,T) = 0, \widetilde{B}(.,T) = 0$ in Ω, so $(\widetilde{y}, \widetilde{p}, \widetilde{\omega}, \widetilde{B}, u, w, v)$ solves a null controllability problem for system (26) with an appropriate right-hand side (f_1, f_2, f_3).

We will prove the following result.

Proposition 1. *Assume that* $(\overline{y}, \overline{p}, \overline{\omega})$ *satisfies* (5) *and* $(\widetilde{y}^0, \widetilde{\omega}^0, \widetilde{B}^0) \in (H \cap L^4(\Omega)^3) \times L^4(\Omega)^3 \times (H \cap L^4(\Omega)^3)$. *Furthermore, assume that*

$$
e^{s\beta^*}(\gamma^*)^{-1/2}(f_1, f_2, f_3) \in \big(L^2(0,T;W^{-1,6}(\Omega)^3)\big)^3.
$$

Then, there exist control functions $u \in L^2(\mathcal{O} \times (0,T))^3, w \in L^2(\mathcal{O} \times (0,T))^3$ *and* $v \in L^2(\mathcal{O} \times (0,T))^3$ *such that if* $(\widetilde{y}, \widetilde{\omega}, \widetilde{B})$ *is the associated solution to* (26), *one has* $(\widetilde{y}, \widetilde{p}, \widetilde{\omega}, \widetilde{B}, u, w, v) \in E$. *In particular,* $\widetilde{y}(.,T) = 0, \widetilde{\omega}(.,T) = 0, \widetilde{B}(.,T) = 0$ *in* Ω.

Proof. The proof is similar to that of Proposition 2 in [16] (see also Proposition 3 in [10]), so in what follows we only give the sketch of the proof.

Let L^* be defined by

$$
L^*(\chi, \kappa, \rho) = (L_1^*(\chi, \kappa, \rho), L_2^*(\chi, \kappa), L_3^*(\chi, \rho))
$$

with

$$\begin{aligned} L_1^*(\chi,\kappa,\rho) &= -\chi_t - \Delta\chi - (D^s\chi)\overline{y} + (D^a\rho)\overline{B} - \mathrm{curl}\kappa - ({}^t\nabla\kappa)\overline{\omega}, \\ L_2^*(\chi,\kappa) &= -\kappa_t - \Delta\kappa - \nabla(\nabla\cdot\kappa) - (\overline{y}\cdot\nabla)\kappa + \kappa - \mathrm{curl}\chi, \\ L_3^*(\chi,\rho) &= -\rho_t - \Delta\rho + (D^s\chi)\overline{B} - (D^a\rho)\overline{y}, \end{aligned}$$

and let us introduce the space

$$\begin{aligned} X_0 = \Big\{ &(\chi,\sigma,\kappa,\rho,\zeta) \in C^2(\overline{Q})^3 \times C^1(\overline{Q}) \times C^2(\overline{Q})^3 \times C^2(\overline{Q})^3 \times C^1(\overline{Q})^1 : \\ &\nabla\cdot\chi = \nabla\cdot\rho = 0 \text{ in } Q, \chi = 0,\ \kappa = 0,\ \rho = 0 \text{ on } \Sigma \Big\}. \end{aligned}$$

Then, we consider the following variational problem: find $(\hat\chi,\hat\sigma,\hat\kappa,\hat\rho,\hat\zeta)$ such that

$$a((\hat\chi,\hat\sigma,\hat\kappa,\hat\rho,\hat\zeta),(\chi,\sigma,\kappa,\rho,\zeta)) = \langle \mathcal{G}, (\chi,\sigma,\kappa,\rho,\zeta)\rangle, \quad \forall(\chi,\sigma,\kappa,\rho,\zeta)\in X_0, \tag{34}$$

where

$$\begin{aligned} &a((\hat\chi,\hat\sigma,\hat\kappa,\hat\rho,\hat\zeta),(\chi,\sigma,\kappa,\rho,\zeta)) \\ &= \iint\limits_Q e^{-2s\beta}(L_1^*(\hat\chi,\hat\kappa,\hat\rho)+\nabla\hat\sigma)\cdot(L_1^*(\chi,\kappa,\rho)+\nabla\sigma)dxdt \\ &\quad + \iint\limits_Q e^{-2s\beta}\gamma L_2^*(\hat\chi,\hat\kappa)\cdot L_2^*(\chi,\kappa)dxdt \\ &\quad + \iint\limits_Q e^{-2s\beta}\left(L_3^*(\hat\chi,\hat\rho)+\nabla\hat\zeta\right)\cdot(L_3^*(\chi,\rho)+\nabla\zeta)\,dxdt \\ &\quad + \iint\limits_Q e^{-2s\beta}\gamma^3\left(\hat\chi 1_{\mathcal{O}}\cdot\chi 1_{\mathcal{O}} + \hat\kappa 1_{\mathcal{O}}\cdot\kappa 1_{\mathcal{O}} + \hat\rho 1_{\mathcal{O}}\cdot\rho 1_{\mathcal{O}}\right)dxdt, \end{aligned}$$

and

$$\begin{aligned} \langle \mathcal{G},(\chi,\sigma,\kappa,\rho,\zeta)\rangle = &\int_0^T \langle f_1,\chi\rangle_{H^{-1}(\Omega)^3,H_0^1(\Omega)^3}\,dt + \int_0^T \langle f_2,\kappa\rangle_{H^{-1}(\Omega)^3,H_0^1(\Omega)^3}\,dt \\ &+ \int_0^T \langle f_3,\rho\rangle_{H^{-1}(\Omega)^3,H_0^1(\Omega)^3}\,dt \\ &+ \int_\Omega \left(\widetilde{y}^0\cdot\chi(0) + \widetilde{\omega}^0\cdot\kappa(0) + \widetilde{B}^0\cdot\rho(0)\right)dx. \end{aligned}$$

From the Carleman inequality (28) applied to functions of X_0, which implies that $a(\cdot,\cdot)$ is a scalar product on X_0. Therefore, we can consider the space X, the completion of X_0 with respect to the norm associated to $a(\cdot,\cdot)$ (denoted by $\|\cdot\|_X$). Then X is a Hilbert space and $a(\cdot,\cdot)$ is well-defined, continuous and definite positive on X. Furthermore, thanks to (28), we see that the linear form $(\chi,\sigma,\kappa,\rho,\zeta)\mapsto \langle\mathcal{G},(\chi,\sigma,\kappa,\rho,\zeta)\rangle$ is well-defined and bounded on X. Consequently, in view of Lax-Milgram's lemma, there exists a unique solution $(\hat\chi,\hat\sigma,\hat\kappa,\hat\rho,\hat\zeta)$ of (34).

Let $(\hat y,\hat\omega,\hat B)$ and $(\hat u,\hat w,\hat v)$ be given by

$$\begin{cases} \hat y = e^{-2s\beta}\left(L_1^*(\hat\chi,\hat\kappa,\hat\rho)+\nabla\hat\sigma\right) & \text{in } Q, \\ \hat\omega = e^{-2s\beta}\gamma L_2^*(\hat\chi,\hat\kappa) & \text{in } Q, \\ \hat B = e^{-2s\beta}\left(L_3^*(\hat\chi,\hat\rho)+\nabla\hat\zeta\right) & \text{in } Q, \\ (\hat u,\hat w,\hat v) = -e^{-2s\beta}\gamma^3\left(\hat\chi 1_{\mathcal{O}},\hat\kappa 1_{\mathcal{O}},\hat\rho 1_{\mathcal{O}}\right) & \text{in } Q. \end{cases} \tag{35}$$

Then, it is readily seen that they satisfy

$$\begin{aligned}&\iint\limits_Q e^{2s\beta}\left(|\hat{y}|^2+\gamma^{-1}|\hat{\omega}|^2+|\hat{B}|^2\right)dxdt\\&+\iint\limits_Q e^{2s\beta}\gamma^{-3}\left(|\hat{u}1_{\mathcal{O}}|^2+|\hat{w}1_{\mathcal{O}}|^2+|\hat{v}1_{\mathcal{O}}|^2\right)dxdt\\=\;&a((\hat{\chi},\hat{\sigma},\hat{\kappa},\hat{\rho}),(\hat{\chi},\hat{\sigma},\hat{\kappa},\hat{\rho}))<+\infty.\end{aligned}\tag{36}$$

Moreover, we can see from (36) that $(\hat{y},\hat{\omega},\hat{B})\in L^2(Q)^9, \hat{u}1_{\mathcal{O}}\in L^2(Q)^3, \hat{w}1_{\mathcal{O}}\in L^2(Q)^3, \hat{v}1_{\mathcal{O}}\in L^2(Q)^3$. On the other hand, from (34) and (35), we see that $(\hat{y},\hat{\omega},\hat{B})$ together with some pressure $\hat{p}$ is the unique solution of (26) which is defined by the transposition with $u=\hat{u}, w=\hat{w}, v=\hat{v}$.

Finally, we must check that $(\hat{y},\hat{p},\hat{\omega},\hat{B},\hat{u},\hat{w},\hat{v})$ belongs to E. We already know that

$$\begin{aligned}&e^{s\beta}(\hat{y},\gamma^{-1}\hat{\omega},\hat{B})\in L^2(Q)^9,\ e^{2s\beta}\gamma^{-3}(\hat{u}1_{\mathcal{O}},\hat{w}1_{\mathcal{O}},P(\hat{v}1_{\mathcal{O}}))\in L^2(Q)^9,\\&e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_1(\hat{y},\hat{\omega},\hat{B})+\nabla\hat{p}-\hat{u}1_{\mathcal{O}}\big)\in L^2(0,T;W^{-1,6}(\Omega)^3),\\&e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_2(\hat{y},\hat{\omega})-\hat{w}1_{\mathcal{O}}\big)\in L^2(0,T;W^{-1,6}(\Omega)^3),\\&e^{s\beta^*}(\gamma^*)^{-1/2}\big(L_3(\hat{y},\hat{B})-P(\hat{v}1_{\mathcal{O}})\big)\in L^2(0,T;W^{-1,6}(\Omega)^3).\end{aligned}$$

Therefore, it remains to check that

$$\begin{aligned}&e^{s\beta^*/2}(\gamma^*)^{-1/4}\hat{y}\in L^2(0,T;V)\cap L^\infty(0,T;H)\cap L^4(0,T;L^{12}(\Omega)^3),\\&e^{s\beta^*/2}(\gamma^*)^{-1/4}\hat{\omega}\in L^2(0,T;H^1(\Omega)^3)\cap L^\infty(0,T;L^2(\Omega)^3)\cap L^4(0,T;L^{12}(\Omega)^3),\\&e^{s\beta^*/2}(\gamma^*)^{-1/4}\hat{B}\in L^2(0,T;V)\cap L^\infty(0,T;H)\cap L^4(0,T;L^{12}(\Omega)^3).\end{aligned}$$

To this end, let us set

$$\begin{aligned}&(y^*,\omega^*,B^*)=e^{s\beta^*/2}(\gamma^*)^{-1/4}(\hat{y},\hat{\omega},\hat{B}),\ p^*=e^{s\beta^*/2}\hat{p},\\&(f_1^*,f_2^*,f_3^*)=e^{s\beta^*/2}(\gamma^*)^{-1/4}(f_1+\hat{u}1_{\mathcal{O}},f_2+\hat{w}1_{\mathcal{O}},f_3+P(\hat{v}1_{\mathcal{O}})).\end{aligned}$$

Then they satisfy

$$\begin{cases}y_t^*-\Delta y^*+(\overline{y}\cdot\nabla)y^*+(y^*\cdot\nabla)\overline{y}-(\overline{B}\cdot\nabla)B^*-(B^*\cdot\nabla)\overline{B}+\nabla p^*\\\qquad+\nabla(\overline{B}\cdot B^*)-\mathrm{curl}\omega^*=f_1^*+(e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\hat{y} & \text{in } Q,\\\omega_t^*-\Delta\omega^*-\nabla(\nabla\cdot\omega^*)+(\overline{y}\cdot\nabla)\omega^*+(y^*\cdot\nabla)\overline{\omega}\\\qquad+\omega^*-\mathrm{curl}y^*=f_2^*+(e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\hat{\omega} & \text{in } Q,\\B_t^*-\Delta B^*+(\overline{y}\cdot\nabla)B^*+(y^*\cdot\nabla)\overline{B}-(\overline{B}\cdot\nabla)y^*-(B^*\cdot\nabla)\overline{y}\\\qquad=f_3^*+(e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\hat{B} & \text{in } Q,\\\nabla\cdot y^*=\nabla\cdot B^*=0 & \text{in } Q,\\y^*=0,\ \omega^*=0,\ B^*=0 & \text{on } \Sigma,\\(y^*(0),\omega^*(0),B^*(0))=e^{s\beta^*(0)/2}(\gamma(0)^*)^{-1/4}\left(\widetilde{y}^0,\widetilde{\omega}^0,\widetilde{B}^0\right) & \text{in } \Omega.\end{cases}$$

We can see that

$$\begin{aligned}&f_1^*+(e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\hat{y}\in L^2(0,T;H^{-1}(\Omega)^3),\\&f_2^*+(e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\hat{\omega}\in L^2(0,T;H^{-1}(\Omega)^3),\\&f_2^*+(e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\hat{B}\in L^2(0,T;H^{-1}(\Omega)^3).\end{aligned}$$

Moreover, $y^*(0) \in H, \omega^*(0) \in L^2(\Omega)^3$ and $B^*(0) \in H$. Therefore, from the well-known result in [28], we know that

$$\begin{cases} y^* \in L^2(0,T;V) \cap L^\infty(0,T;H) \\ \omega^* \in L^2(0,T;H^1(\Omega)^3) \cap L^\infty(0,T;L^2(\Omega)^3) \\ B^* \in L^2(0,T;V) \cap L^\infty(0,T;H). \end{cases}$$

We now have to prove that $(y^*, \omega^*, B^*) \in \left(L^4(0,T;L^{12}(\Omega)^3)\right)^3$.

To prove $y^* \in L^4(0,T;L^{12}(\Omega)^3)$, we follow the arguments in [10]. To do this, let $b \in L^{\frac{4}{3}}(0,T;L^{\frac{12}{11}}(\Omega)^3)$ and we consider the following Stokes system

$$\begin{cases} -z_t - \Delta z + \nabla h = b & \text{in } Q, \\ \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = 0 & \text{in } \Omega. \end{cases} \tag{37}$$

We know (see [10, Lemma 2], the proof uses regularity properties for the Stokes system [15] and some fine interpolation results [30]) that the system (37) has a unique solution (z, h) satisfying

$$z \in L^2(0,T;W_0^{1,6/5}(\Omega)^3) \cap C([0,T];L^{4/3}(\Omega)^3), \tag{38}$$

which depends continuously on b in these spaces. Then y^* satisfies

$$\iint\limits_Q y^* \cdot b dx dt = \int_\Omega e^{s\beta^*(0)/2}\widetilde{y}^0 \cdot z(0) dx + \int_0^T \langle F_1^*, z \rangle_{W^{-1,6}(\Omega)^3, W_0^{1,6/5}(\Omega)^3} dt.$$

Here

$$\begin{aligned} F_1^* =& f_1^* + (e^{s\beta^*/2}(\gamma^*)^{-1/4})_t \hat{y} - (\overline{y} \cdot \nabla) y^* - (y^* \cdot \nabla)\overline{y} + (\overline{B} \cdot \nabla) B^* \\ &+ (B^* \cdot \nabla)\overline{B} - \nabla(\overline{B} \cdot B^*) + \text{curl}\omega^*, \end{aligned}$$

and (z, q) is the solution to (37) associated to b.

We know that $z(0) \in L^{\frac{4}{3}}(\Omega)^3, \nabla z \in L^2(0,T;L^{\frac{6}{5}}(\Omega)^3)$. Remark that $(y^*, \omega^*, B^*) \in L^2(0,T;L^6(\Omega)^3)^3$, all terms of the previous definition make sense by virtue of (38) and the assumption $\widetilde{y}^0 \in L^4(\Omega)^3$. Therefore,

$$y^* \in \left(L^2(0,T;W_0^{1,6/5}(\Omega)^3) \cap C([0,T];L^{4/3}(\Omega)^3)\right)' = L^4(0,T;L^{12}(\Omega)^3).$$

We remark that, by the same above argument, one obtains

$$(\omega^*, B^*) \in \left(L^4(0,T;L^{12}(\Omega)^3)\right)^2.$$

This ends the proof of Proposition 1. □

3.2. **Local controllability of the semilinear problem.** In this subsection we give the proof of Theorem 1.2 by using similar arguments as in pioneering works [10, 16].

We will use the following inverse mapping theorem (see [2]).

Theorem 3.2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ satisfy $\mathcal{A} \in C^1(\mathcal{B}_1; \mathcal{B}_2)$. Assume that $b_1 \in \mathcal{B}_1, \mathcal{A}(b_1) = b_2$ and that $\mathcal{A}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective. Then, there exists $\varepsilon > 0$ such that, for every $b' \in \mathcal{B}_2$ satisfying $\|b' - b_2\|_{\mathcal{B}_2} < \varepsilon$, there exists a solution of the equation*

$$\mathcal{A}(b) = b', \ b \in \mathcal{B}_1.$$

In our setting, we use this theorem with the spaces $\mathcal{B}_1 = E$, $\mathcal{B}_2 = X \times Y$, where

$$X = \Big(L^2(e^{s\beta^*}(\gamma^*)^{-1/2}(0,T); W^{-1,6}(\Omega)^3)\Big)^3,$$

and

$$Y = H \cap L^4(\Omega)^3 \times L^4(\Omega)^3 \times H \cap L^4(\Omega)^3.$$

Then, we consider the operator

$$\mathcal{A}(\widetilde{y},\widetilde{p},\widetilde{\omega},\widetilde{B},u,w,v) = \Big(\mathcal{A}_1(\widetilde{y},\widetilde{p},\widetilde{\omega},\widetilde{B},u), \mathcal{A}_2(\widetilde{y},\widetilde{\omega},w), \mathcal{A}_3(\widetilde{y},\widetilde{B},v), \widetilde{y}(0), \widetilde{\omega}(0), \widetilde{B}(0)\Big)$$

with

$$\begin{aligned}\mathcal{A}_1(\widetilde{y},\widetilde{p},\widetilde{\omega},\widetilde{B},u) &= L_1(\widetilde{y},\widetilde{\omega},\widetilde{B}) + (\widetilde{y}\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\widetilde{B} + \nabla\widetilde{p} + \frac{1}{2}\nabla(\widetilde{B}\cdot\widetilde{B}) - u1_{\mathcal{O}},\\ \mathcal{A}_2(\widetilde{y},\widetilde{\omega},w) &= L_2(\widetilde{y},\widetilde{\omega}) + (\widetilde{y}\cdot\nabla)\widetilde{\omega} - w1_{\mathcal{O}},\\ \mathcal{A}_2(\widetilde{y},\widetilde{B},v) &= L_3(\widetilde{y},\widetilde{B}) + (\widetilde{y}\cdot\nabla)\widetilde{B} - (\widetilde{B}\cdot\nabla)\widetilde{y} - P(v1_{\mathcal{O}}).\end{aligned}$$

To apply Theorem 3.2, we first check that the operator $\mathcal{A}$ is of class $C^1(\mathcal{B}_1,\mathcal{B}_2)$. Indeed, all terms arising in the definition of $\mathcal{A}$ are linear (and consequently C^1), except for $(\widetilde{y}\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\widetilde{B} + \frac{1}{2}\nabla(\widetilde{B}\cdot\widetilde{B})$, $(\widetilde{y}\cdot\nabla)\widetilde{\omega}$, and $(\widetilde{y}\cdot\nabla)\widetilde{B} - (\widetilde{B}\cdot\nabla)\widetilde{y}$. However, the operators

$$\begin{aligned}((\widetilde{y},\widetilde{p},\widetilde{\omega},\widetilde{B},u,w,v),(\widehat{y},\widehat{p},\widehat{\omega},\widehat{B},\widehat{u},\widehat{w},\widehat{v})) \mapsto \Big(&(\widetilde{y}\cdot\nabla)\widehat{y} - (\widetilde{B}\cdot\nabla)\widehat{B} + \frac{1}{2}\nabla(\widetilde{B}\cdot\widehat{B}),\\ &(\widetilde{y}\cdot\nabla)\widehat{\omega},\ (\widetilde{y}\cdot\nabla)\widehat{B} - (\widetilde{B}\cdot\nabla)\widehat{y}\Big)\end{aligned}$$

are continuous from $\mathcal{B}_1 \times \mathcal{B}_2$ to X. So it suffices to prove their continuity from $\mathcal{B}_1 \times \mathcal{B}_1$ into G_1.

First, notice that

$$e^{s\beta^*/2}(\gamma^*)^{-1/4}(\widetilde{y},\widetilde{\omega},\widetilde{B}) \in \big(L^4(0,T;L^{12}(\Omega)^3)\big)^3 \tag{39}$$

for any $(\widetilde{y},\widetilde{p},\widetilde{\omega},\widetilde{B},u,w,v) \in \mathcal{B}_1$.

The nonlinear term $(\widetilde{y}\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\widetilde{B} + \frac{1}{2}\nabla(\widetilde{B}\cdot\widetilde{B})$: We have

$$\begin{aligned}&\|e^{s\beta^*}(\gamma^*)^{-1/2}\Big((\widetilde{y}\cdot\nabla)\widehat{y} - (\widetilde{B}\cdot\nabla)\widehat{B} + \frac{1}{2}\nabla(\widetilde{B}\cdot\widehat{B})\Big)\|_{L^2(0,T;W^{-1,6}(\Omega)^3)}\\ &\le \mathcal{C}\Big(\|e^{s\beta^*}(\gamma^*)^{-1/2}(\widetilde{y}\otimes\widehat{y}\|_{L^2(0,T;L^6(\Omega)^3)} + \|e^{s\beta^*}(\gamma^*)^{-1/2}(\widetilde{B}\otimes\widehat{B})\|_{L^2(0,T;L^6(\Omega)^3)}\\ &\quad + \|e^{s\beta^*}(\gamma^*)^{-1/2}(\widetilde{B}\cdot\widehat{B})\|_{L^2(0,T;L^6(\Omega)^3)}\Big)\\ &\le \mathcal{C}\Big(\|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{y}\|_{L^4(0,T;L^{12}(\Omega)^3)}\|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{y}\|_{L^4(0,T;L^{12}(\Omega)^3)}\\ &\quad + \|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{B}\|_{L^4(0,T;L^{12}(\Omega)^3)}\|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{B}\|_{L^4(0,T;L^{12}(\Omega)^3)}\Big).\end{aligned}$$

So, it follows from (39) that $(\widetilde{y}\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\widetilde{B} + \frac{1}{2}\nabla(\widetilde{B}\cdot\widetilde{B})$ belongs to the class of C^1.

The nonlinear term $(\widetilde{y}\cdot\nabla)\widehat{\omega}$: We have

$$\begin{aligned}&\|e^{s\beta^*}(\gamma^*)^{-1/2}(\widetilde{y}\cdot\nabla)\widehat{\omega}\|_{L^2(0,T;W^{-1,6}(\Omega)^3)}\\ &\le \mathcal{C}\|e^{s\beta^*}(\gamma^*)^{-1/2}\widetilde{y}\otimes\widehat{\omega}\|_{L^2(0,T;L^6(\Omega)^3)}\\ &\le \mathcal{C}\|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widetilde{y}\|_{L^4(0,T;L^{12}(\Omega)^3)}\|e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{\omega}\|_{L^4(0,T;L^{12}(\Omega)^3)}.\end{aligned}$$

So, from (39) we have that $(\widetilde{y}\cdot\nabla)\widetilde{\omega}$ belongs to the class of C^1.

The nonlinear term: $(\widetilde{y}\cdot\nabla)\widetilde{B}-(\widetilde{B}\cdot\nabla)\widetilde{y}$**:** We have the same estimates as in term $(\widetilde{y}\cdot\nabla)\widetilde{y}-(\widetilde{B}\cdot\nabla)\widetilde{B}$. Hence, this term belongs to the class of C^1.

Therefore, we have proved that $\mathcal{A}\in C^1(\mathcal{B}_1,\mathcal{B}_2)$ with

$$\begin{aligned}&\mathcal{A}'(0,0,0,0,0,0,0)(\widetilde{y},\widetilde{p},\widetilde{\omega},\widetilde{B},u,w,v)\\ &=\Big(L(\widetilde{y},\widetilde{\omega},\widetilde{B})+\nabla\widetilde{p}-u1_{\mathcal{O}},-w1_{\mathcal{O}},-P(v1_{\mathcal{O}})),\widetilde{y}(0),\widetilde{\omega}(0),\widetilde{B}(0)\Big),\end{aligned}$$

for all $(\widetilde{y},\widetilde{p},\widetilde{\omega},\widetilde{B},u,w,v)\in\mathcal{B}_1$.

In view of the null controllability result for the linearized system (8) given in Proposition 1, we can see that $\mathcal{A}'(0,0,0,0,0,0,0)$ is surjective.

As a consequence, we can apply Theorem 3.2 for $b_1=(0,0,0,0,0,0,0), b_2=(0,0,0,0,0,0)$ to get the existence of $\varepsilon>0$ such that if $\|\widetilde{y}(0),\widetilde{\omega}(0),\widetilde{B}(0)\|_Y\le\varepsilon$, then we can find controls u,w,v so that the associated solution to (7) satisfies $\widetilde{y}(.,T)=0,\widetilde{\omega}(.,T)=0,\widetilde{B}(.,T)=0$ in Ω. This completes the proof of Theorem 1.2.

4. Appendix: Some well-known Carleman estimates.

With the weight functions α and ξ defined in (11), we now recall some well-known Carleman estimates, which have been used in our proofs above.

Lemma 4.1. [11] *Let $\mathcal{O}$ be a nonempty open subset of Ω. For all $q\in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$, there exists $\mathcal{C}>0$ depending on Ω and $\mathcal{O}$ such that*

$$\begin{aligned}&s^{-1}\iint\limits_Q e^{-2s\alpha}\xi^{-1}\left(|q_t|^2+|\Delta q|^2\right)dxdt\\ &+s^3\lambda^4\iint\limits_Q e^{-2s\alpha}\xi^3|q|^2dxdt+s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi|\nabla q|^2dxdt\\ &\le\mathcal{C}\Big(\iint\limits_Q e^{-2s\alpha}|q_t+\Delta q|^2dxdt+s^3\lambda^4\iint\limits_{\mathcal{O}\times(0,T)}e^{-2s\alpha}\xi^3|q|^2dxdt\Big)\end{aligned}\tag{40}$$

for any $s\ge\mathcal{C}(T^3+T^4)$ and any $\lambda\ge\mathcal{C}$.

Consider the equation

$$y_t-\Delta y=F_0+\sum_{j=1}^3\partial_jF_j\text{ in }Q,\tag{41}$$

where $F_0,F_1,F_2,F_3\in L^2(Q)$. Then we have the following result.

Lemma 4.2. [20, Theorem 2.2] *Let $\widehat{\mathcal{O}}$ be a nonempty open subset of Ω. There exist $s_0\ge1$, $\lambda_0\ge1$ and a constant $\mathcal{C}>0$ (independent of $s\ge s_0$ and $\lambda\ge\lambda_0$) such that for every $y\in L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega))$ satisfying (41), we have for every $s\ge s_0$ and for every $\lambda\ge\lambda_0$,*

$$s^{-1}\iint\limits_Q e^{-2s\alpha}\xi^{-1}|\nabla y|^2dxdt+s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi|y|^2dxdt$$

$$\leq \mathcal{C}\Big(s\lambda^2 \iint\limits_{\widehat{\mathcal{O}}\times(0,T)} e^{-2s\alpha}\xi|y|^2dxdt + s^{-1/2}\left\|e^{-s\alpha}\xi^{-1/4}y\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}$$

$$+ s^{-2}\lambda^{-2}\iint\limits_{Q} e^{-2s\alpha}\xi^{-2}|F_0|^2dxdt + \sum_{j=1}^{3}\iint\limits_{Q} e^{-2s\alpha}|F_j|^2dxdt\Big). \qquad (42)$$

Recall here that

$$\|y\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} = \left(\|y\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))} + \|y\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))}\right)^{1/2}.$$

Let us now consider the following Stokes system

$$\begin{cases} z_t - \Delta z + \nabla q = f & \text{in } Q, \\ \nabla\cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0 & \text{in } \Omega, \end{cases} \qquad (43)$$

with $z_0 \in V$ and $f \in L^2(0,T;L^2(\Omega)^d)$. Then we have the following result for solutions to (43).

Lemma 4.3. [20] *Let $\mathcal{O}$ be a nonempty open subset of Ω. There exist $s_0 \geq 1, \lambda_0 \geq 1$ and $\mathcal{C} > 0$ such that for $s \geq s_0$ and $\lambda \geq \lambda_0$ and for every solution z to the Stokes system* (43)*, we have*

$$s^{-1}\iint\limits_{Q} e^{-2s\alpha}\xi^{-1}|\nabla \text{curl} z|^2dxdt + s\lambda^2\iint\limits_{Q} e^{-2s\alpha}\xi|\text{curl} z|^2dxdt$$

$$+ \lambda^2\iint\limits_{Q} e^{-2s\alpha}|\nabla z|^2dxdt + s^2\lambda^4\iint\limits_{Q}\xi^2|z|^2dxdt$$

$$\leq \mathcal{C}\Big(\iint\limits_{Q} e^{-2s\alpha}|f|^2dxdt + s^3\lambda^4\iint\limits_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|z|^2dxdt\Big).$$

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GLOBALLY ATTRACTING SOLUTIONS TO IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS OF SOBOLEV TYPE*

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Dedicated to Prof. Manh Hung NGUYEN on the occasion of his 60th birthday

Abstract We study a generalized Cauchy problem associated with a class of impulsive fractional differential inclusions of Sobolev type in Banach spaces. Our aim is to prove the existence of a compact set of globally attracting solutions to the problem in question. An application to fractional partial differential equations subject to impulsive effects is given to illustrate our results.

Key words Globally attracting solution; Impulsive condition; Nonlocal condition; Condensing map; Measure of non-compactness; MNC-estimate

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We are concerned with the following problem in a Banach space X:

$$D_0^\alpha Bu(t) \in Au(t) + F(t,u(t)),\ t \neq t_k, t_k \in (0,+\infty), k \in \Lambda, \tag{0.1}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{0.2}$$

$$u(0) = g(u), \tag{0.3}$$

where $D_0^\alpha, \alpha \in (0,1)$, is the fractional derivative in the Caputo sense, A and B are linear, closed and unbounded operators in X, $\Lambda \subset \mathbb{N}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$. The functions F, g and I_k will be specified in Section 3.

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The study of the Sobolev type equations can be traced back to the work of Barenblat et al [5], in which the authors initiated a model of flow of liquid in fissured rocks, i.e. the equations

$$\partial_t(u - \partial_x^2 u) - \partial_x^2 u = 0.$$

This model then was developed and studied in [7, 25] when the authors considered the abstract nonlinear equation

$$\frac{d}{dt}Bu(t) - Au(t) = f(t, u(t))$$

in Banach spaces, where A and B are unbounded operators.

Recently, as the fractional calculus becomes a powerful tool for describing various physical phenomena such as flows in porous media, oscillations and controls (see, e.g. [17, 23, 26]), fractional differential equations have been considered as an alternative tool in modeling. As a matter of fact, the fractional differential equations of Sobolev type have attracted many researches in the last few years. We refer the reader to [3, 4, 15, 18, 24] for some recent results on solvability and controllability which are close to our work.

As far as the system (0.1)-(0.3) is concerned, the appearance of multi-valued nonlinearity F is motivated by a number of problems: differential equations (DEs) with discontinuous right-hand side ([16]), differential variational inequalities ([28]), feedback controls ([20]), etc. Regarding the impulsive condition in (0.2), this is an effect appeared as the state function stands abrupt changes, which happen frequently in biology and engineering. The non-local condition in (0.3) was first studied in [10] and considered as a better description for initial condition than that in classical Cauchy problem. In applications, the non-local condition is usually in the following forms

$$u(0) = u_0 + \sum_{i=1}^{m} c_i u(t_i), c_i \in \mathbb{R}, t_i > 0,$$

$$u(0) = u_0 + \frac{1}{b}\int_0^b k(s)u(s)ds, b > 0, k \text{ is a real function.}$$

It should be mentioned that, impulsive fractional differential equations (IFrDEs) have been an attractive subject in recent years. Concerning IFrDEs in finite dimensional spaces with initial/boundary conditions, we refer to [33] for solvability and stability of Ulam type results. For a complete reference for studies in this direction, see [30, 32]. In addition, apart from IFrDEs with Caputo derivative, a formulation and existence of solutions for IFrDEs involving Hadamard derivative can be found in [34]. Referring to semilinear IFrDEs in Banach spaces, the authors in [29] gave an explicit way to represent mild solutions. By using this formulation and fixed point approach, a number of existence results has been obtained, see e.g. [29–31].

An important question associated with the problem (0.1)-(0.3) is to address the large-time behavior of its solutions. It should be noted that the theory of global attractors (see, e.g. [11]) does not work in this case due to the lack of semigroup property of solution operator. In addition, the using of Lyapunov function to analyze stability of solutions is impractical due to the difficulty in computing and estimating fractional derivatives, even in finite dimensional case. By this reason, results on large-time behavior of solutions to IFrDEs have been little known in literature. In the recent papers [12, 21, 22], we studied some models of semilinear fractional DEs in Banach spaces involving non-local conditions and impulsive effects, in which the existence of attracting solutions was proved by employing the contraction mapping principle. This approach was introduced by Burton and Furumochi [8, 9] in dealing with stability for ordinary/functional differential equations. However, the techniques used in [12, 21, 22] do not work for our problem in this note since the nonlinear functions F, g and I_k are not Lipschitzian in our settings (see Section 3 and 4).

In the present work, we prove that the problem (0.1)-(0.3) has a compact set of attracting solutions in $\mathcal{PC}([0,+\infty);X)$ (see Section 4). To this end, we will construct a regular measure of non-compactness (MNC), namely χ^* on a closed subspace of $\mathcal{PC}([0,+\infty);X)$, then show that the multi-valued solution operator associated with (0.1)-(0.3) is χ^*-condensing, then it admits a compact fixed point set.

Our work is organized as follows. In the next section, we recall some notions and facts of fractional calculus, including the characteristic solution operators given in [15], and the fixed point theory for condensing multi-valued maps. In Section 3, we make feasible assumptions on (0.1)-(0.3) and prove the solvability on compact intervals. Section 4 is devoted to the main result, in which we define the MNC χ^* and show the existence of a compact set of attracting solutions to our problem. An application to polytope fractional partial differential equations is presented in the last section.

1 Preliminaries

1.1 Fractional calculus

Let $L^1(0,T;X)$ be the space of integrable functions on $[0,T]$, in the Bochner sense.

Definition 1.1 The fractional integral of order $\alpha > 0$ of a function $f \in L^1(0,T;X)$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the Gamma function, provided the integral converges.

Definition 1.2 For a function $f \in C^N([0,T];X)$, the Caputo fractional derivative of order $\alpha \in (N-1,N)$ is defined by

$$D_0^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s)ds.$$

Consider problem

$$D_0^\alpha Bu(t) = Au(t) + f(t),\ t \neq t_k, t_k \in (0,+\infty), k \in \Lambda, \tag{1.1}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{1.2}$$

$$u(0) = g(u). \tag{1.3}$$

Assume that $\mathcal{D}(B) \subset \mathcal{D}(A)$, B is bijective and has a bounded inverse. Let $\{T(t)\}$ be the C_0-semigroup generated by AB^{-1}. Putting $v(t) = Bu(t), t \geq 0$, one can rewrite (1.1)-(1.3) as

$$D_0^\alpha v(t) = AB^{-1}v(t) + f(t),\ t \neq t_k, t_k \in (0,+\infty), k \in \Lambda, \tag{1.4}$$

$$\Delta v(t_k) = BI_k(u(t_k)), \tag{1.5}$$

$$v(0) = Bg(u). \tag{1.6}$$

Now employing the formulation of solutions to impulsive fractional differential equations established in [29], we get

$$\begin{aligned} u(t) =&S_\alpha(t)Bg(u) + \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(u(t_k)) \\ &+ \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds, t>0, \end{aligned} \tag{1.7}$$

where $S_\alpha(t)$ and $P_\alpha(t)$ are given by

$$S_\alpha(t)x = \int_0^\infty B^{-1}\phi_\alpha(\theta)T(t^\alpha\theta)xd\theta,$$
$$P_\alpha(t)x = \alpha\int_0^\infty B^{-1}\theta\phi_\alpha(\theta)T(t^\alpha\theta)xd\theta,$$

here ϕ_α is a probability density function defined on $(0,\infty)$, that is, $\phi_\alpha(\theta)\geq 0$ and $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$. Moreover, ϕ_α has the expression

$$\phi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!}\Gamma(n\alpha)\sin(n\pi\alpha).$$

Let $\{U(t)\}_{t\geq 0}$ is a family of bounded linear operators on X. Then we say that $U(\cdot)$ is norm continuous iff the map $t\mapsto U(t)$ is continuous on $(0,\infty)$. If $U(t)\in L(X)$ is a compact operator for each $t>0$ then $U(\cdot)$ is said to be compact.

Lemma 1.3 Let $T(\cdot)$ be the C_0-semigroup generated by AB^{-1}. If $T(\cdot)$ is uniformly bounded, i.e. $\sup_{t\geq 0}\|T(t)\| < +\infty$, then we have the following properties:

(1) If the semigroup $T(\cdot)$ is norm continuous, then $S_\alpha(\cdot)$ and $P_\alpha(\cdot)$ are norm continuous as well;

(2) If B^{-1} is a compact operator or $T(\cdot)$ is a compact semigroup then $S_\alpha(\cdot)$ and $P_\alpha(\cdot)$ are compact.

Proof The proof of the first part follows the same lines as in [21, Lemma 2.1]. For the second part, if B^{-1} is compact then the compactness of $S_\alpha(\cdot)$ and $P_\alpha(\cdot)$ was proved in [15, Lemma 3.2]. Moreover, if $T(\cdot)$ is a compact semigroup then $S_\alpha(\cdot)$ and $P_\alpha(\cdot)$ are compact due to the arguments in [35, Lemma 3.4].

Let $\Phi(t,s)$ be a family of bounded linear operators on X for $t,s\in[0,T], s\leq t$. The following result was proved in [27, Lemma 1].

Lemma 1.4 Assume that Φ satisfies the following conditions:

(Φ1) There exists a function $\rho\in L^q(J), q>1$ such that $\|\Phi(t,s)\|\leq\rho(t-s)$ for all $t,s\in[0,T], s\leq t$;

(Φ2) $\|\Phi(t,s)-\Phi(r,s)\|\leq\epsilon$ for $0\leq s\leq r-\epsilon, r<t=r+h\leq T$ with $\epsilon=\epsilon(h)\to 0$ as $h\to 0$.

Then the operator $\mathbf{S}: L^{q'}(0,T;X)\to C([0,T];X)$ defined by

$$(\mathbf{S}g)(t) := \int_0^t \Phi(t,s)g(s)ds$$

sends any bounded set to an equicontinuous one, where q' is the conjugate of q $(1/q'+1/q=1)$.

Define a linear operator

$$Q_\alpha : L^p([0,T];X) \to C([0,T];X),$$
$$Q_\alpha(f)(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds. \tag{1.8}$$

By the Hölder inequality, we see that Q_α is a bounded operator. Now using the last two lemmas, we have the following result.

Proposition 1.5 If the semigroup $T(\cdot)$ generated by AB^{-1} is uniformly bounded and norm continuous, then the operator Q_α defined by (1.8) maps any bounded set in $L^p(0,T;X)$ into an equicontinuous set in $C([0,T];X)$.

Proof See [21, Proposition 2.3].

1.2 Measure of noncompactness and condensing multivalued maps

Let E be a Banach space. Denote

$$\mathcal{P}(E) = \{B \subset E : B \neq \emptyset\},$$
$$\mathcal{P}_b(E) = \{B \in \mathcal{P}(E) : B \text{ is bounded}\},$$
$$K(E) = \{B \in \mathcal{P}(E) : B \text{ is compact}\},$$
$$K_v(E) = \{B \in K(E) : B \text{ is convex}\}.$$

We will use the following definition of measure of noncompactness. (see [20])

Definition 1.6 A function $\beta : \mathcal{P}_b(E) \to \mathbb{R}^+$ is called *a measure of noncompactness* (MNC) on E if

$$\beta(\overline{co}\,\Omega) = \beta(\Omega) \text{ for every } \Omega \in \mathcal{P}_b(E),$$

where $\overline{co}\,\Omega$ is the closure of the convex hull of Ω. An MNC β is called

i) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}_b(E),\ \Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \le \beta(\Omega_1)$;

ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in E, \Omega \in \mathcal{P}_b(E)$;

iii) algebraically semi-additive if $\beta(\Omega_0 + \Omega_1) \le \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}_b(E)$;

iv) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is *the Hausdorff* MNC $\chi(\cdot)$, which is defined as follows, for $\Omega \in \mathcal{P}_b(E)$ put

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \ \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Let $T \in L(E)$, i.e. T is a bounded linear operator on E. Then one can define the χ-norm of T as follows

$$\|T\|_\chi = \inf\{\beta > 0 : \chi(T(B)) \le \beta \cdot \chi(B) \text{ for all } B \in \mathcal{P}_b(E)\}. \tag{1.9}$$

It is known that (see [20])

- $\|T\|_\chi = \chi(T(\mathbf{B}_1))$ with $\mathbf{B}_1$ being a unit ball in E.
- $\|T\|_\chi \le \|T\|_{L(E)}$.

- $\|T\|_\chi = 0$ iff T is a compact operator.

We need the following result, which is an MNC-estimate. Its proof can be found in [20].

Proposition 1.7 ([20]) If $\{w_n\} \subset L^1(0,T;E)$ such that

$$\|w_n(t)\|_E \leq \nu(t), \text{ for a.e. } t \in [0,T],$$

for some $\nu \in L^1(0,T)$, then we have

$$\chi(\{\int_0^t w_n(s)ds\}) \leq 2\int_0^t \chi(\{w_n(s)\})ds$$

for $t \in [0,T]$.

We also need the following MNC-estimate for the case of uncountable sets.

Proposition 1.8 ([2]) Let $D \subset L^1(0,T;E)$ such that

(1) $\|\xi(t)\|_E \leq \nu(t)$, for all $\xi \in D$ and for a.e. $t \in [0,T]$,

(2) $\chi(D(t)) \leq q(t)$, for a.e. $t \in [0,T]$,

where $\nu, q \in L^1(0,T)$. Then

$$\chi(\int_0^t D(s)ds) \leq 4\int_0^t q(s))ds,$$

here

$$\int_0^t D(s)ds = \{\int_0^t \xi(s)ds : \xi \in D\}.$$

We are in a position to collect some notions and facts of set-valued analysis. Let Y be a metric space.

Definition 1.9 A multivalued map (multimap) $\mathcal{F} : Y \to \mathcal{P}(E)$ is said to be:

i) upper semicontinuous (u.s.c) if $\mathcal{F}^{-1}(V) = \{y \in Y : \mathcal{F}(y) \cap V \neq \emptyset\}$ is a closed subset of Y for every closed set $V \subset E$;

ii) weakly upper semicontinuous (weakly u.s.c) if $\mathcal{F}^{-1}(V)$ is closed subset of Y for all weakly closed set $V \subset E$;

iii) closed if its graph $\Gamma_\mathcal{F} = \{(y,z) : z \in \mathcal{F}(y)\}$ is a closed subset of $Y \times E$;

iv) compact if $\mathcal{F}(Y)$ is relatively compact in E;

v) quasi-compact if its restriction to any compact subset $A \subset Y$ is compact.

The following lemmas give criteria for checking if a given multimap is (weakly) u.s.c.

Lemma 1.10 *([20], Theorem 1.1.12)* Let $G : Y \to \mathcal{P}(E)$ be a closed quasi-compact multimap with compact values. Then G is u.s.c.

Lemma 1.11 *([6], Proposition 2)* Let X be a Banach space and Ω be a nonempty subset of another Banach space. Assume that $\mathcal{G} : \Omega \to \mathcal{P}(X)$ is a multimap with weakly compact, convex values. Then $\mathcal{G}$ is weakly u.s.c if and only if $\{x_n\} \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in \mathcal{G}(x_n)$ implies $y_n \rightharpoonup y_0 \in \mathcal{G}(x_0)$, up to a subsequence.

We now recall the concept of condensing multimaps ([20]).

Definition 1.12 A multimap $\mathcal{F} : Z \subseteq E \to \mathcal{P}(E)$ is said to be condensing with respect to an MNC β (β-condensing) if for any bounded set $\Omega \subset Z$, the relation

$$\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let β be a monotone nonsingular MNC in E. The application of the topological degree theory for condensing maps (see, e.g. [1, 20]) yields the following fixed point principle, which will be use to prove the existence result for (0.1)-(0.3).

Theorem 1.13 ([20, Corollary 3.3.1]) Let $\mathcal{M}$ be a bounded convex closed subset of E and let $\mathcal{F} : \mathcal{M} \to K_v(\mathcal{M})$ be a u.s.c and β-condensing multimap. Then the fixed point set $\mathrm{Fix}(\mathcal{F}) = \{x \in \mathcal{M} : x \in \mathcal{F}(x)\}$ is a nonempty and compact.

2 Existence of solutions on compact intervals

Given $T > 0$, we denote by $\mathcal{PC}([0,T];X)$ the space of functions $u : [0,T] \to X$ such that u is continuous on $[0,T]\backslash\{t_k : k \in \Lambda\}$ and for each $t_k \in [0,T], k \in \Lambda$, there exist

$$u(t_k^-) = \lim_{t \to t_k^-} u(t); \quad u(t_k^+) = \lim_{t \to t_k^+} u(t)$$

and $u(t_k) = u(t_k^-)$. Then $\mathcal{PC}([0,T];X)$ endowed with the norm

$$\|u\|_{\mathcal{PC}} := \sup_{t \in [0,T]} \|u(t)\|,$$

is a Banach space. Let χ be the Hausdorff MNC in X, $\chi_{\mathcal{PC}}$ the Hausdorff MNC in $\mathcal{PC}([0,T];X)$.

We recall the following facts (see [19]): for each bounded set $D \subset \mathcal{PC}([0,T];X)$, one has

- $\chi(D(t)) \leq \chi_{\mathcal{PC}}(D)$, for all $t \in [0,T]$, where $D(t) := \{x(t) : x \in D\}$.
- If D is an equicontinuous set on each interval $(t_k, t_{k+1}] \subset [0,T]$, then

$$\chi_{\mathcal{PC}}(D) = \sup_{t \in [0,T]} \chi(D(t)).$$

To prove existence results for problem (0.1)-(0.3), we make the following assumptions:

(**A**) *AB^{-1} is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$ which is norm continuous.*

(**F**) *$F : [0,T] \times X \to K_v(X)$ is a multimap satisfying that:*

1. *The multimap $F(\cdot, v)$ admits a strongly measurable selection for each $v \in X$ and the multimap $F(t, \cdot)$ is u.s.c for a.e. $t \in (0,T)$;*
2. *There exist functions $m \in L^p(0,T)$, $p > \frac{1}{\alpha}$ and Ψ_F being a real-valued, continuous and nondecreasing function, such that*

$$\|F(t,v)\| \leq m(t)\Psi_F(\|v\|),$$

for all $v \in X$ and for a.e. $t \in (0,T)$, here $\|F(t,v)\| = \sup\{\|\xi\| : \xi \in F(t,v)\}$;

3. *If B^{-1} and $T(\cdot)$ are non-compact, then for any bounded sets $D \subset X$, we have*

$$\chi(F(t,D)) \leq k(t)\chi(D),$$

for a.e. $t \in (0,T)$, where $k \in L^p(0,T)$ is a nonnegative function.

(**G**) *The nonlocal function* $g : \mathcal{PC}([0,T];X) \to \mathcal{D}(B)$ *obeys the following conditions:*

1. $Bg : \mathcal{PC}([0,T];X) \to X$ *is continuous and*

$$\|Bg(u)\| \le \Psi_g(\|u\|_{\mathcal{PC}}),$$

for all $u \in \mathcal{PC}([0,T];X)$*, where* Ψ_g *is a continuous and nondecreasing function on* $\mathbb{R}^+$*;*

2. *There exists* $\eta \ge 0$ *such that*

$$\chi(Bg(D)) \le \eta \chi_{\mathcal{PC}}(D),$$

for all bounded set $D \subset \mathcal{PC}([0,T];X)$.

(**I**) *The operator* $I_k : X \to \mathcal{D}(B)$ *satisfies:*

1. $BI_k : X \to X$ *is continuous and there exists a real-valued, continuous, nondecreasing function* Ψ_I *and a nonnegative sequence* $\{l_k\}_{k\in\Lambda}$ *such that*

$$\|BI_k(x)\|_X \le l_k\, \Psi_I(\|x\|), \text{ for all } x \in X, k \in \Lambda;$$

2. *There exists a nonnegative sequence* $\{\mu_k\}_{k\in\Lambda}$ *such that*

$$\chi(BI_k(D)) \le \mu_k \chi(D),$$

for all bounded subset $D \subset X$.

3. *The sequence* $\{t_k\}_{k\in\Lambda}$ *satisfies* $\inf_{k\in\Lambda}\{t_{k+1} - t_k\} > 0$.

For $u \in \mathcal{PC}([0,T];X)$, we denote

$$\mathcal{P}_F^p(u) = \{f \in L^p(0,T;X) : f(t) \in F(t,u(t))\}.$$

Motivated by formula (1.7), we introduce the following definition for integral solutions to (0.1)-(0.3).

Definition 2.1 A function $u \in \mathcal{PC}([0,T];X)$ is said to be an integral solution of problem (0.1)-(0.3) on the interval $[0,T]$ iff there exists a function $f \in \mathcal{P}_F^p(u)$ such that

$$u(t) = S_\alpha(t)Bg(u) + \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(u(t_k)) + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds, \tag{2.1}$$

for any $t \in [0,T]$.

We now define the *solution operator*

$$\mathcal{F} : \mathcal{PC}([0,T];X) \to \mathcal{P}(\mathcal{PC}([0,T];X))$$

as follows

$$\mathcal{F}(u)(t) = S_\alpha(t)Bg(u) + \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(u(t_k)) + \left\{ \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds : f \in \mathcal{P}_F^p(u)\right\}. \tag{2.2}$$

Since F has convex values, so does $\mathcal{P}_F^p$. This implies that $\mathcal{F}$ has convex values as well. On the other hand, u is an integral solution of (0.1)-(0.3) if it is a fixed point of the solution operator $\mathcal{F}$.

To establish the existence result, we need some properties of $\mathcal{P}_F^p$.

Lemma 2.2 Under the assumption (**F**), the multimap $\mathcal{P}_F^p$ is well-defined and weakly u.s.c.

Proof We first prove the weakly u.s.c property by using Lemma 1.11. Let $\{u_n\} \subset \mathcal{PC}([0,T];X)$ such that $u_n \to u^*, f_n \in \mathcal{P}_F^p(u_n)$. We see that $\{f_n(t)\} \subset C(t) := \overline{F(t,\{u_n(t)\})}$, and $C(t)$ is a compact set for a.e $t \in (0,T)$. Furthermore, by (**F**)(2), $\{f_n\}$ is integrably bounded (bounded by an L^p-integrable function). Therefore $\{f_n\}$ is weakly compact in $L^p(0,T;X)$ (see [13]). Let $f_n \rightharpoonup f^*$. Then by Mazur's lemma (see, e.g. [14]), there are $\widetilde{f}_n \in \mathrm{co}\{f_i : i \geq n\}$ such that $\widetilde{f}_n \to f^*$ in $L^p(0,T;X)$ and then $\widetilde{f}_n(t) \to f^*(t)$ for a.e $t \in (0,T)$, up to a subsequence. Since F has compact values, the upper semicontinuous of $F(t,\cdot)$ means that

$$F(t,u_n(t)) \subset F(t,u^*(t)) + B_\epsilon,$$

for all large n, here $\epsilon > 0$ is given and B_ϵ is the ball in X centered at origin with radius ϵ. So

$$f_n(t) \in F(t,u^*(t)) + B_\epsilon,$$

for a.e. $t \in (0,T)$, and the same inclusion holds for $\widetilde{f}_n(t)$ thanks to the convexity of $F(t,u^*(t))+B_\epsilon$. Accordingly,

$$f^*(t) \in F(t,u^*(t)) + B_\epsilon,$$

for a.e. $t \in (0,T)$. Since ϵ is arbitrary, one gets $f^* \in \mathcal{P}_F^p(u^*)$.

It remains to show that for each $v \in \mathcal{PC}([0,T];X)$, $\mathcal{P}_F^p(v) \neq \emptyset$. Taking (**I**)(3) into account, we see that there are at most a finite number of $t_k \in [0,T]$. Then one can find a sequence $\{v_n\}$ of step functions which converges uniformly to v on $[0,T]$. Then for each n there exists a strongly measurable function f_n such that $f_n(t) \in F(t,v_n(t))$, thanks to (**F**)(1). That is, $\{f_n(t)\} \subset C(t)$, where $C(t) = \overline{F(t,\{v_n(t)\})}$ is a compact set, thanks to the upper-continuity of $F(t,\cdot)$. Using the same argument as in the first part, we see that $\{f_n\}$ is a weakly compact in $L^p(0,T;X)$ and $f_n \rightharpoonup f \in L^p(0,T;X)$ and $f(t) \in F(t,v(t))$ for a.e. $t \in (0,T)$. That is $f \in \mathcal{P}_F^p(v)$. The proof is complete.

Lemma 2.3 Under the assumptions (**A**) and (**F**), the composition

$$Q_\alpha \circ \mathcal{P}_F^p : \mathcal{PC}([0,T];X) \to \mathcal{P}(\mathcal{PC}([0,T];X))$$

is a u.s.c. multimap with compact values, where Q_α is defined by (1.8).

Proof The proof is proceeded in two steps.

Step 1: $Q_\alpha \circ \mathcal{P}_F^p$ is a closed multimap. Let

$$\{u_n\} \subset \mathcal{PC}([0,T];X), u_n \to u^*; z_n \in Q_\alpha \circ \mathcal{P}_F^p(u_n) \text{ and } z_n \to z^*.$$

We show that $z^* \in Q_\alpha \circ \mathcal{P}_F^p(u^*)$. Take $f_n \in \mathcal{P}_F^p(u_n)$ such that

$$z_n(t) = Q_\alpha(f_n)(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f_n(s) ds. \tag{2.3}$$

By Lemma 2.2 we get that $f_n \rightharpoonup f^* \in L^p(0,T;X)$ and $f^* \in \mathcal{P}_F^p(u^*)$. Since Q_α is linear and continuous, we have $Q_\alpha(f_n) \rightharpoonup Q_\alpha(f^*)$. In addition, $C(t) = \{f_n(t) : n \geq 1\}$ is relatively compact, and then

$$\begin{aligned}\chi(\{Q_\alpha(f_n)(t)\}) &\leq \chi\Big(\Big\{\int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f_n(s)ds\Big\}\Big)\\ &\leq 2\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\chi(\{f_n(s)\})ds = 0,\end{aligned}$$

according to Proposition 1.7. Due to Proposition 1.5, $\{Q_\alpha(f_n)\}$ is equicontinuous. Then by the Arzela - Ascoli theorem, we have $\{Q_\alpha(f_n)\}$ is relatively compact. Therefore one has $Q_\alpha(f_n) \to Q_\alpha(f^*)$. So it follows from (2.3) that

$$z^*(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f^*(s)ds = Q_\alpha(f^*)(t),$$

for all $t \in [0,T]$, where $f^* \in \mathcal{P}_F^p(u^*)$, thus $z^* \in Q_\alpha \circ \mathcal{P}_F^p(u^*)$.

Step 2: $Q_\alpha \circ \mathcal{P}_F^p$ is a quasi-compact multimap. Let $\mathcal{K} \subset \mathcal{PC}([0,T];X)$ be a compact set and $\{z_n\} \subset Q_\alpha \circ \mathcal{P}_F^p(\mathcal{K})$. We prove that $\{z_n\}$ is relatively compact in $C([0,T];X)$, and hence in $\mathcal{PC}([0,T];X)$. Let $\{u_k\} \subset \mathcal{K}$ such that $z_n \in Q_\alpha \circ \mathcal{P}_F^p(u_n)$. Then one can assume that $u_n \to u^*$ in $\mathcal{PC}([0,T];X)$ up to a subsequence. Take $f_n \in \mathcal{P}_F^p(u_n)$ such that $z_n(t) = Q_\alpha(f_n)(t)$, for all $t \in [0,T]$. Since $\{f_n(s)\} \subset F(s, \overline{\{u_n(s)\}})$, one sees that $\{f_n(s)\}$ is relatively compact for a.e. $s \in (0,T)$. Thus $\{Q_\alpha(f_n)(t)\}$ is a compact set for all $t \in [0,T]$. In addition, $\{Q_\alpha(f_n)\}$ is equicontinuous due to Proposition 1.5, then $\{z_n\}$ is relatively compact in $C([0,T];X)$.

Thus the conclusion follows from Step 1, Step 2 and Lemma 1.10.

Lemma 2.4 Let the hypotheses (**A**), (**F**), (**G**) and (**I**) hold. Then the solution operator $\mathcal{F}$ satisfies

$$\chi_{\mathcal{PC}}(\mathcal{F}(D)) \leq \Big[\Big(\eta + \sum_{t_k\in(0,T)} \mu_k\Big) S_\alpha^T + 4\sup_{t\in(0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)ds\Big]\chi_{\mathcal{PC}}(D),$$

for all bounded set $D \subset \mathcal{PC}([0,T];X)$, here $S_\alpha^T = \sup_{t\in[0,T]}\|S_\alpha(t)\|$.

Proof Let $D \subset \mathcal{PC}([0,T];X)$ be a bounded set. Then we have

$$\mathcal{F}(D) = \mathcal{F}_1(D) + \mathcal{F}_2(D) + \mathcal{F}_3(D),$$

where

$$\begin{aligned}&\mathcal{F}_1(u)(t) = S_\alpha(t)Bg(u),\\ &\mathcal{F}_2(u)(t) = \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(u(t_k)),\\ &\mathcal{F}_3(u)(t) = \Big\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds : f \in \mathcal{P}_F^p(u), t\in[0,T]\Big\}.\end{aligned}$$

So

$$\chi_{\mathcal{PC}}(\mathcal{F}(D)) \leq \chi_{\mathcal{PC}}(\mathcal{F}_1(D)) + \chi_{\mathcal{PC}}(\mathcal{F}_2(D)) + \chi_{\mathcal{PC}}(\mathcal{F}_3(D)).$$

For $z_1, z_2 \in \mathcal{F}_1(D)$, there exist $u_1, u_2 \in D$ such that

$$\begin{aligned}&z_1(t) = S_\alpha(t)Bg(u_1),\\ &z_2(t) = S_\alpha(t)Bg(u_2), t\in[0,T],\end{aligned}$$

then

$$\begin{aligned}\|z_1(t) - z_2(t)\| &\le \|S_\alpha(t)\| \; \|Bg(u_1) - Bg(u_2)\| \\ &\le S_\alpha^T \; \|Bg(u_1) - Bg(u_2)\|, \; t \in [0, T].\end{aligned}$$

It follows that

$$\|z_1 - z_2\|_{\mathcal{PC}} \le S_\alpha^T \; \|Bg(u_1) - Bg(u_2)\|.$$

Thus

$$\chi_{\mathcal{PC}}(\mathcal{F}_1(D)) \le S_\alpha^T \cdot \chi(Bg(D)).$$

Employing **(G)**(2), we have

$$\chi_{\mathcal{PC}}(\mathcal{F}_1(D)) \le \eta S_\alpha^T \cdot \chi_{\mathcal{PC}}(D). \tag{2.4}$$

Now let $z_1, z_2 \in \mathcal{F}_2(D)$, one can find $u_1, u_2 \in D$ such that

$$z_1(t) - z_2(t) = \sum_{t_k \in (0,T)} S_\alpha(t - t_k) B[I_k(u_1(t_k)) - I_k(u_2(t_k))], t \in [0, T].$$

Hence

$$\|z_1 - z_2\|_{\mathcal{P}C} \le S_\alpha^T \sum_{t_k \in (0,T)} \|BI_k(u_1(t_k)) - BI_k(u_2(t_k))\|.$$

This inequality implies that

$$\begin{aligned}\chi_{\mathcal{P}C}(\mathcal{F}_2(D)) &\le S_\alpha^T \sum_{t_k \in (0,T)} \chi(BI_k(D(t_k))) \\ &\le S_\alpha^T \sum_{t_k \in (0,T)} \mu_k \chi(D(t_k)) \\ &\le \Big(S_\alpha^T \sum_{t_k \in (0,T)} \mu_k\Big) \chi_{\mathcal{P}C}(D),\end{aligned} \tag{2.5}$$

thanks to (**I**)(2).

Regarding $\mathcal{F}_3(D)$, for $t \in [0, T]$, we have

$$\begin{aligned}\chi(\mathcal{F}_3(D)(t)) &= \chi\big(\int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) \mathcal{P}_F^p(D)(s) ds\big) \\ &\le 4 \int_0^t (t-s)^{\alpha-1} \chi\big(P_\alpha(t-s) \mathcal{P}_F^p(D)(s)\big) ds,\end{aligned} \tag{2.6}$$

due to Proposition 1.8. If B^{-1} or $T(\cdot)$ is compact, so is $P_\alpha(\cdot)$ due to Lemma 1.3. Then $\chi(\mathcal{F}_3(D)(t)) = 0$, thanks to the fact that $\chi\big(P_\alpha(t-s)\mathcal{P}_F^p(D)(s)\big) = 0$ for $s \in (0, t)$. In the opposite case, we have

$$\chi\big(P_\alpha(t-s)\mathcal{P}_F^p(D)(s)\big) \le \|P_\alpha(t-s)\|_\chi \, \chi(\mathcal{P}_F^p(D)(s)) \le \|P_\alpha(t-s)\|_\chi k(s) \chi(D(s)).$$

Plugging this in (2.6), we get

$$\begin{aligned}\chi(\mathcal{F}_3(D)(t)) &\le 4 \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s) \chi(D(s)) ds \\ &\le \left(4 \sup_{t \in (0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s) ds\right) \chi_{\mathcal{PC}}(D).\end{aligned} \tag{2.7}$$

We observe that $\mathcal{P}_F^p(D)$ is bounded in $L^p(0,T;X)$ since D is bounded in $\mathcal{PC}([0,T];X)$. By Proposition 1.5, the set

$$\mathcal{F}_3(D) = Q_\alpha \circ \mathcal{P}_F^p(D)$$

is equicontinuous in $C([0,T];X)$. Thus

$$\chi_{\mathcal{PC}}(\mathcal{F}_3(D)) = \sup_{t\in[0,T]} \chi(\mathcal{F}_3(D)(t)).$$

In view of (2.7), one has

$$\chi_{\mathcal{PC}}(\mathcal{F}_3(D)) \le \big(4 \sup_{t\in(0,T]} \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)ds\big)\chi_{\mathcal{PC}}(D). \tag{2.8}$$

Combining (2.4), (2.5) and (2.8), we arrive at

$$\chi_{\mathcal{PC}}(\mathcal{F}(D)) \le \left[\Big(\eta + \sum_{t_k\in(0,T)} \mu_k\Big)S_\alpha^T + 4\sup_{t\in(0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)ds\right]\chi_{\mathcal{PC}}(D).$$

The proof is complete.

Theorem 2.5 Assume that the hypotheses of Lemma 2.4 hold. Then the problem (0.1)-(0.3) has at least one integral solution in $\mathcal{PC}([0,T];X)$, provided that

$$\Big(\eta + \sum_{t_k\in(0,T)} \mu_k\Big)S_\alpha^T + 4\sup_{t\in(0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)ds < 1, \tag{2.9}$$

and

$$\begin{aligned}\liminf_{r\to\infty}\frac{1}{r}\bigg[\Big(\Psi_g(r) + \Psi_I(r)\sum_{t_k\in(0,T)} l_k\Big)S_\alpha^T \\ + \Psi_F(r)\sup_{t\in(0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)ds\bigg] < 1,\end{aligned} \tag{2.10}$$

where $S_\alpha^T = \sup_{t\in[0,T]}\|S_\alpha(t)\|$.

Proof By Lemma 2.3 and the continuity of Bg and BI_k, one sees that $\mathcal{F}$ is u.s.c with compact and convex values.

By (2.9) and Lemma 2.4, we obtain the $\chi_{\mathcal{PC}}$-condensing property for $\mathcal{F}$. In order to apply Theorem 1.13, it remains to show that $\mathcal{F}(B_R) \subset B_R$ for some $R>0$, where B_R is the closed ball in $\mathcal{PC}([0,T];X)$ centered at 0 with radius R.

Assume to the contrary that there exists a sequence $\{v_n\} \subset \mathcal{PC}([0,T];X)$ such that $\|v_n\|_{\mathcal{PC}} \le n$ and $z_n \in \mathcal{F}(v_n)$ with $\|z_n\|_{\mathcal{PC}} > n$. From the formulation of $\mathcal{F}$, one can find $f_n \in \mathcal{P}_F^p(v_n)$ such that

$$\begin{aligned} z_n(t) =& S_\alpha(t)Bg(v_n) + \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(v_n(t_k)) \\ &+ \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f_n(s)ds.\end{aligned}$$

Then

$$\begin{aligned}\|z_n(t)\| \le& \|S_\alpha(t)\|\|Bg(v_n)\| + \sum_{t_k\in(0,T)} \|S_\alpha(t-t_k)\|\|BI_k(v_n(t_k))\| \\ &+ \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\|f_n(s)\|ds\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in[0,T]} \|S_\alpha(t)\| \Psi_g(\|v_n\|_{\mathcal{PC}} + \sum_{t_k\in(0,T)} l_k \Psi_I(\|v_n(t_k)\|)\Big) \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) \Psi_F(\|v_n(s)\|) ds \\
&\leq S_\alpha^T \Big(\Psi_g(n) + \sum_{t_k\in(0,T)} l_k \Psi_I(\|v_n\|_{\mathcal{PC}})\Big) \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) \Psi_F(\|v_n\|_{\mathcal{PC}}) ds \\
&\leq S_\alpha^T \Big(\Psi_g(n) + \Psi_I(n) \sum_{t_k\in(0,T)} l_k\Big) \\
&\quad + \Psi_F(n) \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
n < \|z_n\|_{\mathcal{PC}} \leq & S_\alpha^T \Big(\Psi_g(n) + \Psi_I(n) \sum_{t_k\in(0,T)} l_k\Big) \\
& + \Psi_F(n) \sup_{t\in(0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
1 < \frac{1}{n}\|z_n\|_{\mathcal{PC}} \leq & \frac{1}{n}\Big[S_\alpha^T \Big(\Psi_g(n) + \Psi_I(n) \sum_{t_k\in(0,T)} l_k\Big) \\
& + \Psi_F(n) \sup_{t\in(0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) ds\Big].
\end{aligned}
$$

Passing to the limit in the last inequality, one gets a contradiction. The proof is complete.

3 Globally attracting solutions

In this section, we prove the existence of attracting integral solution to problem (0.1)-(0.3). To this end, we consider the function space

$$
\mathcal{PC}_0 = \{u \in \mathcal{PC}([0,+\infty); X) : \lim_{t\to\infty} u(t) = 0\}
$$

with the norm

$$
\|u\|_\infty = \sup_{t\geq 0} \|u(t)\|,
$$

where $\mathcal{PC}([0,+\infty); X)$ is defined similarly to $\mathcal{PC}([0,T]; X)$ as $T = +\infty$. Then $\mathcal{PC}_0$ is a Banach space. In this section, the multimap $\mathcal{P}_F^p$ is defined as follows: for $u \in \mathcal{PC}([0,+\infty); X)$,

$$
\mathcal{P}_F^p(u) = \left\{ f \in L^p_{loc}(\mathbb{R}^+; X) : f(t) \in F(t, u(t)) \text{ for a.e. } t \in \mathbb{R}^+ \right\}.
$$

Denote by π_T the restriction operator on $\mathcal{PC}_0$, that is, $\pi_T(x)$ is the restriction of x on $[0,T]$. Then the function

$$
\chi_\infty(D) = \sup_{T>0} \chi_{\mathcal{PC}}(\pi_T(D)) \tag{3.1}
$$

is an MNC on $\mathcal{PC}_0$, here we recall that $\chi_{\mathcal{PC}}$ is the Hausdorff MNC on $\mathcal{PC}([0,T]; X)$. Argued as in [2], χ_∞ is not a regular MNC on $\mathcal{PC}_0$. We will define a regular one on this space. Let

$$
d_T(D) = \sup_{x\in D} \sup_{t\geq T} \|x(t)\|, \tag{3.2}
$$

$$d_\infty(D) = \lim_{T\to\infty} d_T(D), \tag{3.3}$$

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D). \tag{3.4}$$

The following result is important for our purpose.

Lemma 3.1 The MNC χ^* defined by (3.4) is regular on $\mathcal{PC}_0$.

Proof Let $D \subset \mathcal{PC}_0$ be a bounded set such that $\chi^*(D) = 0$. It is obvious that $\pi_T(D)$ is relatively compact in $\mathcal{PC}([0,T];X)$. We show that D is relatively compact in $\mathcal{PC}_0$.

For $\epsilon > 0$, since $d_\infty(D) = 0$ one can take $T > 0$ such that $\sup_{t\geq T}\|u(t)\| < \frac{\epsilon}{2}$, for all $u \in D$. This means that

$$\|u - \pi_T(u)\|_\infty < \frac{\epsilon}{2}, \text{ for all } u \in D,$$

here $\pi_T(u)$ agrees with a function in $\mathcal{PC}_0$ in the following manner

$$\pi_T(u) = \begin{cases} u(t), & t \in [0,T], \\ 0, & t > T. \end{cases}$$

Now since $\pi_T(D)$ is a compact set in $\mathcal{PC}([0,T];X)$, we can write

$$\pi_T(D) \subset \bigcup_{i=1}^{N} B_T(u_i; \frac{\epsilon}{2}), \tag{3.5}$$

where $u_i \in \mathcal{PC}([0,T];X), i = 1, ..., N$, the notation $B_T(u;r)$ stands for the ball in $\mathcal{PC}([0,T];X)$ centered at u with radius r. Defining

$$\hat{u}_i(t) = \begin{cases} u_i(t), & t \in [0,T], \\ 0, & t > T, \end{cases}$$

then $\{\hat{u}_i\}_{i=1}^N$ belong to $\mathcal{PC}_0$. We assert that

$$D \subset \bigcup_{i=1}^{N} B_\infty(\hat{u}_i; \epsilon),$$

here $B_\infty(u;r)$ is the ball in $\mathcal{PC}_0$ with center u and radius r. Indeed, let $u \in D$ then by (3.5), there is a number $k \in \{1, ..., N\}$ such that

$$\|\pi_T(u) - u_k\|_{\mathcal{PC}([0,T];X)} < \frac{\epsilon}{2}.$$

This implies that

$$\|\pi_T(u) - \hat{u}_k\|_\infty < \frac{\epsilon}{2}.$$

Then

$$\|u - \hat{u}_k\|_\infty \leq \|u - \pi_T(u)\|_\infty + \|\pi_T(u) - \hat{u}_k\|_\infty \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $u \in B_\infty(\hat{u}_k; \epsilon)$ and we have $D \subset \bigcup_{i=1}^N B_\infty(\hat{u}_i; \epsilon)$. Hence D is relatively compact in $\mathcal{PC}_0$. The proof is complete.

We now prove that $\mathcal{F}$ keeps $\mathcal{PC}_0$ invariant, i.e. $\mathcal{F}(\mathcal{PC}_0) \subset \mathcal{PC}_0$, and $\mathcal{F}$ is χ^*-condensing on $\mathcal{PC}_0$. In order to get attracting solutions to problem (0.1)-(0.3), we have to replace (**A**), (**F**), (**G**) and (**I**) by stronger ones.

(**A***) *The semigroup* $\{T(t)\}_{t\geq 0}$ *satisfies* (**A**) *and the operator families* $\{S_\alpha(t);\ P_\alpha(t)\}_{t\geq 0}$ *are asymptotically stable, that is,*

$$\lim_{t\to\infty}\|S_\alpha(t)\| = 0, \quad \lim_{t\to\infty}\|P_\alpha(t)\| = 0.$$

(**F***) $F : \mathbb{R}^+ \times X \to K_v(X)$ *satisfies* (**F**) *for every* $T > 0$*, with* $m, k \in L^p_{loc}(\mathbb{R}^+)$ *and* $\Psi_F(r) \leq r$ *for all* $r \geq 0$.

(**G***) *The function* $g : \mathcal{PC}([0, +\infty); X) \to \mathcal{D}(B)$ *satisfies* (**G**) *for any* $T > 0$.

(**I***) *The jump functions* $I_k : X \to \mathcal{D}(B)$ *satisfies* (**I**) *with* $\sum_{k\in\Lambda} l_k < +\infty$ *and* $\sum_{k\in\Lambda}\mu_k < +\infty$.

The following proposition show a case, in which (**A***) is satisfied.

Proposition 3.2 Assume that the semigroup $\{T(t)\}_{t\geq 0}$ generated by AB^{-1} is norm continuous and exponentially stable, i.e., there are positive numbers a, M such that

$$\|T(t)\| \leq Me^{-at}.$$

Then there exist two positive number C_S and C_P such that

$$\|S_\alpha(t)\| \leq M\|B^{-1}\| \min\left(1, C_S t^{-\alpha}\right), \tag{3.6}$$

$$\|P_\alpha(t)\| \leq M\|B^{-1}\| \min\left(\frac{1}{\Gamma(\alpha)}, C_P t^{-2\alpha}\right), \ \ \forall t > 0. \tag{3.7}$$

Proof The proof follows the same lines as those in [2].

Lemma 3.3 Let (**A***), (**F***), (**G***), (**I***) hold. Then $\mathcal{F}(\mathcal{PC}_0) \subset \mathcal{PC}_0$ provided that

$$\vartheta = \sup_{t>0}\int_0^{\delta t}\|P_\alpha(t-s)\|\, m(s)ds < +\infty, \tag{3.8}$$

$$\kappa = \sup_{t>0}\int_{\delta t}^{t}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\, m(s)ds < +\infty, \tag{3.9}$$

for some $\delta \in (0, 1)$.

Proof We recall that

$$\begin{aligned}\mathcal{F}(u)(t) =& S_\alpha(t)Bg(u) + \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(u(t_k)) \\ &+ \left\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds : f \in \mathcal{P}^p_F(u)\right\}, t > 0.\end{aligned}$$

Let $u \in \mathcal{PC}_0$ such that $R = \|u\|_\infty > 0$ and $z \in \mathcal{F}(u)$. We prove that $z \in \mathcal{PC}_0$, i.e. $z(t) \to 0$ as $t \to +\infty$.

Let $\epsilon > 0$ be given. Then there exists $T_1 > 0$ such that

$$\|u(t)\| \leq \epsilon, \forall t > T_1. \tag{3.10}$$

From the assumption that $\sum_{k\in\Lambda} l_k < +\infty$, there exists $N_0 \in \mathbb{N}$ such that $\sum_{k>N_0} l_k \leq \epsilon$. Now for $t > 0$,

$$\begin{aligned}\|z(t)\| \leq& \|S_\alpha(t)\|\|Bg(u)\| \\ &+ \sum_{k\leq N_0}\|S_\alpha(t-t_k)\|\,\|BI_k(u(t_k))\| + \sum_{k>N_0}\|S_\alpha(t-t_k)\|\,\|BI_k(u(t_k))\|\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,\|f(s)\|ds\\
&\le \|S_\alpha(t)\|\Psi_g(R)+\Psi_I(R)\sum_{k\le N_0}\|S_\alpha(t-t_k)\|\,l_k+S_\alpha^\infty\Psi_I(R)\sum_{k>N_0}l_k\\
&+\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,m(s)\,\Psi_F(\|u(s)\|)ds\\
&=E_1(t)+E_2(t)+E_3(t),
\end{aligned}
$$

where $S_\alpha^\infty=\sup\limits_{t\ge 0}\|S_\alpha(t)\|$, and

$$
\begin{aligned}
E_1(t)&=\|S_\alpha(t)\|\Psi_g(R),\\
E_2(t)&=\Psi_I(R)\sum_{k\le N_0}\|S_\alpha(t-t_k)\|\,l_k+S_\alpha^\infty\Psi_I(R)\sum_{k>N_0}l_k,\\
E_3(t)&=\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,m(s)\,\Psi_F(\|u(s)\|)ds.
\end{aligned}
$$

By ($\mathbf{A}^*$) there is $T_2>0$ such that

$$\|S_\alpha(t)\|\le\epsilon, \|P_\alpha(t)\|\le\epsilon, \forall t>T_2, \tag{3.11}$$

so

$$E_1(t)\le\epsilon\,\Psi_g(R),\ \forall t>T_2. \tag{3.12}$$

In addition,

$$E_2(t)\le\epsilon\Big(\sum_{k\le N_0}l_k+S_\alpha^\infty\Big)\Psi_I(R),\ \forall t>T_2+t_{N_0}. \tag{3.13}$$

Concerning $E_3(t)$, for $t>\frac{T_1}{\delta}$ one has

$$
\begin{aligned}
E_3(t)&=\Big(\int_0^{\delta t}+\int_{\delta t}^t\Big)(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,m(s)\,\Psi_F(\|u(s)\|)ds\\
&\le\Psi_F(R)\int_0^{\delta t}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,m(s)ds\\
&\quad+\Psi_F(\epsilon)\int_{\delta t}^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,m(s)ds\\
&\le\frac{R}{[(1-\delta)t]^{1-\alpha}}\int_0^{\delta t}\|P_\alpha(t-s)\|\,m(s)ds\\
&\quad+\epsilon\int_{\delta t}^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,m(s)ds
\end{aligned}
$$

thanks to (3.10) and the fact that $\delta t>T_1$. Now choosing $T_3>\frac{T_1}{\delta}$ such that

$$\frac{R}{[(1-\delta)t]^{1-\alpha}}<\epsilon,\ \forall t>T_3,$$

we get

$$E_3(t)\le(\vartheta+\kappa)\epsilon, \tag{3.14}$$

where ϑ,κ are given in (3.8)-(3.9). Combining (3.12)-(3.14) yields

$$\|z(t)\|\le\epsilon\big[\Psi_g(R)+\big(\sum_{k\le N_0}l_k+S_\alpha^\infty\big)\Psi_I(R)+\vartheta+\kappa\big],$$

for all $t > \max\{T_2 + t_{N_0}, T_3\}$. So the last inequality ensures that $z \in \mathcal{PC}_0$. The proof is complete.

Lemma 3.4 Let (**A***), (**F***), (**G***) and (**I***) hold. If $\vartheta < +\infty$ and $\max\{\kappa, \ell\} < 1$, where ϑ, κ are given in (3.8)-(3.9) and

$$\ell = \Big(\eta + \sum_{k\in\Lambda} \mu_k\Big) S_\alpha^\infty + 4 \sup_{t>0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s)ds, \tag{3.15}$$

then $\mathcal{F}$ is χ^*-condensing on $\mathcal{PC}_0$.

Proof By the hypotheses and Lemma 3.3, one can consider the solution operator $\mathcal{F} : \mathcal{PC}_0 \to \mathcal{P}(\mathcal{PC}_0)$. Let $D \subset \mathcal{PC}_0$ be a bounded set. Taking $r > 0$ such that $\|u\|_\infty \le r$, $\forall u \in D$. We have $\pi_T(D)$ bounded in $\mathcal{PC}([0,T]; X)$.

By the same arguments as in the proof of Lemma 2.4, one has

$$\chi_{\mathcal{PC}}\big(\pi_T(\mathcal{F}(D))\big) \le l_T \cdot \chi_{\mathcal{PC}}(\pi_T(D)),$$

where

$$l_T = \bigg[\Big(\eta + \sum_{t_k\in(0,T)} \mu_k\Big) S_\alpha^T + 4 \sup_{t\in(0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s)ds\bigg],$$

where $\|\cdot\|_\chi$ is the χ-norm of a bounded linear operator defined by (1.9). This implies

$$\chi_\infty(\mathcal{F}(D)) \le \ell \cdot \chi_\infty(D). \tag{3.16}$$

It remains to estimate $d_\infty(D)$. For each $z \in \mathcal{F}(D)$, there exists $u \in D$ and $f \in \mathcal{P}_F^p(u)$ such that

$$\begin{aligned} z(t) =& S_\alpha(t)Bg(u) + \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(u(t_k)) \\ &+ \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)ds, \forall t > 0. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{F}_1(u)(t) &= S_\alpha(t)Bg(u), \\ \mathcal{F}_2(u)(t) &= \sum_{0<t_k<t} S_\alpha(t-t_k)BI_k(u(t_k)), \\ \mathcal{F}_3(u)(t) &= \bigg\{ \int_0^{\delta t} (t-s)^{\alpha-1} P_\alpha(t-s) f(s)ds : f \in \mathcal{P}_F^p(u) \bigg\}, \\ \mathcal{F}_4(u)(t) &= \bigg\{ \int_{\delta t}^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)ds : f \in \mathcal{P}_F^p(u) \bigg\}, \end{aligned}$$

for $t > 0$. Then

$$\mathcal{F}(D) = \mathcal{F}_1(D) + \mathcal{F}_2(D) + \mathcal{F}_3(D) + \mathcal{F}_4(D). \tag{3.17}$$

We first show that

$$d_\infty(\mathcal{F}_1(D)) = d_\infty(\mathcal{F}_2(D)) = d_\infty(\mathcal{F}_3(D)) = 0 \tag{3.18}$$

by arguing that for any $\epsilon > 0$, there exists $T > 0$ such that for all $z \in \mathcal{F}_i(D)$, $i \in \{1, 2, 3\}$, $\|z(t)\| < C\epsilon$ for $t \ge T$, where $C = C(r) > 0$.

Let $z \in \mathcal{F}_1(D)$, then one can take $u \in D$ such that $z(t) = S_\alpha(t)Bg(u)$. We have

$$\|z(t)\| \le \|S_\alpha(t)\| \Psi_g(\|u\|_\infty) \le \|S_\alpha(t)\| \Psi_g(r).$$

The last inequality implies that for all $z \in \mathcal{F}_1(D)$, $\|z(t)\| < \epsilon\,\Psi_g(r)$ for $t \geq T_1 > 0$ thanks to the fact that $\|S_\alpha(t)\| \to 0$ as $t \to +\infty$.

Regarding $\mathcal{F}_2(D)$, we observe that, for $z = \mathcal{F}_2(u), u \in D$,

$$\begin{aligned}
\|z(t)\| &\leq \sum_{0<t_k<t} \|S_\alpha(t-t_k)\| l_k \Psi_I(\|u(t_k)\|) \\
&\leq \Psi_I(r) \sum_{0<t_k<t} \|S_\alpha(t-t_k)\| l_k \\
&\leq \Psi_I(r) \left(\sum_{k \leq N_0} \|S_\alpha(t-t_k)\| l_k + \sum_{k>N_0} \|S_\alpha(t-t_k)\| l_k \right) \\
&\leq \Psi_I(r) \left(\sum_{k \leq N_0} \|S_\alpha(t-t_k)\| l_k + S_\alpha^\infty \sum_{k>N_0} l_k \right),
\end{aligned}$$

where $N_0 \in \mathbb{N}$ such that $\sum_{k>N_0} l_k < \epsilon$. Noticing that $\|S_\alpha(t-t_k)\| < \epsilon$ for all $t \geq T_1 + t_{N_0}$, one gets

$$\|z(t)\| \leq \epsilon\,\Psi_I(r) \left(\sum_{k\in\Lambda} l_k + S_\alpha^\infty \right), \forall t > T_1 + t_{N_0}, \forall z \in \mathcal{F}_2(D).$$

Now for $z = \mathcal{F}_3(u), u \in D$, by $(\mathbf{F}^*)$ we have

$$\begin{aligned}
\|z(t)\| &\leq \int_0^{\delta t} (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) \|u(s)\| ds \\
&\leq \frac{r}{[(1-\delta)t]^{1-\alpha}} \int_0^{\delta t} \|P_\alpha(t-s)\| m(s) ds \\
&\leq \frac{r\,\vartheta}{[(1-\delta)t]^{1-\alpha}} < \epsilon\, r\vartheta, \forall t \geq T_2 > 0,
\end{aligned}$$

where ϑ is defined by (3.8).

We are in a position to deal with $d_\infty(\mathcal{F}_4(D))$. For $z = \mathcal{F}_4(u), u \in D$, one has

$$\begin{aligned}
\|z(t)\| &\leq \int_{\delta t}^{t} (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) \|u(s)\| ds \\
&\leq \left(\int_{\delta t}^{t} (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) ds \right) \sup_{s \geq \delta t} \|u(s)\| \\
&\leq \kappa \sup_{s\geq\delta t} \|u(s)\| \leq \kappa \sup_{u \in D} \sup_{s \geq \delta t} \|u(s)\|, \ \forall t > 0,
\end{aligned}$$

where κ is given by (3.9). Taking $T \in (0, \delta t]$, we see that

$$\|z(t)\| \leq \kappa \sup_{u\in D} \sup_{s \geq T} \|u(s)\| = \kappa \cdot d_T(D), \forall t \geq T.$$

Therefore

$$\sup_{z \in \mathcal{F}_4(D)} \sup_{t \geq T} \|z(t)\| \leq \kappa \cdot d_T(D),$$

and then by the definition of d_∞,

$$d_\infty(\mathcal{F}_4(D)) \leq \kappa \cdot d_\infty(D). \tag{3.19}$$

It follows from (3.17)-(3.19) that

$$d_\infty(\mathcal{F}(D)) \leq \kappa \cdot d_\infty(D).$$

Combining with (3.16), we arrive at

$$\begin{aligned}\chi^*(\mathcal{F}(D)) &= \chi_\infty(\mathcal{F}(D)) + d_\infty(\mathcal{F}(D)) \\ &\le \max\{\kappa, \ell\}\,(\chi_\infty(D) + d_\infty(D)) \\ &= \max\{\kappa, \ell\} \cdot \chi^*(D).\end{aligned}$$

The proof is complete.

The following theorem is our main result.

Theorem 3.5 Let (**A***), (**F***), (**G***) and (**I***) hold. Then problem (0.1)-(0.3) possesses a compact set of globally attracting solutions, provided that $\vartheta < +\infty$ and $\max\{\ell, \rho\} < 1$, where ϑ is defined by (3.8), ℓ is given in (3.15) and

$$\rho = \liminf_{r\to\infty} \frac{1}{r}\left[\Big(\Psi_g(r) + \Psi_I(r)\sum_{k\in\Lambda} l_k\Big)S_\alpha^\infty\right] + \sup_{t>0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)ds. \quad (3.20)$$

Proof By (3.20), we follows the same arguments as in the proof of Theorem 2.5 to get a closed ball $\mathbf{B}_R = B(0,R)$ in $\mathcal{PC}_0$ such that $\mathcal{F}(\mathbf{B}_R) \subset \mathbf{B}_R$. From now on, we consider $\mathcal{F}$ as a multimap from $\mathbf{B}_R$ into itself. Notice that the condition $\rho < 1$ implies $\kappa < 1$. Then by Lemma 3.4, $\mathcal{F}$ is χ^*-condensing. It remains to show that $\mathcal{F}$ is a u.s.c. multimap. Rewriting $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where

$$\mathcal{F}_1(u)(t) = S_\alpha(t)Bg(u) + \sum_{0<t<t_k} S_\alpha(t-t_k)BI_k(u(t_k)),$$

$$\mathcal{F}_2(u)(t) = \left\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds : f \in \mathcal{P}_F^p(u)\right\},$$

we see that $\mathcal{F}_1$ is continuous, thanks to the continuity of Bg and BI_k. We will prove that $\mathcal{F}_2$ is u.s.c. by using Lemma 1.10. Let $\{u_n\} \subset \mathbf{B}_R$ converge to u^* and $z_n \in \mathcal{F}_2(u_n)$ be such that $z_n \to z^*$ (the convergence in the norm of $\mathcal{PC}_0$). We check that $z^* \in \mathcal{F}_2(u^*)$, i.e. $z^*(t) \in \mathcal{F}_2(u^*)(t), \forall t > 0$. But this can be proceeded by the same arguments as in the proof of Lemma 2.3. Now we testify the quasi-compactness. Let $K \subset \mathbf{B}_R$ be a compact set and $\{z_n\} \subset \mathcal{F}_2(K)$. Then one can take $\{u_n\} \subset K$ and $f_n \in \mathcal{P}_F^p(u_n)$ such that

$$z_n(t) = \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f_n(s)ds, t > 0.$$

Arguing as in the proof of Lemma 2.3, we get that $\{\pi_T(z_n)\}$ is relatively compact for any $T > 0$, i.e.

$$\chi_\infty(\{z_n\}) = \sup_{T>0} \chi_{\mathcal{PC}}(\{\pi_T(z_n)\}) = 0.$$

Now using the estimate of d_∞ as in the proof of Lemma 3.4, one obtains

$$d_\infty(\{z_n\}) \le \kappa\, d_\infty(\{u_n\}).$$

This implies

$$\chi^*(\{z_n\}) = \chi_\infty(\{z_n\}) + d_\infty(\{z_n\}) \le \kappa\,\chi^*(\{u_n\}) = 0$$

thanks to the compactness of $\{u_n\}$. Hence $\chi^*(\{z_n\}) = 0$ and by the regularity of χ^*, $\{z_n\}$ is relatively compact. The proof is complete.

4 An application

This section is devoted to an application of the obtained abstract results to a systems of fractional PDEs. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. We are concerned with the following problem

$$\partial_t^\alpha u(t,x) - \partial_t^\alpha \Delta_x u(t,x) - \Delta_x u(t,x) = f(t,x), \tag{4.1}$$

$$f(t,x) \in \text{co}\{f_1(t,u(t,x)),...,f_m(t,u(t,x))\}, x\in\Omega, t>0, t\neq t_k, k\in\mathbb{N}, \tag{4.2}$$

$$u(t,x) = 0,\ x\in\partial\Omega, t>0, \tag{4.3}$$

$$u(t_k^+,x) = u(t_k^-,x) + \int_\Omega H_k(x,y)u(t_k,y)dy,\ x\in\Omega, \tag{4.4}$$

$$u(0,x) = v(x) + \int_0^b\int_\Omega G(s,x,y)u(s,y)dyds,\ x\in\Omega. \tag{4.5}$$

In this system, $\partial_t^\alpha, \alpha\in(\frac{1}{2},1)$, stands for the Caputo fractional derivative with respect to t, Δ_x is the Laplacian with respect to x, and

$$\text{co}\{f_1,...,f_m\} = \left\{\sum_{i=1}^m \mu_i f_i:\ \mu_i\geq 0, \sum_{i=1}^m \mu_i = 1\right\}.$$

Let $X = L^2(\Omega)$, $A = \Delta$ (the Laplacian) with $\mathcal{D}(A) = H^2(\Omega)\cap H_0^1(\Omega)$. Let $\{\lambda_n\}_{n\geq 1}$ be the eigenvalues of $-A$ with corresponding eigenvectors $\{e_n\}_{n\geq1}$. Then we know that $0<\lambda_1<\lambda_2<...<\lambda_n<...$ with $\lambda_n\to+\infty$ as $n\to+\infty$, moreover

$$Au = -\sum_{n=1}^\infty \lambda_n\langle u,e_n\rangle e_n,$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product in X. Now consider $B = I-\Delta$ with $\mathcal{D}(B)=\mathcal{D}(A)$. We see that B has the following representation

$$Bu = \sum_{n=1}^\infty (1+\lambda_n)\langle u,e_n\rangle e_n.$$

Therefore

$$B^{-1}u = \sum_{n=1}^\infty \frac{1}{1+\lambda_n}\langle u,e_n\rangle e_n,$$
$$AB^{-1}u = \sum_{n=1}^\infty \frac{-\lambda_n}{1+\lambda_n}\langle u,e_n\rangle e_n.$$

This implies that the semigroup $T(\cdot)$ generated by AB^{-1} can be expressed by

$$T(t)u = \sum_{n=1}^\infty e^{\frac{-\lambda_n}{1+\lambda_n}t}\langle u,e_n\rangle e_n.$$

Obviously, $\|T(t)\|\leq e^{-\beta t}, \forall t\geq 0$ with $\beta = \dfrac{\lambda_1}{1+\lambda_1}>0$. So one gets the asymptotic stability of the characteristic solution operators $S_\alpha(\cdot), P_\alpha(\cdot)$ and $(\mathbf{A}^*)$ is satisfied. Furthermore, by Proposition 3.2

$$\begin{aligned}\|S_\alpha(t)\| &\leq \|B^{-1}\|\min\left(1, C_S t^{-\alpha}\right),\\ \|P_\alpha(t)\| &\leq \|B^{-1}\|\min\left(\frac{1}{\Gamma(\alpha)}, C_P t^{-2\alpha}\right)\ \text{ for all } t>0.\end{aligned} \tag{4.6}$$

In particular, $S_\alpha^\infty = \sup_{t\geq 0}\|S_\alpha(t)\| \leq \|B^{-1}\|$.

Let $F:\mathbb{R}^+\times X\to \mathcal{P}(X)$ be the multimap defined by

$$F(t,v)(x) = \text{co}\{f_1(t,v(x)),...,f_m(t,v(x))\}.$$

We assume that $f_i:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}, i=1,...,m$, are continuous functions such that

$$|f_i(t,z)| \leq m(t)|z|, \forall (t,z)\in\mathbb{R}^+\times\mathbb{R}, \tag{4.7}$$

where $m\in BC(\mathbb{R}^+;\mathbb{R}^+)$, the space of continuous bounded functions on $\mathbb{R}^+$, so that $I_0^\alpha m \in BC(\mathbb{R}^+;\mathbb{R}^+)$, i.e.

$$I_0^\alpha m(t) = O(1) \text{ as } t\to+\infty. \tag{4.8}$$

It is easily seen that for each $(t,v)\in\mathbb{R}^+\times X$, $F(t,v)$ is a closed bounded subset of the finite dimensional space $X_m = \text{span}\{f_1(t,v(\cdot)),...,f_m(t,v(\cdot))\}$. So $F(t,v)$ is a compact set in X, that is, F has compact values. By the continuity of $f_i, i=1,..,m$, one can check that $F(t,\cdot)$ is a u.s.c. multimap, i.e. for v_n converging to v in X and for $\epsilon>0$,

$$F(t,v_n)\subset F(t,v)+\epsilon B(0,1), \forall n > N(\epsilon),$$

with $N(\epsilon)\in\mathbb{N}$ and $B(0,1)$ being the unit ball in X. We observe that B^{-1} is compact, then (**F***) is satisfied since we have

$$\|F(t,v)\| \leq m(t)\|v\|,$$

thanks to (4.7).

Consider the jump functions I_k defined by

$$I_k(v)(x) = \int_\Omega H_k(x,y)v(y)dy.$$

Suppose that $H_k:\Omega\times\Omega\to\mathbb{R}, k=1,2,...$ are measurable functions such that H_k together with $\Delta_x H_k$ belong to $L^2(\Omega\times\Omega)$. Denoting

$$h_k(x,y) = H_k(x,y) - \Delta_x H_k(x,y),$$

then BI_k has the form

$$BI_k(v)(x) = \int_\Omega h_k(x,y)v(y)dy,$$

and it is a Hilbert-Schmidt operator. In particular, BI_k is compact. We deduce that I_k satisfies (**I**)(2) with $\mu_k=0$. In addition, one can check that I_k satisfies (**I**)(1) with

$$l_k = \|h_k\|_{L^2(\Omega\times\Omega)},\ \Psi_I(r) = r, \forall r\geq 0.$$

Then (**I***) is fulfilled if we assume $\sum_{k=1}^\infty l_k < \infty$.

Regarding the nonlocal function, put

$$g(w)(x) = v(x) + \int_0^b\int_\Omega G(s,x,y)w(s,y)dyds,\ w\in\mathcal{PC}([0,+\infty);X).$$

We make an assumption that $v\in H^2(\Omega)$ and $G:[0,b]\times\Omega\times\Omega\to\mathbb{R}$ is a measurable function with $G(t,\cdot,\cdot),\Delta_x G(t,\cdot,\cdot)\in L^2(\Omega\times\Omega)$. Then by putting

$$\tilde{G}(s,x,y) = (I-\Delta_x)G(s,x,y),$$

we have

$$Bg(w)(x) = v(x) - \Delta v(x) + \int_0^b\int_\Omega \tilde{G}(s,x,y)w(s,y)dyds.$$

It follows that

$$\begin{aligned}\|Bg(w)\| &\le \|v\|_{H^2} + \int_0^b \|\tilde{G}(s,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}\|w(s,\cdot)\|ds \\ &\le \|v\|_{H^2} + \left(\int_0^b \|\tilde{G}(s,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}ds\right)\|w\|_\infty.\end{aligned}$$

Thus $(\mathbf{G})(1)$ is satisfied with

$$\Psi_g(r) = \|v\|_{H^2} + \left(\int_0^b \|\tilde{G}(s,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}ds\right) r.$$

Since the operator K defined by

$$K(v)(x) = \int_\Omega \tilde{G}(s,x,y)v(y)dy$$

is a Hilbert-Schmidt operator for fixed $s \in [0,b]$, we see that for any bounded set $D \in \mathcal{PC}([0,+\infty);X)$, $K(D(s))$ is relatively compact in X. Hence the set $Bg(D)$ presented by

$$Bg(D) = Bv + \int_0^b K(D(s))ds$$

is relatively compact as well, thank to the fact that (see Proposition 1.8)

$$\chi(Bg(D)) \le 4\int_0^b \chi(K(D(s)))ds = 0.$$

So $(\mathbf{G})(2)$ is testified with $\eta = 0$.

We are now in a position to clarify the conditions in Theorem 3.5, i.e.

$$\vartheta < +\infty,\ \ell < 1,\ \rho < 1.$$

According to the above settings for (4.1)-(4.5), we get

$$\begin{aligned}\ell &= \Big(\eta + \sum_{k\in\Lambda}\mu_k\Big)S_\alpha^\infty + 4\sup_{t>0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)ds = 0, \\ \rho &= \left(\int_0^b \|\tilde{G}(s,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}ds + \sum_{k=1}^\infty \|h_k\|_{L^2(\Omega\times\Omega)}\right)S_\alpha^\infty \\ &\qquad + \sup_{t>0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)ds.\end{aligned}$$

Let $\phi(t) = \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)ds$. By the estimate for P_α in (4.6) we have

$$\phi(t) \le \frac{\|B^{-1}\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)ds = \|B^{-1}\|I_0^\alpha m(t) = O(1) \text{ as } t\to+\infty,$$

thanks to (4.8). So $\phi_\infty = \sup_{t>0}\phi(t) < +\infty$.

Now we check that $\vartheta = \sup_{t>0}\int_0^{\frac{t}{2}} \|P_\alpha(t-s)\|m(s)ds < +\infty$ (take $\delta = \frac{1}{2}$). Putting $\psi(t) = \int_0^{\frac{t}{2}} \|P_\alpha(t-s)\|m(s)ds$, we show that $\lim_{t\to+\infty}\psi(t) = 0$. Indeed, by the estimate (4.7) and the fact that $m \in BC(\mathbb{R}^+;\mathbb{R}^+)$, we obtain

$$\psi(t) \le \|B^{-1}\|\int_0^{\frac{t}{2}} (t-s)^{-2\alpha}m(s)ds$$

$$\leq \|B^{-1}\| \left(\frac{t}{2}\right)^{-2\alpha} \int_0^{\frac{t}{2}} m(s)ds \leq \|B^{-1}\| \left(\frac{t}{2}\right)^{-2\alpha+1} \|m\|_\infty$$
$$\to 0 \text{ as } t \to +\infty.$$

One can give an example of m satisfying (4.8). Let $m(t) = \dfrac{\mu}{1+t^\alpha}$, $t \geq 0$, then

$$\begin{aligned}
I_0^\alpha m(t) &= \frac{\mu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{ds}{1+s^\alpha} \\
&= \frac{\mu}{\Gamma(\alpha)} \left(\int_0^{\frac{t}{2}} (t-s)^{\alpha-1} \frac{ds}{1+s^\alpha} + \int_{\frac{t}{2}}^t (t-s)^{\alpha-1} \frac{ds}{1+s^\alpha} \right) \\
&\leq \frac{\mu}{\Gamma(\alpha)} \left(\left(\frac{t}{2}\right)^{\alpha-1} \int_0^{\frac{t}{2}} \frac{ds}{1+s^\alpha} + \frac{1}{1+\left(\frac{t}{2}\right)^\alpha} \int_{\frac{t}{2}}^t (t-s)^{\alpha-1} ds \right) \\
&= \frac{\mu}{\Gamma(\alpha)} \left(\frac{t}{2}\right)^{\alpha-1} \int_0^{\frac{t}{2}} \frac{ds}{1+s^\alpha} + \frac{\mu}{\Gamma(1+\alpha)} \frac{\left(\frac{t}{2}\right)^\alpha}{1+\left(\frac{t}{2}\right)^\alpha}.
\end{aligned}$$

So

$$\lim_{t\to+\infty} I_0^\alpha m(t) \leq \frac{\mu}{(1-\alpha)\Gamma(\alpha)} + \frac{\mu}{\Gamma(1+\alpha)}.$$

Summing up, the problem (4.1)-(4.5) possesses a compact set of globally attracting solutions if

$$\rho = \left(\int_0^b \|\tilde{G}(s,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)} ds + \sum_{k=1}^\infty \|h_k\|_{L^2(\Omega\times\Omega)} \right) S_\alpha^\infty + \phi_\infty < 1.$$

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Some Generalizations of Fixed Point Theorems in Partially Ordered Metric Spaces and Applications to Partial Differential Equations with Uncertainty

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Abstract Some generalized contractions using altering distances in partially ordered metric spaces are investigated and their applications to fuzzy partial differential equations are considered. Starting from the Banach contraction principle, our theorems presented here generalize, extend, and improve different results existing in the literature on the existence of coincidence points for a pair of mappings. In terms of their applicability, this might constitute the first paper dealing with the solvability of fuzzy partial differential equations from the point of view of considering the structure of the fuzzy number space as a partially ordered space. Under the generalized contractive-like property over comparable items, which is weaker than the Lipschitz condition, we show that the existence of just a lower or an upper solution is enough to prove the existence and uniqueness of two types of fuzzy solutions in the sense of gH-differentiability.

Keywords Contractive-like mapping principle · Well-posed boundary value problems · Fuzzy partial hyperbolic differential equations · Generalized Hukuhara derivatives

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1 Introduction

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. The Banach contraction principle is widely considered as the source of fixed point theory. It is a very popular tool to deal with the existence problems in many branches of mathematical analysis. There has been a large number of generalizations of the Banach contraction principle. In particular, an interesting aspect is to deduce the existence and uniqueness of fixed point for self-maps on a metric space by altering distances between the points with the use of a certain control function. These control functions were introduced by Khan et al. in [16] and then applied in many works as, for instance, [3, 9, 14, 27, 34], where some fixed point theorems were investigated with the help of such altering distance functions.

Recently, a new technique was proposed in order to weaken the requirements on the contraction property by considering metric spaces endowed with a partial ordering. This approach was initiated by Ran and Reurings in [33] with some applications to matrix equations. It was later refined and extended in [28] by Nieto and Rodríguez-López and applied to periodic boundary value problems for ordinary differential equations (ODEs). Following this direction, in this paper, we generalize some fixed point theorems in partially ordered sets of Amini-Harandi and Emami [3] by using altering distances. With the help of the weak contractivity coefficient function $\beta \in \mathcal{S} := \mathcal{S}_0 \cup \{1_{[0,\infty)}\}$, where $\mathcal{S}_0$ is the class of functions $\beta : [0, \infty) \to [0, 1)$ that satisfy the condition

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0,$$

and $1_{[0,\infty)}$ is the indicator function on $[0, +\infty)$, i.e., $1_{[0,\infty)}(t) = 1$ for all $t \in [0, \infty)$, and $1_{[0,\infty)} = 0$, otherwise, we weaken the required conditions by considering weak contractions of Harjani and Sadarangani [14], and Nashine and Samet [27].

Since the base space does not necessarily have a vectorial structure, these fixed point theorems can be applied to prove the existence of solutions to ODEs, and partial differential equations (PDEs) in abstract spaces. We note that the space of fuzzy numbers is not a Banach space, but it is a quasilinear space having a partial ordering. Hence, there have been some recent results on the existence of solutions to fuzzy ODEs (see [25, 29, 36]) as applications of fixed point theory in partially ordered metric spaces.

In this paper, besides giving some new generalized results on the existence of coincidence points for a pair of mappings in partially ordered sets, we also show their applications in the field of fuzzy PDEs to illustrate the usability of our obtained results. The problem considered is

$${}_kD_{xy}u(x, y) = f(x, y, u(x, y)), \quad (x, y) \in J := [0, a] \times [0, b], \quad k = 1, 2, \tag{1}$$

with condition

$$u(x, 0) = \eta_1(x), \quad x \in [0, a], \quad u(0, y) = \eta_2(y), \quad y \in [0, b], \tag{2}$$

where $u : J \to \mathbb{R}_{\mathcal{F}}$ is a fuzzy-valued mapping and ${}_kD_{xy}$ (for $k = 1, 2$) represents the gH-partial derivatives operators. This boundary value problem was considered in some previous research works [2, 20–24], in which the authors proved the validity of Picard's theorem. In these results, the Lipschitz contractivity of the function f is vital for the existence of the fuzzy solution. If f is just continuous or even not continuous, the situation is far different and some necessary conditions must be imposed in order to guarantee the existence of solutions (in the case of crisp ODEs we can see [3, 14, 15, 27, 28], and in the fuzzy case, we refer to [1, 29, 30, 36]).

In this paper, we show that, under the assumption of nondecreasing monotonicity and weak-contractivity of the mapping f only over comparable elements, the existence of just a lower or an upper solution is enough to guarantee the existence and uniqueness of two types of fuzzy solutions to the Problem (1)–(2). Some previous significant results for ODEs have been investigated in [29, 30, 36]. Our results presented here give some new approaches on the existence of two types of fuzzy solutions for some class of fuzzy PDEs under the gH-differentiability. One difficulty to be faced in the study of this problem is the existence of gH-differences, which also allows us to obtain a new solution to fuzzy PDEs with decreasing length of its support. In this case, the qualitative solutions may be better in comparison with those of crisp PDEs. Our results extend to a class of fuzzy PDEs some existing results for fuzzy ODEs by Alikhani and Bahrami [1], Nieto and Rodríguez-López [29], and Villamizar-Roa et al. [36].

The remainder of this paper is organized as follows. Section 2 presents our main results (Theorems 1 and 2), in which we prove the existence of coincidence points for a pair of mappings in a partially ordered metric space, and, in particular, we deduce a fixed point theorem. Our method is mainly based on the generalized contractive-like condition. Section 3 provides some results on the existence and uniqueness of solution for fuzzy partial differential equations as an effective application of our theorems presented in Section 2. Some necessary preliminaries about fuzzy analysis and gH-derivatives are shown in Sections 3.1 and 3.2. The boundary value problem of interest is stated in Section 3.3, and the study of the solvability of this problem is also included. Finally, some conclusions and future directions are discussed in Section 4.

2 Generalized Coincidence and Fixed Point Theorems

In this section, we provide some definitions and new results related to generalized coincidence and fixed point theorems in partially ordered metric spaces.

For $x \in \mathbb{R}$, $[x]$ is the greatest integer function or integer value, gives the largest integer less than or equal to x (the floor function).

By $\hat{C}([0,\infty))$, we denote the space of all nonnegative and continuous functions $\phi : [0,\infty) \to [0,\infty)$, for which the following property holds

$$\phi(t) = 0 \quad \text{if and only if} \quad t = 0.$$

Definition 1 [14] A nondecreasing function ψ in $\hat{C}([0,\infty))$ is called an altering distance function on $[0,\infty)$.

Some examples of altering distance functions on $[0,\infty)$ are t^2; $\ln(1+t)$; $t^2 - \ln(1+t^2)$.

Definition 2 [27] Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a metric space. We say that X is regular if, for an arbitrary nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x$ in X, then $x_n \le x$ for all $n \in \mathbb{N}$.

Definition 3 [14] If $(X, \le)$ is a partially ordered set and $f : X \to X$, we say that f is monotone nondecreasing (resp., nonincreasing) if $x, y \in X$, $x \le y$ implies $f(x) \le f(y)$ (resp., $f(y) \le f(x)$).

Definition 4 [27] Let $(X, \leq)$ be a partially ordered set and let f, g be mappings from X to itself such that $f(X) \subset g(X)$. We say that f is weakly increasing with respect to g if, for all $x \in X$, we have $f(x) \leq f(y)$ for all $y \in g^{-1}(f(x))$, where

$$g^{-1}(f(x)) := \{u \in X \mid g(u) = f(x)\}.$$

Definition 5 [27] Let (X, d) be a metric space and $f, g : X \to X$. The pair $\{f, g\}$ is said to be compatible if $\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x$ for some $x \in X$.

In this section, we extend the main results in [3, 14, 27] to get a generalized fixed point theorem in partially ordered metric spaces.

Theorem 1 *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f, g : X \to X$ be given mappings satisfying the following assumptions:*

i) $f(X) \subset g(X)$.
ii) *f is weakly increasing with respect to g.*
iii) *One of the two following conditions holds:*

   (a) *X is a regular metric space and $g(X)$ is a closed subspace of (X, d), or*
   (b) *f and g are continuous and the pair (f, g) is compatible.*

iv) *There exist a function $\beta \in \mathcal{S}$, $\phi \in \hat{C}([0, \infty))$, and ψ a strictly increasing altering distance function such that the following inequality holds*

$$\begin{aligned}\psi\left(d(f(x), f(y))\right) \leq\ & \beta\left(d(g(x), g(y))\right)\psi\left(d(g(x), g(y))\right) \\ & -\gamma\left(d(g(x), g(y))\right)\phi\left(d(g(x), g(y))\right)\end{aligned} \tag{3}$$

for all $(x, y) \in X \times X$ satisfying that $g(x)$ and $g(y)$ are comparable, where

$$\gamma(t) = [\beta(t)] \quad \textit{for all } t \in [0, \infty).$$

Then, there exists a coincidence point x of f and g in X, i.e., $f(x) = g(x)$.

Proof We proceed in several steps.

Step 1. Firstly, we contribute a nondecreasing sequence $\{g(x_n)\}$ in X.

Let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, we can construct a sequence $\{x_n\}$ in X defined by

$$g(x_{n+1}) = f(x_n) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Since $x_1 \in g^{-1}(f(x_0))$, $x_2 \in g^{-1}(f(x_1))$ and f is weakly increasing with respect to g, we obtain

$$g(x_1) = f(x_0) \leq f(x_1) = g(x_2) \leq f(x_2) = g(x_3) \leq \cdots$$

Therefore, by recurrence, we obtain a nondecreasing sequence

$$g(x_1) \leq g(x_2) \leq g(x_3) \leq \cdots \leq g(x_n) \leq g(x_{n+1}) \leq \cdots$$

Since $g(x_n) \leq g(x_{n+1})$ for $n \geq 1$, it follows from (3) that

$$\begin{aligned} \psi(d(g(x_{n+1}), g(x_{n+2}))) &= \psi(d(f(x_n), f(x_{n+1}))) \\ &\leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))) \\ &\quad -\gamma(d(g(x_n), g(x_{n+1})))\phi(d(g(x_n), g(x_{n+1}))) \\ &\leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))) \\ &\leq \psi(d(g(x_n), g(x_{n+1}))) \end{aligned}$$

for all $n \geq 1$. Hence, we have

$$\psi(d(g(x_{n+1}), g(x_{n+2}))) \leq \psi(d(g(x_n), g(x_{n+1}))) \quad \text{for all } n \geq 1.$$

Due to the strictly increasing character of the function ψ, $\{d(g(x_n), g(x_{n+1}))\}$ is a non-increasing and bounded from below sequence in $\mathbb{R}$. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = r. \tag{4}$$

We will prove that $r = 0$. In fact, from the continuity property of ψ and ϕ, we have

$$\lim_{n\to\infty} \psi(d(g(x_n), g(x_{n+1}))) = \psi\left(\lim_{n\to\infty} d(g(x_n), g(x_{n+1}))\right) = \psi(r)$$

and

$$\lim_{n\to\infty} \phi(d(g(x_n), g(x_{n+1}))) = \phi\left(\lim_{n\to\infty} d(g(x_n), g(x_{n+1}))\right) = \phi(r).$$

If $\beta = 1_{[0,\infty)}$, then $\gamma(t) = [\beta(t)] = 1$ for all $t \geq 0$. In this case, it follows from (3) that the following estimation holds

$$\psi(d(g(x_{n+1}), g(x_{n+2}))) \leq \psi(d(g(x_n), g(x_{n+1}))) - \phi(d(g(x_n), g(x_{n+1}))) \text{ for all } n \geq 1.$$

By taking limits on both sides when $n \to \infty$, we get

$$\psi(r) \leq \psi(r) - \phi(r),$$

which implies that $0 \leq -\phi(r)$, and using that $\phi \in \hat{C}([0, \infty))$, we obtain $\phi(r) = 0$ and $r = 0$.

On the other hand, if $\beta \in \mathcal{S}_0$, from (3) and the inequalities $g(x_n) \leq g(x_{n+1})$, $n \geq 1$, we have

$$\begin{aligned} \psi(d(g(x_{n+1}), g(x_{n+2}))) &= \psi(d(f(x_n), f(x_{n+1}))) \\ &\leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))), \quad n \geq 1. \end{aligned}$$

By contradiction method, we assume that $r > 0$. It permits to affirm, from (4), the non-increasing character of the sequence $\{d(g(x_n), g(x_{n+1}))\}$ and the properties of ψ, that $\psi(d(g(x_n), g(x_{n+1}))) > 0$ for $n \geq 1$. Hence

$$\frac{\psi(d(g(x_{n+1}), g(x_{n+2})))}{\psi(d(g(x_n), g(x_{n+1})))} \leq \beta(d(g(x_n), g(x_{n+1}))) < 1$$

for $n \geq 1$. By taking limits on both sides of this equation, it leads to

$$\lim_{n\to\infty} \beta(d(g(x_n), g(x_{n+1}))) = 1.$$

Taking into account that $\beta \in \mathcal{S}_0$, the previous condition implies that $\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0$, which is a contradiction.

Hence, in both cases, we have that $r = 0$ and thus, $\{g(x_n)\}$ is a nondecreasing sequence satisfying that

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0. \tag{5}$$

Step 2. Next, we prove that $\{g(x_n)\}$ is a Cauchy sequence.
Case 1: If there exists an $n \in \mathbb{N}$ such that $g(x_n) = g(x_{n+1})$, then, from (3), we have

$$\psi(d(f(x_n), f(x_{n+1}))) \leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))) = 0.$$

This inequality implies, by the properties of ψ, that $f(x_n) = f(x_{n+1})$ or $g(x_{n+1}) = g(x_{n+2})$. So, for all $m \geq n$, we have that $g(x_m) = g(x_n)$. It obviously shows that $\{g(x_n)\}$ is a Cauchy sequence.
Case 2: Assume that all the successive terms of $\{g(x_n)\}$ are different, that is, $g(x_n) \neq g(x_{n+1})$ for every $n \in \mathbb{N}$. We prove that

$$\limsup_{m\to\infty} \sup_{n\geq m} d(g(x_n), g(x_m)) = 0.$$

Indeed, suppose that $\limsup_{m\to\infty} \sup_{n\geq m} d(g(x_n), g(x_m)) \neq 0$ and select $\varepsilon > 0$ such that

$$\limsup_{m\to\infty} \sup_{n\geq m} d(g(x_n), g(x_m)) > \varepsilon.$$

Then, we can choose two subsequences $\{g(x_{n_k})\}$, $\{g(x_{m_k})\}$ of $\{g(x_n)\}$ such that $n_k \geq m_k > k$ and

$$d(g(x_{n_k}), g(x_{m_k})) > \varepsilon. \tag{6}$$

For each fixed m_k, we choose n_k to be the smallest number such that $n_k \geq m_k$ satisfying (6). Note that (6) implies, in fact, that $n_k > m_k$. Hence, it follows that $n_k - 1 \geq m_k$ and

$$d(g(x_{n_k-1}), g(x_{m_k})) \leq \varepsilon.$$

Then, we get

$$\begin{aligned} \varepsilon < d(g(x_{n_k}), g(x_{m_k})) &\leq d(g(x_{n_k}), g(x_{n_k-1})) + d(g(x_{n_k-1}), g(x_{m_k})) \\ &\leq d(g(x_{n_k}), g(x_{n_k-1})) + \varepsilon. \end{aligned} \tag{7}$$

Taking into account (5) and letting $k \to \infty$ in (7), we have

$$\lim_{k\to\infty} d(g(x_{n_k}), g(x_{m_k})) = \varepsilon. \tag{8}$$

Since

$$\begin{aligned} d(g(x_{n_k}), g(x_{m_k})) &\leq d(g(x_{n_k}), g(x_{n_k-1})) + d(g(x_{n_k-1}), g(x_{m_k-1})) \\ &\quad + d(g(x_{m_k-1}), g(x_{m_k})), \quad k \geq 1, \end{aligned} \tag{9}$$

using (5), (6), and passing to the limit inferior when $k \to \infty$ in the inequality (9), we obtain

$$\liminf_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) \geq \varepsilon. \tag{10}$$

On the other hand, from the estimation

$$\begin{aligned} d(g(x_{n_k-1}), g(x_{m_k-1})) &\leq d(g(x_{n_k}), g(x_{n_k-1})) + d(g(x_{n_k}), g(x_{m_k})) \\ &\quad + d(g(x_{m_k-1}), g(x_{m_k})), \quad k \geq 1, \end{aligned}$$

we get, from (5) and (8), that

$$\limsup_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) \le \varepsilon. \tag{11}$$

Thus, by combining (10) and (11), we have

$$\lim_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) = \varepsilon. \tag{12}$$

Now $m_k \le n_k$ implies $m_k - 1 \le n_k - 1$ and, thus, $g(x_{m_k-1}) \le g(x_{n_k-1})$. Applying the inequality (3) once again, we have

$$\begin{aligned}\psi(d(g(x_{n_k}), g(x_{m_k}))) &= \psi(d(f(x_{n_k-1}), f(x_{m_k-1})))\\ &\le \beta(d(g(x_{n_k-1}), g(x_{m_k-1})))\psi(d(g(x_{n_k-1}), g(x_{m_k-1})))\\ &\quad -\gamma(d(g(x_{n_k-1}), g(x_{m_k-1})))\phi(d(g(x_{n_k-1}), g(x_{m_k-1}))).\end{aligned} \tag{13}$$

If $\beta = 1_{[0,\infty)}$, then $\gamma(t) = \beta(t) = 1$ for all $t \ge 0$. Since ψ and ϕ are continuous, by passing to the limit as $k \to \infty$ in (13), we have $\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon)$, that is, $0 \le -\phi(\varepsilon)$. Hence, by the properties of ϕ, it follows that $\phi(\varepsilon) = 0$ and $\varepsilon = 0$.

On the other hand, if $\beta \in \mathcal{S}_0$, then $0 \le \beta(t) < 1$ for all $t \ge 0$. Denote $t_k = d(g(x_{n_k-1}), g(x_{m_k-1}))$ for $k \ge 1$. Since $\{\beta(t_k)\} \subset [0, 1]$ and $[0, 1]$ is a compact set in $\mathbb{R}$, then there exists a subsequence $\{\beta(t_{k_j})\}$ converging to $\lambda \in [0, 1]$. Therefore, by choosing subsequences if necessary, we assume that

$$\lim_{k\to\infty} \beta(d(g(x_{n_k-1}), g(x_{m_k-1}))) = \lambda \in [0, 1].$$

If $\lambda = 1$, then $\lim_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) = 0$, which implies that $\varepsilon = 0$. If $0 \le \lambda < 1$, then, from

$$\begin{aligned}\psi(d(g(x_{n_k}), g(x_{m_k}))) &= \psi(d(f(x_{n_k-1}), f(x_{m_k-1})))\\ &\le \beta(d(g(x_{n_k-1}), g(x_{m_k-1})))\psi(d(g(x_{n_k-1}), g(x_{m_k-1}))), \quad k \ge 1,\end{aligned}$$

by passing to the limit as $k \to \infty$ and using the continuity property of ψ, we have that $\psi(\varepsilon) \le \lambda\psi(\varepsilon)$, or, equivalently, $(1 - \lambda)\psi(\varepsilon) \le 0$. Hence, $\psi(\varepsilon) = 0$ and $\varepsilon = 0$.

Therefore, it follows that

$$\lim_{m\to\infty} \sup_{n\ge m} d(g(x_n), g(x_m)) = 0.$$

Thus, $\{g(x_n)\}$ is a Cauchy sequence in (X, d).

Step 3. We prove the existence of a coincidence point of f and g.

Case 1: Assume that X is a regular metric space and that $g(X)$ is a closed subspace of (X, d). Then $(g(X), d)$ is a complete metric subspace of (X, d). Since $\{g(x_n)\}$ is a Cauchy sequence in $(g(X), d)$, there exists $u = g(z) \in g(X)$ such that $g(x_n) \to u = g(z)$ as $n \to \infty$. Since $\{g(x_n)\}$ is a nondecreasing sequence and X is regular, then $g(x_n) \le g(z)$ for all $n \in \mathbb{N}$. Applying (3) once again, we have

$$\begin{aligned}0 &\le \psi(d(f(z), g(x_{n+1})))\\ &= \psi(d(f(z), f(x_n)))\\ &\le \beta(d(g(z), g(x_n)))\psi(d(g(z), g(x_n))) - \gamma(d(g(z), g(x_n)))\phi(d(g(z), g(x_n)))\\ &\le \beta(d(g(z), g(x_n)))\psi(d(g(z), g(x_n)))\\ &\le \psi(d(g(z), g(x_n))).\end{aligned}$$

By using the property of continuity of ψ and letting $n \to \infty$, we get

$$\psi(d(f(z), g(z))) = 0.$$

It clearly follows that $d(f(z), g(z)) = 0$ and $f(z) = g(z)$.
Case 2: Assume that f and g are continuous and that the pair (f, g) is compatible.

Since $\{g(x_n)\}$ is a Cauchy sequence in a complete metric space (X, d), there exists $z \in X$ such that $g(x_n) \to z$ and $f(x_n) = g(x_{n+1}) \to z$, as $n \to \infty$. Since f, g are continuous, we get

$$\lim_{n\to\infty} g(g(x_n)) = g(z); \quad \lim_{n\to\infty} f(g(x_n)) = f(z); \quad \lim_{n\to\infty} g(f(x_n)) = g(z).$$

Since $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ and the pair (f, g) is compatible, it follows that

$$\lim_{n\to\infty} d(g(f(x_n)), f(g(x_n))) = 0.$$

Thus, from

$$0 \le d(g(z), f(z)) \le d(g(z), g(g(x_{n+1}))) + d(g(f(x_n)), f(g(x_n))) + d(f(g(x_n), f(z)))$$

and letting $n \to \infty$, we have that $d(g(z), f(z)) = 0$, i.e., $f(z) = g(z)$.

In consequence, z is a coincidence point of f and g and the theorem is proved. □

Remark 1 Theorem 1 is actually an extension of some previous results in [3] and [27]. Indeed,

1. If we choose $\beta(\cdot) = 1_{[0,\infty)}(\cdot)$, then we get the context of Theorem 2.4 and Theorem 2.6 in [27];
2. If we choose $\beta \in \mathcal{S}_0$, then $\gamma(t) = [\beta(t)] = 0$ for all $t \in [0, \infty)$. Hence, we receive a generalized result connected to Theorem 2.1 in [3], with ψ an altering distance function and g a generalized function defined on X.

Theorem 2 *Assume that $(X, \le)$ is a partially ordered set and that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping. Assume that:*

i) *There exists $\beta \in \mathcal{S}$ such that*

$$\psi(d(f(x), f(y))) \le \beta(d(x, y))\psi(d(x, y)) - \gamma(d(x, y))\phi(d(x, y)) \tag{14}$$

for all $x \le y$ in X, where ψ is a strictly increasing altering distance function, $\phi \in \hat{C}([0, \infty))$ and $\gamma(t) = [\beta(t)]$ for all $t \in [0, \infty)$.

ii) *There exists $x_0 \in X$ such that $x_0 \le f(x_0)$ or $f(x_0) \le x_0$.*

iii) *One of the two following conditions holds:*

   (a) *X is a regular metric space; or*
   (b) *f is continuous.*

Then f has a fixed point in X, that is, there exists a point $z \in X$ such that $f(z) = z$. Furthermore, if

$$\textit{for each } y, z \in X, \textit{ there exists } x \in X \textit{ which is comparable both to } y \textit{ and } z, \tag{15}$$

then the fixed point of f is unique.

Proof If $f(x_0) = x_0$, then x_0 is a fixed point of f. We consider the case when $x_0 < f(x_0)$, that is, $x_0 \le f(x_0)$ but $x_0 \ne f(x_0)$. Since f is a nondecreasing mapping, by induction method, we construct a sequence

$$x_0 < f(x_0) \le f^2(x_0) \le \cdots \le f^n(x_0) \le f^{n+1}(x_0) \le \cdots$$

Set $x_{n+1} = f(x_n)$ for all $n \geq 0$. We have that $\{x_n\}$ is a nondecreasing sequence in X. The existence of a fixed point for the mapping f is proved similarly to the proof of Theorem 1 when g is the identity mapping from X to itself, i.e., $g = Id_X$.

Now, we prove the uniqueness of the fixed point. Indeed, assume that y and z are two fixed points of f. From hypothesis (15), there exists a point $x \in X$ which is comparable both to y and z. From the monotonicity property of f, this implies that, for each $n \in \mathbb{N}$, $f^n(x)$ is comparable both to $f^n(y) = y$ and $f^n(z) = z$. Therefore, by applying the inequality (14), we have

$$\begin{aligned}\psi(d(z, f^n(x))) &= \psi(d(f^n(z), f^n(x))) \\ &\leq \beta(d(f^{n-1}(z), f^{n-1}(x)))\psi(d(f^{n-1}(z), f^{n-1}(x))) \\ &\quad -\gamma(d(f^{n-1}(z), f^{n-1}(x)))\phi(d(f^{n-1}(z), f^{n-1}(x))) \\ &\leq \beta(d(f^{n-1}(z), f^{n-1}(x)))\psi(d(f^{n-1}(z), f^{n-1}(x))) \\ &\leq \psi(d(f^{n-1}(z), f^{n-1}(x))) \\ &= \psi(d(z, f^{n-1}(x))), \quad n \in \mathbb{N}, \ n \geq 2.\end{aligned}$$

Denote $\tau_n = d(z, f^n(x)) \in [0, \infty)$, $n \in \mathbb{N}$, $n \geq 1$. By the strict monotonicity of ψ, it follows that $0 \leq \tau_n \leq \tau_{n-1}$, $n \in \mathbb{N}$, $n \geq 2$. Consequently, the sequence τ_n is nonnegative and decreasing. So there exists $r \geq 0$ such that $\lim_{n\to\infty} \tau_n = r$. We prove that $r = 0$.

Case 1: If $\beta = 1_{[0,\infty)}$, then $\beta(t) = 1$ for all $t \geq 0$ and $\gamma(t) = 1$ for all $t \geq 0$. From (13), we have

$$\psi(\tau_n) \leq \psi(\tau_{n-1}) - \phi(\tau_{n-1}), \quad n \in \mathbb{N}, \ n \geq 2.$$

Passing to the limit as $n \to \infty$, by the continuity of the mappings ψ and ϕ, we have $\psi(r) \leq \psi(r) - \phi(r)$ and $\phi(r) = 0$. That implies $r = 0$.

Case 2: If $\beta \in \mathcal{S}_0$, from (13), we get

$$\psi(\tau_n) \leq \beta(\tau_{n-1})\psi(\tau_{n-1}), \quad n \in \mathbb{N}, n \geq 2. \tag{16}$$

By choosing subsequences if necessary, we assume that

$$\lim_{n\to\infty} \beta(\tau_n) = \lambda \in [0, 1],$$

which allows to deduce, by letting $n \to \infty$ in (16), that $\psi(r) \leq \lambda\psi(r)$, that is, $\psi(r)(1 - \lambda) \leq 0$. If $\lambda < 1$, then $\psi(r) = 0$, i.e., $r = 0$. If $\lambda = 1$, then $\lim_{n\to\infty} \beta(\tau_n) = 1$. From the properties of the function $\beta \in \mathcal{S}_0$, one gets $\lim_{n\to\infty} \tau_n = 0$. By the uniqueness of the limit, we prove that $r = 0$.

By applying analogous arguments, we have $\lim_{n\to\infty} d(y, f^n(x)) = 0$. It follows that

$$0 \leq d(y, z) \leq d(y, f^n(x)) + d(f^n(x), z) \to 0 \quad \text{as } n \to \infty.$$

This means that $y = z$. It completes the proof. □

Remark 2 Theorem 2 is also connected with some previous results:

1. If we choose $\beta(\cdot) = 1_{[0,\infty)}(\cdot)$, we receive again Theorems 2.1, 2.2, and 2.3 in [14], with weaker conditions on the function ϕ (here, ϕ is not necessarily nondecreasing on $[0, \infty)$).
2. If we choose $\beta \in \mathcal{S}_0$, we obtain a generalized result connected to Theorem 2.1 in [3], with ψ a strictly increasing altering distance function.
3. If we choose $\beta \in \mathcal{S}_0$ and $\psi = Id_{[0,\infty)}$ the identity mapping, one has again Theorem 2.1 in [3].

Remark 3 It is well-known that the hypothesis (15) is equivalent to the following hypothesis in [28]:

for each $y, z \in X$, there exists in X a lower bound or an upper bound of y, z.

Remark 4 From the proof of Theorem 2, we deduce that, if z is a fixed point of f, then $\lim_{n\to\infty} d(f^n(x), z) = 0$ for any $x \in X$ comparable to z.

Remark 5 We can affirm from the proof of Theorem 2 that, in order to obtain the existence of a unique fixed point for some function f, it is not necessary for the function f to be continuous. Instead of the condition of continuity, we can consider the requirement that the space X is regular. This restriction is valid in the case where X is the space of fuzzy sets on $\mathbb{R}$ (see [29]).

In the next section, we investigate some applications of these fixed point theorems to prove the existence of solution for a class of fuzzy partial differential equations.

3 Application to Fuzzy Partial Differential Equations

3.1 Fuzzy Partially Ordered Metric Spaces

Let $\mathbb{R}_{\mathcal{F}}$ be the space of fuzzy sets on $\mathbb{R}$ that are nonempty subsets $\{(x, u(x)) : x \in \mathbb{R}\}$ in $\mathbb{R} \times [0, 1]$ of certain functions $u : \mathbb{R} \to [0, 1]$ being normal, fuzzy-convex, upper semi-continuous, and compact-supported.

Let $u \in \mathbb{R}_{\mathcal{F}}$. The α-cuts or level sets of u are defined by

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\} \quad \text{for each } 0 < \alpha \leq 1,$$

which are nonempty, compact, and convex subsets of $\mathbb{R}$ for all $0 < \alpha \leq 1$. The same properties hold for $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$, which is called the support of u. For $u \in \mathbb{R}_{\mathcal{F}}$, we denote the parametric form of u by $[u]^\alpha = [u_{l\alpha}, u_{r\alpha}]$ for all $0 \leq \alpha \leq 1$, and $\text{len}([u])^\alpha = u_{r\alpha} - u_{l\alpha}$.

In $\mathbb{R}_{\mathcal{F}}$, we define the supremum metric d_∞ as follows

$$d_\infty(u, v) = \sup_{0\leq\alpha\leq1} d_H\left([u]^\alpha, [v]^\alpha\right) \quad \text{for all } u, v \in \mathbb{R}_{\mathcal{F}},$$

where d_H is the Hausdorff metric in the set consisting of all nonempty, compact, and convex subsets of $\mathbb{R}$. It is well-known that $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is a complete metric space (see, for instance, [19]).

The addition and the multiplication by a scalar in the space of fuzzy numbers $\mathbb{R}_{\mathcal{F}}$ is defined levelsetwise, that is, for all $u, v \in \mathbb{R}_{\mathcal{F}}$, $\alpha \in [0, 1]$, and $k \in \mathbb{R}$,

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [ku]^\alpha = k[u]^\alpha.$$

In the special case where $k = -1$, $(-1)[u]^\alpha = (-1)[u_{l\alpha}, u_{r\alpha}] = [-u_{r\alpha}, -u_{l\alpha}]$.

If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, we call $w = u \ominus v$ the Hukuhara difference (or H-difference) of u and v. If $u \ominus v$ exists, then $[u \ominus v]^\alpha = [u_{l\alpha} - v_{l\alpha}, u_{r\alpha} - v_{r\alpha}]$ for all $0 \leq \alpha \leq 1$.

Lemma 1 [17] *For all $u, v, w, e \in \mathbb{R}_{\mathcal{F}}$, if the H-differences $u \ominus v$, $w \ominus e$ exist, then*

$$d_\infty(u \ominus v, w \ominus e) \leq d_\infty(u, w) + d_\infty(v, e).$$

Definition 6 [29] In $\mathbb{R}_{\mathcal{F}}$, a partial ordering can be defined as follows:

$$x \leq y \quad \text{if} \quad x_{l\alpha} \leq y_{l\alpha} \text{ and } x_{r\alpha} \leq y_{r\alpha} \qquad \text{for all } \alpha \in [0, 1],$$

where $x, y \in \mathbb{R}_{\mathcal{F}}$, $[x]^\alpha = [x_{l\alpha}, x_{r\alpha}]$, $[y]^\alpha = [y_{l\alpha}, y_{r\alpha}]$, $\alpha \in [0, 1]$.

Lemma 2 [29] *Some properties of fuzzy sets with respect to the partial ordering $\leq$ are:*

1) *If $x \leq y$, then $x + z \leq y + z$ for $x, y, z \in \mathbb{R}_{\mathcal{F}}$.*
2) *For every nondecreasing sequence $\{x_n\} \subset \mathbb{R}_{\mathcal{F}}$, if $x_n \to x$ in $\mathbb{R}_{\mathcal{F}}$, then $x_n \leq x$ for all $n \in \mathbb{N}$.*
3) *Every pair of elements of $\mathbb{R}_{\mathcal{F}}$ has an upper bound and a lower bound in $\mathbb{R}_{\mathcal{F}}$.*

Lemma 3 *If $u, v, w \in \mathbb{R}_{\mathcal{F}}$ are such that $w \leq v$ and the H-differences $u \ominus v$, $u \ominus w$ exist, then $u \ominus v \leq u \ominus w$.*

Proof It is clear that $w_{l\alpha} \leq v_{l\alpha}$ and $w_{r\alpha} \leq v_{r\alpha}$, imply that $u_{l\alpha} - v_{l\alpha} \leq u_{l\alpha} - w_{l\alpha}$ and $u_{r\alpha} - v_{r\alpha} \leq u_{r\alpha} - w_{r\alpha}$ for all $\alpha \in [0, 1]$. □

For $J \subset \mathbb{R}^2$, we denote by $C(J, \mathbb{R}_{\mathcal{F}})$ the space of all continuous functions defined on J and fuzzy-valued in $\mathbb{R}_{\mathcal{F}}$. Set

$$H_\lambda(u, v) = \sup_{(x,y)\in J} \left\{ d_\infty(u(x, y), v(x, y))e^{-\lambda(x+y)} \right\}$$

for $u, v \in C(J, \mathbb{R}_{\mathcal{F}})$, where $\lambda > 0$. It is easy to see that $(C(J, \mathbb{R}_{\mathcal{F}}), H_\lambda)$ is a complete metric space [19].

Definition 7 Consider $f, g \in C(J, \mathbb{R}_{\mathcal{F}})$. We say that $f \leq g$ in $C(J, \mathbb{R}_{\mathcal{F}})$ if and only if $f(x, y) \leq g(x, y)$ for all $(x, y) \in J$. That means $f_{l\alpha}(x, y) \leq g_{l\alpha}(x, y)$ and $f_{r\alpha}(x, y) \leq g_{r\alpha}(x, y)$ for all $\alpha \in [0, 1]$ and $(x, y) \in J$.

Some of the following properties of fuzzy-valued continuous functions with respect to the partial ordering $\leq$ are inferred directly from the corresponding properties of fuzzy numbers in $(\mathbb{R}_{\mathcal{F}}, \leq)$ given in Lemma 2.

Lemma 4 *Let $(\mathbb{R}_{\mathcal{F}}, \leq)$ be the space of fuzzy numbers equipped with the partial ordering defined, then we have*

1) *$(C(J, \mathbb{R}_{\mathcal{F}}), \leq)$ is a partial ordered space;*
2) *$(C(J, \mathbb{R}_{\mathcal{F}}), H_\lambda)$ is a regular metric space;*
3) *Every pair of elements of $C(J, \mathbb{R}_{\mathcal{F}})$ has an upper bound and a lower bound in $C(J, \mathbb{R}_{\mathcal{F}})$.*

Proof These properties have been established briefly in [29]. We include their proofs for the sake of completeness. The proofs of property 1) and property 3) are obvious, since they are true in $\mathbb{R}_{\mathcal{F}}$. So we can proceed for each $(x, y) \in J$, and these properties are satisfied in $C(J, \mathbb{R}_{\mathcal{F}})$ (note that we can select the upper and lower bounds to be continuous). Hence, we only give the proof of property 2).

2) Indeed, assume that $\{u_n\} \subset C(J, \mathbb{R}_{\mathcal{F}})$ is a nondecreasing sequence and convergent to u in $C(J, \mathbb{R}_{\mathcal{F}})$, then $\{u_n(x, y)\}$ is a nondecreasing sequence in $\mathbb{R}_{\mathcal{F}}$ for every $(x, y) \in J$. Moreover, for each $(x, y) \in J$,

$$e^{-\lambda(x+y)} d_\infty\left(u_n(x, y), u(x, y)\right) \leq \sup_J \left\{ d_\infty(u_n(x, y), u(x, y)) e^{-\lambda(x+y)} \right\} = H_\lambda(u_n, u).$$

Since $\lim_{n\to\infty} H_\lambda(u_n, u) = 0$, we have $\lim_{n\to\infty} d_\infty(u_n(x, y), u(x, y)) = 0$, or $u_n(x, y)$ converges to $u(x, y)$ in $\mathbb{R}_{\mathcal{F}}$ for every $(x, y) \in J$. From Lemma 2, we have $u_n(x, y) \leq u(x, y)$ for all $n \in \mathbb{N}$ and every $(x, y) \in J$. □

3.2 Some Preliminaries on Fuzzy Analysis

For $u, v \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara difference [4] (or gH-difference) of u and v, denoted by $u \ominus_{gH} v$ is defined as the element $w \in \mathbb{R}_{\mathcal{F}}$ such that

$$u \ominus_{gH} v = w \quad \Longleftrightarrow \quad \text{(i)}\ \ u = v + w \quad \text{or} \quad \text{(ii)}\ \ v = u + (-1)w.$$

Notice that, if $u \ominus v$ exists, then $u \ominus_{gH} v = u \ominus v$. If (i) and (ii) are satisfied simultaneously, then w is a crisp number. Also, $u \ominus_{gH} u = \hat{0}$ and if $u \ominus_{gH} v$ exists, it is unique.

The generalized Hukuhara partial derivatives (gH-p-derivatives, for short) of a fuzzy-valued mapping $f : I \subset \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ are defined in Definitions 2.9 and 3.4 in [2]. Denote by $C^2(I, \mathbb{R}_{\mathcal{F}})$ the set of all functions $f \in C(I, \mathbb{R}_{\mathcal{F}})$ which have gH-p-derivatives up to order 2 with respect to x and y continuous on I.

Definition 8 [2] Let $f : I \to \mathbb{R}_{\mathcal{F}}$ be gH-p-differentiable with respect to x at $(x_0, y_0) \in I$. We say that f is (i)-gH differentiable with respect to x at $(x_0, y_0) \in I$ if

$$[f_x(x_0, y_0)]^\alpha = [\partial_x f_{l\alpha}(x_0, y_0), \partial_x f_{r\alpha}(x_0, y_0)] \quad \forall \alpha \in [0, 1]$$

and that f is (ii)-gH differentiable with respect to x at $(x_0, y_0) \in I$ if

$$[f_x(x_0, y_0)]^\alpha = [\partial_x f_{r\alpha}(x_0, y_0), \partial_x f_{l\alpha}(x_0, y_0)] \quad \forall \alpha \in [0, 1].$$

The (i) and (ii)-gH derivatives of f with respect to y are defined similarly.

Definition 9 Let $f \in C^2(I, \mathbb{R}_{\mathcal{F}})$ and f_y be gH-p-differentiable at $(x_0, y_0) \in I$ with respect to x and do not have any switching points on I. We say that

a) f_{xy} is in type 1 of gH-derivatives (denote ${}_1D_{xy} f$) if the type of gH-derivatives of both f and f_y are the same. Then, for $\alpha \in [0, 1]$,

$$\left[{}_1D_{xy} f(x_0, y_0)\right]^\alpha = \left[\partial_{xy} f_{l\alpha}(x_0, y_0), \partial_{xy} f_{r\alpha}(x_0, y_0)\right].$$

b) f_{xy} is in type 2 of gH-derivatives (denote ${}_2D_{xy} f$) if the type of gH-derivatives of both f and f_y are different. Then, for $\alpha \in [0, 1]$,

$$\left[{}_2D_{xy} f(x_0, y_0)\right]^\alpha = \left[\partial_{xy} f_{r\alpha}(x_0, y_0), \partial_{xy} f_{l\alpha}(x_0, y_0)\right].$$

It is a well-known result that, if f is continuous on U, then f is integrable on U. Moreover, we have the following properties.

Lemma 5 *Let U be a compact subset of $\mathbb{R}^2$, $u \leq v$ in $C(U, \mathbb{R}_{\mathcal{F}})$. Then*

$$\int_U u(x, y)dxdy \leq \int_U v(x, y)dxdy.$$

Proof From the definition of the fuzzy Aumann integral [19], we have

$$\left[\int_U u(x,y)dxdy\right]^\alpha = \left[\int_U u_{l\alpha}(x,y)dxdy, \int_U u_{r\alpha}(x,y)dxdy\right]$$

and

$$\left[\int_U v(x,y)dxdy\right]^\alpha = \left[\int_U v_{l\alpha}(x,y)dxdy, \int_U v_{r\alpha}(x,y)dxdy\right]$$

for every $\alpha \in [0,1]$.

Since $u \leq v$ in $C(U, \mathbb{R}_\mathcal{F})$, then $u(x,y) \leq v(x,y) \in \mathbb{R}_\mathcal{F}$ for all $(x,y) \in U$. That means, from Definition 6, that $(u(x,y))_{l\alpha} \leq (v(x,y))_{l\alpha}$, $(u(x,y))_{r\alpha} \leq (v(x,y))_{r\alpha}$ for all $\alpha \in [0,1]$. It implies that

$$\int_U u_{l\alpha}(x,y)dxdy \leq \int_U v_{l\alpha}(x,y)dxdy, \quad \int_U u_{r\alpha}(x,y)dxdy \leq \int_U v_{r\alpha}(x,y)dxdy$$

for all $\alpha \in [0,1]$. From Definition 6, we deduce that $\int_U u(x,y)dxdy \leq \int_U v(x,y)dxdy$. □

3.3 Statement of the Problems

In this part, we prove some new results on the existence of a unique solution for fuzzy partial differential equations with local boundary conditions by applying the theory presented in Section 2.

For arbitrary positive real numbers a, b, we denote $J_a = [0,a]$, $J_b = [0,b]$, $J = J_a \times J_b$. We recall Problem (1)–(2) with $\eta_1(\cdot) \in C(J_a, \mathbb{R}_\mathcal{F})$, $\eta_2(\cdot) \in C(J_b, \mathbb{R}_\mathcal{F})$ being given functions such that $\eta_1(0) = \eta_2(0)$ and the difference $\eta_2(y) \ominus \eta_1(0)$ exists for all $y \in J_b$ and the function $f : J \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ has no switching points. This boundary value problem has been considered in some references such as [2, 20–22]. In these papers, the authors prove the Picard's theorem for Problem (1)–(2), i.e., when f is Lipschitz continuous, the problem has a unique fuzzy solution. By weakening the Lipschitz condition, now the function f only needs to satisfy a generalized contractive-like condition between comparable items, and we also prove the existence of fuzzy solutions.

For $(x,y) \in J$, let $I_{xy}f(x,y,u)$ denote the integral $\int_0^y \int_0^x f(s,t,u(s,t))dsdt$. We change the order of integration with respect to the notation in [22], since, in the derivatives ${}_kD_{xy}$, we first calculate a derivative with respect to y and then with respect to x, so that we integrate in the reverse order.

Lemma 6 [22] *Assume that f is a continuous function on $J \times \mathbb{R}_\mathcal{F}$ and that $u(\cdot,\cdot) \in C^2(J, \mathbb{R}_\mathcal{F})$ satisfies Problem (1)–(2) in J. Then $u(\cdot,\cdot)$ satisfies the following integral equations:*

1) *If $k = 1$ then $u(x,y) = p(x,y) + I_{xy}f(x,y,u)$ for $(x,y) \in J$; or*
2) *If $k = 2$ then $u(x,y) = p(x,y) \ominus (-1)I_{xy}f(x,y,u)$ for $(x,y) \in J$,*

where

$$p(x,y) = \eta_1(x) + \eta_2(y) \ominus \eta_1(0). \tag{17}$$

Definition 10 A function $u \in C(J, \mathbb{R}_\mathcal{F})$ is called an integral solution of type 1 of the Problem (1)–(2) if it satisfies the following integral equation

$$u(x,y) = p(x,y) + I_{xy}f(x,y,u) \quad \text{for all } (x,y) \in J$$

and $u \in C(J, \mathbb{R}_{\mathcal{F}})$ is called an integral solution of type 2 of the Problem (1)–(2) if it satisfies the following integral equation

$$u(x, y) = p(x, y) \ominus (-1) I_{xy} f(x, y, u) \quad \text{for all } (x, y) \in J,$$

where $p(\cdot, \cdot)$ is defined by (17).

Remark 6 Notice that Definition 10 makes sense via Lemma 6.

Definition 11 A fuzzy function $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ is called a (k)-lower $(k = 1, 2)$ solution of the Problem (1)–(2) if

$$\begin{aligned} &{}_k D_{xy}\mu(x, y) \leq f(x, y, \mu(x, y)), \quad (x, y) \in J, \\ &\mu(x, 0) \leq \eta_1(x), \quad x \in J_a, \qquad \mu(0, y) \leq \eta_2(y), \quad y \in J_b, \qquad \mu(0, 0) = \eta_1(0). \end{aligned}$$

Analogously, a fuzzy function $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ is called a (k)-upper $(k = 1, 2)$ solution of the Problem (1)–(2) if

$$\begin{aligned} &{}_k D_{xy}\mu(x, y) \geq f(x, y, \mu(x, y)), \quad (x, y) \in J, \\ &\mu(x, 0) \geq \eta_1(x), \quad x \in J_a, \qquad \mu(0, y) \geq \eta_2(y), \quad y \in J_b, \qquad \mu(0, 0) = \eta_1(0). \end{aligned}$$

Remark 7 The first steps in the theory of lower and upper solutions have been given by Picard for PDEs and ODEs [31, 32]. In both cases, the existence of a solution is guaranteed from a monotone iterative technique. Dragoni [10, 11] are the first ones that recognize explicitly the central role of lower and upper solutions for ordinary differential equations with Dirichlet boundary value conditions. In the monograph of Bernfeld and Lakshmikantham [5], Ladde et al. [18] the theory of the method of lower and upper solutions and the monotone iterative technique are presented in details.

In this paper, the existence of lower solutions or upper solutions of considered problem is used as a sufficient condition in generalized contractive-like theorems in Section 2 to ensure the existence and uniqueness of two types of fuzzy solutions to the Problem (1)–(2). For more about the method of lower and upper solutions, we refer the reader to the classical work of Mawhin [26] and the surveys in this field of De Coster and Habets [6–8] in which we can find historical and bibliographical references together with recent results and open problems.

3.4 Existence and Uniqueness of Fuzzy Solutions

Lemma 7 *For an arbitrary strictly increasing altering distance function γ and for all positive real numbers a, b, there exists $\lambda > 0$ such that the function*

$$\Phi(t) = \gamma(t) - \gamma\left(\frac{1}{\lambda^2}\left(1 - e^{-\lambda a}\right)\left(1 - e^{-\lambda b}\right)t\right), \quad t \in [0, \infty),$$

belongs to $\hat{C}([0, \infty))$.

Proof From the continuity of γ, Φ is a continuous function on $[0, \infty)$. Choose $\lambda > 0$ such that

$$\frac{1}{\lambda^2}\left(1 - e^{-\lambda a}\right)\left(1 - e^{-\lambda b}\right) < 1.$$

Then, for all $t \geq 0$, we have $\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t \leq t$. Since γ is increasing, it follows that $\gamma\left(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t\right) \leq \gamma(t)$ for all $t \geq 0$. Hence $\Phi(t) \geq 0$ for all $t \geq 0$.

Now, we consider $t > 0$. From $\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t < t$ and the strict increase property of γ, it implies that $\Phi(t) > 0$. It follows that, if $\Phi(t) = 0$, then $t = 0$ (and conversely). It completes the proof. □

Theorem 3 *Let f be a continuous function that satisfies the following two hypotheses:*

(h_1) *$f : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is nondecreasing in the third variable, i.e., if $\nu \leq \xi \in \mathbb{R}_{\mathcal{F}}$, then $f(x, y, \nu) \leq f(x, y, \xi)$ for all $(x, y) \in J$.*

(h_2) *f is weakly contractive over comparable elements, that is, for some altering distance function ψ and $\phi \in \hat{C}([0, \infty))$, the following estimation*

$$\psi(d_\infty(f(x, y, \nu), f(x, y, \xi))) \leq \psi(d_\infty(\nu, \xi)) - \phi(d_\infty(\nu, \xi))$$

holds for all $(x, y) \in J$, $\nu \leq \xi$ in $\mathbb{R}_{\mathcal{F}}$.

Suppose that there exists a (1)*-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ for the Problem* (1)–(2)*. Then the Problem* (1)–(2) *has a unique integral solution of type* 1 *on J.*

Proof Define the operator $T_1 : C(J, \mathbb{R}_{\mathcal{F}}) \to C(J, \mathbb{R}_{\mathcal{F}})$ by

$$(T_1 u)(x, y) = p(x, y) + I_{xy} f(x, y, u), \quad (x, y) \in J, \tag{18}$$

for $u \in C(J, \mathbb{R}_{\mathcal{F}})$, where $p(\cdot, \cdot)$ is defined by (17).

Step 1: We prove that T_1 is a nondecreasing operator in $C(J, \mathbb{R}_{\mathcal{F}})$.

Assume that $u \leq v$ in $C(J, \mathbb{R}_{\mathcal{F}})$, which means $u(s, t) \leq v(s, t)$ for all $(s, t) \in J$. From hypothesis (h_1), that is, the nondecreasing character of f with respect to the third variable, we have that $f(s, t, u(s, t)) \leq f(s, t, v(s, t))$ for all $(s, t) \in J$. Then, from Lemma 5, we have

$$I_{xy} f(x, y, u) \leq I_{xy} f(x, y, v) \quad \text{for } (x, y) \in J.$$

It means that $(T_1 u)(x, y) \leq (T_1 v)(x, y)$ for all $(x, y) \in J$. Hence, $T_1 u \leq T_1 v$.

Step 2: Now, we prove that

$$d_\infty(f(x, y, \nu), f(x, y, \eta)) \leq d_\infty(\nu, \eta) \quad \text{for all } \nu \leq \eta \text{ in } \mathbb{R}_{\mathcal{F}} \text{ and } (x, y) \in J.$$

Indeed, assume that $\nu \leq \eta$ in $\mathbb{R}_{\mathcal{F}}$ but $d_\infty(\nu, \eta) < d_\infty(f(x, y, \nu), f(x, y, \eta))$ for some $(x, y) \in J$. Due to the nondecrease property of ψ, we have

$$\psi(d_\infty(\nu, \eta)) \leq \psi(d_\infty(f(x, y, \nu), f(x, y, \eta))). \tag{19}$$

On the other hand, from the hypothesis (h_2), we have

$$\begin{aligned} \psi(d_\infty(f(x, y, \nu), f(x, y, \eta))) &\leq \psi(d_\infty(\nu, \eta)) - \phi(d_\infty(\nu, \eta)) \\ &\leq \psi(d_\infty(\nu, \eta)) \end{aligned} \tag{20}$$

for all $\nu \leq \eta$ in $\mathbb{R}_{\mathcal{F}}$. From (19) and (20), one has

$$\psi(d_\infty(\nu, \eta)) = \psi(d_\infty(f(x, y, \nu), f(x, y, \eta))).$$

It follows from (20) that $0 \leq -\phi(d_\infty(\nu, \eta))$ or $\phi(d_\infty(\nu, \eta)) = 0$. Thanks to $\phi \in \hat{C}([0, \infty))$, that implies $d_\infty(\nu, \eta) = 0$. Hence

$$\psi(d_\infty(f(x, y, \nu), f(x, y, \eta))) = \psi(d_\infty(\nu, \eta)) = 0.$$

It implies $d_\infty(f(x, y, \nu), f(x, y, \eta)) = 0$, leading to a contradiction.

Step 3: We check the generalized contractive-like property of the operator T_1.

For all $u \leq v$ in $C(J, \mathbb{R}_{\mathcal{F}})$, we have $u(x, y) \leq v(x, y)$ for all $(x, y) \in J$. It is known from Step 2 that

$$d_\infty(f(x, y, u(x, y)), f(x, y, v(x, y))) \leq d_\infty(u(x, y), v(x, y)) \quad \text{for all } (x, y) \in J.$$

Thus

$$\begin{aligned} d_\infty((T_1u)(x, y), (T_1v)(x, y)) &= d_\infty\left(p(x, y) + I_{xy}f(x, y, u), p(x, y) + I_{xy}f(x, y, v)\right) \\ &= d_\infty\left(I_{xy}f(x, y, u), I_{xy}f(x, y, v)\right) \\ &\leq \int_0^y \int_0^x d_\infty(f(s, t, u(s, t)), f(s, t, v(s, t)))dsdt \\ &\leq \int_0^y \int_0^x d_\infty(u(s, t), v(s, t))dsdt \\ &\leq \int_0^y \int_0^x H_\lambda(u, v)e^{\lambda(s+t)}dsdt \\ &= \frac{1}{\lambda^2}H_\lambda(u, v)(e^{\lambda x} - 1)(e^{\lambda y} - 1). \end{aligned}$$

Then, for all $(x, y) \in J$, we have

$$d_\infty((T_1u)(x, y), (T_1v)(x, y))e^{-\lambda(x+y)} \leq \frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda x})(1 - e^{-\lambda y}).$$

Therefore

$$H_\lambda(T_1u, T_1v) \leq \frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b}). \tag{21}$$

For an arbitrary strictly increasing altering distance function γ, from (21), we have

$$\begin{aligned} \gamma(H_\lambda(T_1u, T_1v)) &\leq \gamma\left(\frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right) \\ &= \gamma(H_\lambda(u, v)) - \left[\gamma(H_\lambda(u, v)) - \gamma\left(\frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right)\right]. \end{aligned}$$

Denote $\Phi(t) = \gamma(t) - \gamma\left(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t\right)$, $t \in [0, \infty)$. From Lemma 7, there exists $\lambda > 0$ such that Φ belongs to $\hat{C}([0, \infty))$ and

$$\gamma(H_\lambda(T_1u, T_1v)) \leq \gamma(H_\lambda(u, v)) - \Phi(H_\lambda(u, v)) \quad \text{for all } u \leq v \text{ in } C(J, \mathbb{R}_{\mathcal{F}}).$$

This means that the operator T_1 satisfies the contractive-like property.

Step 4: Since there exists a (1)-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ for the Problem (1)–(2), then

$$\begin{aligned} \mu_{l\alpha}(x, y) &\leq \mu_{l\alpha}(x, 0) + \mu_{l\alpha}(0, y) - \mu_{l\alpha}(0, 0) + \int_0^y \int_0^x f_{l\alpha}(s, t, \mu(s, t))dsdt \\ &\leq (\eta_1)_{l\alpha}(x) + (\eta_2)_{l\alpha}(y) - (\eta_1)_{l\alpha}(0) + \int_0^y \int_0^x f_{l\alpha}(s, t, \mu(s, t))dsdt, \\ \mu_{r\alpha}(x, y) &\leq \mu_{r\alpha}(x, 0) + \mu_{r\alpha}(0, y) - \mu_{r\alpha}(0, 0) + \int_0^y \int_0^x f_{r\alpha}(s, t, \mu(s, t))dsdt \\ &\leq (\eta_1)_{r\alpha}(x) + (\eta_2)_{r\alpha}(y) - (\eta_1)_{r\alpha}(0) + \int_0^y \int_0^x f_{r\alpha}(s, t, \mu(s, t))dsdt, \end{aligned}$$

for $\alpha \in [0, 1]$ and $(x, y) \in J$, so that

$$\mu(x, y) \leq \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I_{xy}f(x, y, \mu) = (T_1\mu)(x, y)$$

for all $(x, y) \in J$. It follows that $\mu \leq T_1\mu$ in $C(J, \mathbb{R}_{\mathcal{F}})$.

It is easy to see from Steps 1–4 that the operator T_1 satisfies all the hypotheses of Theorem 2 in case $\beta = 1_{[0,\infty)}$. In consequence, T_1 has a fixed point in $C(J, \mathbb{R}_{\mathcal{F}})$. Note that $C(J, \mathbb{R}_{\mathcal{F}})$ satisfies that every pair of elements of $C(J, \mathbb{R}_{\mathcal{F}})$ have an upper bound and a lower bound in $C(J, \mathbb{R}_{\mathcal{F}})$ (Lemma 4). It follows that the operator T_1 has a unique fixed point, which is the unique integral solution of type 1 to Problem (1)–(2). □

Remark 8 The existence of an integral solution of type 1 is guaranteed by the weakly non-decreasing character and the generalized weak contractivity property of function f. The existence of an integral solution of type 2 is more difficult to obtain due to the requirement of the existence of Hukuhara differences.

We denote

$$\hat{C}(J, \mathbb{R}_{\mathcal{F}}) = \{u \in C(J, \mathbb{R}_{\mathcal{F}}) : p(x, y) \ominus (-1)I_{xy} f(x, y, u) \text{ exists for all } (x, y) \in J\},$$

where $p(x, y)$ is defined by (17).

Lemma 8 *Consider $(C(J, \mathbb{R}_{\mathcal{F}}), d)$ a complete metric space. If f is a continuous function and $\hat{C}(J, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$, then $(\hat{C}(J, \mathbb{R}_{\mathcal{F}}), d)$ is a complete metric space.*

Proof Let $\{u_m\}_{m=1}^{\infty}$ be a sequence in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$ converging towards u (in $C(J, \mathbb{R}_{\mathcal{F}})$). Then, for all $(x, y) \in J$, the following differences exist

$$p(x, y) \ominus (-1)I_{xy} f(x, y, u_m).$$

For simplicity of exposition, let

$$F(u_m)(x, y) = (-1)I_{xy} f(x, y, u_m).$$

From Proposition 21 in [35], we know that, for each fixed $(x, y) \in J$,

$$\begin{cases} \text{len}[p(x, y)]^{\alpha} \geq \text{len}[F(u_m)(x, y)]^{\alpha}, \quad 0 \leq \alpha \leq 1, \\ (p(x, y))_{l\alpha} - (F(u_m)(x, y))_{l\alpha} \text{ is monotonically increasing in } \alpha \in [0, 1], \\ (p(x, y))_{r\alpha} - (F(u_m)(x, y))_{r\alpha} \text{ is monotonically decreasing in } \alpha \in [0, 1]. \end{cases}$$

Since f is continuous and $\{u_m\}_{m=1}^{\infty}$ converges uniformly to u, then

$$\text{len}\left[\int_0^y \int_0^x f(s, t, u_m(s, t))dsdt\right]^{\alpha}$$

is convergent towards

$$\text{len}\left[\int_0^y \int_0^x f(s, t, u(s, t))dsdt\right]^{\alpha}$$

for each $\alpha \in [0, 1]$. Therefore, $\text{len}[F(u_m)(x, y)]^{\alpha}$ converges to $\text{len}[F(u)(x, y)]^{\alpha}$, where

$$F(u)(x, y) = (-1)I_{xy} f(x, y, u) = (-1)\int_0^y \int_0^x f\,(s, t, u(s, t))\,dsdt.$$

Hence, from the inequality

$$\text{len}[p(x, y)]^{\alpha} \geq \text{len}[F(u_m)(x, y)]^{\alpha}, \quad 0 \leq \alpha \leq 1,$$

we derive that, for each fixed $(x, y) \in J$,

$$\text{len}[p(x, y)]^{\alpha} \geq \text{len}[F(u)(x, y)]^{\alpha}, \quad 0 \leq \alpha \leq 1.$$

Moreover, for arbitrary $0 \leq \alpha \leq \gamma \leq 1$, we have

$$(p(x, y))_{l\alpha} - (F(u_m)(x, y))_{l\alpha} \leq (p(x, y))_{l\gamma} - (F(u_m)(x, y))_{l\gamma}.$$

Taking the limits when $m \to \infty$ and using similar arguments as above, we receive

$$(p(x, y))_{l\alpha} - (F(u)(x, y))_{l\alpha} \leq (p(x, y))_{l\gamma} - (F(u)(x, y))_{l\gamma}.$$

By analogous arguments, one has

$$(p(x, y))_{r\alpha} - (F(u)(x, y))_{r\alpha} \geq (p(x, y))_{r\gamma} - (F(u)(x, y))_{r\gamma}$$

for all $0 \leq \alpha \leq \gamma \leq 1$.

Therefore, the difference

$$p(x, y) \ominus (-1)I_{xy} f(x, y, u)$$

exists for all $(x, y) \in J$. It shows that $u \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ and $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$ is a closed subset of the space $C(J, \mathbb{R}_{\mathcal{F}})$. Since $(C(J, \mathbb{R}_{\mathcal{F}}), d)$ is a complete metric space, $(\hat{C}(J, \mathbb{R}_{\mathcal{F}}), d)$ is also a complete metric space. □

By changing the solution space to $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$, we can prove the existence of solution of type 2 to the Problem (1)–(2).

Theorem 4 *Let f be a continuous function satisfying the hypotheses* (h_1)–(h_2) *in Theorem 3. Moreover, suppose that the following hypotheses are fulfilled:*

(h_3) $\hat{C}(J, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$.

(h_4) *If $u \in C(J, \mathbb{R}_{\mathcal{F}})$ satisfies that $u \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$, then the Hukuhara difference*

$$p(x, y) \ominus (-1)I_{xy} f(x, y, v)$$

also exists for every $(x, y) \in J$, where

$$v(x, y) = p(x, y) \ominus (-1)I_{xy} f(x, y, u), \quad (x, y) \in J.$$

Suppose that there exists a (2)*-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}}) \cap \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ for the Problem* (1)–(2)*. Then the Problem* (1)–(2) *has an integral solution of type* 2 *on J.*

Furthermore, if the following condition holds:

(h_5) *For each pair $u, v \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ fixed, there exists $\xi \in C(J, \mathbb{R}_{\mathcal{F}})$ an upper or a lower bound of u, v such that the Hukuhara difference $p(x, y) \ominus (-1)I_{xy} f(x, y, \xi)$ exists for all $(x, y) \in J$,*

then the Problem (1)–(2) *has a unique integral solution of type* 2 *on J.*

Proof By the hypothesis (h_3), $\hat{C}(J, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$ and it is clear that, for every $u \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$, the Hukuhara difference $p(x, y) \ominus (-1)I_{xy} f(x, y, u)$ exists for all $(x, y) \in J$. By the assumption (h_4), it is reasonable to build the operator $T_2 : \hat{C}(J, \mathbb{R}_{\mathcal{F}}) \to \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ defined by

$$(T_2 u)(x, y) = p(x, y) \ominus (-1)I_{xy} f(x, y, u), \quad (x, y) \in J.$$

Similarly to Step 2 in the proof of Theorem 3, we receive from hypotheses (h_1)–(h_2) that

$$d_\infty(f(x, y, v), f(x, y, \eta)) \leq d_\infty(v, \eta)$$

for all $v \leq \eta$ in $\mathbb{R}_{\mathcal{F}}$ and $(x, y) \in J$.

Using analogous arguments as in the proof of (21) and combining with Lemma 1, for all $u \le v$ in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$, we have

$$\begin{aligned}
&d_\infty((T_2u)(x, y), (T_2v)(x, y)) \\
&= d_\infty\left(p(x, y) \ominus (-1)I_{xy}f(x, y, u), p(x, y) \ominus (-1)I_{xy}f(x, y, v)\right) \\
&\le d_\infty(I_{xy}f(x, y, u), I_{xy}f(x, y, v)) \\
&\le \frac{1}{\lambda^2}H_\lambda(u, v)(e^{\lambda x} - 1)(e^{\lambda y} - 1),
\end{aligned}$$

and it follows that

$$H_\lambda(T_2u, T_2v) \le \frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b}). \tag{22}$$

Now, assume that $u \le v$ in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$. We need to indicate the nondecreasing character of the operator T_2, proving that $T_2u \le T_2v$. Since $u(s, t) \le v(s, t)$ for all $(s, t) \in J$, and using the hypothesis of the nondecreasing character of f in the third variable, we have $f(s, t, u(s, t)) \le f(s, t, v(s, t))$ for all $(s, t) \in J$. It follows from Lemma 5 that

$$\int_0^y\int_0^x f(s, t, u(s, t))dsdt \le \int_0^y\int_0^x f(s, t, v(s, t))dsdt,$$

or

$$(-1)\int_0^y\int_0^x f(s, t, v(s, t))dsdt \le (-1)\int_0^y\int_0^x f(s, t, u(s, t))dsdt$$

for all $(x, y) \in J$. Hence, by Lemma 3, since the differences involved exist, we have

$$\begin{aligned}
(T_2v)(x, y) &= p(x, y) \ominus (-1)\int_0^y\int_0^x f(s, t, v(s, t))dsdt \\
&\ge p(x, y) \ominus (-1)\int_0^y\int_0^x f(s, t, u(s, t))dsdt = (T_2u)(x, y)
\end{aligned}$$

for all $(s, t) \in J$, and the consequence is that T_2 is a nondecreasing operator on $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$.

From (22), for an arbitrary strictly increasing altering distance function γ, we have

$$\begin{aligned}
\gamma(H_\lambda(T_2u, T_2v)) &\le \gamma\left(\frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right) \\
&= \gamma(H_\lambda(u, v)) - \left[\gamma(H_\lambda(u, v)) - \gamma\left(\frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right)\right].
\end{aligned}$$

Denote $\Phi(t) = \gamma(t) - \gamma\left(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t\right)$, $t \in [0, \infty)$. Then, from Lemma 7, there exists $\lambda > 0$ such that Φ is in $\hat{C}([0, \infty))$ and T_2 satisfies the generalized contractive-like condition

$$\gamma(H_\lambda(T_2u, T_2v)) \le \gamma(H_\lambda(u, v)) - \Phi(H_\lambda(u, v)) \quad \text{for all } u, v \in \hat{C}(J, \mathbb{R}_{\mathcal{F}}) \text{ with } u \le v.$$

Next, since there exists a (2)-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}}) \cap \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ for the Problem (1)–(2), we prove that $\mu \le T_2\mu$. Note that the difference

$$(T_2\mu)(x, y) = p(x, y) \ominus (-1)\int_0^y\int_0^x f(s, t, \mu(s, t))dsdt$$

exists for all $(x, y) \in J$, since $\mu \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$.

Besides, from ${}_2D_{xy}\mu(x, y) \le f(x, y, \mu(x, y))$, we deduce that

$$\int_0^y \int_0^x {}_2D_{xy}\mu(s, t)dsdt \le \int_0^y \int_0^x f(s, t, \mu(s, t))dsdt$$

for all $(x, y) \in J$. The previous inequality together with $\mu(x, 0) \le \eta_1(x)$, $\mu(0, y) \le \eta_2(y)$, and $\mu(0, 0) = \eta_1(0)$, implies that

$$\begin{aligned}\mu_{r\alpha}(x, y) &\le \mu_{r\alpha}(x, 0) + \mu_{r\alpha}(0, y) - \mu_{r\alpha}(0, 0) + \int_0^y \int_0^x f_{l\alpha}(s, t, \mu(s, t))dsdt \\ &\le (\eta_1)_{r\alpha}(x) + (\eta_2)_{r\alpha}(y) - (\eta_1)_{r\alpha}(0) + \int_0^y \int_0^x f_{l\alpha}(s, t, \mu(s, t))dsdt, \\ \mu_{l\alpha}(x, y) &\le \mu_{l\alpha}(x, 0) + \mu_{l\alpha}(0, y) - \mu_{l\alpha}(0, 0) + \int_0^y \int_0^x f_{r\alpha}(s, t, \mu(s, t))dsdt \\ &\le (\eta_1)_{l\alpha}(x) + (\eta_2)_{l\alpha}(y) - (\eta_1)_{l\alpha}(0) + \int_0^y \int_0^x f_{r\alpha}(s, t, \mu(s, t))dsdt\end{aligned}$$

for $\alpha \in [0, 1]$ and $(x, y) \in J$, which proves that

$$\begin{aligned}\mu(x, y) &\le \eta_1(x) + \eta_2(y) \ominus \eta_1(0) \ominus (-1)\int_0^y \int_0^x f(s, t, \mu(s, t))dsdt \\ &= p(x, y) \ominus (-1)\int_0^y \int_0^x f(s, t, \mu(s, t))dsdt = (T_2\mu)(x, y)\end{aligned}$$

for all $(x, y) \in J$. Therefore, $\mu \le T_2\mu$ in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$.

Because of Lemma 8, since $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$ is a closed subspace of $C(J, \mathbb{R}_{\mathcal{F}})$, then $(\hat{C}(J, \mathbb{R}_{\mathcal{F}}), H_\lambda)$ is a complete metric space. Besides, the properties 1) and 2) in Lemma 4 are valid in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$. Then the operator T_2 satisfies all the hypotheses of Theorem 2 in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$. Hence, T_2 has a fixed point in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$. The uniqueness of fixed point comes from the existence of an upper or a lower bound in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$ for each pair of fixed elements in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$, which comes from ($h_5$). This completes the proof. □

Theorem 5 *The conclusions of Theorems 3 and 4 are still valid if instead of a* (k)*-lower solution, a* (k)*-upper solution* ($k = 1, 2$) *of Problem* (1)–(2) *is supposed to be exist.*

Proof If μ is a (1)-upper solution to the Problem (1)–(2), then

$$\mu(x, y) \ge \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I_{xy} f(x, y, \mu(x, y)) = (T_1\mu)(x, y)$$

for all $(x, y) \in J$, from which it follows that $\mu \ge T_1\mu$. Hence, the existence of a unique integral solution of type 1 for Problem (1)–(2) is derived from Theorem 2. The proof of the solvability of Problem (1)–(2) with a unique integral solution of type 2 is obtained similarly by taking a (2)-upper solution μ in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$. □

Finally, we prove the existence of solutions to Problem (1)–(2) by applying the generalized results obtained in Section 2 for the case $\beta \in \mathcal{S}_0$.

In the space $C(J, \mathbb{R}_{\mathcal{F}})$, we consider the metric

$$d(u, v) = \sup_{(x,y)\in J} \{d_\infty(u(x, y), v(x, y))\}.$$

Due to the compactness of J in $\mathbb{R}^2$, it is easy to see that $(C(J, \mathbb{R}_{\mathcal{F}}), d)$ is a complete metric space.

For an arbitrary altering distance function η, we denote by $\mathcal{B}_\eta$ the class of functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

i) φ is monotonic increasing.
ii) $\varphi(t) < t$ for $t > 0$.
iii) The function $\beta : [0, \infty) \to [0, 1)$ defined as $\beta(t) = \begin{cases} \frac{\varphi \circ \eta(t)}{\eta(t)}, & t > 0, \\ 0, & t = 0 \end{cases}$ is in $\mathcal{S}_0$.

Theorem 6 *Consider Problem* (1)–(2)*, with a continuous function f satisfying the hypothesis* (h_1)*, and suppose that there exist a strictly increasing altering distance function ψ satisfying $\psi(t) \le t$ if $t > 0$, and $\varphi \in \mathcal{B}_\psi$ such that the following inequality holds*

$$d_\infty(f(x, y, u(x, y)), f(x, y, v(x, y))) \le \frac{1}{ab}\varphi\left(\psi(d_\infty(u(x, y), v(x, y)))\right), \quad (x, y) \in J, \tag{23}$$

for $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$. Then the existence of a (1)*-lower solution (or a* (1)*-upper solution) $\mu \in C^2(J, \mathbb{R}_\mathcal{F})$ for the Problem* (1)–(2) *provides the existence of a unique integral solution of type* 1 *to the Problem* (1)–(2).

Proof Consider the operator $T_1 : (C(J, \mathbb{R}_\mathcal{F}), d) \to (C(J, \mathbb{R}_\mathcal{F}), d)$ defined by (18).

Using (h_1) and following the same reasoning as in Step 1 of Theorem 3, we obtain the nondecreasing character of the operator T_1 in $C(J, \mathbb{R}_\mathcal{F})$.

For all $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$, we have, from (23),

$$\begin{aligned} d_\infty((T_1u)(x, y), (T_1v)(x, y)) &= d_\infty(I_{xy}f(x, y, u(x, y)), I_{xy}f(x, y, v(x, y))) \\ &\le \int_0^y \int_0^x d_\infty(f(s, t, u(s, t)), f(s, t, v(s, t)))dsdt \\ &\le \frac{1}{ab}\int_0^y \int_0^x \varphi\left(\psi(d_\infty(u(x, y), v(x, y)))\right) dsdt. \end{aligned}$$

Since $d_\infty(u(x, y), v(x, y)) \le d(u, v)$ for all $(x, y) \in J$, by using the nondecrease property of ψ and φ, we get $\psi(d_\infty(u(x, y), v(x, y))) \le \psi(d(u, v))$ and

$$\varphi(\psi(d_\infty(u(x, y), v(x, y)))) \le \varphi(\psi(d(u, v)))$$

for all $(x, y) \in J$. It follows, for all $(x, y) \in J$, that

$$\begin{aligned} d_\infty((T_1u)(x, y), (T_1v)(x, y)) &\le \frac{1}{ab}\varphi\left(\psi(d(u, v))\right) \int_0^y \int_0^x dsdt \\ &= \frac{1}{ab}xy\varphi\left(\psi(d(u, v))\right) \le \varphi\left(\psi(d(u, v))\right). \end{aligned}$$

Thus, for $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$,

$$d(T_1u, T_1v) \le \varphi\left(\psi(d(u, v))\right).$$

From the nondecreasing character of ψ, we get, for $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$,

$$\begin{aligned} \psi\left(d(T_1u, T_1v)\right) &\le \psi\left(\varphi\left(\psi(d(u, v))\right)\right) \le \varphi\left(\psi(d(u, v))\right) \\ &= \frac{\varphi\left(\psi(d(u, v))\right)}{\psi(d(u, v))}\psi(d(u, v)) = \beta(d(u, v))\psi(d(u, v)), \end{aligned}$$

if $d(u,v) > 0$, and the inequality is trivially valid if $d(u,v) = 0$. Here, we have

$$\beta(t) = \begin{cases} \frac{\varphi \circ \psi(t)}{\psi(t)} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

which belongs to $\mathcal{S}_0$, by hypothesis.

Finally, let $\mu \in C^2(J, \mathbb{R}_\mathcal{F})$ be a (1)-lower solution for the Problem (1)–(2). It is clear again that $\mu \le T_1\mu$, since $\mu(x,y) \le \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I_{xy} f(x,y,\mu) = (T_1\mu)(x,y)$, $(x,y) \in J$. Similarly, if there exists a (1)-upper solution μ for the Problem (1)–(2), then we have $\mu \ge T_1\mu$. Note that $(C(J, \mathbb{R}_\mathcal{F}), d)$ is also regular.

Overall, the operator T_1 satisfies all the hypotheses of Theorem 2 in case $\beta \in \mathcal{S}_0$. In consequence, T_1 has a fixed point in $C(J, \mathbb{R}_\mathcal{F})$. Noticing that every pair of elements of $C(J, \mathbb{R}_\mathcal{F})$ has an upper and a lower bound, it follows that the operator T_1 has a unique fixed point. □

Theorem 7 *Consider Problem* (1)–(2) *with* f *continuous satisfying the hypotheses* (h_1)*,* (h_3)*,* (h_4) *and suppose that there exist a strictly increasing altering distance function* ψ *satisfying* $\psi(t) \le t$ *if* $t > 0$*, and* $\varphi \in \mathcal{B}_\psi$ *such that the inequality* (23) *holds for* $u \le v$ *in* $C(J, \mathbb{R}_\mathcal{F})$.

Then the existence of a (2)*-lower solution (or a* (2)*-upper solution)* $\mu \in C^2(J, \mathbb{R}_\mathcal{F}) \cap \hat{C}(J, \mathbb{R}_\mathcal{F})$ *for the Problem* (1)–(2) *provides the existence of a fuzzy integral solution of type* 2 *to the Problem* (1)–(2).

Furthermore, if the condition (h_5) *holds, then the Problem* (1)–(2) *has a unique integral solution of type* 2 *on* J.

Proof Using analogous arguments for the operator T_2 in Theorem 4, we deduce the existence of a (unique) integral solution of type 2 to the Problem (1)–(2). □

Example 1 Denote $\mathbb{R}_\mathcal{F}{}^+ = \{z \in \mathbb{R}_\mathcal{F} : \hat{0} \le z\}$, where $\hat{0}$ is defined by $\hat{0}(t) = 1$ if $t = 0$ and $\hat{0}(t) = 0$ in other cases. In this example, we consider the following fuzzy partial hyperbolic equation under generalized Hukuhara derivatives

$$\begin{cases} {}_kD_{xy}u = f(x,y,u(x,y)), & (x,y) \in J = [0,a] \times [0,b], \\ u(x,0) = 0, & x \in J_a, \\ u(0,y) = 0, & y \in J_b, \end{cases} \tag{24}$$

where $f : J \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}{}^+$. Note that $u(0,0) = 0$ is deduced for a solution.

Theorem 8 *Consider* $f : J \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}{}^+$ *continuous and nondecreasing with respect to the third variable and suppose that, if* $u \le v$ *in* $C(J, \mathbb{R}_\mathcal{F})$*, then*

$$\begin{aligned} & d_\infty(f(x,y,u(x,y)), f(x,y,v(x,y))) \\ & \le \frac{1}{ab} \ln\left(1 + \min\{d_\infty^2(u(x,y), v(x,y)), d_\infty(u(x,y), v(x,y))\}\right) \end{aligned} \tag{25}$$

for all $(x,y) \in J$*. Then Problem* (24) *has a unique nonnegative fuzzy integral solution of type* 1*. In addition to the hypotheses, if* (h_3) *and* (h_4) *are satisfied, then Problem* (24) *has a nonnegative integral solution of type* 2 *(unique if* (h_5) *holds).*

Proof Consider the cone $P = \{u \in C(J, \mathbb{R}_{\mathcal{F}}) : u \geq \hat{0}\}$, where we also denote by $\hat{0}$ the constant function equal to $\hat{0}$ at any point. Obviously, (P, d) is a complete metric space (and regular). The operator T_1 defined as $(T_1 u)(x, y) = I_{xy} f(x, y, u)$ is nondecreasing and maps P into itself since $f(x, y, u(x, y))$ is a nonnegative continuous function for each $u \in P$. Besides, $T_1(\hat{0}) \geq \hat{0}$ ($\hat{0}$ is a lower solution). From Theorem 6 with $\varphi(t) = \ln(1+t)$, $\psi(t) = \min\{t^2, t\}$, we derive the conclusion.

Note that the condition $f : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}{}^+$ can be relaxed to $f : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ if we impose that $f(x, y, \hat{0}) \geq \hat{0}$ for every $(x, y) \in J$, due to the nondecreasing character of T_1, which yields $T_1 u \geq T_1(\hat{0}) \geq \hat{0}$ for $u \in P$.

Note also that, in this example, the weak solution of type 2 is sought in the space of functions $u \in C(J, \mathbb{R}_{\mathcal{F}})$ such that $u \geq \hat{0}$ and $f(x, y, u(x, y))$ is crisp for every $(x, y) \in J$, so condition (h_4) (and, hence, (h_3)) is satisfied if $f(x, y, z)$ is crisp for each $(x, y) \in J$ and $z \in \mathbb{R}_{\mathcal{F}}$ crisp. Under this restriction, (h_5) also holds since, given $u, v \geq \hat{0}$, we can take as a crisp lower bound of u, v the constant function $\hat{0}$. □

4 Conclusions

In this study, we have firstly presented some new generalized theorems on fixed points for nondecreasing mappings from a partially ordered metric space to itself. These results develop some previous results of [3, 14, 27] and admit them as special cases. Secondly, we have investigated the existence and uniqueness of fuzzy solutions to a boundary value problem for a class of fuzzy partial hyperbolic equation under generalized Hukuhara derivatives. Via these results, the function placed in the right-hand side of the equation does not need to be Lipschitz continuous. In spite of this condition, f is only demanded to satisfy a generalized contractive-like condition. However, a hypothesis of existing a lower or upper solution of considered problem is required. In real world applications, the use of lower and upper solutions method is hampered by the difficulty to exhibit such functions. This method does not require to find a solution of a boundary value problem but find lower and upper solutions. This replacement reminds us to the Liapunov's second method. Furthermore, in many theorems, the assumptions at hand provide lower and upper solutions and their use simplifies the argument. The questions arise whether it is easy to recognize that a set of assumptions provides such lower and upper solutions? Is it easy to find them? In general, there is no clue to finding these solutions. This drawback motivated more works to study the way to construct the lower as well as upper solutions in differential equations theory. Some efforts to offer a construction of lower and upper solutions can be seen Lemma 1.5.2 in [15] for initial value problems of first order ordinary differential equations, Chapters VI to X in [8] for showing how to build in specific cases appropriate lower and upper solutions of some classes of two points boundary value problems. For partial differential equations, we can cite here some works [12, 13]. This observation is our primary motivation in future work for stating conditions that ensure a given function is a lower or an upper solution of our considered problems.

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ALMOST PERIODIC SOLUTIONS OF PERIODIC LINEAR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Prof. Nguyen Manh Hung on the occasion of his 60th birthday

Abstract. We study conditions for the abstract periodic linear functional differential equation $\dot{x} = Ax + F(t)x_t + f(t)$ to have almost periodic with the same structure of frequencies as f. The main conditions are stated in terms of the spectrum of the monodromy operator associated with the equation and the frequencies of the forcing term f. The obtained results extend recent results on the subject. A discussion on how the results could be extended to the case when A depends on t is given.

1. Introduction

In this paper we consider the existence and uniqueness of almost periodic solutions with the same structure of spectrum as f in equations of the following form

$$\frac{dx(t)}{dt} = Ax(t) + F(t)x_t + f(t), \quad x \in \mathbb{X}, t \in \mathbb{R}, \tag{1.1}$$

where the (unbounded) linear operator A generates a strongly continuous semigroup and the bounded linear operator $F(t)$ is periodic and is defined as follows, $x_t \in C_r := C([-r, 0], \mathbb{X})$, $x_t(\theta) := x(t+\theta)$, $r > 0$ is a given positive real number, $F(t)\varphi := \int_{-r}^{0} d\eta(t,s)\varphi(s)$, $\forall \varphi \in C_r$, $\eta(t,\cdot) : C_r \to L(\mathbb{X})$ is periodic in t, of bounded variation, and $\sup_t \|F(t)\| < \infty$, and f is a $\mathbb{X}$-valued almost periodic function. A discussion on how the results could be extended to the case when A depends on t periodically will be given at the end of the paper.

In the theory of ordinary differential equations one of the questions that are of interest to many researchers is when exist periodic solutions to equations of the form

$$\frac{dx}{dt} = B(t)x + f(t), t \in \mathbb{R}, x \in \mathbb{C}^n, \tag{F}$$

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where f is periodic, and $B(t)$ is a $n \times n$-matrix that is periodic with the same period as $f(t)$. A famous Massera's Theorem ([11]) says that Eq. (F) has a periodic solution with the same period as B and f if and only if it has a solution that is bounded on the positive half line. In addition, the periodic solution is unique if 1 is not an eigenvalue of the monodromy operator. Since then there have been many efforts to extend this classic result to various classes of equations and functions (see e.g. [1, 2, 5, 6, 16, 17, 18, 19, 20, 23]). We refer the reader to some recent developments [5, 8, 19, 20, 23] and their references for more recent information in this direction. We note that the results on the existence of periodic solutions are usually proved via the existence of fixed points of the monodromy operator (or, period map) (see e.g. [2, 10, 23]). Among the research methods used in this direction we note that when f is almost periodic the monodromy operator method is no longer applicable because the system is no longer periodic. Instead, one uses a new method that is based on the concept of *evolution semigroups* associated with the evolutionary processes generated by the equations. Also, the requirement that the period of the solutions be the same as that of the forcing term f will be understood as a requirement on the frequencies of the solutions that are not more than those of f. This justifies the introduction of the concept of *spectrum of a function* that allows us to measure the set of frequencies of a function on the real line. As is known, a fundamental technique of research in the ODE and FDE is variation-of-constants formulas (VCF) in the phase space. In the case of abstract functional differential equations, the VCF in the phase space is no longer valid. Instead, a weak version may make sense. In this short paper we will recall briefly these concepts and related results in the next section. We will present an extension of the Massera's Theorem for almost periodic solutions of Eq. (1.1) (Theorems 3.3 and 3.4). We prove that the condition of existence of bounded solutions could be removed and the equations always have a unique almost periodic solutions with frequencies as f if the part of spectrum of the monodromy operator on the unit circle does not intersect the spectrum of f. To our best knowledge the results obtained in this paper extends some previous ones in [1, 5, 18], and complements many other results in [1, 15, 16, 17, 19, 22]. In [17] the authors showed that if A generates a compact C_0-semigroup the existence of almost periodic solutions to Eq. (1.1) could be reduced to the finite dimensional case of ODE, so the problem could be thoroughly studied. The novelty of our results obtained in this paper is that we study the problem when A generates any C_0-semigroup, (and even more generally, when A is a family of operators that generates a periodic evolutionary process). This makes the part of spectrum on the unit circle more complicated and the nature of the problem is not of finite dimension. Finally, we give a discussion on how the obtained results could be extended to the case when A may depend on time t periodically. In this case without the variation-of-constants in the phase space the main results are still true though their proofs will be adjusted.

2. Preliminaries

2.1. **Notation.** Throughout the paper we will use the following notations: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ stand for the sets of natural, integer, real, complex numbers, respectively. Γ denotes the unit circle in the complex plane $\mathbb{C}$. For any complex number z the notation $\Re z$ stands for its real part. $\mathbb{X}$ will denote a given complex Banach space. Given two Banach spaces $\mathbb{X}, \mathbb{Y}$ by $L(\mathbb{X}, \mathbb{Y})$ we will denote the space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. As usual, $\sigma(T), \rho(T), R(\lambda, T)$ are the notations of the spectrum, resolvent set and resolvent of the operator T. The notations $BC(\mathbb{R}, \mathbb{X}), BUC(\mathbb{R}, \mathbb{X}), AP(\mathbb{X})$ will stand for the spaces of all $\mathbb{X}$-valued bounded continuous, bounded uniformly continuous functions on $\mathbb{R}$ and its subspace of almost periodic (in Bohr's sense) functions, respectively.

2.2. **Circular Spectrum of Functions.** Below we will introduce a transform of a function $g \in L^\infty(\mathbb{R}, \mathbb{X})$ on the real line that leads to a concept of spectrum of a function. This spectrum coincides with the set of $\overline{e^{isp(g)}}$ if in addition g is uniformly continuous, where $sp(g)$ denotes the Beurling spectrum of g. All results mentioned below on the circular spectrum of a function could be found in [14].

Let $g \in L^\infty(\mathbb{R}, \mathbb{X})$. Consider the complex function $\mathcal{S}g(\lambda)$ in $\lambda \in \mathbb{C}\backslash\Gamma$ defined as

$$\mathcal{S}g(\lambda) := R(\lambda, S)g, \quad \lambda \in \mathbb{C}\backslash\Gamma. \tag{2.1}$$

Since S is a translation, this transform is an analytic function in $\lambda \in \mathbb{C}\backslash\Gamma$.

Definition 2.1. The *circular spectrum* of $g \in L^\infty(\mathbb{R}, \mathbb{X})$ is defined to be the set of all $\xi_0 \in \Gamma$ such that $\mathcal{S}g(\lambda)$ has no analytic extension into any neighborhood of ξ_0 in the complex plane. This spectrum of g is denoted by $\sigma(g)$ and will be called for short *the spectrum of* g if this does not cause any confusion. We will denote by $\rho(g)$ the set $\Gamma\backslash\sigma(g)$.

Proposition 2.2. Let $\{g_n\}_{n=1}^\infty \subset L^\infty(\mathbb{R}, \mathbb{X})$ such that $g_n \to g \in L^\infty(\mathbb{R}, \mathbb{X})$, and let Λ be a closed subset of the unit circle. Then the following assertions hold:

- i) $\sigma(g)$ is closed.
- ii) If $\sigma(g_n) \subset \Lambda$ for all $n \in \mathbb{N}$, then $\sigma(g) \subset \Lambda$.
- iii) $\sigma(\mathcal{A}g) \subset \sigma(g)$ for every bounded linear operator $\mathcal{A}$ acting in $BUC(\mathbb{R}, \mathbb{X})$ that commutes with S.
- iv) If $\sigma(g) = \emptyset$, then $g = 0$.

Proof. For i), ii) and iv) the proofs are given in [14]. For iii) the proof is obvious from the definition of the circular spectrum. □

Corollary 2.3. Let Λ be a closed subset of the unit circle and $\mathcal{F}$ be one of the function spaces $BUC(\mathbb{R}, \mathbb{X}), AP(\mathbb{X})$. Then, the set

$$\Lambda_{\mathcal{F}}(\mathbb{X}) := \{g \in \mathcal{F} | \ \sigma(g) \subset \Lambda\} \tag{2.2}$$

is a closed subspace of $\mathcal{F}$.

Lemma 2.4. *Let Λ be a closed subset of the unit circle and $\mathcal{F}$ be one of the function spaces $BUC(\mathbb{R},\mathbb{X}), AP(\mathbb{X})$. Then, the translation operator S leaves the space $\Lambda_{\mathcal{F}}(\mathbb{X})$ invariant. Moreover,*

$$\sigma(S|_{\Lambda_{\mathcal{F}}(\mathbb{X})}) = \Lambda. \tag{2.3}$$

Below we will recall the concept of Beurling spectrum of a function. We denote by F the Fourier transform, i.e.

$$(Ff)(s) := \int_{-\infty}^{+\infty} e^{-ist} f(t)dt \tag{2.4}$$

($s \in \mathbb{R}, f \in L^1(\mathbb{R})$). Then the *Beurling spectrum* of $u \in BUC(\mathbb{R},\mathbb{X})$ is defined to be the following set

$$\begin{aligned} sp(u) \ := \ \{\xi \in \mathbb{R} : \forall \epsilon > 0 \ \ \exists f \in L^1(\mathbb{R}), \\ suppFf \subset (\xi - \epsilon, \xi + \epsilon), f * u \neq 0\} \end{aligned}$$

where

$$f * u(s) := \int_{-\infty}^{+\infty} f(s-t)u(t)dt.$$

The following result is a consequence of the Weak Spectral Mapping Theorem that relates the circular spectrum and Beurling spectrum of a uniformly continuous function.

Corollary 2.5. Let $g \in BUC(\mathbb{R},\mathbb{X})$. Then

$$\sigma(g) = \overline{e^{isp(g)}}. \tag{2.5}$$

2.3. **Almost periodic functions.** A subset $E \subset \mathbb{R}$ is said to be *relatively dense* if there exists a number $l > 0$ (*inclusion length*) such that every interval $[a, a + l]$ contains at least one point of E. Let f be a continuous function on $\mathbb{R}$ taking values in a complex Banach space $\mathbb{X}$. f is said to be *almost periodic in the sense of Bohr* if to every $\epsilon > 0$ there corresponds a relatively dense set $T(\epsilon, f)$ (*of ϵ-periods*) such that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| \leq \epsilon, \ \forall \tau \in T(\epsilon, f).$$

If f is almost periodic function, then (approximation theorem [9, Chap. 2]) it can be approximated uniformly on $\mathbb{R}$ by a sequence of trigonometric polynomials, i.e., a sequence of functions in $t \in \mathbb{R}$ of the form

$$P_n(t) := \sum_{k=1}^{N(n)} a_{n,k} e^{i\lambda_{n,k} t}, \quad n = 1, 2, ...; \lambda_{n,k} \in \mathbb{R}, a_{n,k} \in \mathbb{X}, t \in \mathbb{R}. \tag{2.6}$$

Of course, every function which can be approximated by a sequence of trigonometric polynomials is almost periodic. Specifically, the exponents of the trigonometric polynomials (i.e., the reals $\lambda_{n,k}$ in

(2.6)) can be chosen from the set of all reals λ (*Fourier exponents*) such that the following integrals (*Fourier coefficients*)

$$a(\lambda, f) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}dt$$

are different from 0. As is known, there are at most countably such reals λ, the set of which will be denoted by $\sigma_b(f)$ and called *Bohr spectrum* of f. Throughout the paper we will use the relation $sp(f) = \overline{\sigma_b(f)}$.

If $g \in BUC(\mathbb{R}, \mathbb{X})$ with countable $\sigma(g)$, then its Beurling spectrum $sp(g)$ is also countable by Corollary 2.5. Therefore, if $\mathbb{X}$ does not contain any space isomorphic to c_0 (the space of all numerical sequences converging to zero), the function g is almost periodic (see e.g. [9]). If $\mathbb{X}$ is convex it does not contain c_0.

2.4. **Evolutionary processes and the associated evolution semigroups.**

Definition 2.6. Let $(U(t,s))_{t\geq s}$ be a two-parameter family of bounded operators in a Banach space $\mathbb{X}$. Then, it is called an evolutionary process if

i) $U(t,t) = I$ for all $t \in \mathbb{R}$,
ii) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,
iii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbb{X}$,
iv) $\|U(t,s)\| < Ne^{\omega(t-s)}$ for some positive N, ω independent of $t \geq s$.

An evolutionary process is called *1-periodic* if

$$U(t+1, s+1) = U(t,s), \text{ for all } t \geq s.$$

Recall that for a given 1-periodic evolutionary process $(U(t,s))_{t\geq s}$ the following operator

$$M(t) := U(t, t-1), t \in \mathbb{R}$$

is called *monodromy operator* (or sometime *period map, Poincaré map*). Thus we have a family of monodromy operators. We will denote $M := M(0)$. The nonzero eigenvalues of $M(t)$ are called *characteristic multipliers*. An important property of monodromy operators is stated in the following lemma whose proof can be found in [7, 8].

Lemma 2.7. *Under the notation as above the following assertions hold:*

i) $M(t+1) = M(t)$ *for all* t*; characteristic multipliers are independent of time, i.e. the nonzero eigenvalues of* $M(t)$ *coincide with those of* M,
ii) $\sigma(M(t))\backslash\{0\} = \sigma(M)\backslash\{0\}$, *i.e., it is independent of* t,
iii) *If* $\lambda \in \rho(M)$, *then the resolvent* $R(\lambda, M(t))$ *is strongly continuous,*
iv) *If* $\mathcal{M}$ *denotes the operator of multiplication by* $M(t)$ *in any one of the function spaces* $BUC(\mathbb{R}, \mathbb{X})$ *or* $AP(\mathbb{X})$, *then*

$$\sigma(\mathcal{M})\backslash\{0\} \subset \sigma(M)\backslash\{0\}. \tag{2.7}$$

Given an evolutionary process $(U(t,s))_{t\geq s}$, the following semigroup $(T^h)_{h\geq 0}$ is called its associated evolution semigroup

$$T^h g := U(t,t-h)g(t-h),\ t\in\mathbb{R}, g\in BUC(\mathbb{R},\mathbb{X}). \tag{2.8}$$

In general, the evolution semigroup associated with a 1-periodic evolutionary process may not be strongly continuous in the whole space $BUC(\mathbb{R},\mathbb{X})$, but in a closed subspace F that includes all elements of $AP(\mathbb{X})$ and mild solutions in the above sense (see e.g. [1], [18]). To describe the evolution semigroup associated with a given $(U(t,s))_{t\geq s}$ we consider the following integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi, \text{ for all } \ t\geq s, \tag{2.9}$$

where f is an element of $BUC(\mathbb{R},\mathbb{X})$. We recall the following linear operator $\mathcal{L} : D(\mathcal{L}) \subset BUC(\mathbb{R},\mathbb{X}) \to BUC(\mathbb{R},\mathbb{X})$, where $D(\mathcal{L})$ consists of all solutions of Eq.(2.9) $u(\cdot)\in BUC(\mathbb{R},\mathbb{X})$ with some $f\in BUC(\mathbb{R},\mathbb{X})$. If $u\in D(\mathcal{L})$, then we define $\mathcal{L}u(\cdot) := f$. This operator $\mathcal{L}$ is well defined as a singled-valued operator and is obviously an extension of the differential operator $d/dt - A$ (see e.g. [16]). Below, by abuse of notation, we will use the same notation $\mathcal{L}$ to designate its restriction to closed subspaces of $BUC(\mathbb{R},\mathbb{X})$ if this does not make any confusion.

If $(T(t))_{t\geq 0}$ is a C_0-semigroup in a Banach space $\mathbb{X}$, then $U(t,s) := T(t-s)$ determines a 1-periodic evolutionary process.

2.5. **Mild solutions of Eq.(1.1) and a variation of constants formula.**

Definition 2.8. A continuous function $u(\cdot)$ on $\mathbb{R}$ is said to be a mild solution on $\mathbb{R}$ of Eq.(1.1) with initial $\phi\in C_r$, and is denoted by $u(\cdot,s,\phi,f)$ if $u_s=\phi$ and for all $t>s$

$$u(t) = T(t-s)\phi(0) + \int_s^t T(t-\xi)[F(\xi)u_\xi + f(\xi)]d\xi. \tag{2.10}$$

A function $u\in BC(\mathbb{R},\mathbb{X})$ is said to be a mild solution of (1.1) on $\mathbb{R}$ if

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\xi)[F(\xi)u_\xi + f(\xi)]d\xi, \text{ for all } t\geq s. \tag{2.11}$$

Below we will denote by $\mathcal{F}$ the operator acting on $BUC(\mathbb{R},\mathbb{X})$ defined by the formula

$$\mathcal{F}u(\xi) := F(\xi)u_\xi,\ \forall u\in BUC(\mathbb{R},\mathbb{X}).$$

The following results can be verified directly following the lines in [1, 12, 18].

Lemma 2.9. *Let $(T^h)_{h\geq 0}$ be the evolution semigroup associated with a given strongly continuous semigroup $(T(t))_{t\geq s}$ and $\mathcal{S}$ denote the space of all elements of $BUC(\mathbb{R},\mathbb{X})$ at which $(T^h)_{h\geq 0}$ is strongly continuous. Then the following assertions hold true:*

i) *Every mild solution $u\in BUC(\mathbb{R},\mathbb{X})$ of Eq.(1.1) is an element of $\mathcal{S}$,*
ii) *$AP(\mathbb{X})\subset\mathcal{S}$,*

iii) *For the infinitesimal generator $\mathcal{G}$ of $(T^h)_{h\geq 0}$ in the space $\mathcal{S}$ one has the relation: $\mathcal{G}g = -\mathcal{L}g$ if $g \in D(\mathcal{G})$.*

For bounded uniformly continuous mild solutions $x(\cdot)$ the following characterization is very useful:

Theorem 2.10. *$x(\cdot)$ is a bounded uniformly continuous mild solution of Eq.(1.1) if and only if $\mathcal{L}x(\cdot) = \mathcal{F}x(\cdot) + f$.*

As is well known, the homogeneous equation associated with (1.1) generates an evolutionary process $(U(t,s))_{t\geq s}$ in the space $C_r = C([-r,0],\mathbb{X})$. In fact,

$$U(t,s) : C_r \ni \phi \mapsto u_t \in C_r, \tag{2.12}$$

where u is the solution of the equation

$$\begin{aligned} u(\tau) &= T(\tau - s)\phi(0) + \int_s^\tau T(\tau-\xi)F(\xi)u_\xi d\xi,\ \tau \geq s, \\ u_s &= \phi. \end{aligned}$$

We introduce a function Γ^n defined by

$$\Gamma^n(\theta) = \begin{cases} (n\theta + 1)I, & -1/n \leq \theta \leq 0 \\ \\ 0, & \theta < -1/n, \end{cases}$$

where n is any positive integer and I is the identity operator on $\mathbb{X}$. Since the evolutionary process $(U(t,s))_{t\geq s}$ is strongly continuous, the C_r-valued function $U(t,s)\Gamma^n f(s)$ is continuous in $s \in (-\infty, t]$ whenever $f \in \mathrm{BC}(\mathbb{R},\mathbb{X})$.

The following theorem, whose proof could be found in [17], is a variation of constant formula for solutions of (1.1) in the phase space C_r:

Theorem 2.11. *The segment $u_t(s,\phi;f)$ of solution $u(\cdot,s,\phi,f)$ of (1.1) satisfies the following relation in C_r:*

$$u_t(s,\phi;f) = U(t,s)\phi + \lim_{n\to\infty}\int_s^t U(t,\xi)\Gamma^n f(\xi)d\xi, \qquad t \geq s. \tag{2.13}$$

Moreover, the above limit exists uniformly for bounded $|t-s|$.

3. Existence of almost periodic solutions of Eq.(1.1)

The result below is an upper estimate of the spectrum of a mild solution to (1.1) that is a key to understand the behavior of a bounded and uniformly continuous mild solution of (1.1).

Lemma 3.1. *Let u be a bounded and uniformly continuous mild solution of the equation (1.1). Then, the following estimate holds*

$$\sigma(u) \subset \sigma_\Gamma(M) \cup \sigma(f). \tag{3.1}$$

where $\sigma_\Gamma(M) := \{z \in \mathbb{C} : \ |z| = 1, z \in \sigma(M)\}$.

Proof. By the formula (2.13)

$$u_t \ = \ U(t,t-1)u_{t-1} + \lim_{n\to\infty} \int_{t-1}^{t} U(t,s)\Gamma^n f(s)ds, \tag{3.2}$$

and the limit exists uniformly for all bounded t. First, as f is uniformly continuous and bounded we can see that the function

$$A : \mathbb{R} \ni t \mapsto \lim_{n\to\infty} \int_{t-1}^{t} U(t,s)\Gamma^n f(s)ds \in C_r \tag{3.3}$$

is also bounded and uniformly continuous. We can check easily the valadity of the identity

$$\lambda R(\lambda, S)S(-1) = R(\lambda, S) + S(-1),$$

for any $|\lambda| \neq 1$, where $S(t)$ stands for the translation group, and $S := S(1)$. Note that the operator $\mathcal{M}$ of multiplication by $M(t)$ commutes with S since the evolutionary process $(U(t,s))_{t\geq s}$ is 1-periodic. Below we will denote by ω the function $\mathbb{R} \ni t \mapsto u_t \in C_r$. Then, from the identity (3.2) one has (for all $\lambda \neq 0$ and $|\lambda| \neq 1$)

$$\lambda R(\lambda, S)\omega \ = \ \lambda R(\lambda, S)\mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A.$$

Therefore,

$$\begin{aligned} \lambda R(\lambda, S)\omega - \mathcal{M}R(\lambda, S)\omega \ &= \ \mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A, \\ (\lambda - \mathcal{M})R(\lambda, S)\omega \ &= \ \mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A. \end{aligned}$$

As shown in [14, Lemma 5.3] for each fixed $n \in \mathbb{N}$

$$\sigma(G_n f) \subset \sigma(f),$$

where

$$G_n f(t) := \int_{t-1}^{t} U(t,s)\Gamma^n f(s)ds.$$

As the limit in the formula (2.13) is uniform in t we can see that $\sigma(A) \subset \sigma(f)$. Finally, if $\lambda_0 \notin (\sigma_\Gamma(M) \cup \sigma(f))$, then near λ_0 the following holds

$$R(\lambda, S)\omega = R(\lambda, \mathcal{M})(\mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A). \tag{3.4}$$

This shows that the complex function $R(\lambda, S)\omega$ is defined as an analytic function in a neighborhood of λ_0.

We will show further that this yields that the function $R(\lambda, S)\omega(0)$ is also defined and analytic in a neighborhood of λ_0. In fact, before we proceed that we introduce $p : C_r \to \mathbb{X}$ defined as

$p(w) := w(0)$. If so, with our above notations $p\circ\omega = u$, and $p\circ S^k\omega = S^k u$ for all $k\in\mathbb{N}$. If $|\lambda|>1$ we have

$$\begin{aligned} p\circ R(\lambda,S)\omega &= \lambda^{-1}p\circ(I-S/\lambda)^{-1}\omega \\ &= \lambda^{-1}p\circ\left(\sum_{k=0}^{\infty}S^k/\lambda^k\right)\omega \\ &= \lambda^{-1}\left(\sum_{k=0}^{\infty}S^k/\lambda^k\right)u \\ &= R(\lambda,S)u. \end{aligned}$$

Note that for simplicity we make an abuse of notation by denoting also by S the translation in the function space $BUC(\mathbb{R},\mathbb{X})$ as well as in $BUC(\mathbb{R},C_r)$. Similarly, for $\lambda\neq 0$ and $|\lambda|<1$ we can show that $p\circ R(\lambda,S)\omega = R(\lambda,S)u$. Hence, the transform $R(\lambda,S)u$ of the function u has $p\circ R(\lambda,S)\omega$ as an analytic extension in a neighborhood of λ_0. This shows that (3.1) holds true, finishing the proof of the lemma. □

Next, we recall some concepts and results in [20]. Note that although the proofs could be found in [20] we would like to give some new ones that seem to be simpler and would be more convenient to the reader.

Let us consider the subspace $\mathcal{N}\subset BUC(\mathbb{R},\mathbb{X})$ (or $AP(\mathbb{X})$, respectively) consisting of all functions $v\in BUC(\mathbb{R},\mathbb{X})$ (or $AP(\mathbb{X})$, respectively) such that

$$\sigma(v)\subset S_1\cup S_2\ , \tag{3.5}$$

where S_1, S_2 are disjoint closed subsets of the unit circle Γ.

Lemma 3.2. *Under the above notations and assumptions the function space $\mathcal{N}$ can be split into a direct sum $\mathcal{N}=\mathcal{N}_1\oplus\mathcal{N}_2$ such that $v\in\mathcal{N}_i$ if and only if $\sigma(v)\subset S_i$ for $i=1,2$. Moreover, any bounded linear operator in $BUC(\mathbb{R},\mathbb{X})$ (or $AP(\mathbb{X})$, respectively), that commutes with the translation S, leaves invariant $\mathcal{N}$ as well as $\mathcal{N}_j$, $j=1,2$.*

Proof. By Lemma 2.4 and the Riezs spectral projection the space $\mathcal{N}$ could be split into the direct sum $\mathcal{N}=\mathcal{N}_1\oplus\mathcal{N}_2$ with $\mathcal{N}_1$ is the image of the projection

$$P := \frac{1}{2i\pi}\int_\gamma R(\lambda, S|_{\mathcal{N}})d\lambda\ ,$$

where γ is a positively oriented contour enclosing S_1 and disjoint from S_2. We have

$$\sigma(S|_{\mathcal{N}_1})\subset S_1;\ \sigma(S|_{\mathcal{N}_2})\subset S_2.$$

Therefore, if $v\in\mathcal{N}_i$, $(i=1,2)$ by the definition of the circular spectrum it is easy to see that

$$\sigma(v)\subset\sigma(S|_{\mathcal{N}_i})\subset S_i.$$

The second claim is obvious as any bounded linear operator in $BUC(\mathbb{R},\mathbb{X})$ (or $AP(\mathbb{X})$, respectively) that commutes with S must commute with P, so it leaves the spaces $\mathcal{N},\mathcal{N}_1,\mathcal{N}_2$ invariant. □

Theorem 3.3. *(Decomposition Theorem) Let the following condition be satisfied*

i) *Eq.(1.1) has a mild solution* $u \in BUC(\mathbb{R},\mathbb{X})$ *(or in* $AP(\mathbb{X})$*, respectively)*

ii)

$$\sigma_\Gamma(M) \setminus \sigma(f) \text{ be closed.} \tag{3.6}$$

Then there exists a mild solution w *of Eq.(1.1) in* $BUC(\mathbb{R},\mathbb{X})$ *(or* $AP(\mathbb{X})$*, respectively) such that*

$$\sigma(w) \subset \sigma(f), \tag{3.7}$$

that is unique if

$$\sigma_\Gamma(M) \cap \sigma(f) = \emptyset. \tag{3.8}$$

Proof. By Lemma 3.1

$$\sigma(u) \subset \sigma_\Gamma(M) \cup \sigma(f). \tag{3.9}$$

Let us denote by Λ the set $\sigma_\Gamma(M)\cup\sigma(f)$, S_1 the set $\sigma(f)$ and S_2 the set $\sigma_\Gamma(M) \setminus \sigma(f)$, respectively. Thus, these two sets are closed and disjoint subsets of the unit circle Γ, so by Lemma 3.2 there exists the projection P from $\mathcal{N}$ onto $\mathcal{N}_1$ which is commutative with $\mathcal{F}$ and T^h. Since u is a mild solution of (1.1) if and only if $u \in D(\mathcal{L})$ and

$$\mathcal{L}u = \mathcal{F}u + f, \tag{3.10}$$

by Lemma 2.9 we have

$$\mathcal{L}u = -\mathcal{G}u,$$

so this yields

$$\begin{aligned} P\mathcal{L}u &= -P\mathcal{G}u \\ &= -P \lim_{h\to 0^+} \frac{T^h u - u}{h} \\ &= -\lim_{h\to 0^+} P\frac{T^h u - u}{h} \\ &= -\lim_{h\to 0^+} \frac{T^h Pu - Pu}{h} \\ &= -\mathcal{G}Pu \\ &= \mathcal{L}Pu. \end{aligned}$$

Since $Pf = f$ and P commutes with $\mathcal{F}$,

$$\begin{aligned} P\mathcal{L}u &= P\mathcal{F}u + Pf \\ \mathcal{L}Pu &= \mathcal{F}Pu + f. \end{aligned}$$

By Theorem 2.10 this shows $w := Pu \in \mathcal{N}_1$ is a mild solution of Eq. (1.1) that has circular spectrum $\sigma(Pu) \subset S_1 = \sigma(f)$. Next, if condition (3.8) holds, then the uniqueness of such a solution in $\mathcal{N}_1$ is clear. In fact, suppose that there is another mild solution $v \in BUC(\mathbb{R}, \mathbb{X})$ (or in $AP(\mathbb{X})$, respectively) to Eq.(1.1) such that $\sigma(v) \subset \sigma(f)$, then $w - v$ is a mild solution of the homogeneous equation corresponding to Eq.(1.1), so $\sigma(w - v) \subset \sigma_\Gamma(M)$. As $\sigma(v) \subset \sigma(f)$, by (3.8) this yields that $\sigma(w - v) = \emptyset$, and because of this $w - v = 0$. This completes the proof of the theorem. □

Recall that the set of all real numerical sequences that are convergent to zero is a Banach space with sup-norm that is denoted by c_0. As a consequence of the above theorem we obtain the following main result of the paper.

Theorem 3.4. *Assume that Eq. (1.1) has a bounded uniformly continuous mild solution u, and Condition (3.6) of Theorem 3.3 is satisfied. Moreover, let the space $\mathbb{X}$ not contain c_0 and $\sigma(f)$ be countable. Then there exists an almost periodic mild solution w to Eq.(1.1) such that $\sigma(w) \subset \sigma(f)$. Furthermore, if (3.8) holds, then such a solution w is unique.*

Proof. The proof is obvious in view of [9, Theorem 4, p.92] and Theorem 3.3. □

Below we will relax the condition on the existence of a bounded uniformly continuous mild solutions when a condition (3.8) is satisfied.

Theorem 3.5. *Under the above notation assume that*

$$\sigma_\Gamma(M) \cap \sigma(f) = \emptyset \tag{3.11}$$

holds. Then there exists a unique almost periodic mild solution w to Eq. (1.1) such that $\sigma(w) \subset \sigma(f)$.

Proof. Consider the difference equation

$$w(t) = M(t)w(t-1) + g(t), \ t \in \mathbb{R}, \tag{3.12}$$

where for all $t \in \mathbb{R}$

$$\begin{aligned} M(t) &:= U(t, t-1), \\ g(t) &:= \lim_{n\to\infty} \int_{t-1}^{t} U(t,s)\Gamma^n f(s) ds. \end{aligned}$$

First, we note that g is almost periodic function taking values in C_r. In fact, for each $n \in \mathbb{N}$ the function

$$F_n : \mathbb{R} \ni t \mapsto \Gamma^n f(t) \in C_r$$

is an almost periodic function with $\sigma(F_n) \subset \sigma(f)$. Next, by [14, Lemma 5.3] the function

$$F : \mathbb{R} \ni t \mapsto \int_{t-1}^{t} U(t,\xi) F_n(\xi) d\xi$$

is also almost periodic, and $\sigma(F) \subset \sigma(F_n) \subset \sigma(f)$. Therefore, g is almost periodic and $\sigma(g) \subset \sigma(f)$.

By [14, Theorem 4.7] if (3.11) holds there exists a unique almost periodic solution w to (3.12) such that $\sigma(w) \subset \sigma(f)$. Our next goal is to prove that there exists a mild solution u of Eq. (1.1) such that $u_n = w(n)$ for all $n \in \mathbb{Z}$. For each fixed $n \in \mathbb{Z}$ consider the unique mild solution to Eq. (1.1) on the interval $[n, n+1]$ that is generated by the equation

$$\begin{aligned} u(t) &= T(t-n)[w(n)](0) + \int_n^t T(t-\eta)[F(\eta)u_\eta + f(\eta)]d\eta, \ t \in [n, n+1], \\ u_n &= w(n). \end{aligned}$$

This solution exists uniquely on the interval $[n, n+1]$ for each $n \in \mathbb{Z}$. By the Variation-of-Constants formula (2.13)

$$u_t = U(t,n)w(n) + \lim_{m\to\infty} \int_n^t U(t,s)\Gamma^m f(s)ds, \qquad t \geq n. \tag{3.13}$$

Therefore, if $t = n+1$ we have that $u_{n+1} = w(n+1)$. This means that we obtain a mild solution u of Eq. (1.1) that is defined on each interval $[n, n+1]$ by (3.13) so that it coincides with w at each integer n. Therefore, the sequence $w(n) = u_n$ is almost periodic. This yields that $u(n) = u_n(0)$ is an almost periodic sequence. We are going to prove that u is almost periodic function. The proof will follow a well known idea in [4] that are used in [1, 5] as well. For the completeness we present it below.

As $w(\cdot)$ and f are almost periodic, so is the function $g : \mathbf{R} \ni t \mapsto (w(t), f(t)) \in C \times \mathbf{X}$ (see [9, p.6]). As is known, the sequence $\{g(n)\} = \{(w(n), f(n))\}$ is almost periodic. Hence, for every positive ϵ the following set is relatively dense (see [4, p. 163-164])

$$T := \mathbf{Z} \cap T(g, \epsilon),$$

where $T(g,\epsilon) := \{\tau \in \mathbf{R} : \sup_{t\in\mathbf{R}} \|g(t+\tau) - g(t)\| < \epsilon\}$, i.e., the set of ϵ periods of g. Hence, for every $m \in T$ we have

$$\begin{aligned} \|f(t+m) - f(t)\| &< \epsilon, \forall t \in \mathbf{R}, \\ \|w(n+m) - w(n)\| &< \epsilon, \forall n \in \mathbf{Z}. \end{aligned}$$

Since u is a solution to Eq.(2.10), for $0 \leq s < 1$ and all $n \in \mathbf{N}$, we have

$$\begin{aligned} \|u(n+m+s) - u(n+s)\| &\leq \|T(s)\| \cdot \|w(n+m) - w(n)\| \\ &\quad + \int_0^s \|T(s-\xi)\| \big[\sup_t \|F(t)\| \cdot \|u_{n+m+\xi} - u_{n+\xi}\| \\ &\quad + \|f(n+m+\xi) - f(n+\xi)\|\big] d\xi \\ &\leq Ne^\omega \|w(n+m) - w(n)\| + Ne^\omega \int_0^s \big[\|F\| \\ &\qquad \times \|u_{n+m+\xi} - u_{n+\xi}\| + \|f(n+m+\xi) - f(n+\xi)\|\big] d\xi. \end{aligned}$$

Hence

$$\|x_{n+m+s} - x_{n+s}\| \leq Ne^{\omega}\|w(n+m) - w(n)\|$$
$$+Ne^{\omega}\int_0^s \big[\|F\|\cdot\|x_{n+m+\xi} - x_{n+\xi}\| + \|f(n+m+\xi) - f(n+\xi)\|\big]d\xi.$$

Using the Gronwall inequality we can show that

$$\|u_{n+m+s} - u_{n+s}\| \leq \epsilon M, \tag{3.14}$$

where M is a constant which depends only on $\sup_t\|F(t)\|, N, \omega$. This shows that m is a ϵM-period of the function $x(\cdot)$. Finally, since T is relatively dense for every ϵ, we see that $x(\cdot)$ is an almost periodic mild solution of Eq.(1.1). Once the almost periodicity of u was proved we are are able to apply the Decomposition Theorem 3.3 to finish the proof of this theorem. □

4. Discussion: Variation-of-constant formula in the phase space and further extension

Our results in the previous section could be extended to a bit more general case of periodic equations. Namely, let us consider equations of the form

$$\frac{du}{dt} = A(t)u + F(t)u_t + f(t),\ t \in \mathbb{R}, \tag{4.1}$$

where the family of (possibly unbounded) operators $A(t)$ generates a 1-periodic evolutionary process and $F(t)$ is a 1-periodic family of bounded operators as in (1.1), and f is an almost periodic function taking values in $\mathbb{X}$.

The presentation of our proofs of the results in the previous section relies on the variation-of-constants formula (2.13) in the phase space C_r that allows us to easily outline the ideas. In turn, we have made use of the formula available in the case when $A(t)$ is independent of t although our results could be true even if $A(t)$ may depend on t periodically with the same period as that of $F(t)$.

As shown in [5, Lemma 4.1], there is a way to get around with the variation-of-constant formula (2.13). Below is a version of Lemma 4.1 from [5] that could be used to extend our results in the previous section to the general case of equations (4.1). We consider the following Cauchy Problem for each given $t \in \mathbb{R}$

$$\begin{aligned} y(\xi) &= \int_{t-1}^{\xi} V(\xi,\eta)[F(\eta)y_\eta + f(\eta)]d\eta,\ \xi \geq t-1, \\ y_{t-1} &= 0 \in C_r, \end{aligned}$$

where $(V(t,s))_{t\geq s}$ is a 1-periodic evolutionary process generated by the homogeneous equation

$$\frac{du}{dt} = A(t)u,$$

Let us define $v : \mathbb{R} \ni t \mapsto y_t \in C_r$. We define the operator $L : BUC(\mathbb{R},\mathbb{X}) \ni f \mapsto v$.

Lemma 4.1. *The operator L is well defined operator in $BUC(\mathbb{R}, \mathbb{X})$ that is linear and continuous and commutes with the translation S.*

Proof. Since the proof could be easily adapted from that of [5, Lemma 4.1] details will be omitted. □

From the definition of the function v we can verify that if u is a mild solution of (1.1) on the real line, then

$$u_t = U(t, t-1)u_{t-1} + v(t), \ t \in \mathbb{R}.$$

Therefore, the circular spectrum of u could be estimated as below

Lemma 4.2.

$$\sigma(u) \subset \sigma_\Gamma(M) \cup \sigma(f).$$

Proof. Since L is linear, bounded and commutes with S we have $\sigma(v) = \sigma(Lf) \subset \sigma(f)$. The rest of the proof is similar to that of Lemma 3.1. □

All main results of the previous section, Theorems 3.3, 3.4 and 3.5 will follow if we adjust the technique of decomposition as discussed in [20] to periodic evolutionary processes.

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Liouville type theorem for nonlinear elliptic system involving Grushin operator

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ABSTRACT

We study the degenerate elliptic system of the form

$$\begin{cases} -\Delta_G u = v^p \\ -\Delta_G v = u^q \end{cases} \quad \text{on } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $\Delta_G := \Delta_x + |x|^{2\alpha}\Delta_y$ is the Grushin operator, $\alpha \geq 0$ and $p \geq q > 1$. We establish some Liouville type results for stable solutions of the system. In particular, we prove the comparison principle – a crucial step to establish such results. As consequences, we obtain a Liouville type theorem for the scalar equation and provide a counterpart of the previous result in C. Cowan (2013) [7].

1. Introduction

In this paper, we study the Liouville type theorem for stable classical solutions of the semilinear degenerate elliptic system

$$\begin{cases} -\Delta_G u = v^p \\ -\Delta_G v = u^q \end{cases} \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \tag{1.1}$$

where $\Delta_G u = \Delta_x u + |x|^{2\alpha}\Delta_y u$ is the Grushin operator, Δ_x and Δ_y are Laplace operators with respect to $x \in \mathbb{R}^{N_1}$ and $y \in \mathbb{R}^{N_2}$. Here we always assume that $\alpha \geq 0$ and $p \geq q > 1$. Recall that G_α is elliptic for $|x| \neq 0$ and degenerates on the manifold $\{0\} \times \mathbb{R}^{N_2}$. This operator was introduced in [16] (see also Baouendi [1])

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and has attracted the attention of many mathematicians. In the special case $\alpha = 1$, this operator is close related to the Heisenberg Laplacian in $H^n = \mathbb{C}^n \times \mathbb{R}$ (see e.g., [3,4]).

We start by noting that, in the case $\alpha = 0$, the system (1.1) is reduced to the Lane–Emden system

$$\begin{cases} -\Delta u = v^p \\ -\Delta v = u^q \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which has received considerably attention in the last decade (see e.g., [22,27,26,29,7,13] and references therein). For this system, the so-called Lane–Emden conjecture says that there has no positive classical solution if and only if the pair (p, q) lies below the Sobolev critical hyperbola, i.e.,

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}.$$

So far, this conjecture has been proved for the case $N \le 4$, see e.g., [22,27,26,29]. When $N \ge 5$, there have been some partial results concerning the nonexistence of positive classical solution, see [29,5,21]. For the class of positive radial solutions, the Lane–Emden conjecture was solved by Mitidieri [22], Serrin and Zou [28].

Recently, the Liouville type theorem for a special class of solutions – the so-called stable solutions – has been studied by many mathematicians, see [12,30,9,11] for Lane–Emden equation and [7,13,18,19] for Lane–Emden system. In particular, Cowan [7] has obtained the following result.

Theorem A. (Cowan [7])

i) *Suppose* $2 < q \le p$ *and*

$$N - 2 - \frac{4p+4}{pq-1}\left(\sqrt{\frac{pq(q+1)}{p+1}} + \sqrt{\frac{pq(q+1)}{p+1} - \sqrt{\frac{pq(q+1)}{p+1}}}\right) < 0. \tag{1.3}$$

Then there is no positive stable solution of (1.2). *In particular, there is no positive stable solution of* (1.2) *for any* $2 \le q \le p$ *if* $N \le 10$.

ii) *Suppose* $1 < q \le 2$, $\sqrt{\frac{pq(q+1)}{p+1}} - \sqrt{\frac{pq(q+1)}{p+1} - \sqrt{\frac{pq(q+1)}{p+1}}} < \frac{q}{2}$ *and* (1.3). *Then there is no positive stable solution of* (1.2).

The main tools in [7] are comparison principle, integral estimates via stability assumption and boostrap argument. This result was then partially extended in [19,18,17] to the case of weighted Lane–Emden system.

We now turn to the general case $\alpha \ge 0$. It is well known that when $\alpha > 0$, the operator Δ_G belongs to the wide class of subelliptic operators studied by Franchi et al. in [14] (see also [3,4]). Let us recall some related results for the scalar version of (1.1), i.e., for the equation $-\Delta_G u = u^p$. The Liouville type theorem has been recently proved by Monticelli [24] for nonnegative classical solutions, and by Yu [31] for nonnegative weak solutions. The optimal condition on the range of the exponent is $p < \frac{N_\alpha+2}{N_\alpha-2}$, where

$$N_\alpha := N_1 + (1+\alpha)N_2$$

is called the homogeneous dimension. The main tool in [24,31] is the Kelvin transform combined with technique of moving planes. Before that, Dolcetta and Cutrì [10] established the Liouville-type theorem for nonnegative super-solutions under the condition $p \le \frac{N_\alpha}{N_\alpha-2}$ (see also [8]). However, to our best knowledge, there has not any work treating the system (1.1) for the case $\alpha > 0$.

The purpose of this paper is to classify the stable solutions of (1.1). Before stating our main results, let us recall the definition of such solutions motivated by [23], see also [7,13].

Definition. A positive solution $(u,v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ of (1.1) is called stable if there are positive smooth functions ξ, η such that

$$\begin{cases} -\Delta_G \xi = p v^{p-1} \eta \\ -\Delta_G \eta = q u^{q-1} \xi \end{cases} \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

Our first result concerning the classification of stable solutions of (1.1) is the following.

Theorem 1.1.

i) Suppose that $\frac{4}{3} < q \le p$ *and*

$$N_\alpha < 2 + \frac{4p+4}{pq-1}\left(\sqrt{\frac{pq(q+1)}{p+1}} + \sqrt{\frac{pq(q+1)}{p+1} - \sqrt{\frac{pq(q+1)}{p+1}}}\right). \tag{1.5}$$

Then there is no stable positive solution of (1.1). In particular, the assertion is true if $N_\alpha \le 10$.

ii) In the case $1 < q \le \max(\frac{4}{3}, p)$*, in addition to (1.5), we assume that*

$$\sqrt{\frac{pq(q+1)}{p+1}} - \sqrt{\frac{pq(q+1)}{p+1} - \sqrt{\frac{pq(q+1)}{p+1}}} < \frac{q}{2}. \tag{1.6}$$

Then the system (1.1) has also no stable positive solution.

Notice immediately that Theorem A is a direct consequence of Theorem 1.1 when $\alpha = 0$.

In the above theorem, one sees that in the case $1 < q \le \max(\frac{4}{3}, p)$, there is additional assumption (1.6) due to the restriction of technique. Motivated by the idea in [17] concerning the inverse comparison principle, we obtain the second result without assumption (1.6).

Theorem 1.2. *Suppose that* $1 < q \le \max(\frac{4}{3}, p)$ *and*

$$N_\alpha < 2 + \left(2 + \frac{2(q+1)}{pq-1} + \frac{4(2-q)}{p+q-2}\right)\left(\sqrt{\frac{pq(q+1)}{p+1}} + \sqrt{\frac{pq(q+1)}{p+1} - \sqrt{\frac{pq(q+1)}{p+1}}}\right). \tag{1.7}$$

Then there is no bounded stable positive solution of (1.1). In particular, the system (1.1) has no bounded stable positive solution if $N_\alpha \le 2 + 2\left(\sqrt{2} + \sqrt{2 - \sqrt{2}}\right) \simeq 6.359$.

As consequences, let us consider the scalar equation

$$-\Delta_G u = u^p \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \tag{1.8}$$

Recall that, see e.g. [11], a classical solution of (1.8) is called stable if

$$p\int\limits_{\mathbb{R}^N} u^{p-1}\phi^2 dxdy \le \int\limits_{\mathbb{R}^N} |\nabla_G \phi|^2 dxdy, \text{ for all } \phi \in C_c^1(\mathbb{R}^N). \tag{1.9}$$

In particular, it follows from (2.1) when $p = q, u = v$, that (1.9) is a consequence of the notion of stability for the system.

The following is a corollary of Theorems 1.1 and 1.2.

Corollary 1.3. *Assume $p > 1$ and*

$$N_\alpha - 2 - \frac{4}{p-1}(p + \sqrt{p^2 - p}) < 0. \tag{1.10}$$

Then the problem (1.8) has no stable positive solution. In particular, there is no stable positive solution of (1.8) if $N_\alpha \le 10$.

Remark 1.4. In the case $1 < p \le \frac{4}{3}$, the non-existence of bounded stable solutions of (1.8) with $\alpha = 0$ was proved in [17]. This result is improved in our Corollary 1.3 without restriction on the boundedness of solutions.

To prove Theorems 1.1 and 1.2, we borrow crucially the idea of Cowan [7] who established Theorem A. The key in the proof is the comparison principle and nonlinear integral estimates. The former was proved by Souplet [29] (see also Bidaut-Véron [2] for the proof in bounded domain with additional assumption), and it is shown to be very useful to study qualitative properties of solutions of elliptic system, see e.g., [13,7,18,25]. However, the techniques used to prove the comparison principle in the previous works for the Laplace operator do not seem applicable to the system (1.1) because the operator Δ_G is no longer symmetry and it degenerates on the manifold $\{0\} \times \mathbb{R}^{N_2}$. Then, in this paper, we establish the comparison principle for Grushin operators by developing the idea in [6]. In addition, the L^1-estimate to the boostrap iteration in [7] does not work in the case of Grushin operator, we instead switch to the L^2-estimate in the boostrap argument. We also employ the idea in [17] to prove the “inverse” comparison principle which is crucial to handle the case $1 < q \le \frac{4}{3}$. Remark also that the method used in the present paper can be applied to study the weighted systems, and to more general class of degenerate operator, such as the Δ_λ operator (see [15,20]) of the form

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2 \partial_{x_i}), \quad \lambda = (\lambda_1, \ldots, \lambda_N) : \mathbb{R}^N \to \mathbb{R}^N.$$

Here $\lambda_i : \mathbb{R}^N \to \mathbb{R}, i = 1, ..., N$ are nonnegative continuous functions satisfying some properties such that Δ_λ is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^N$.

We finish this section by describing briefly outline of the proof. Suppose that (u, v) is a stable positive solution of (1.1).

Step 1. From the definition, we give a stability criterion under the integral form.

$$\sqrt{pq}\int\limits_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q-1}{2}} \phi^2 dxdy \le \int\limits_{\mathbb{R}^N} |\nabla_G \phi|^2 dxdy, \text{ for all } \phi \in C_c^1(\mathbb{R}^N),$$

where $\nabla_G := (\nabla_x, |x|^\alpha \nabla_y)$ denotes the Grushin gradient.

Step 2. Establish the comparison principle

$$\frac{v^{p+1}}{p+1} \le \frac{u^{q+1}}{q+1}$$

without assumptions on the boundedness and the stability of solutions, and the inverse comparison principle

$$u \leq \|v\|_\infty^{\frac{p-q}{q+1}} v$$

for bounded solutions.

Step 3. Let $\theta = \frac{pq(q+1)}{p+1}$ and $\sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} < t < \sqrt{\theta} + \sqrt{\theta - \sqrt{\theta}}$. Using the comparison principle we prove

$$\int_{\mathbb{R}^N} v^p u^{2t-1} \phi^2 dxdy \leq C \int_{\mathbb{R}^N} u^{2t} \big(|\nabla_G \phi|^2 + |\Delta_G \phi| \phi \big) dxdy.$$

for all $\phi \in C_c^2(\mathbb{R}^N; [0,1])$. Moreover, for bounded solutions in the case $1 < q \leq \max(p; \frac{4}{3})$, we obtain

$$\int_{B_R \times \mathbf{B}_{R^{1+\alpha}}} u^2 dxdy \leq C R^{N_\alpha - 2 - \frac{2(q+1)}{pq-1} - \frac{4(2-q)}{p+q-2}},$$

where B_R (resp. $\mathbf{B}_R$) denotes the ball of radius R centered at the origin of $\mathbb{R}^{N_1}$ (resp. $\mathbb{R}^{N_2}$).

Step 4. We finally use L^2-estimates for Grushin operator and apply the bootstrap iteration to obtain the desired results.

The paper is organized as follows. In Section 2, we establish some technical lemmas. The proof of Theorems 1.1, 1.2 and Corollary 1.3 are given in Section 3.

2. Some technical lemmas

In this section, we shall prove some auxiliary results concerning the system (1.1). We first establish a stability inequality.

Lemma 2.1. *Assume that (u, v) is a positive stable solution of the system (1.1). Then for $\phi, \psi \in C_c^1(\mathbb{R}^N)$, we have*

$$\sqrt{pq} \int_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q-1}{2}} |\phi\psi| dxdy \leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla_G \phi|^2 + |\nabla_G \psi|^2) dxdy.$$

In particular, if $\psi = \phi$ then we have

$$\sqrt{pq} \int_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q-1}{2}} \phi^2 dxdy \leq \int_{\mathbb{R}^N} |\nabla_G \phi|^2 dxdy. \tag{2.1}$$

Proof. We follow the idea in [7,13]. Let $\phi, \psi \in C_c^1(\mathbb{R}^N)$. Multiplying the first equation in (1.4) by $\frac{\phi^2}{\xi}$ we get

$$-\int_{\mathbb{R}^N} \Delta_G \xi . \frac{\phi^2}{\xi} dxdy = \int_{\mathbb{R}^N} p v^{p-1} \eta \frac{\phi^2}{\xi} dxdy.$$

Using the integration by parts and Young's inequality $2ab - b^2 \leq a^2$ to obtain

$$-\int_{\mathbb{R}^N} \Delta_G \xi . \frac{\phi^2}{\xi} dxdy = \int_{\mathbb{R}^N} \left(2\frac{\phi}{\xi} \nabla_G \phi \cdot \nabla_G \xi - |\nabla_G \xi|^2 \frac{\phi^2}{\xi^2} \right) dxdy \leq \int_{\mathbb{R}^N} |\nabla_G \phi|^2 dxdy.$$

Consequently,

$$\int_{\mathbb{R}^N} pv^{p-1}\eta\frac{\phi^2}{\xi}dxdy \le \int_{\mathbb{R}^N} |\nabla_G\phi|^2 dxdy. \tag{2.2}$$

By the same argument, we also have

$$\int_{\mathbb{R}^N} qu^{q-1}\xi\frac{\psi^2}{\eta}dxdy \le \int_{\mathbb{R}^N} |\nabla_G\psi|^2 dxdy. \tag{2.3}$$

Finally, adding (2.2), (2.3) and using again the Young inequality for the left hand side, we obtain the desired result. □

In what follows, the constant C does not depend on a positive parameter R and may change from line to line. The following lemma is more or less known, we provide here the proof because we couldn't find a satisfactory one in the literature.

Lemma 2.2. *Suppose that* (u, v) *is a positive solution of* (1.1) *with* $p \ge q > 1$. *Then, for* $R > 0$ *there exists* $C > 0$ *independent of* R *such that*

$$\int_{B_R\times\mathbf{B}_{R^{1+\alpha}}} v^p dxdy \le CR^{N_\alpha-2-\frac{2(p+1)}{pq-1}}, \tag{2.4}$$

and

$$\int_{B_R\times\mathbf{B}_{R^{1+\alpha}}} u^q dxdy \le CR^{N_\alpha-2-\frac{2(q+1)}{pq-1}}. \tag{2.5}$$

Proof. Let $\chi_j \in C_c^\infty(\mathbb{R};[0,1]), j = 1,2$ such that $\chi_j = 1$ on $[-1,1]$ and $\chi_j = 0$ outside $[-2^{1+(j-1)\alpha}, 2^{1+(j-1)\alpha}]$.

For $R > 0$, put $\varphi_R(x,y) = \chi_1(\frac{|x|}{R})\chi_2(\frac{|y|}{R^{1+\alpha}})$. Then, there exists $C > 0$ independent of R such that

$$|\nabla_x\varphi_R| \le \frac{C}{R}, \quad |\nabla_y\varphi_R| \le \frac{C}{R^{1+\alpha}},$$
$$|\Delta_x\varphi_R| \le \frac{C}{R^2}, \quad |\Delta_y\varphi_R| \le \frac{C}{R^{2(1+\alpha)}}.$$

Let $m \ge 2$ be a fixed constant which will be chosen sufficiently large later on. Multiplying the first equation in (1.1) by φ_R^m and integrating over $B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}$ to arrive at

$$\int_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} -\Delta_G u\varphi_R^m dxdy = \int_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} v^p\varphi_R^m dxdy. \tag{2.6}$$

On the other hand, using the integration by parts and Hölder's inequality we get

$$\begin{aligned}\int_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} -\Delta_G u\varphi_R^m dxdy &\le \frac{C_m}{R^2}\int_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} u\varphi_R^{m-2} dxdy \\ &\le C_m R^{\frac{N_\alpha}{q'}-2}\left(\int_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} u^q\varphi_R^{(m-2)q} dxdy\right)^{\frac{1}{q}},\end{aligned} \tag{2.7}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Combining (2.6) and (2.7) to obtain

$$\int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} v^p\varphi_R^m dxdy \le C_m R^{\frac{N_\alpha}{q'}-2}\left(\int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} u^q\varphi_R^{(m-2)q}dxdy\right)^{\frac{1}{q}}. \tag{2.8}$$

The same argument also yields, for $k \ge 2$, that

$$\int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} u^q\varphi_R^k dxdy \le C_k R^{\frac{N_\alpha}{p'}-2}\left(\int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} v^p\varphi_R^{(k-2)p}dxdy\right)^{\frac{1}{p}}, \tag{2.9}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Note that $pq > 1$, then we choose $k = (m-2)q$ and m large such that $(k-2)p = ((m-2)q-2)p > m$. Hence, (2.8) and (2.9) give

$$\int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} v^p\varphi_R^m dxdy \le C_m R^{\frac{N_\alpha}{q'}-2+\frac{N_\alpha}{qp'}-\frac{2}{q}}\left(\int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} v^p\varphi_R^m dxdy\right)^{\frac{1}{pq}}. \tag{2.10}$$

Therefore, (2.4) is deduced from (2.10) and a simple computation. Similarly, (2.5) follows from (2.4) and (2.9). □

By interpolation argument, we also have

Corollary 2.3. *Under the assumptions of Lemma 2.2, for $0 \le t < p, 0 \le \tau < q$, there is $C > 0$ independent of R such that*

$$\int\limits_{B_R\times\mathbf{B}_{R^{1+\alpha}}} v^t dxdy \le CR^{N_\alpha-\frac{2(q+1)}{pq-1}t},$$

and

$$\int\limits_{B_R\times\mathbf{B}_{R^{1+\alpha}}} u^\tau dxdy \le CR^{N_\alpha-\frac{2(p+1)}{pq-1}\tau}.$$

We next generalize the comparison property in [29,6,2] for the system (1.1) without stability assumption.

Lemma 2.4. *Suppose that (u, v) is a smooth positive solution of (1.1) with $1 < q \le p$. Then there holds*

$$\frac{v^{p+1}}{p+1} \le \frac{u^{q+1}}{q+1}. \tag{2.11}$$

Proof. Let $\sigma = \frac{q+1}{p+1} \le 1$ and $l = \sigma^{-\frac{1}{p+1}}$. The inequality (2.11) is equivalent to

$$v \le lu^\sigma. \tag{2.12}$$

Put $w = v - lu^\sigma$. A simple computation gives

$$\Delta_x w = \Delta_x v - l\sigma\Delta_x u.u^{\sigma-1} - l\sigma(\sigma-1)|\nabla_x u|^2 u^{\sigma-2},$$
$$|x|^{2\alpha}\Delta_y w = |x|^{2\alpha}\left(\Delta_y v - l\sigma\Delta_y u.u^{\sigma-1} - l\sigma(\sigma-1)|\nabla_y u|^2 u^{\sigma-2}\right).$$

Therefore,

$$\begin{aligned}\Delta_G w &= \Delta_G v - l\sigma\Delta_G u.u^{\sigma-1} - l\sigma(\sigma-1)|\nabla_G u|^2 u^{\sigma-2}\\ &\geq \Delta_G v - l\sigma\Delta_G u.u^{\sigma-1} = -u^q + l\sigma u^{\sigma-1}v^p\\ &= u^{\sigma-1}\left(-u^{q+1-\sigma} + l\sigma v^p\right) = u^{\sigma-1}\left(\left(\frac{v}{l}\right)^p - (u^\sigma)^p\right).\end{aligned} \tag{2.13}$$

We now prove (2.12) by contradiction. Suppose that $M = \sup_{\mathbb{R}^N} w > 0$ ($M \leq +\infty$).

Case 1: the supremum of w is attained at infinity.

Choose the cut-off function $\chi \in C_c^\infty(\mathbb{R}^N; [0,1])$ and let $\phi(x,y) = \chi^m(x,y)$. Here $m > 0$ will be chosen later. Since $\nabla\chi$ and $\Delta\chi$ are bounded, there is a constant $C > 0$ such that

$$|\Delta\phi| \leq C\phi^{\frac{m-2}{m}},\ \frac{|\nabla\phi|^2}{\phi} \leq C\phi^{\frac{m-2}{m}}. \tag{2.14}$$

Let $\phi_R(x,y) = \phi(\frac{x}{R}, \frac{y}{R^{1+\alpha}})$, $w_R = \phi_R w$ then

$$\sup_{\mathbb{R}^N} w_R(x,y) = \max_{\mathbb{R}^N} w_R(x,y) \to M \text{ as } R \to \infty. \tag{2.15}$$

Take (x_R, y_R) such that $\max_{\mathbb{R}^N} w_R(x,y) = w_R(x_R, y_R)$. This implies that

$$\nabla_G w_R(x_R, y_R) = 0,\ \Delta_G w_R(x_R, y_R) \leq 0. \tag{2.16}$$

In what follows, all the estimates are taken at the point (x_R, y_R). First, using $\nabla w_R = 0$ at (x_R, y_R) we have

$$0 = \nabla_G w_R = \nabla_G\phi_R\, w + \phi_R\, \nabla_G w.$$

Thus,

$$\nabla_G w = -\phi_R^{-1}\nabla_G\phi_R\, w.$$

Since $\Delta_G w_R \leq 0$ at the point (x_R, y_R), we obtain

$$0 \geq \Delta_G w_R = \Delta_G\phi_R w + 2\nabla_G\phi_R \cdot \nabla_G w + \phi_R\Delta_G w.$$

Hence,

$$\phi_R\Delta_G w \leq (2\phi_R^{-1}|\nabla_G\phi_R|^2 - \Delta_G\phi_R)w. \tag{2.17}$$

Combining (2.14) and (2.17), one has

$$\phi_R\Delta_G w \leq \frac{C}{R^2}\phi_R^{\frac{m-2}{m}} w. \tag{2.18}$$

Recall that $v - lu^\sigma = w$. Then for $w > 0$, it is easy to see that

$$\frac{v^p}{w^p} - \frac{(lu^\sigma)^p}{w^p} \geq 1 \text{ or equivalently } (\frac{v}{l})^p - (u^\sigma)^p \geq \frac{w^p}{l^p}. \tag{2.19}$$

It follows from (2.13), (2.18) and (2.19) that

$$\phi_R u^{\sigma-1} w^p \le \frac{C}{R^2} \phi_R^{\frac{m-2}{m}} w.$$

Recall that the constant C is independent of R. Consequently,

$$\phi_R^{\frac{m+2}{m}} u^{\sigma-1} w^p \le \frac{C}{R^2} \phi_R w.$$

By choosing $m = \frac{2}{p-1}$ (or $p = \frac{m+2}{m}$), we get

$$u^{\sigma-1} w_R^p \le \frac{C}{R^2} w_R, \quad \text{or } u^{\sigma-1} w_R^{p-1} \le \frac{C}{R^2}. \tag{2.20}$$

We recall that all the above estimates are taken at the point (x_R, y_R). Note that $\sigma - 1 \le 0$. If the sequence $u(x_R, y_R)$ is bounded, then $u^{\sigma-1}(x_R, y_R) w_R^{p-1}(x_R, y_R) \ge C w_R^{p-1}(x_R, y_R)$ where $C > 0$ is independent of R. This together with (2.20) follow $w_R^{p-1}(x_R, y_R) \le \frac{C}{R^2}$. Let $R \to \infty$ we have contradiction.

If the sequence $u(x_R, y_R)$ is unbounded, up to a subsequence, we may assume that

$$\lim_{R \to +\infty} u(x_R, y_R) = +\infty.$$

Since $p \ge q > 1$, there exists $\varepsilon > 0$ small enough such that $pq - 1 - \varepsilon(q+1) > 0$ and $p > 1 + \varepsilon$. For $0 < b < a$, using the mean value theorem we have

$$\begin{aligned} a^p - b^p &= (a^{1+\varepsilon})^{\frac{p}{1+\varepsilon}} - (b^{1+\varepsilon})^{\frac{p}{1+\varepsilon}} \\ &\ge \frac{p}{1+\varepsilon} (b^{1+\varepsilon})^{\frac{p}{1+\varepsilon}-1} (a^{1+\varepsilon} - b^{1+\varepsilon}) \\ &= \frac{p}{1+\varepsilon} (b^{p-\varepsilon-1}) (a^{1+\varepsilon} - b^{1+\varepsilon}) \\ &\ge \frac{p}{1+\varepsilon} (b^{p-\varepsilon-1}) (a-b)^{1+\varepsilon}. \end{aligned}$$

Choosing $a = \frac{1}{l} v(x_R, y_R), b = u^\sigma(x_R, y_R)$ and using (2.13), we arrive at

$$\begin{aligned} \Delta_G w(x_R, y_R) &\ge C u^{\sigma-1}(x_R, y_R) u^{\sigma(p-1-\varepsilon)}(x_R, y_R) w^{1+\varepsilon} \\ &= C u^{\frac{pq-1-\varepsilon(q+1)}{p+1}}(x_R, y_R) w^{1+\varepsilon}(x_R, y_R) \ge C w^{1+\varepsilon}(x_R, y_R), \end{aligned} \tag{2.21}$$

where $C > 0$ (independent of R) and in the last inequality we have used the unboundedness of the sequence $u(x_R, y_R)$.

Inserting (2.21) into (2.18) and choosing $\frac{m+2}{m} = 1 + \varepsilon$, we obtain

$$w_R^\varepsilon(x_R, y_R) \le \frac{C}{R^2}. \tag{2.22}$$

It suffices to take $R \to +\infty$ in (2.22) to get the contradiction.

Case 2: there is (x^0, y^0) *such that* $M = \sup_{\mathbb{R}^N} w = w(x^0, y^0) > 0$.

The estimates (2.13) and (2.19) imply that

$$\Delta_G w(x^0, y^0) \ge C u^{\sigma-1}(x^0, y^0) w^p(x^0, y^0) > 0.$$

Hence, there exists at least an index j such that $\frac{\partial^2 w}{\partial x_j^2}(x^0, y^0) > 0$ or $\frac{\partial^2 w}{\partial y_j^2}(x^0, y^0) > 0$. This contradicts $w(x^0, y^0) = \sup_{\mathbb{R}^N} w$. The proof is complete. □

Combining the proof of Lemma 2.4 with the idea in [17], we have the inverse comparison principle as follows.

Lemma 2.5. *Suppose that* (u, v) *is a smooth positive solution of* (1.1) *with* $1 < q \le p$ *and* v *is bounded. Then* u *is bounded and satisfies*

$$u \le \|v\|_\infty^{\frac{p-q}{q+1}} v. \tag{2.23}$$

Proof. Let $l = \|v\|_\infty^{\frac{p-q}{q+1}}$ and $w = u - lv$. Then

$$\Delta_G w = -v^p + lu^q = -\frac{v^p}{\|v\|_\infty^p}\|v\|_\infty^p + lu^q \ge -\frac{v^q}{\|v\|_\infty^q}\|v\|_\infty^p + lu^q = l\Big(u^q - (lv)^q\Big). \tag{2.24}$$

In order to obtain the proof, it suffices to use the arguments as in Lemma 2.4 by noting that (2.13) is replaced by (2.24). The detail is then omitted. □

Using Lemma 2.4 and following the proof in [13, Proposition 2], we get

Lemma 2.6. *Let* $\theta = \frac{pq(q+1)}{p+1}$ *and*

$$\sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} < t < \sqrt{\theta} + \sqrt{\theta - \sqrt{\theta}}. \tag{2.25}$$

Then we have

$$\int_{\mathbb{R}^N} v^p u^{2t-1}\phi^2 dxdy \le C \int_{\mathbb{R}^N} u^{2t}(|\nabla_G \phi|^2 + |\Delta_G \phi|\phi) dxdy,$$

for all $\phi \in C_c^2(\mathbb{R}^N)$ *satisfying* $0 \le \phi \le 1$. *Here* C *does not depend on* (u, v).

Proof. It follows from Lemma 2.1 with the test function $u^t\phi$ that

$$\begin{aligned}\sqrt{pq}\int_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q-1}{2}} u^{2t}\phi^2 dxdy &\le \int_{\mathbb{R}^N} |\nabla_G(u^t\phi)|^2 dxdy \\ &= t^2\int_{\mathbb{R}^N} |\nabla_G u|^2 u^{2t-2}\phi^2 dxdy + \int_{\mathbb{R}^N} u^{2t}|\nabla_G\phi|^2 dxdy - \frac{1}{2}\int_{\mathbb{R}^N} u^{2t}\Delta_G(\phi^2) dxdy \end{aligned}\tag{2.26}$$

Multiplying the first equation in (1.1) by $u^{2t-1}\phi^2$ and integrating by parts to arrive at

$$(2t-1)\int_{\mathbb{R}^N} |\nabla_G u|^2 u^{2t-2}\phi^2 dxdy - \frac{1}{2t}\int_{\mathbb{R}^N} u^{2t}\Delta_G(\phi^2) dxdy = \int_{\mathbb{R}^N} v^p u^{2t-1}\phi^2 dxdy. \tag{2.27}$$

Thus, from (2.26) and (2.27) one has

$$\sqrt{pq}\int_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q-1}{2}} u^{2t}\phi^2 dxdy \le \frac{t^2}{2t-1}\int_{\mathbb{R}^N} v^p u^{2t-1}\phi^2 dxdy + C\int_{\mathbb{R}^N} u^{2t}(|\nabla_G\phi|^2 + |\Delta_G\phi|\phi) dxdy \tag{2.28}$$

To end the proof, it is enough to apply Lemma 2.4 to the left hand side of (2.28) and remark that $\theta - \frac{t^2}{2t-1} = \sqrt{\frac{pq(q+1)}{p+1}} - \frac{t^2}{2t-1} > 0$. □

The following lemma is a key step to deal with the case $1 < q \le \max(p; \frac{4}{3})$ in the proof of Theorem 1.2.

Lemma 2.7. *Let (u, v) be a positive solution of (1.1) with $1 < q \le \max(p; \frac{4}{3})$. If v is bounded, then*

$$\int_{B_R \times \mathbf{B}_{R^{1+\alpha}}} u^2 dxdy \le CR^{N_\alpha - 2 - \frac{2(q+1)}{pq-1} - \frac{4(2-q)}{p+q-2}}.$$

Proof. Let $\chi \in C_c^\infty(\mathbb{R}^N; [0,1])$ be a cut-off function satisfying $\chi = 1$ on $B_1 \times \mathbf{B}_1$ and $\chi = 0$ outside $B_2 \times \mathbf{B}_{2^{1+\alpha}}$. Put $\varphi_R(x,y) = \chi(\frac{x}{R}, \frac{y}{R^{1+\alpha}})$. Note that (2.28) is still true without the assumption (2.25). Taking $\phi = \varphi_R^m$, $t = 1$ in (2.28) and then applying Lemma 2.4 to the right hand side to obtain

$$\sqrt{pq} \int_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q+3}{2}} \varphi_R^{2m} dxdy \le \sqrt{\frac{p+1}{q+1}} \int_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q+3}{2}} \varphi_R^{2m} dxdy + \frac{C}{R^2} \int_{\mathbb{R}^N} u^2 \varphi_R^{2m-2} dxdy. \tag{2.29}$$

Since $pq > \frac{p+1}{q+1}$, the estimate (2.29) yields

$$\int_{\mathbb{R}^N} v^{\frac{p-1}{2}} u^{\frac{q+3}{2}} \varphi_R^{2m} dxdy \le \frac{C}{R^2} \int_{\mathbb{R}^N} u^2 \varphi_R^{2m-2} dxdy. \tag{2.30}$$

By applying Lemma 2.5 to the left hand side of (2.30), we have

$$\int_{\mathbb{R}^N} u^{\frac{p+q+2}{2}} \varphi_R^{2m} dxdy \le \frac{C}{R^2} \int_{\mathbb{R}^N} u^2 \varphi_R^{2m-2} dxdy. \tag{2.31}$$

Recall that $1 < q \le \max(p, \frac{4}{3})$, then $\sigma := \frac{p+q-2}{p+2-q} \in (0,1)$ and $2 = \sigma q + (1-\sigma)\frac{p+q+2}{2}$. We estimate the integral in the right hand side of (2.31) by using Hölder's inequality as follows

$$\begin{aligned} \int_{\mathbb{R}^N} u^2 \varphi_R^{2m-2} dxdy &\le \left(\int_{\mathbb{R}^N} u^{\frac{p+q+2}{2}} \varphi_R^{2m} dxdy \right)^{1-\sigma} \left(\int_{\mathbb{R}^N} u^q \varphi_R^{2m-\frac{2}{\sigma}} dxdy \right)^{\sigma} \\ &\le \left(\int_{\mathbb{R}^N} u^{\frac{p+q+2}{2}} \varphi_R^{2m} dxdy \right)^{1-\sigma} \left(\int_{B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}} u^q dxdy \right)^{\sigma}, \end{aligned} \tag{2.32}$$

where in the last inequality we have used $0 \le \varphi_R \le 1$ and chosen m large enough such that $m\sigma > 1$.

Combining (2.31), (2.32) and (2.5) to obtain

$$\int_{\mathbb{R}^N} u^2 \varphi_R^{2m-2} dxdy \le CR^{-2\frac{1-\sigma}{\sigma}} R^{N_\alpha - 2 - \frac{2(q+1)}{pq-1}} = CR^{N_\alpha - 2 - \frac{2(q+1)}{pq-1} - \frac{4(2-q)}{p+q-2}}.$$

Finally, note that $\varphi_R = 1$ on $B_R \times \mathbf{B}_{R^{1+\alpha}}$, we finish the proof of Lemma. □

Remark 2.8. The boundedness of v in Lemma 2.7 is only necessary for the use of the fact that $u \le Cv$ in Lemma 2.5. However, if $p = q$, then we have $u = v$ by Lemma 2.4. Thus, in the case $p = q$, Lemma 2.7 is still true without the assumption of boundedness of solutions.

3. Proof of main results

3.1. Proof of Theorem 1.1

Let $\varphi \in C_c^\infty(\mathbb{R}^N;[0,1])$ be a cut-off function such that

$$\varphi = 1 \text{ on } B_1 \times \mathbf{B}_1 \text{ and } \varphi = 0 \text{ outside } B_2 \times \mathbf{B}_{2^{1+\alpha}}. \tag{3.1}$$

Let w be a smooth function and let $k_\alpha = \frac{N_\alpha}{N_\alpha - 2}$. By using Sobolev inequality (see [31]) and integration by parts, we have

$$\begin{aligned}
\left(\int\limits_{B_1\times\mathbf{B}_1} w^{2k_\alpha}dxdy \right)^{\frac{1}{2k_\alpha}} &\le \left(\int\limits_{B_2\times\mathbf{B}_{2^{1+\alpha}}} (w\varphi)^{2k_\alpha}dxdy \right)^{\frac{1}{2k_\alpha}} \\
&\le C\left(\int\limits_{B_2\times\mathbf{B}_{2^{1+\alpha}}} |\nabla_G(w\varphi)|^2 dxdy \right)^{\frac{1}{2}} \\
&= C\left(\int\limits_{B_2\times\mathbf{B}_{2^{1+\alpha}}} |\nabla_G(w)|^2\varphi^2 + w^2|\nabla_G\varphi|^2 + \frac{1}{2}\nabla_G(w^2)\cdot\nabla_G(\varphi^2)dxdy \right)^{\frac{1}{2}} \\
&= C\left(\int\limits_{B_2\times\mathbf{B}_{2^{1+\alpha}}} |\nabla_G(w)|^2\varphi^2 + w^2|\nabla_G\varphi|^2 + \frac{1}{2}w^2(-\Delta_G(\varphi^2))dxdy \right)^{\frac{1}{2}} \\
&\le C\left(\int\limits_{B_2\times\mathbf{B}_{2^{1+\alpha}}} |\nabla_G(w)|^2 + w^2dxdy \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\left(\int\limits_{B_1\times\mathbf{B}_1} w^{2k_\alpha}dxdy \right)^{\frac{1}{k_\alpha}} \le C \int\limits_{B_2\times\mathbf{B}_{2^{1+\alpha}}} (|\nabla_G(w)|^2 + w^2)dxdy.$$

A scaling argument follows that

$$\begin{aligned}
\left(\int\limits_{B_R\times\mathbf{B}_{R^{1+\alpha}}} w^{2k_\alpha}dxdy \right)^{\frac{1}{k_\alpha}} \le CR^{2+N_\alpha\left(\frac{1}{k_\alpha}-1\right)} &\int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} |\nabla_G(w)|^2dxdy \\
&+ CR^{N_\alpha\left(\frac{1}{k_\alpha}-1\right)} \int\limits_{B_{2R}\times\mathbf{B}_{(2R)^{1+\alpha}}} w^2dxdy. \qquad (3.2)
\end{aligned}$$

Suppose that (u,v) is a positive stable solution of (1.1). Set

$$w = u^t \text{ for } \sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} < t < \sqrt{\theta} + \sqrt{\theta - \sqrt{\theta}}.$$

Then,

$$|\nabla_G w|^2 = t^2 |\nabla_G u|^2 u^{2t-2}.$$

Let $\varphi_R = \varphi\left(\frac{x}{R}, \frac{y}{R^{1+\alpha}}\right)$ where φ is given in (3.1). Then,

$$\begin{aligned}\int\limits_{B_R \times \mathbf{B}_{R^{1+\alpha}}} |\nabla_G(w)|^2 dxdy &= C \int\limits_{B_R \times \mathbf{B}_{R^{1+\alpha}}} |\nabla_G u|^2 u^{2t-2} dxdy \\ &\leq C \int\limits_{B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}} |\nabla_G u|^2 u^{2t-2} \varphi_R^2 dxdy. \end{aligned} \tag{3.3}$$

Multiplying the first equation in (1.1) by $u^{2t-1}\varphi_R^2$ and using the integration by parts, we obtain

$$\begin{aligned}\int\limits_{B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}} |\nabla_G u|^2 u^{2t-2} \varphi_R^2 dxdy &= \frac{1}{2t-1} \int\limits_{B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}} v^p u^{2t-1} \varphi_R^2 dxdy \\ &+ \frac{1}{2t(2t-1)} \int\limits_{B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}} u^{2t} \Delta_G(\varphi_R^2) dxdy. \end{aligned} \tag{3.4}$$

The estimations (3.3), (3.4) and Lemma 2.6 imply that

$$\int\limits_{B_R \times \mathbf{B}_{R^{1+\alpha}}} |\nabla_G(w)|^2 dxdy \leq CR^{-2} \int\limits_{B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}} w^2 dxdy.$$

This together with (3.2) give

$$\left(\int\limits_{B_R \times \mathbf{B}_{R^{1+\alpha}}} w^{2k_\alpha} \right)^{\frac{1}{k_\alpha}} dxdy \leq CR^{N_\alpha\left(\frac{1}{k_\alpha}-1\right)} \int\limits_{B_{2R} \times \mathbf{B}_{(2R)^{1+\alpha}}} w^2 dxdy. \tag{3.5}$$

In the following, we need the assumption (1.6) which is rewritten as

$$\sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} < \frac{q}{2}, \qquad \theta = \frac{pq(q+1)}{p+1}.$$

It is easy to see that $\theta \geq q^2$ and the function $\theta \mapsto \sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}}$ is decreasing. Then

$$\frac{1}{2} = \lim_{\theta \to +\infty} \left(\sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} \right) \leq \sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} \leq q - \sqrt{q^2 - q}.$$

On the other hand, if $q > \frac{4}{3}$, then $q - \sqrt{q^2 - q} \leq \frac{q}{2}$. Therefore, the condition (1.6) is always fulfilled in case (i).

Fix a real positive number τ satisfying

$$2(\sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}}) \leq 2\tau < q. \tag{3.6}$$

Let m be a non-negative integer satisfying $\tau k_\alpha^{m-1} < \sqrt{\theta} + \sqrt{\theta - \sqrt{\theta}} \leq \tau k_\alpha^m$. We construct an increasing geometric sequence

$$\sqrt{\theta}-\sqrt{\theta-\sqrt{\theta}}<t_1<t_2<...<t_m<\sqrt{\theta}+\sqrt{\theta-\sqrt{\theta}}$$

given by

$$2t_1=2\tau k, 2t_2=2\tau k k_\alpha, ..., 2t_m=2\tau k k_\alpha^{m-1},$$

where $k\in[1,k_\alpha]$ is chosen such that t_m is arbitrarily close to $\sqrt{\theta}+\sqrt{\theta-\sqrt{\theta}}$.

To simplify notations below, we use $R_n=2^nR$. By using (3.5) and an induction argument, we obtain

$$\begin{aligned}
\left(\int\limits_{B_R\times\mathbf{B}_{R^{1+\alpha}}} u^{2t_mk_\alpha}dxdy\right)^{\frac{1}{t_mk_\alpha}} &\le C\left(R^{N_\alpha\left(\frac{1}{k_\alpha}-1\right)}\right)^{\frac{1}{t_m}}\left(\int\limits_{B_{R_1}\times\mathbf{B}_{(R_1)^{1+\alpha}}} u^{2t_m}dxdy\right)^{\frac{1}{t_m}}\\
&= CR^{N_\alpha\left(\frac{1}{k_\alpha t_m}-\frac{1}{t_m}\right)}\left(\int\limits_{B_{R_1}\times\mathbf{B}_{(R_1)^{1+\alpha}}} u^{2t_{m-1}k_\alpha}dxdy\right)^{\frac{1}{t_{m-1}k_\alpha}}\\
&\le CR^{N_\alpha\left(\frac{1}{k_\alpha t_m}-\frac{1}{t_1}\right)}\left(\int\limits_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} u^{2t_1}dxdy\right)^{\frac{1}{t_1}}\\
&= CR^{N_\alpha\left(\frac{1}{k_\alpha t_m}-\frac{1}{\tau k}\right)}\left(\int\limits_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} u^{2\tau k}dxdy\right)^{\frac{1}{\tau k}}.
\end{aligned} \tag{3.7}$$

For the last integral, we shall use Hölder's inequality, (3.5) and Corollary 2.3 to obtain

$$\begin{aligned}
\int\limits_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} u^{2\tau k}dxdy &\le \left(\int\limits_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} u^{2\tau k_\alpha}dxdy\right)^{\frac{k}{k_\alpha}}\left(\int\limits_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} dxdy\right)^{1-\frac{k}{k_\alpha}}\\
&\le C\left(R^{N_\alpha\left(\frac{1}{k_\alpha}-1\right)}\int\limits_{B_{R_{m+1}}\times\mathbf{B}_{(R_{m+1})^{1+\alpha}}} u^{2\tau}dxdy\right)^k R^{N_\alpha\left(1-\frac{k}{k_\alpha}\right)}\\
&\le C\left(R^{N_\alpha\left(\frac{1}{k_\alpha}-1\right)}.R^{N_\alpha-\frac{2p+2}{pq-1}.2\tau}\right)^k R^{N_\alpha\left(1-\frac{k}{k_\alpha}\right)}\\
&= C\left(R^{\frac{N_\alpha}{k_\alpha}-\frac{2p+2}{pq-1}.2\tau}\right)^k R^{N_\alpha\left(1-\frac{k}{k_\alpha}\right)}.
\end{aligned}$$

Consequently,

$$\left(\int\limits_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} u^{2\tau k}dxdy\right)^{\frac{1}{\tau k}} \le CR^{\frac{N_\alpha}{\tau k_\alpha}-\frac{2p+2}{pq-1}.2}.R^{N_\alpha\left(1-\frac{k}{k_\alpha}\right)\frac{1}{\tau k}} = CR^{\frac{N_\alpha}{\tau k}-\frac{2p+2}{pq-1}.2}. \tag{3.8}$$

Substituting (3.8) into the last inequality of (3.7), one has

$$\left(\int_{B_R\times\mathbf{B}_{R^{1+\alpha}}} u^{2t_m k_\alpha}dxdy\right)^{\frac{1}{t_m k_\alpha}} \le CR^{\frac{N_\alpha}{k_\alpha t_m}-\frac{2p+2}{pq-1}\cdot 2}. \tag{3.9}$$

Recall that $k_\alpha = \frac{N_\alpha}{N_\alpha-2}, \theta = \frac{pq(q+1)}{p+1}$. Under Assumption (1.5), we choose $k \in [1, k_\alpha]$ such that t_m close to $\sqrt{\theta}+\sqrt{\theta-\sqrt{\theta}}$ and then the exponent in the right hand side of (3.9) is negative. Let $R \to +\infty$ in (3.9), we obtain the contradiction.

For the second assertion in (i), by adopting the proof in [7, Remark 3], we have $\frac{p+1}{pq-1}(\sqrt{\theta}+\sqrt{\theta-\sqrt{\theta}}) > 2$ for any $\frac{4}{3} < q \le p$. Then (1.5) is fulfilled if $N_\alpha \le 10$. The proof is finished.

3.2. *Proof of Theorem 1.2*

The proof of Theorem 1.2 is similar to that of Theorem 1.1. Then we give here the sketch of proof and the detail is omitted.

We emphasize that Lemma 2.7 plays an important role in the proof. To avoid using the condition $\sqrt{\theta}-\sqrt{\theta-\sqrt{\theta}} < \frac{q}{2}$ as in Theorem 1.1, we replace the constant τ in (3.6) by the one satisfying

$$2(\sqrt{\theta}-\sqrt{\theta-\sqrt{\theta}}) \le 2\tau < 2.$$

Then Lemma 2.7 follows that

$$\begin{aligned}\int_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} u^{2\tau}dxdy &\le \left(\int_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} u^2 dxdy\right)^{\tau}\left(\int_{B_{R_m}\times\mathbf{B}_{(R_m)^{1+\alpha}}} dxdy\right)^{1-\tau}\\ &\le CR^{\left(N_\alpha-2-\frac{2(q+1)}{pq-1}-\frac{4(2-q)}{p+q-2}\right)\tau}R^{N_\alpha(1-\tau)}\\ &= CR^{N_\alpha-\left(2-\frac{2(q+1)}{pq-1}-\frac{4(2-q)}{p+q-2}\right)\tau}.\end{aligned}$$

By the same argument as in the proof of Theorem 1.1, we obtain the following estimate

$$\left(\int_{B_R\times\mathbf{B}_{R^{1+\alpha}}} u^{2t_m k_\alpha}dxdy\right)^{\frac{1}{t_m k_\alpha}} \le CR^{\frac{N_\alpha}{k_\alpha t_m}-2-\frac{2(q+1)}{pq-1}-\frac{4(2-q)}{p+q-2}}. \tag{3.10}$$

By the assumption (1.7), we choose $k \in [1, k_\alpha]$ such that t_m close to $\sqrt{\theta}+\sqrt{\theta-\sqrt{\theta}}$ and then the right hand side of (3.10) tends to zero as $R \to +\infty$ and obtain the contradiction.

For the rest of proof. We shall show that, for all $1 < q \le \max(\frac{4}{3}; p)$,

$$F(p,q) := \left(2+\frac{2(q+1)}{pq-1}+\frac{4(2-q)}{p+q-2}\right)\left(\sqrt{\theta}+\sqrt{\theta-\sqrt{\theta}}\right) > 2\left(\sqrt{2}+\sqrt{2-\sqrt{2}}\right).$$

Indeed, for $p \ge q > 1$, we have

$$\frac{2(q+1)}{pq-1} \ge \frac{2}{p-1} \quad \text{and } \theta > \frac{2p}{p+1}.$$

Combining with the fact that $f(z) := \sqrt{z}+\sqrt{z-\sqrt{z}}$ is increasing in $(1,\infty)$, we deduce that

$$F(p,q) > \left(2+\frac{2}{p-1}\right)\left(\sqrt{\frac{2p}{p+1}}+\sqrt{\frac{2p}{p+1}-\sqrt{\frac{2p}{p+1}}}\right), \quad \text{for all } 1<q\le \max(\frac{4}{3};p). \tag{3.11}$$

Denote $s = \frac{2p}{p+1}$ with $s \in (1,2)$, then (3.11) is written as

$$F(p,q) > g(s) := \frac{s}{s-1}\left(\sqrt{s}+\sqrt{s-\sqrt{s}}\right), \quad \text{for all } 1<q\le \max(\frac{4}{3};p).$$

A simple estimate yields

$$g'(s) = -\frac{1}{(s-1)^2}\left[f(s) - \frac{s(s-1)}{2\sqrt{s}\sqrt{s-\sqrt{s}}}\left(f(s)-\frac{1}{2}\right)\right] < 0, \quad \text{for all } s\in(1,2)$$

where we used

$$\frac{s(s-1)}{2\sqrt{s}\sqrt{s-\sqrt{s}}} < 1, \quad \text{for all } s\in(1,2).$$

This implies

$$F(p,q) > g(2) = 2\left(\sqrt{2}+\sqrt{2-\sqrt{2}}\right) \quad \text{for all } 1<q\le \max(\frac{4}{3};p).$$

Thus, (1.7) is true if $N_\alpha \le 2+2\left(\sqrt{2}+\sqrt{2-\sqrt{2}}\right)$. The proof is complete.

Remark 3.1. In Theorem 1.1(i), by taking $p=q$ and letting $p\to\infty$ we see that the right-hand side of (1.5) tends to 10. Hence,

$$\inf_{4/3<q\le p}\left(2+\frac{4p+4}{pq-1}\left(\sqrt{\frac{pq(q+1)}{p+1}}+\sqrt{\frac{pq(q+1)}{p+1}-\sqrt{\frac{pq(q+1)}{p+1}}}\right)\right) = 10.$$

Similarly, by taking $q=1$ and letting $p\to\infty$ in the right-hand side of (1.7), one see that

$$\inf_{1<q\le\max(\frac{4}{3},p)}\left(2+\left(2+\frac{2(q+1)}{pq-1}+\frac{4(2-q)}{p+q-2}\right)\left(\sqrt{\frac{pq(q+1)}{p+1}}+\sqrt{\frac{pq(q+1)}{p+1}-\sqrt{\frac{pq(q+1)}{p+1}}}\right)\right)$$
$$= 2+2\left(\sqrt{2}+\sqrt{2-\sqrt{2}}\right).$$

3.3. Proof of Corollary 1.3

Recall that the stability of the solution of (1.1) implies the estimate (2.1) which is exactly, when $p = q, u = v$, the definition of stable solution of (1.8). Let $p=q$, then $u=v$ by Lemma 2.4 and the system (1.1) becomes the scalar equation (1.8).

Case 1. $q > \frac{4}{3}$. It follows from Theorem 1.1 (with $p=q$) that the problem (1.8) has no stable positive solution provided (1.10).

Case 2. $1 < q \le \frac{4}{3}$. By using Remark 2.8, it is easy to see that if $p=q$, then the first assertion of Theorem 1.2 is still true without the assumption of boundedness of solutions. It means that problem (1.8) has no stable positive solution provided (1.10).

In both cases, there is no stable positive solution of (1.8) if $N_\alpha \le 10$. The proof is finished.

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