An Introduction to Homogenization and G-convergence

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This paper contains the notes of five lectures concerning an introduction to Homogenization and G-convergence, delivered on September 6-8, 1993 as a part of the "School on Homogenization" at the ICTP, Trieste. The main topics treated are the following ones:

I. Homogenization of second order linear elliptic operators
II. Homogenization of monotone operators
III. G-convergence; H-convergence.

0. Introduction

Composite materials (fibred, stratified, porous,...) play an important role in many branches of Mechanics, Physics, Chemistry and Engineering. Typically, in such materials, the physical parameters (such as conductivity, elasticity coefficients, ...) are discontinuous and oscillate between the different values characterizing each of the components. When these components are intimately mixed, these parameters oscillate very rapidly and the *microscopic* structure becomes complicated.

On the other hand we may think to get a good approximation of the *macroscopic* behaviour of such a heterogeneous material by letting the parameter ε_h, which describes the fineness of the microscopic structure, tend to zero in the equations describing phenomena such as heat conduction and elasticity. It is the purpose of *homogenization theory* to describe these limit processes, when ε_h tends to zero.

More precisely, homogenization deals with the asymptotic analysis of Partial Differential Equations of Physics in heterogeneous materials with a periodic structure, when the characteristic length ε_h of the period tends to zero.

A good model for the study of the physical behaviour of a heterogeneous body with a fine periodic structure, e.g. in electrostatics, magnetostatics, or stationary heat diffusion is given by

$$(0.1) \qquad \begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon_h})Du_h) = f \qquad \text{on } \Omega\,, \\ u_h|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega\,; \end{cases}$$

here Ω is a bounded open subset of $\mathbf{R}^n$ which will be considered as a piece of the heterogeneous material and ε_h is the period of the structure in all directions. The function u_h can be interpreted as the electric potential, magnetic potential, or the temperature, respectively. The coefficients $a(\frac{x}{\varepsilon_h}) = (a_{ij}(\frac{x}{\varepsilon_h}))$ are ε_h-periodic functions and describe the physical properties of the different materials constituting the body (they are the dielectric coefficients, the magnetic permeability and the thermic conductivity coefficients, respectively). The function f is a given source term.

When the period of the structure is very small, a direct numerical approximation of the solution to (0.1) may be very heavy, or even impossible. Then homogenization provides an alternative way of approximating such solutions u_h by means of a function u which solves the problem corresponding to a "homogenized" material

$$\begin{cases} -\mathrm{div}(bDu) = f & \text{on } \Omega\,, \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{0.2}$$

where b is a constant matrix (for a homogeneous material the physical properties does not depend on x). The "homogenized" matrix b may be interpreted as the physical parameters of a homogeneous body, whose behaviour is equivalent, from a "macroscopic" point of view, to the behaviour of the material with the given periodic microstructure (described by (0.1)) (the coefficients b_{ij} are called also "effective" coefficients or "effective" parameters since they describe the macroscopic properties of the medium).

In these lectures we shall consider the asymptotic analysis of the solutions to

$$\begin{cases} -\mathrm{div}(a_h(x, Du_h)) = f & \text{on } \Omega\,, \\ u_h|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

in the following main cases:
(i) $a_h(x,\xi) = a(\frac{x}{\varepsilon_h})\xi$, with $a(x)$ periodic matrix;
(ii) $a_h(x,\xi) = a(\frac{x}{\varepsilon_h}, \xi)$, with $a(\cdot, \xi)$ periodic;
(iii) $a_h(x,\xi) = a^h(x)\xi$, without periodicity assumptions on a^h.

The main references for the homogenization theory of periodic structures are the books by Bensoussan-Lions-Papanicolaou [9], Sanchez Palencia [40], Lions [31], Bakhvalov-Panasenko [6], and Oleinik-Shamaev-Yosifian [37]. Other general references for the theory of the homogenization of partial differential equations are Babuska [4], Bensoussan [7], and Bergman-Lions-Papanicolaou-Murat-Tartar-Sanchez Palencia [10].

Notation

Let $n \in \mathbf{N}$ be fixed. Given $m \in \mathbf{N}$, the elements of $\mathbf{R}^m$ are usually considered as column vectors, $(\cdot,\cdot)$ denotes the scalar product on $\mathbf{R}^m$, and $|\cdot|$ will be the usual euclidean norm. Let $M^{n\times n}$ be denote the set of all $n \times n$ real matrices. Given $M = (M_{ij}) \in M^{n\times n}$ and $\xi \in \mathbf{R}^n$, $M\xi$ is the vector of $\mathbf{R}^n$ with components $(M\xi)_i = \sum_{j=1}^n M_{ij}\xi_j$, $i \in \{1,\dots,n\}$ and $(M\xi,\eta) = \sum_{i,j=1}^n M_{ij}\xi_j\eta_i$ for every ξ, $\eta \in \mathbf{R}^n$. We shall identify $M^{n\times n}$ with $\mathbf{R}^{n^2}$.

We use the symbol $|A|$ to the denote the Lebesgue measure of the set $A \subseteq \mathbf{R}^n$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure.

For every open subset $A \in \mathbf{R}^n$ and $f \in L^1(A)$ we denote by $\mathcal{M}_A(f)$ the average of f (with respect to A) defined as $\mathcal{M}_A(f) = \frac{1}{|A|}\int_A f(x)\,dx$. If no confusion can occur, we shall simply write $\mathcal{M}(f)$.

For what concerns L^p and Sobolev spaces we refer to the Appendix.

I. Homogenization of second order linear elliptic operators

1. Setting of the problem

Let $z_1, \ldots, z_n$ be n linearly independent vectors of $\mathbf{R}^n$, and let P be the parallelogram with sides $z_1, \ldots, z_n$, i.e.,

$$P = \{t_1 z_1 + \ldots + t_n z_n : 0 < t_i < 1 \quad \text{for } i = 1, \ldots, n\}.$$

We say that a function $\varphi : \mathbf{R}^n \to \mathbf{R}^m$ is P*-periodic* if $\varphi(x) = \varphi(x + z_i)$ for every $x \in \mathbf{R}^n$ and for every $i = 1, \ldots, n$. In this case we say that P is a *periodicity cell* of the function φ. For the sake of simplicity (without loss of generality) we shall assume from now on that the periodicity cell P is the unit cube $Y =]0,1[^n$. Hence, if $e_1, \ldots, e_n$ denotes the canonical basis of $\mathbf{R}^n$ then $\varphi : \mathbf{R}^n \to \mathbf{R}^m$ is Y-periodic if $\varphi(x) = \varphi(x + e_i)$ for every $x \in \mathbf{R}^n$ and for every $i = 1, \ldots, n$.

Let us consider the function $a : \mathbf{R}^n \to M^{n\times n}$, with $a(x) = (a_{ij}(x))$ for $x \in \mathbf{R}^n$, satisfying the following properties:

(1.1) a_{ij} is Y-periodic on $\mathbf{R}^n$ for every $i, j = 1, \ldots, n$;

(1.2) $a_{ij} \in L^\infty(\mathbf{R}^n)$ for every $i, j = 1, \ldots, n$;

(1.3) there exists a constant $\alpha > 0$ such that $(a(x)\xi, \xi) = \sum_{i,j=1}^n a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2$ for a.e. $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$.

We then define $a^h : \mathbf{R}^n \to M^{n\times n}$ by

$$a^h(x) = a\left(\frac{x}{\varepsilon_h}\right), \tag{1.4}$$

where (ε_h) is a sequence of positive real numbers converging to 0. Note that the functions a_{ij}^h are $\varepsilon_h Y$-periodic on $\mathbf{R}^n$.

Let Ω be a bounded open subset of $\mathbf{R}^n$ (we shall consider it as a piece of a heterogeneous material). For a fixed $\varepsilon_h > 0$, let us consider the Dirichlet boundary value problem on Ω

$$\begin{cases} -\mathrm{div}(a^h(x)Du_h) = f & \text{on } \Omega, \\ u_h|_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where f is a given smooth function on Ω. Assume $f \in H^{-1,2}(\Omega)$. The variational (weak) formulation of (1.5) becomes then: find $u_h \in H_0^{1,2}(\Omega)$ such that

$$\begin{cases} \int_\Omega (a^h(x)Du_h, Dv)\,dx = \langle f, v\rangle & \text{for every } v \in H_0^{1,2}(\Omega) \\ u_h \in H_0^{1,2}(\Omega) \end{cases} \tag{1.6}$$

(note that this presentation does not require regularity assumptions for the functions a_{ij}. Moreover, we get a priori estimates on u_h which are independent of ε_h and are not based on the regularity of the coefficients).

Let us note that by the Lax-Milgram lemma (see Lemma A.3.1) we have existence and uniqueness of a solution to (1.6). Indeed, let us define the bilinear form $a_1^h : H_0^{1,2}(\Omega) \times H_0^{1,2}(\Omega) \to \mathbf{R}$ by

$$a_1^h(u,v) = \int_\Omega (a^h(x)Du, Dv)\,dx \quad \text{for every } u,\ v \in H_0^{1,2}(\Omega).$$

We observe that the boundedness assumption (1.2) and Hölder's inequality yield immediately

$$|a_1^h(u,v)| \leq c\|u\|_{H_0^{1,2}(\Omega)}\|v\|_{H_0^{1,2}(\Omega)} \quad \text{for every } u,\ v \in H_0^{1,2}(\Omega)$$

(take into account Remark A.1.15). Moreover, the ellipticity condition (1.3) ensures that

$$a_1^h(u,u) \geq \alpha\|u\|^2_{H_0^{1,2}(\Omega)} \quad \text{for every } u \in H_0^{1,2}(\Omega).$$

Hence, a_1^h defines a bilinear continuous and coercive form on $H_0^{1,2}(\Omega)$ and the existence and uniqueness of a solution to (1.6) is guaranteed.

Remark 1.1. Instead of the Dirichlet boundary conditions in (1.5) one can consider also more general boundary conditions; for example, Neumann boundary conditions or mixed boundary conditions. However, to fix the ideas about homogenization, we will consider for the moment Dirichlet boundary conditions.

Let us come back to the Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}(a^h(x)Du_h) = f \quad \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{1.7}$$

We can associate to a^h the (second order elliptic) operator $\mathcal{A}^h : H_0^{1,2}(\Omega) \to H^{-1,2}(\Omega)$ defined by

$$\mathcal{A}^h u = -\mathrm{div}(a^h Du)\,,$$

and (1.7) can be written also in the form

$$\begin{cases} \mathcal{A}^h u_h = f \quad \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{1.8}$$

Now, let us consider the sequence (u_h) of solutions to (1.8) corresponding to the sequence (ε_h). Let us note that our assumptions on a^h guarantee that

$$\|u_h\|_{H_0^{1,2}(\Omega)} \le c\,,$$

where c is a constant independent of h (for more details see Section 2 and Section 4). Therefore, there exist a subsequence $(u_{\sigma(h)})$ of (u_h) and a function $u_0 \in H_0^{1,2}(\Omega)$ such that

$$u_{\sigma(h)} \rightharpoonup u_0 \qquad \text{weakly in } H_0^{1,2}(\Omega)\,.$$

At this point it is natural to ask whether u_0 solves a boundary value problem of the type (1.8), i.e.,

$$\begin{cases} -\mathrm{div}(b(x)Du_0) = f \qquad \text{on } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases}$$

The aim of the next sections is to answer at this question. We shall *construct a second order elliptic operator $\mathcal{B}$ such that (u_h) converges to u_0 (in an appropriate topology), where u_0 is the solution to*

$$\begin{cases} \mathcal{B}u_0 = f \quad \text{on } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,, \end{cases} \tag{1.9}$$

with $\mathcal{B}u = -\mathrm{div}(b(x)Du)$. The operator $\mathcal{B}$ is the so called *homogenized* operator of the family $(\mathcal{A}^h)$ and $b(x)$ the *homogenized coefficients.*

As pointed out in the introduction, this convergence analysis is related to the problem of finding the physical properties of a homogeneous material, whose overall response is close to that of the heterogeneous periodic material (whose physical properties are described by (1.4)), when the size ε_h of the periodicity cell tends to 0.

The problem of passing to the limit in (1.7), when ε_h approaches to 0, is rather delicate (as we will see soon) and requires the introduction of new techniques. The main difficulty lies in the passage to the limit in $(a^h(x)Du_h)$, which is the product of only weakly convergent sequences.

Before attacking the study of the general case, let us consider a simple particular case.

2. An example in dimension 1

Let $\Omega =]x_0, x_1[\subset \mathbf{R}$ and $f \in L^2(\Omega)$. Let (ε_h) be a sequence of positive real numbers converging to 0 and let $a^h(x) = a(\frac{x}{\varepsilon_h})$, where $a : \mathbf{R} \to \mathbf{R}$ is a measurable Y-periodic function satisfying

$$0 < \alpha \le a(x) \le \beta < +\infty \quad \text{a.e. on } \mathbf{R}. \tag{2.1}$$

We consider the Dirichlet boundary value problems

$$\begin{cases} -\frac{d}{dx}(a^h(x)\frac{du_h}{dx}(x)) = f & \text{in } \Omega, \\ u_h(x_0) = u_h(x_1) = 0 \end{cases} \tag{2.2}$$

(for every $\varepsilon_h > 0$, (2.2) is, for example, the stationary heat equation in a 1-dimensional $\varepsilon_h Y$-periodic medium). The weak formulation is then

$$\begin{cases} \int_\Omega a^h \frac{du_h}{dx}\frac{dv}{dx}\, dx = \int_\Omega fv\, dx & \text{for every } v \in H_0^{1,2}(\Omega), \\ u_h \in H_0^{1,2}(\Omega). \end{cases} \tag{2.3}$$

As seen in the previous section, for every fixed ε_h, there exists a unique solution $u_h \in H_0^{1,2}(\Omega)$ to problem (2.3). By taking $v = u_h$ in (2.3) and using Hölder's inequality we get

$$\int_\Omega a^h (\frac{du_h}{dx})^2 dx \le \|f\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)}.$$

Using (2.1) and the Poincaré inequality (see Theorem A.1.14 and Remark A.1.15) we obtain

$$\|u_h\|_{H_0^{1,2}(\Omega)} \le \frac{c}{\alpha}\|f\|_{L^2(\Omega)}, \tag{2.4}$$

where c is a positive constant depending only on Ω. Hence the sequence (u_h) is uniformly bounded in $H_0^{1,2}(\Omega)$. Therefore (see Theorem A.1.3 and Theorem A.1.11) there exist $u_0 \in H_0^{1,2}(\Omega)$ and a subsequence, still denoted by (u_h), such that

$$u_h \rightharpoonup u_0 \qquad \text{weakly in } H_0^{1,2}(\Omega). \tag{2.5}$$

Moreover, the periodicity assumption on a (see Theorem A.1.18) yields

$$a^h \overset{*}{\rightharpoonup} \mathcal{M}(a) = \frac{1}{|Y|}\int_Y a(y)\, dy \quad \text{in } L^\infty(\Omega) \text{ weak* (hence weakly in } L^2(\Omega)) \tag{2.6}$$

(note that $|Y| = 1$). From (2.5), (2.6), and (2.2) it is tempting to believe that in the limit one has

$$\begin{cases} -\frac{d}{dx}(\mathcal{M}(a)\frac{du_0}{dx}) = f & \text{in } \Omega, \\ u_0(x_0) = u_0(x_1) = 0. \end{cases} \tag{2.7}$$

But this is *false* in general, being $\xi^h(x) \equiv a^h(x)\frac{du_h}{dx}(x)$ the product of two sequences converging both *weakly* (see Remark A.1.19). To obtain the correct answer let us proceed as follows: note that by the boundedness of a^h in $L^\infty(\Omega)$ and (2.4) we have that ξ^h is uniformly bounded in $L^2(\Omega)$, and by (2.2) satisfies

$$-\frac{d\xi^h}{dx} = f \quad \text{in } \Omega. \tag{2.8}$$

Hence, ξ^h is uniformly bounded in $H^{1,2}(\Omega)$. Since the injection $H^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see Theorem A.1.12), it follows that one can assume (at least passing to a subsequence) that

$$\xi^h \to \xi^0 \quad \text{strongly in } L^2(\Omega) \tag{2.9}$$

so that

$$\frac{1}{a^h}\xi^h \rightharpoonup \mathcal{M}(\frac{1}{a})\xi^0 \quad \text{weakly in } L^2(\Omega). \tag{2.10}$$

(Note that $0 < \frac{1}{\beta} \leq \frac{1}{a} \leq \frac{1}{\alpha} < +\infty$. Moreover the periodicity assumption on a implies by Theorem A.1.18 that $(\frac{1}{a^h})$ converges in $L^\infty(\Omega)$ weak* to $\mathcal{M}(\frac{1}{a})$ and $\frac{1}{\beta} \leq \mathcal{M}(\frac{1}{a}) \leq \frac{1}{\alpha}$.) But

$$\frac{1}{a^h}\xi^h = \frac{du_h}{dx}\,,$$

so that (2.5) and (2.10) imply

$$\frac{du_0}{dx} = \mathcal{M}(\frac{1}{a})\,\xi^0\,.$$

By passing to the limit in the sense of distributions in $-\frac{d\xi^h}{dx} = f$ we have $-\frac{d\xi^0}{dx} = f$, so that u_0 is the solution to the Dirichlet boundary value problem

$$\begin{cases} -\frac{d}{dx}(\frac{1}{\mathcal{M}(\frac{1}{a})}\,\frac{du_0}{dx}) = f & \text{in } \Omega\,, \\ u_0(x_0) = u_0(x_1) = 0\,. \end{cases} \tag{2.11}$$

The homogenized operator $\mathcal{B}$ associated to $\mathcal{A}^h$ is given by

$$\mathcal{B} = -\frac{1}{\mathcal{M}(\frac{1}{a})}\,\frac{d^2}{dx^2}\,. \tag{2.12}$$

Note that in this case the homogenized operator is related to the harmonic mean (and not to the arithmetic mean) of a (compare (2.7) with (2.12)). Finally, by the uniqueness of the solution to (2.11) it follows that the whole sequence (u_h) converges weakly in $H^{1,2}_0(\Omega)$ to u_0, without extracting a subsequence (use the Urysohn property).

Remark 2.1. Let us note that

$$\mathcal{M}(a) \geq \frac{1}{\mathcal{M}(\frac{1}{a})}$$

with strict inequality in general. This fact follows immediately by the Hölder inequality applied to $\int_Y \sqrt{a}\frac{1}{\sqrt{a}}\,dy$ (recall that $0 < \alpha \leq a(x) \leq \beta < +\infty$ for a.e. $x \in \mathbf{R}$).

3. Asymptotic expansions using multiple scales

For every $h \in \mathbf{N}$, let $a^h : \mathbf{R}^n \to M^{n\times n}$ be the function defined by (1.4). In order to study the asymptotic behaviour of the solutions u_h to

$$\begin{cases} \mathcal{A}^h u_h \equiv -\operatorname{div}(a^h(x) Du_h) = f & \text{on } \Omega\,, \\ u_h|_{\partial\Omega} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

where $f \in L^2(\Omega)$, an efficient technique consists in applying asymptotic expansions using multiple scales (i.e., "slow" and "fast" variables). More precisely, the heuristic device is to suppose that u_h has a two-scale expansion of the form

$$u_h(x) = u_0(x, \frac{x}{\varepsilon_h}) + \varepsilon_h u_1(x, \frac{x}{\varepsilon_h}) + \varepsilon_h^2 u_2(x, \frac{x}{\varepsilon_h}) + \dots\,, \tag{3.2}$$

where the functions $u_i(x, y)$ are Y-periodic in y for every $x \in \Omega$. This means that we postulate the existence of smooth functions $u_i(x, y)$ defined on $\Omega \times \mathbf{R}^n$, Y-periodic in y and independent of ε_h such that the right hand side of (3.2) is an asymptotic expansion of u_h (as well as its derivatives).

Let us note that the two variables x and $\frac{x}{\varepsilon_h}$ take into account the two scales of the homogenization phenomenon; the x variable is the macroscopic variable, whereas the $\frac{x}{\varepsilon_h}$ variable takes into account the "microscopic" geometry.

The method we are going to develop turns out to be very useful to obtain the *right answers* in the study of the limit behaviour of the solutions to problem (3.1) (but also for more general cases). The proof of the *correctness* of the formulas obtained by this method can sometimes be made directly, but in general other tools will be needed (for example, the use of particular test functions and a compensated compactness lemma).

Remark 3.1. It is technically complicated to keep track of the *boundary* conditions when seeking u_h in the form (3.2) and this is actually the source of serious technical difficulties in justifying the method. The method will nevertheless give the "right answer" because it will turn out that, in this sort of problems, the boundary conditions are somewhat irrelevant.

The idea of the method is (simply) to insert (3.2) in equation (3.1) and to identify powers of ε_h . In order to present these computations in a simple form, given a smooth function $\Phi(x, y)$ of two variables, define the function $\Phi_h(x)$ of one variable by

$$\Phi_h(x) = \Phi(x, \frac{x}{\varepsilon_h})$$

and note that

$$\frac{\partial \Phi_h}{\partial x_i}(x) = \big(\frac{\partial \Phi}{\partial x_i} + \frac{1}{\varepsilon_h}\frac{\partial \Phi}{\partial y_i}\big)\,(x, \frac{x}{\varepsilon_h})\,.$$

Then, one can write

$$\mathcal{A}^h \Phi_h = \big[(\varepsilon_h^{-2}\mathcal{A}_0 + \varepsilon_h^{-1}\mathcal{A}_1 + \varepsilon_h^0 \mathcal{A}_2)\Phi\big](x, \frac{x}{\varepsilon_h})\,, \tag{3.3}$$

where

$$\begin{aligned}
\mathcal{A}_0 &= -\sum_{i,j=1}^n \frac{\partial}{\partial y_i}\big(a_{ij}(y)\frac{\partial}{\partial y_j}\big) \\
\mathcal{A}_1 &= -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\big(a_{ij}(y)\frac{\partial}{\partial y_j}\big) - \sum_{i,j=1}^n \frac{\partial}{\partial y_i}\big(a_{ij}(y)\frac{\partial}{\partial x_j}\big) \\
\mathcal{A}_2 &= -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\big(a_{ij}(y)\frac{\partial}{\partial x_j}\big)\,.
\end{aligned}$$

By using (3.2) and (3.3), the equation (3.1) becomes, under the assumptions that a and the $u_i(x,y)$ are smooth,

$$(\mathcal{A}_0 u_0)(x, \frac{x}{\varepsilon_h}) = 0 \qquad \text{on } \Omega\,, \tag{3.4}$$

$$(\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0)(x, \frac{x}{\varepsilon_h}) = 0 \qquad \text{on } \Omega\,, \tag{3.5}$$

$$(\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0)(x, \frac{x}{\varepsilon_h}) = f(x) \qquad \text{on } \Omega\,; \tag{3.6}$$

of course one can (formally) proceed:

$$(\mathcal{A}_0 u_3 + \mathcal{A}_1 u_2 + \mathcal{A}_2 u_1 + \mathcal{A}_3 u_0)(x, \frac{x}{\varepsilon_h}) = 0 \qquad \text{on } \Omega \quad \text{etc.}$$

Let us see that the homogenized operator can be constructed from (3.4), (3.5) and (3.6), which will be done in the sequel.

Because of the Y-periodicity of $u_i(x,\cdot)$, around any point x the function $z \mapsto u_i(z, \frac{z}{\varepsilon_h})$ behaves like $z \mapsto u_i(x, \frac{z}{\varepsilon_h})$. Hence we shall determine u_i by means of the following problems where x is now a fixed parameter:

$$\begin{cases} (\mathcal{A}_0 u_0)(x,\cdot) = 0 \qquad \text{on } Y\,, \\ u_0(x,\cdot) \quad Y\text{-periodic (i.e., } u_0(x,\cdot) \text{ has the same values on the opposite faces of } Y); \end{cases} \tag{3.7}$$

$$\begin{cases} (\mathcal{A}_0 u_1)(x,\cdot) = -(\mathcal{A}_1 u_0)(x,\cdot) \qquad \text{on } Y\,, \\ u_1(x,\cdot) \quad Y\text{-periodic;} \end{cases} \tag{3.8}$$

$$\begin{cases} (\mathcal{A}_0 u_2)(x,\cdot) = f(x) - (\mathcal{A}_1 u_1 + \mathcal{A}_2 u_0)(x,\cdot) \qquad \text{on } Y\,, \\ u_2(x,\cdot) \quad Y\text{-periodic.} \end{cases} \tag{3.9}$$

Let us consider problems (3.7)-(3.9) in the framework of weak solutions.

Let us start by proving an existence result for a boundary value problem on the unit cube. Let $H^{1,2}_{\text{per}}(Y)$ denote the subset of $H^{1,2}(Y)$ of functions u which have the same trace on the opposite faces of Y. Moreover, we denote by $H^{1,2}_{\sharp}(Y)$ the subset of $H^{1,2}(\Omega)$ of all the functions u with mean value zero which have the same trace on the opposite faces of Y.

Lemma 3.2. *Let $F \in (H^{1,2}_{\text{per}}(Y))^*$. Then*

$$\begin{cases} \int_Y (a(y) D\varphi, D\psi)\, dy = \langle F, \psi \rangle \qquad \textit{for every } \psi \in H^{1,2}_{\text{per}}(Y)\,, \\ \varphi \in H^{1,2}_{\sharp}(Y) \end{cases} \tag{3.10}$$

admits a unique solution if and only if

$$\langle F, 1 \rangle = 0\,. \tag{3.11}$$

Proof. Condition (3.11) is clearly necessary since $\int_Y (a(y)D\varphi, D\psi)\,dy = 0$ if ψ is constant. Note that $H^{1,2}_\sharp(Y)$ is a closed subset of $H^{1,2}(Y)$, and therefore a Hilbert space. Moreover, by the Poincaré-Wirtinger inequality (see Theorem A.1.14) $\|Dv\|_{L^2(Y;\mathbf{R}^n)}$ defines a norm on $H^{1,2}_\sharp(Y)$ equivalent to the norm $\|v\|_{H^{1,2}(Y)}$. Let us consider $a_1 : H^{1,2}_\sharp(Y) \times H^{1,2}_\sharp(Y) \to \mathbf{R}$ defined by

$$a_1(\varphi, \psi) = \int_Y (a(y)D\varphi, D\psi)\,dy\,.$$

Clearly, a_1 is a bilinear form. Moreover, by the boundedness assumption of a it follows that

$$|a_1(\varphi,\psi)| \le c\|D\varphi\|_{L^2(Y;\mathbf{R}^n)}\|D\psi\|_{L^2(Y;\mathbf{R}^n)}$$

for every φ, $\psi \in H^{1,2}_\sharp(Y)$. Therefore, a_1 is continuous. Moreover, the ellipticity condition satisfied by a implies immediately that a_1 is coercive. Therefore, by the Lax-Milgram lemma there exists a unique function $\varphi \in H^{1,2}_\sharp(Y)$ satisfying

$$\int_Y (a(y)D\varphi, D\psi)\,dy = \langle F, \psi\rangle \qquad \text{for every } \psi \in H^{1,2}_\sharp(Y)\,.$$

Since $\langle F, 1\rangle = 0$ it turns out that φ satisfies also

$$\int_Y (a(y)D\varphi, D\psi)\,dy = \langle F, \psi\rangle \qquad \text{for every } \psi \in H^{1,2}_{\text{per}}(Y)\,.$$

Hence, (3.10) admits a unique solution in $H^{1,2}_\sharp(Y)$. □

Let us apply this lemma to the solution of (3.7), (3.8) and (3.9).

Step 1: Study of (3.7).

Let us look for a solution to the problem (3.7), i.e.,

$$\begin{cases} (\mathcal{A}_0 u_0)(x,\cdot) = 0 & \text{on } Y\,, \\ u_0(x,\cdot) & Y\text{-periodic} \end{cases}$$

(note that the periodicity condition plays the role of boundary conditions). By using the Green formula and by taking into account the periodicity assumptions, one proves easily that problem (3.7) is equivalent to the following one: find $u_0(x,\cdot) \in H^{1,2}_{\text{per}}(Y)$ such that

$$\int_Y (a(y)Du_0, D\psi)\,dy = 0 \quad \text{for every } \psi \in H^{1,2}_{\text{per}}(Y)\,. \tag{3.12}$$

Let us recall that x plays the role of a parameter (hence Du_0 is the gradient with respect to y). By Lemma 3.2 we can conclude that $u_0(x,\cdot) \in H^{1,2}_{\text{per}}(Y)$ is determined by (3.12) up to a constant. By taking $\psi = u_0(x,\cdot)$ in (3.12) and by using the ellipticity condition satisfied by a, it follows immediately that $u_0(x,\cdot)$=costant, i.e.,

$$u_0(x,y) = u_0(x)\,. \tag{3.13}$$

Step 2: Study of (3.8).

Using (3.13), problem (3.8) reduces to

$$\text{(3.14)} \qquad \begin{cases} (\mathcal{A}_0 u_1)(x,\cdot) = \sum_{i,j=1}^n \big(\frac{\partial}{\partial y_i} a_{ij}(\cdot)\big) \frac{\partial u_0}{\partial x_j}(x) & \text{on } Y\,, \\ u_1(x,\cdot) \quad Y\text{-periodic.} \end{cases}$$

This is still a problem in y, where x is a parameter. Due to the separation of variables on the right hand side of (3.14), we are able to represent u_1 in a simple form. Let us note that by Green's formula and the periodicity assumptions the weak formulation of (3.14) becomes: find $u_1(x,\cdot) \in H^{1,2}_{\text{per}}(Y)$ such that

$$\text{(3.15)} \qquad \int_Y (a(y)Du_1, D\psi) dy = -\sum_{i,j=1}^n \frac{\partial u_0}{\partial x_j} \int_Y a_{ij} D_i \psi \, dy \quad \text{for every } \psi \in H^{1,2}_{\text{per}}(Y)\,.$$

Let us consider for $k = 1, \ldots, n$ the problem

$$\text{(3.16)} \qquad \begin{cases} \int_Y (a(y)Dw^k(y), D\psi(y))\, dy = -\int_Y (a(y)e_k, D\psi(y))\, dy & \text{for every } \psi \in H^{1,2}_\sharp(Y)\,, \\ w^k \in H^{1,2}_\sharp(Y)\,. \end{cases}$$

Note that the function $F^k : H^{1,2}_\sharp(Y) \to \mathbf{R}$ defined by $\psi \mapsto F^k(\psi) = -\int_Y (a(y)e_k, D\psi)\, dy$ is a linear and continuous function on $H^{1,2}_\sharp(Y)$. By the Lax-Milgram lemma for every $k = 1, \ldots, n$ there exists a unique solution to (3.16). Then the general solution to (3.15) becomes

$$\text{(3.17)} \qquad u_1(x,y) = \sum_{k=1}^n w^k(y) \frac{\partial u_0(x)}{\partial x_k} + \widetilde{u}_1(x)\,,$$

where $\widetilde{u}_1$ is an additive constant (function of the parameter x).

Step 3: Study of (3.9).

We now consider (3.9) where we think of u_2 as the unknown, x being a parameter. Let us consider the function $F : H^{1,2}_{\text{per}}(Y) \to \mathbf{R}$ defined by

$$\langle F, \psi \rangle = \int_Y f(x) \partial x_i(y)\, dy + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \int_Y a_{ij}(y) \frac{\partial u_1}{\partial y_j} \psi \, dy$$

$$- \sum_{i,j=1}^n \int_Y a_{ij}(y) \frac{\partial u_1}{\partial x_j} \frac{\partial \psi}{\partial y_i}\, dy + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \int_Y a_{ij}(y) \frac{\partial u_0}{\partial x_j} \psi \, dy\,.$$

We note that problem (3.9) is equivalent (the proof is analogous to the previous ones) to find $u_2(x,\cdot) \in H^{1,2}_{\text{per}}(Y)$ such that

$$\int_Y (a(y)Du_2, D\psi)\, dy = \langle F, \psi \rangle \qquad \text{for every } \psi \in H^{1,2}_{\text{per}}(Y)\,.$$

By virtue of Lemma 3.2, u_2 exists if and only if

$$\text{(3.18)} \qquad \langle F, 1 \rangle = 0\,.$$

Condition (3.18) is the homogenized equation we are looking for. Indeed, by taking $\psi = 1$ and the expression of u_1 into account, (3.18) becomes

$$\text{(3.19)} \qquad -\sum_{i,k=1}^n b_{ik} \frac{\partial^2 u_0}{\partial x_i \partial x_k} = f \qquad \text{on } \Omega\,,$$

where

$$b_{ik} = \frac{1}{|Y|}\int_Y (a_{ik}(y) + \sum_{j=1}^{n} a_{ij}(y)\frac{\partial w^k(y)}{\partial x_j})\, dy \tag{3.20}$$

(recall that $|Y| = 1$ and will be therefore successively omitted). The equation (3.19) is the *homogenized* equation and the coefficients b_{ij} are the *homogenized coefficients*. We will prove (see Proposition 4.2) that the homogenized matrix is symmetric if a has this property and satisfies an ellipticity condition like a. Finally, to obtain a well posed problem for u_0, we only need a boundary condition for u_0. From (3.1) and (3.2) we obtain $u_0|_{\partial\Omega}(x) = 0$ on $\partial\Omega$. Note that this relation is formal, but it will be rigorously proved below.

Remark 3.3. Let us note that the preceeding considerations can be summarized as follows: if we postulate an expansion of the form (3.2), the first term u_0 is determined as a solution to the equation (3.19) with the boundary condition $u_0|_{\partial\Omega}(x) = 0$ on $\partial\Omega$. The formal rule (which will be justified below) to compute the homogenized coefficients is as follows:

i) solve (3.16) on the unit cell Y, for $k = 1, \dots, n$;

ii) b_{ik} is given by (3.20).

We shall prove in Theorem 4.1 that (u_h) converges (in an suitable topology) to the function u_0 given above.

Remark 3.4. Let us conclude with some remarks on the homogenized operator.

(i) In the one-dimensional case one has $w^k = w$ solution to

$$-\frac{d}{dy}(a(y)\frac{dw}{dy}) = \frac{da(y)}{dy}\,;$$

hence, $a(y)\frac{dw}{dy} = -a(y) + c$. The condition on w to be periodic implies that $\int_Y(-1 + \frac{c}{a(y)})dy = 0$, i.e.,

$$-1 + c\mathcal{M}(\frac{1}{a}) = 0 \qquad \frac{dw}{dy} = -1 + \frac{c}{a(y)}\,.$$

Then, the homogenized coefficient has, according to (3.20), the form

$$b = \int_Y (a(y) - a(y) + c)\, dy = c = \frac{1}{\mathcal{M}(\frac{1}{a})}\,,$$

and we find (2.12).

(ii) Let us note that the homogenized coefficients have the form

$$b_{ik} = \mathcal{M}(a_{ik}) + \sum_{j=1}^{n}\mathcal{M}(a_{ij}\frac{\partial w^k}{\partial y_j})\,.$$

As we have already seen in the one dimensional case

$$\sum_{j=1}^{n} a_{ij}^h D_j u_h \quad \not\rightharpoonup \quad \sum_{j=1}^{n}\mathcal{M}(a_{ij})D_j u_0$$

and $\mathcal{M}(a_{ij}\frac{\partial w^k}{\partial y_j})$ appears as a "corrector".

4. Homogenization (symmetric case)

By $\mathcal{S}_\sharp$ we denote the set of all functions $a : \mathbf{R}^n \to M^{n\times n}$ such that $a(x) = (a_{ij}(x))$, $i,j = 1,\dots,n$ is Y-periodic and satisfies the following properties:

(4.1) $a_{ij} \in L^\infty(\mathbf{R}^n)$ for every $i,j = 1,\dots,n$;

(4.2) $a_{ij} = a_{ji}$ on $\mathbf{R}^n$ for every $i,j = 1,\dots,n$;

(4.3) there exists a constant $\alpha > 0$ such that $(a(x)\xi,\xi) = \sum_{i,j=1}^n a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2$ for a.e. $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$.

Given $a \in \mathcal{S}_\sharp$, we consider the following Dirichlet boundary value problems on the bounded open subset Ω of $\mathbf{R}^n$:

$$\begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon_h})Du_h) = f_h & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,, \end{cases} \tag{4.4}$$

where (f_h) is a sequence of functions converging strongly in $H^{-1,2}(\Omega)$ to f, and (ε_h) is a sequence of positive real numbers converging to 0.

In this section we shall prove the convergence, as (ε_h) tends to 0, of the solutions u_h to (4.4) to the solution u_0 of the following homogenized problem

$$\begin{cases} -\mathrm{div}(b\,Du_0) = f & \text{on } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{4.5}$$

The constant matrix $b = (b_{ij})$ is defined by

$$b_{ik} = \int_Y (a_{ik}(y) + \sum_{j=1}^n a_{ij}(y)\frac{\partial w^k(y)}{\partial y_j})\,dy\,, \tag{4.6}$$

where w^k is the unique solution to the local problem

$$\begin{cases} \int_Y (a(y)(e_k + Dw^k(y)), Dv(y))\,dy = 0 & \text{for every } v \in H_\sharp^{1,2}(Y)\,, \\ w^k \in H_\sharp^{1,2}(Y)\,. \end{cases} \tag{4.7}$$

More precisely, we shall present here Tartar's proof (known as the energy method) of the following convergence theorem of De Giorgi and Spagnolo (see [44], [23]). (Note that some homogenization results for (4.4) are proven also in [4] and [5].)

Theorem 4.1. *Let $a \in \mathcal{S}_\sharp$ and let (ε_h) be a sequence of positive real numbers converging to 0. Assume that (f_h) converges strongly in $H^{-1,2}(\Omega)$ to f. Let u_h and u_0 be the solutions to (4.4) and (4.5), respectively. Then,*

$$u_h \rightharpoonup u_0 \qquad \textit{weakly in } H_0^{1,2}(\Omega)\,. \tag{4.8}$$

$$a(\frac{x}{\varepsilon_h})Du_h \rightharpoonup b\,Du_0 \qquad \textit{weakly in } L^2(\Omega;\mathbf{R}^n)\,. \tag{4.9}$$

Proof. Recall that $a^h(x) = a(\frac{x}{\varepsilon_h}) \in M^{n\times n}$ for every $x \in \mathbf{R}^n$. The weak formulation of the Dirichlet boundary value problems (4.4) becomes then

$$\begin{cases} \int_\Omega (a^h(x)Du_h, Dv)\,dx = \langle f_h, v\rangle \quad \text{for every } v \in H_0^{1,2}(\Omega)\,, \\ u_h \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{4.10}$$

By taking $v = u_h$ in (4.10), and by taking (4.3) into account we have

$$\alpha \int_\Omega |Du_h|^2 dx \le \int_\Omega (a^h(x)Du_h, Du_h)\,dx \le \|f_h\|_{H^{-1,2}(\Omega)} \|u_h\|_{H_0^{1,2}(\Omega)} \le c\|u_h\|_{H_0^{1,2}(\Omega)}\,,$$

where c is a constant independent of h. By Remark A.1.15 this implies that

$$\|u_h\|_{H_0^{1,2}(\Omega)} \le C\,, \tag{4.11}$$

where C is a constant independent of h. Consider now the vector in $\mathbf{R}^n$ defined as

$$\xi^h(x) = a^h(x)Du_h(x) \quad \text{on } \Omega\,, \tag{4.12}$$

i.e., $\xi_i^h(x) = \sum_{j=1}^n a_{ij}^h(x)\frac{\partial u_h(x)}{\partial x_j}$ for every $i = 1, \ldots, n$. Since (4.1) and (4.11) hold, we get immediately

$$\|\xi^h\|_{L^2(\Omega;\mathbf{R}^n)} \le C'\,, \tag{4.13}$$

where C' is a constant independent of h. Therefore, there exist $u_* \in H_0^{1,2}(\Omega)$, $\xi^* \in L^2(\Omega;\mathbf{R}^n)$ and two subsequences, still denoted by (u_h), (ξ^h) such that

$$u_h \rightharpoonup u_* \quad \text{weakly in } H_0^{1,2}(\Omega)\,, \tag{4.14}$$

$$\xi^h \rightharpoonup \xi^* \quad \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,. \tag{4.15}$$

Now, by writing (4.10) in the form

$$\int_\Omega (\xi^h, Dv)\,dx = \langle f_h, v\rangle \qquad \text{for every } v \in H_0^{1,2}(\Omega)\,,$$

we can pass to the limit for any fixed $v \in H_0^{1,2}(\Omega)$ and we get

$$\int_\Omega (\xi^*, Dv)\,dx = \langle f, v\rangle \qquad \text{for every } v \in H_0^{1,2}(\Omega) \tag{4.16}$$

(note that here the weak convergence in $H^{-1,2}(\Omega)$ of (f_h) to f would suffice). Let us suppose that

$$\xi^*(x) = b\,Du_*(x) \qquad \text{for a.e. } x \in \Omega\,. \tag{4.17}$$

Then, (4.16) shows that $u_* \in H_0^{1,2}(\Omega)$ satisfies the weak formulation of the problem (4.5). By the uniqueness of the solution to problem (4.5), we may conclude that $u_* = u_0$ (in Proposition 4.2 we prove that b satisfies the same ellipticity conditions as a; hence, the solution of (4.5) is unique). Therefore *we have only to prove (4.17).* This will be done by means of the so called "energy method" (developed by

L. Tartar) which is based on the introduction of test functions of a special suitable form (let us underline that they have to be enough to identify the limit problem. As pointed out in other occasions, the main difficulty lies in the passage to the limit in products of only weakly convergent sequences).

Let us consider the local problem (4.7) and let $w^k \in H^{1,2}_\sharp(Y)$ be the solution to (4.7). Let us still denote by w^k its Y-periodic extension to $\mathbf{R}^n$. By Lemma A.1.16 it turns out that $w^k \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$. Let us define for every $k = 1, \dots, n$ the sequence of functions

$$w_h^k(x) = x_k + \varepsilon_h w^k(\frac{x}{\varepsilon_h}) = (e_k, x) + \varepsilon_h w^k(\frac{x}{\varepsilon_h}) \quad \text{for a.e. } x \in \mathbf{R}^n \tag{4.18}$$

(note that this function is in fact the sum of the first terms of the expansion $u_0(x) + \varepsilon_h u_1(x, \frac{x}{\varepsilon_h})$ with $u_0(x) = (e_k, x)$ and $\widetilde{u}_1(x) = 0$). The periodicity property of this function yields easily that

$$\begin{cases} w_h^k \to x_k & \text{strongly in } L^2(\Omega), \quad (\text{as } h \to \infty) \\ Dw_h^k \rightharpoonup e_k & \text{weakly in } L^2(\Omega; \mathbf{R}^n), \quad (\text{as } h \to \infty). \end{cases} \tag{4.19}$$

Moreover, by Lemma A.1.17 (with $g(y) = a(y)(e_k + Dw^k(y))$), the functions w_h^k satisfy the equations

$$-\mathrm{div}(a^h(x) Dw_h^k(x)) = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^n). \tag{4.20}$$

Then, by multiplying (4.20) by any function $v \in H_0^{1,2}(\Omega)$ we have

$$\int_\Omega (a^h(x) Dw_h^k, Dv)\, dx = 0. \tag{4.21}$$

In order to avoid difficulties with the boundary condition, let us take a function $\varphi \in C_0^\infty(\Omega)$ and let us write (4.10) with $v = \varphi w_h^k \in H_0^{1,2}(\Omega)$ and (4.21) with $v = \varphi u_h \in H_0^{1,2}(\Omega)$. We have then

$$\begin{aligned} &\int_\Omega (a^h(x) Du_h, (D\varphi) w_h^k)\, dx + \int_\Omega (a^h(x) Du_h, (Dw_h^k)\varphi)\, dx = \langle f_h, \varphi w_h^k \rangle \\ &\int_\Omega (a^h(x) Dw_h^k, (D\varphi) u_h)\, dx + \int_\Omega (a^h(x) Dw_h^k, (Du_h)\varphi)\, dx = 0 \end{aligned} \tag{4.22}$$

Since $a_{ij} = a_{ji}$ for every $i, j \in \{1, \dots, n\}$, we have that

$$\int_\Omega (a^h(x) Dw_h^k, (Du_h)\varphi)\, dx = \int_\Omega (a^h(x) Du_h, (Dw_h^k)\varphi)\, dx.$$

Therefore, by subtracting the second equation in (4.22) from the first one, we get

$$\int_\Omega (a^h(x) Du_h, (D\varphi) w_h^k)\, dx - \int_\Omega (a^h(x) Dw_h^k, (D\varphi) u_h)\, dx = \langle f_h, \varphi w_h^k \rangle \tag{4.23}$$

for every $\varphi \in C_0^\infty(\Omega)$. Now we are in the position to pass to the limit in (4.23) as $h \to \infty$, since each term is the scalar product in $L^2(\Omega; \mathbf{R}^n)$ of an element which converges weakly and another which converges strongly in $L^2(\Omega; \mathbf{R}^n)$. Indeed

$$\xi^h(x) = a^h(x) Du_h \rightharpoonup \xi^* \qquad \text{weakly in } L^2(\Omega; \mathbf{R}^n)$$

(by (4.15)), and

$$(D\varphi) w_h^k \to (D\varphi) x_k \quad \text{strongly in } L^2(\Omega; \mathbf{R}^n)$$

(note that $(D\varphi)$ is fixed). Moreover,

$$(a^h Dw_h^k)_i(x) = \sum_{j=1}^n a_{ij}(\frac{x}{\varepsilon_h})\frac{\partial w_h^k}{\partial x_j}(x) = \sum_{j=1}^n \big(a_{ij}(\cdot)(\delta_{jk} + \frac{\partial w^k}{\partial y_j}(\cdot))\big)(\frac{x}{\varepsilon_h})$$

for every $i \in \{1, \ldots, n\}$. Hence,

$$(a^h Dw_h^k)_i \rightharpoonup \int_Y (a_{ik}(y) + \sum_{j=1}^n a_{ij}(y)\frac{\partial w^k(y)}{\partial y_j})\, dy = (4.6) = b_{ik} \quad \text{weakly in } L^2(\Omega)\,. \tag{4.24}$$

Finally, (4.14) and Rellich's theorem imply that

$$(D\varphi)u_h \to (D\varphi)u_* \qquad \text{strongly in } L^2(\Omega;\mathbf{R}^n)\,.$$

Since by (4.1) the sequence (w_h^k) converges to x_k weakly in $H_0^{1,2}(\Omega)$ and by assumption (f_h) converges to f strongly in $H^{-1,2}(\Omega)$, we can finally assert that

$$\int_\Omega (\sum_{i=1}^n \xi_i^*(D_i\varphi)x_k - \sum_{i=1}^n b_{ik}(D_i\varphi)u_*)\, dx = \langle f, \varphi x_k\rangle\,. \tag{4.25}$$

Moreover, by (4.16) with $v = \varphi x_k$, the previous equation becomes

$$\int_\Omega \sum_{i=1}^n (\xi_i^* x_k - b_{ik}u_*)(D_i\varphi)\, dx = \int_\Omega \sum_{i=1}^n \xi_i^* D_i(\varphi x_k)\, dx \quad \text{for every } \varphi \in C_0^\infty(\Omega)\,,$$

and we get for every $k = 1, \ldots, n$

$$\int_\Omega (\xi_k^* - \sum_{i=1}^n b_{ik}D_i u_*)\varphi\, dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega)\,.$$

Since the last equation holds for every $\varphi \in C_0^\infty(\Omega)$, we get that

$$\xi_k^* = \sum_{i=1}^n b_{ik}D_i u_* \qquad \text{a.e. on } \Omega$$

for every $k = 1, \ldots, n$. By the simmetry of the matrix b, which is shown in the next proposition, the proof of (4.17) is accomplished. Since the homogenized operator is uniquely defined and we have uniqueness of the solution to (4.5) we may conclude that the convergences

$$\begin{aligned} u_h &\rightharpoonup u_0 \qquad \text{weakly in } H_0^{1,2}(\Omega) \\ a^h Du_h &\rightharpoonup bDu_0 \qquad \text{weakly in } L^2(\Omega;\mathbf{R}^n) \end{aligned}$$

hold for the whole sequence, and not only for the above extracted subsequence. □

Proposition 4.2. *Let $a : \mathbf{R}^n \to M^{n\times n}$ be a function in $\mathcal{S}_\sharp$. Let b be the constant matrix defined by (4.6). Then b is still symmetric and satisfies the same ellipticity condition as a, i.e.,*

(i) $b_{ik} = b_{ki}$ *for every* $i, k = 1, \dots n$;

(ii) $(b\xi, \xi) = \sum_{i,k=1}^{n} b_{ik}\xi_k\xi_i \geq \alpha|\xi|^2$ *for every* $\xi \in \mathbf{R}^n$.

Proof. Let us prove (i). Fix k and s in $\{1, \dots, n\}$ and let $v = w^s$ in (4.7). We obtain

$$\int_Y (a(y)(e_k + Dw^k(y)), Dw^s(y))\, dy = 0\,.$$

By adding to both sides the quantity $\int_Y (a(y)(e_k + Dw^k(y)), e_s)\, dy$ we get

$$\int_Y (a_{sk}(y) + \sum_{j=1}^{n} a_{sj}(y)\frac{\partial w^k(y)}{\partial y_j})\, dy = \int_Y (a(y)(e_k + Dw^k(y)), (e_s + Dw^s(y)))\, dy\,,$$

i.e.,

$$b_{sk} = \int_Y (a(y)(e_k + Dw^k(y)), (e_s + Dw^s(y)))\, dy\,.$$

Since $a(x)$ is symmetric on $\mathbf{R}^n$, the proof of (i) is accomplished.

Let us show (ii). Given $\xi \in \mathbf{R}^n$, let us define the sequence of functions

$$v_h(x) = \sum_{k=1}^{n} \xi_k w_h^k(x) \quad \text{for a.e. } x \in \mathbf{R}^n\,,$$

where $w_h^k(x)$, $k = 1, \dots, n$ are the functions defined by (4.18). Note that by (4.19) and (4.24) we have

$$\text{(4.26)} \qquad \begin{cases} v_h \to \sum_{k=1}^{n} \xi_k x_k = (\xi, x) & \text{strongly in } L^2(\Omega)\,, \\ Dv_h \rightharpoonup \xi & \text{weakly in } L^2(\Omega; \mathbf{R}^n)\,, \\ (a^h Dv_h)_i \rightharpoonup (b\xi)_i & \text{weakly in } L^2(\Omega), \text{ for every } i = 1, \dots n\,. \end{cases}$$

Moreover by (4.20) we obtain $-\mathrm{div}(a^h Dv_h) = 0$ in $\mathcal{D}'(\mathbf{R}^n)$. Let us show that

$$\text{(4.27)} \qquad \int_\Omega (a^h(x)Dv_h, Dv_h)\varphi\, dx \to \int_\Omega (b\xi, \xi)\varphi\, dx \quad \text{for every } \varphi \in C_0^\infty(\Omega)\,.$$

Note that

$$\begin{aligned} \int_\Omega (a^h(x)Dv_h, Dv_h)\varphi\, dx &= -\int_\Omega (a^h(x)Dv_h, D\varphi)v_h\, dx - \langle \mathrm{div}(a^h Dv_h), \varphi v_h\rangle \\ &= -\int_\Omega (a^h Dv_h, D\varphi)v_h\, dx\,. \end{aligned}$$

By virtue of (4.26) the last integral converges to

$$-\int_\Omega (b\xi, D\varphi)(\xi, x)\, dx = \int_\Omega (b\xi, \xi)\varphi\, dx\,,$$

which proves (4.27) (note that this result can be obtained also immediately by the compensated compactness lemma (see Lemma A.2.1 with $g_h = a^h Dv_h$ and $u_h = v_h$).

Let us note that the ellipticity condition of a implies that

$$\int_{\Omega}(a^h(x)Dv_h, Dv_h)\varphi\, dx \geq \int_{\Omega}\alpha|Dv_h|^2\varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Now, by passing to the limit as $h \to \infty$, (4.27) and the weak lower semicontinuity of the norm in $L^2(\Omega; \mathbf{R}^n)$ imply that

$$\int_{\Omega}(b\xi, \xi)\varphi\, dx \geq \alpha \int_{\Omega}|\xi|^2\varphi\, dx \tag{4.28}$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, which implies immediately (ii). □

Let us state now some results which follow with minor modifications from the previous homogenization result.

Let $a_0 : \mathbf{R}^n \to \mathbf{R}$ be a Y-periodic function belonging to $L^\infty(\mathbf{R}^n)$. Moreover assume that there exists a constant $\beta > 0$ such that $a_0 \geq \beta$ a.e. on $\mathbf{R}^n$. Then we have:

Corollary 4.3. *Let $a \in \mathcal{S}_\sharp$, a_0 be as defined above, and let (ε_h) be a sequence of positive real numbers converging to 0. Assume that (f_h) converges to f strongly in $H^{-1,2}(\Omega)$. For every $h \in \mathbf{N}$, let u_h be the solution to the Dirichlet boundary value problem*

$$\begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon_h})Du_h) + a_0(\frac{x}{\varepsilon_h})u_h = f_h & \text{on } \Omega, \\ u_h \in H_0^{1,2}(\Omega). \end{cases} \tag{4.29}$$

Then

$$u_h \rightharpoonup u_0 \qquad \text{weakly in } H_0^{1,2}(\Omega),$$

where u_0 is the unique solution to the homogenized problem

$$\begin{cases} -\mathrm{div}(bDu_0) + \mathcal{M}(a_0)u_0 = f & \text{on } \Omega, \\ u_0 \in H_0^{1,2}(\Omega). \end{cases} \tag{4.30}$$

The constant matrix b is defined by (4.6) and $\mathcal{M}(a_0)$ is the mean value of a_0 on Y.

Proof. It follows easily from Theorem 4.1 and the fact that

$$\int_{\Omega} a_0(\frac{x}{\varepsilon_h})u_h v\, dx \to \int_{\Omega}\mathcal{M}(a_0)u_0 v\, dx$$

for every $v \in H_0^{1,2}(\Omega)$ (note that $a_0(\frac{x}{\varepsilon_h}) \overset{*}{\rightharpoonup} \mathcal{M}(a_0)$ in $L^\infty(\Omega)$ weak*, and hence we have $a_0(\frac{x}{\varepsilon_h})u_h \rightharpoonup \mathcal{M}(a_0)u_0$ weakly in $L^2(\Omega)$). □

Corollary 4.4. *Assume $\partial\Omega$ be Lipschitz. Let $a \in \mathcal{S}_\sharp$, a_0 be as defined above, and let (ε_h) be a sequence of positive real numbers converging to 0. Assume that (f_h) converges to f weakly in $L^2(\Omega)$. For every $h \in \mathbf{N}$, let u_h be the solution to the Neumann boundary value problem*

$$\begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon_h})Du_h) + a_0(\frac{x}{\varepsilon_h})u_h = f_h & \text{on } \Omega, \\ (a(\frac{x}{\varepsilon_h})Du_h, \nu) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.31}$$

where ν *denotes the unit outer normal to the boundary* $\partial\Omega$. *Then*

$$u_h \rightharpoonup u_0 \qquad \textit{weakly in } H^{1,2}(\Omega)\,,$$

where u_0 *is the unique solution to the problem*

$$\begin{cases} -\mathrm{div}(bDu_0) + \mathcal{M}(a_0)u_0 = f & \textit{on } \Omega\,, \\ (bDu_0, \nu) = 0 \quad \textit{on } \partial\Omega\,. \end{cases} \tag{4.32}$$

The constant matrix b *is defined by (4.6) and* $\mathcal{M}(a_0)$ *is the mean value of* a_0 *on* Y.

Proof. Let us use as above the notation $a^h(x) = a(\frac{x}{\varepsilon_h})$ and set $a_0^h(x) = a_0(\frac{x}{\varepsilon_h})$. By weak solution of (4.31) we mean a function $u_h \in H^{1,2}(\Omega)$ satisfying

$$\int_\Omega \big((a^h(x)Du_h, Dv) + a_0^h(x)u_h v\big)dx = \int_\Omega f_h v\,dx \qquad \text{for every } v \in H^{1,2}(\Omega)\,. \tag{4.33}$$

Proceeding as in Theorem 4.1 and taking into account that $a_0(x) \geq \beta > 0$ for a.e. $x \in \mathbf{R}^n$, we get that the sequence (u_h) is uniformly bounded in $H^{1,2}(\Omega)$ and therefore converges (up to a subsequence) weakly in $H^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to a function u_*. Moreover, by the periodicity property of a_0 we have that (a_0^h) converges to $\mathcal{M}(a_0)$ in $L^\infty(\Omega)$ weak*. We then obtain instead of the equation (4.16) the following relation

$$\int_\Omega \big((\xi^*, Dv) + \mathcal{M}(a_0)u_* v\big)\,dx = \int_\Omega fv\,dx \qquad \text{for every } v \in H^{1,2}(\Omega) \tag{4.34}$$

(note that ξ^* is the weak limit in $L^2(\Omega;\mathbf{R}^n)$ of the sequence (ξ^h) defined as $a^h(x)Du_h(x)$). This shows that the functions ξ^* and u_* satisfy a certain equation and an associated boundary condition as in the classical Neumann boundary value problem. Therefore, to conclude the proof it remains to show that

$$\xi^*(x) = bDu_*(x) \qquad \text{for a.e. } x \in \Omega\,.$$

But the proof of this relation is of course the same as in Theorem 4.1, since it is a local property independent of the boundary conditions. □

Remark 4.5. The example studied in Section 2 shows that in general (Du_h) *does not converge strongly in* $L^2(\Omega;\mathbf{R}^n)$ to Du_0. Indeed, assume for a moment that the solutions u_h and u_0 to (2.3) and (2.11) respectively, satisfy

$$\frac{du_h}{dx} \to \frac{du_0}{dx} \qquad \text{strongly in } L^2(\Omega)\,. \tag{4.35}$$

Then,

$$a^h(x)\frac{du_h}{dx} \rightharpoonup \mathcal{M}(a)\frac{du_0}{dx} \qquad \text{weakly in } L^2(\Omega)\,; \tag{4.36}$$

and one would be able to pass to the limit directly in the equation

$$\int_\Omega a^h \frac{du_h}{dx}\frac{dv}{dx}dx = \int_\Omega fv\,dx$$

and obtain

$$\int_\Omega \mathcal{M}(a)\frac{du_0}{dx}\frac{dv}{dx}\,dx = \int_\Omega fv\,dx \qquad \text{for every } v \in H^{1,2}(\Omega)\,.$$

But this is not the limit equation (see also Remark 2.1) and we get a contradiction.

Therefore, (4.8.) cannot be improved without adding extra terms (of the "corrector" type). In [9] (Chapter 1, Section 5) one can find the proof of the following corrector result.

Theorem 4.6. *Let us assume that the hypotheses of Theorem 4.1 hold true. Moreover, assume that*

i) $f_h, f \in L^2(\Omega)$;

ii) w^k *defined by (4.7) belongs to* $W^{1,\infty}(Y)$ *for every* $k = 1, \dots, n$.

Then

$$Du_h = Du_0 + P^h Du_0 + r_h \qquad \textit{with } r_h \to 0 \textit{ strongly in } L^2(\Omega; \mathbf{R}^n),$$

where the matrix $P^h(x) = (P^h_{ik}(x))$ *is defined by* $P^h_{ik}(x) = \frac{\partial w^k}{\partial x_i}(\frac{x}{\varepsilon_h})$.

Remark 4.7. Note that from a numerical point of view correctors are important since the weak $H^{1,2}$-convergence is not completely satisfactory. Correctors give a "good" approximation of Du_h, since it is an approximation in the strong topology of $L^2(\Omega; \mathbf{R}^n)$ (the term $P^h Du_0$ "corrects" rapid oscillations of the gradient of $(u_h - u_0)$).

Furthermore, the corrector result turns out to be a basic tool in the study of the asymptotic behaviour of the bounded solutions u_h to quasilinear equations of the form

$$-\text{div}(a(\frac{x}{\varepsilon_h})Du_h) + \gamma u_h = H^h(x, u_h, Du_h),$$

where $a \in \mathcal{S}_\sharp$, $\gamma > 0$ and the Hamiltonians $H^h = H^h(x, s, \xi)$ are measurable in x, continuous in the pair $[s, \xi]$ and have, for example, quadratic growth in ξ (see [8], where also the case a non-symmetric has been considered).

II. Homogenization of monotone operators

5. Homogenization and correctors for monotone operators

Let us deal now with the homogenization of a sequence of nonlinear monotone operators $\mathcal{A}^h : H^{1,2}_0(\Omega) \to H^{-1,2}(\Omega)$ of the form

$$\mathcal{A}^h u = -\text{div}(a(\frac{x}{\varepsilon_h}, Du)),$$

where $a(x, \cdot)$ is Y-periodic and satisfies suitable assumptions of uniform strict monotonicity and uniform Lipschitz-continuity. The results presented here are contained in [47] (see also [2] and [3]).

By $\mathcal{N}_\sharp$ we denote the set of all functions $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ such that for every $\xi \in \mathbf{R}^n$, $a(\cdot, \xi)$ is Lebesgue measurable and Y-periodic and satisfies the following properties: there exist two constants $0 < \alpha \le \beta < +\infty$ such that

(5.1) (strict monotonicity) $(a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2) \ge \alpha |\xi_1 - \xi_2|^2$

(5.2) (Lipschitz-continuity) $|a(x, \xi_1) - a(x, \xi_2)| \le \beta |\xi_1 - \xi_2|$

for a.e. $x \in \mathbf{R}^n$ and for every ξ_1, $\xi_2 \in \mathbf{R}^n$. Moreover

(5.3) $a(x, 0) = 0$ for a.e. $x \in \mathbf{R}^n$.

Remark 5.1. Note that $a(x,\xi) = a(x)\xi$, where $a : \mathbf{R}^n \to M^{n\times n}$ is Y-periodic and satisfies (4.1) and (4.3) (without any symmetry assumption) belongs to $\mathcal{N}_\sharp$. Therefore we shall deduce from a homogenization result proven for $\mathcal{N}_\sharp$ a homogenization result for a sequence of operators of the form $\mathcal{A}^h u = -\mathrm{div}(a(\frac{x}{\varepsilon_h})Du)$, where a is not necessarily symmetric.

Given $a \in \mathcal{N}_\sharp$, for every $\varepsilon_h > 0$ and $f_h \in H^{-1,2}(\Omega)$ let us consider the following Dirichlet boundary value problem on the bounded open subset Ω of $\mathbf{R}^n$:

$$\begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du_h)) = f_h \quad \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{5.4}$$

Remark 5.2. By a classical result in existence theory for boundary value problems defined by monotone operators (see Theorem A.3.2) for every $f_h \in H^{-1,2}(\Omega)$ and for every $\varepsilon_h > 0$ there exists a unique solution $u_h \in H_0^{1,2}(\Omega)$ to (5.4). Indeed, let us consider the operator $\mathcal{A}^h : H_0^{1,2}(\Omega) \to H^{-1,2}(\Omega)$ defined by

$$\mathcal{A}^h u = -\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du))\,.$$

By (5.1) we have that

$$\begin{aligned} &\langle \mathcal{A}^h u_1 - \mathcal{A}^h u_2, u_1 - u_2\rangle = \\ &= \int_\Omega (a(\frac{x}{\varepsilon_h}, Du_1) - a(\frac{x}{\varepsilon_h}, Du_2), Du_1 - Du_2)\,dx \,\geq\, \alpha \int_\Omega |Du_1 - Du_2|^2 dx \end{aligned}$$

for every u_1, $u_2 \in H_0^{1,2}(\Omega)$, which guarantees that $\mathcal{A}^h$ is a strictly monotone and coercive map on $H_0^{1,2}(\Omega)$ (take into account (5.3)). Moreover, by (5.2) we get

$$\|\mathcal{A}^h u_1 - \mathcal{A}^h u_2\|_{H^{-1,2}(\Omega)} \,\leq\, \beta \|u_1 - u_2\|_{H_0^{1,2}(\Omega)}$$

for every u_1, $u_2 \in H_0^{1,2}(\Omega)$ which proves the continuity of $\mathcal{A}^h$. Therefore, by Theorem A.3.2 we have existence and uniqueness of a solution to (5.4)

In this section we shall prove the following homogenization result:

Theorem 5.3. *Let $a \in \mathcal{N}_\sharp$ and let (ε_h) be a sequence of positive real numbers converging to 0. Assume that (f_h) converges strongly in $H^{-1,2}(\Omega)$ to f. Let (u_h) be the solutions to (5.4). Then,*

$$\begin{aligned} &u_h \rightharpoonup u_0 \qquad \textit{weakly in } H_0^{1,2}(\Omega)\,, \\ &a(\frac{x}{\varepsilon_h}, Du_h) \rightharpoonup b(Du_0) \qquad \textit{weakly in } L^2(\Omega;\mathbf{R}^n)\,, \end{aligned}$$

where u_0 is the unique solution to the homogenized problem

$$\begin{cases} -\mathrm{div}(b(Du_0)) = f \quad \textit{on } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{5.5}$$

The operator $b : \mathbf{R}^n \to \mathbf{R}^n$ is defined for every $\xi \in \mathbf{R}^n$ by

$$b(\xi) = \int_Y a(y, \xi + Dw^\xi(y))\,dy\,, \tag{5.6}$$

where w^ξ is the unique solution to the local problem

$$\begin{cases} \int_Y (a(y,\xi + Dw^\xi(y)), Dv(y))\,dy = 0 \quad \textit{for every } v \in H_\sharp^{1,2}(Y)\,, \\ w^\xi \in H_\sharp^{1,2}(Y)\,. \end{cases} \tag{5.7}$$

Remark 5.4. Proceeding analogously as in Remark 5.2 one can prove the existence and uniqueness of a solution to the local problem (5.7). It can be shown directly by using the definition of b and the properties satisfied by a, that $b : \mathbf{R}^n \to \mathbf{R}^n$ is monotone and continuous on $\mathbf{R}^n$ (hence, by Theorem A.3.2 maximal monotone). Furthermore, it will be seen in the sequel that the operator b satisfies strict monotonicity properties like a (this implies in particular the uniqueness of the solution to (5.5)).

Let us show that b is monotone. Given ξ_1, $\xi_2 \in \mathbf{R}^n$, by the definition of b there exist $w^{\xi_i} \in H^{1,2}_\sharp(Y)$ $i = 1, 2$ satisfying

$$\int_Y (a(y, \xi_i + Dw^{\xi_i}(y)), Dv)\, dy = 0 \qquad \text{for every } v \in H^{1,2}_\sharp(Y) \tag{5.8}$$

and

$$b(\xi_i) = \int_Y a(y, \xi_i + Dw^{\xi_i}(y))\, dy\,.$$

Therefore, by taking (5.8) and (5.1) into account, we get

$$\begin{aligned}
(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) &= (\int_Y a(y, \xi_1 + Dw^{\xi_1}(y))\, dy - \int_Y a(y, \xi_2 + Dw^{\xi_2}(y))\, dy, \xi_1 - \xi_2) \\
&= \int_Y (a(y, \xi_1 + Dw^{\xi_1}(y)) - a(y, \xi_2 + Dw^{\xi_2}(y)), \xi_1 - \xi_2)\, dy \\
&= \int_Y (a(y, \xi_1 + Dw^{\xi_1}(y)) - a(y, \xi_2 + Dw^{\xi_2}(y)), (\xi_1 - Dw^{\xi_1}(y)) - (\xi_2 + Dw^{\xi_2}(y)))\, dy \\
&\geq 0\,;
\end{aligned}$$

this proves that b is monotone.

Let us prove that for every ξ_1, $\xi_2 \in \mathbf{R}^n$ we have

$$|b(\xi_1) - b(\xi_2)| \leq \frac{\beta^2}{\alpha} |\xi_1 - \xi_2|\,. \tag{5.9}$$

Let $w^{\xi_i} \in H^{1,2}_\sharp(Y)$ $i = 1, 2$ satisfying

$$\int_Y (a(y, \xi_i + Dw^{\xi_i}(y)), Dv)\, dy = 0 \qquad \text{for every } v \in H^{1,2}_\sharp(Y) \tag{5.10}$$

and

$$b(\xi_i) = \int_Y a(y, \xi_i + Dw^{\xi_i}(y))\, dy\,.$$

Then, by taking (5.2), (5.1) and (5.10) into account, we get

$$\begin{aligned}
|b(\xi_1) - b(\xi_2)|^2 &= |\int_Y a(y, \xi_1 + Dw^{\xi_1}(y))\, dy - \int_Y a(y, \xi_2 + Dw^{\xi_2}(y))\, dy|^2 \\
&\leq \Big(\int_Y |a(y, \xi_1 + Dw^{\xi_1}(y)) - a(y, \xi_2 + Dw^{\xi_2}(y))|\, dy\Big)^2 \\
&\leq \Big(\beta \int_Y |(\xi_1 + Dw^{\xi_1}(y)) - (\xi_2 + Dw^{\xi_2}(y))|\, dy\Big)^2 \\
&\leq \beta^2 \Big(\int_Y |(\xi_1 + Dw^{\xi_1}(y)) - (\xi_2 + Dw^{\xi_2}(y))|^2\, dy\Big) \\
&\leq \frac{\beta^2}{\alpha} \Big(\int_Y (a(y, \xi_1 + Dw^{\xi_1}(y)) - a(y, \xi_2 + Dw^{\xi_2}(y)), (\xi_1 + Dw^{\xi_1}(y)) - (\xi_2 + Dw^{\xi_2}(y))\, dy\Big) \\
&\leq \frac{\beta^2}{\alpha} (b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \leq \frac{\beta^2}{\alpha} |b(\xi_1) - b(\xi_2)| |\xi_1 - \xi_2|\,,
\end{aligned}$$

and (5.9) follows.

Proof of Theorem 5.3. By Remark 5.2, for every $h \in \mathbf{N}$, there exists a unique solution u_h to the problem

$$\begin{cases} \int_\Omega (a(\frac{x}{\varepsilon_h}, Du_h), Dv)\, dx = \langle f_h, v\rangle \qquad \text{for every } v \in H_0^{1,2}(\Omega)\,, \\ u_h \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{5.11}$$

By taking $v = u_h$ in (5.11) and by means of the assumptions (5.1) and (5.3) (take into account also that (f_h) is uniformly bounded in $H^{-1,2}(\Omega)$), we get immediately

$$\|u_h\|_{H_0^{1,2}(\Omega)} \leq c\,, \tag{5.12}$$

where c is a constant independent of h. Let us define

$$\xi^h = a(\frac{x}{\varepsilon_h}, Du_h)\,.$$

By (5.2), (5.3) and (5.12) we obtain that

$$\|\xi^h\|_{L^2(\Omega;\mathbf{R}^n)} \leq C\,, \tag{5.13}$$

where C is a constant independent of h. Therefore, there exist $u_* \in H_0^{1,2}(\Omega)$ and $\xi^* \in L^2(\Omega;\mathbf{R}^n)$ and two subsequences, still denoted by (u_h) and (ξ^h), such that

$$\begin{aligned} u_h &\rightharpoonup u_* \qquad \text{weakly in } H_0^{1,2}(\Omega)\,, \\ \xi^h &\rightharpoonup \xi^* \qquad \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,. \end{aligned} \tag{5.14}$$

By passing to the limit in (5.11) we get (in the sense of distributions)

$$-\mathrm{div}\xi^* = f \qquad \text{on } \Omega$$

(note that here the weak convergence in $H^{-1,2}(\Omega)$ of (f_h) to f would suffice). If we show that

$$\xi^* = b(Du_*) \qquad \text{a.e. on } \Omega\,,$$

then by the uniqueness of the solution to problem (5.5) we have to conclude that $u_* = u_0$. Arguing as in the proof of Theorem 4.1 we obtain then that the convergences

$$\begin{aligned} u_h &\rightharpoonup u_* \qquad \text{weakly in } H_0^{1,2}(\Omega)\,, \\ \xi^h &\rightharpoonup \xi^* \qquad \text{weakly in } L^2(\Omega;\mathbf{R}^n) \end{aligned}$$

hold for the whole sequence, and not only for the above extracted subsequence. Therefore, the proof of Theorem 5.3 is accomplished if we show that $\xi^* = b(Du_*)$ a.e. on Ω.

In order to prove that $\xi^* = b(Du_*)$ a.e. on Ω we define a sequence of suitable functions $w_h^\eta \in H^{1,2}(\Omega)$, $\varepsilon_h Y$-periodic, in the following way. Given $\eta \in \mathbf{R}^n$, let us consider a solution $w^\eta \in H_\sharp^{1,2}(Y)$ to problem (5.7). Let us still denote by w^η its Y-periodic extension to $\mathbf{R}^n$. It can be proved (see Lemma A.1.16) that $w^\eta \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ and

$$\int_{\mathbf{R}^n} (a(x, \eta + Dw^\eta(x)), Dv(x))\, dx = 0$$

for every $v \in C_0^\infty(\mathbf{R}^n)$ (see Lemma A.1.17). Let us define

$$w_h^\eta(x) = (\eta, x) + \varepsilon_h w^\eta(\frac{x}{\varepsilon_h}) \qquad \text{for a.e. } x \in \mathbf{R}^n . \tag{5.15}$$

The periodicity properties of this function and of a yield easily that

$$\begin{cases} w_h^\eta \rightharpoonup (\eta, x) & \text{weakly in } H^{1,2}(\Omega) , \\ Dw_h^\eta \rightharpoonup \eta & \text{weakly in } L^2(\Omega; \mathbf{R}^n) , \\ a(\frac{x}{\varepsilon_h}, Dw_h^\eta(x)) = a(\cdot, \eta + Dw^\eta(\cdot))(\frac{x}{\varepsilon_h}) \rightharpoonup b(\eta) & \text{weakly in } L^2(\Omega; \mathbf{R}^n) . \end{cases} \tag{5.16}$$

By the monotonicity of a we have

$$\int_\Omega (a(\frac{x}{\varepsilon_h}, Du_h(x)) - a(\frac{x}{\varepsilon_h}, Dw_h^\eta(x)), Du_h(x) - Dw_h^\eta(x))\varphi(x)\, dx \ \geq\ 0$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. By passing to the limit as h tends to ∞, the compensated compactness lemma A.2.1 implies that

$$\int_\Omega (\xi^*(x) - b(\eta), Du_*(x) - \eta)\varphi(x)\, dx \ \geq\ 0$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ (note that $-\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du_h)) = f_h$, and (f_h) converges to f strongly in $H^{-1,2}(\Omega)$; moreover $-\mathrm{div}(a(\frac{x}{\varepsilon_h}, Dw_h^\eta)) = 0$ for every $h \in \mathbf{N}$, and (5.16) hold). Therefore, for every $\eta \in \mathbf{R}^n$ we have

$$(\xi^*(x) - b(\eta), Du_*(x) - \eta) \ \geq\ 0 \qquad \text{for a.e. } x \in \Omega . \tag{5.17}$$

In particular, if we denote by (η_m) a countable dense subset of $\mathbf{R}^n$, (5.17) yields that

$$(\xi^*(x) - b(\eta_m), Du_*(x) - \eta_m) \ \geq\ 0 \qquad \text{for a.e. } x \in \Omega, \text{ for every } m \in \mathbf{N} . \tag{5.18}$$

This implies by the continuity of b (see Remark 5.4) that

$$(\xi^*(x) - b(\eta), Du_*(x) - \eta) \ \geq\ 0 \qquad \text{for a.e. } x \in \Omega \text{ and for every } \eta \in \mathbf{R}^n .$$

By taking the maximal monotonicity of b into account the last inequality guarantees that $\xi^*(x) = b(Du_*(x))$ for a.e. $x \in \Omega$, which was our goal. ☐

Proposition 5.5. *The operator $b : \mathbf{R}^n \to \mathbf{R}^n$ defined by (5.6) satisfies the following property:*

$$(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \ \geq\ \alpha|\xi_1 - \xi_2|^2 \tag{5.19}$$

for every ξ_1, $\xi_2 \in \mathbf{R}^n$.

Proof. Let $\xi_i \in \mathbf{R}^n$, $i = 1, 2$. For every $i = 1, 2$ let us consider the sequence of functions $w_h^{\xi_i} \in H^{1,2}(\Omega)$ such that

$$\begin{cases} w_h^{\xi_i} \rightharpoonup (\xi_i, x) & \text{weakly in } H^{1,2}(\Omega) , \\ Dw_h^{\xi_i} \rightharpoonup \xi_i & \text{weakly in } L^2(\Omega; \mathbf{R}^n) , \\ a(\frac{x}{\varepsilon_h}, Dw_h^{\xi_i}(x)) = a(\cdot, \xi_i + Dw^{\xi_i}(\cdot))(\frac{x}{\varepsilon_h}) \rightharpoonup b(\xi_i) & \text{weakly in } L^2(\Omega; \mathbf{R}^n) . \end{cases} \tag{5.20}$$

By the monotonicity of a it follows that

$$\int_\Omega (a(\frac{x}{\varepsilon_h}, Dw_h^{\xi_1}) - a(\frac{x}{\varepsilon_h}, Dw_h^{\xi_2}), Dw_h^{\xi_1} - Dw_h^{\xi_2})\varphi(x)\,dx \geq \alpha \int_\Omega |Dw_h^{\xi_1} - Dw_h^{\xi_2}|^2 \varphi(x)\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. By passing to the limit as h tends to ∞, the compensated compactness lemma (used on the left hand side) and the weak lower semicontinuity of the norm in $L^2(\Omega;\mathbf{R}^n)$ (applied on the right hand side) ensure that

$$\int_\Omega (b(\xi_1) - b(\xi_2), D\xi_1 - D\xi_2)\varphi(x)\,dx \geq \alpha \int_\Omega |\xi_1 - \xi_2|^2 \varphi(x)\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, which implies (5.19). □

Finally, let us mention that a corrector result for the case $a \in \mathcal{N}_\sharp$ has been proven in [22]. It can be stated as follows:

Theorem 5.6. *Assume that the hypotheses of Theorem 5.3 hold true. Let u_h be the solutions to the equations (5.4) and let u_0 be the solution to problem (5.5). Then*

$$Du_h = p_h(\cdot, M_h Du_0) + r_h \qquad \textit{with } r_h \to 0 \textit{ strongly in } L^2(\Omega;\mathbf{R}^n)\,.$$

Here, for every $\varepsilon_h > 0$, the function $p_h : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is defined by $p_h(x,\xi) = \xi + Dw^\xi(\frac{x}{\varepsilon_h})$, where w^ξ is the unique solution to the local problem (5.7). Moreover, for every $\varphi \in L^2(\Omega;\mathbf{R}^n)$ the function $M_h\varphi : \mathbf{R}^n \to \mathbf{R}^n$ is defined by

$$(M_h\varphi)(x) = \sum_{i\in I_h} 1_{Y_h^i}(x) \frac{1}{|Y_h^i|} \int_{Y_h^i} \varphi(y)\,dy\,,$$

where $Y_h^i = \varepsilon_h(i+Y)$ (for $i \in \mathbf{Z}^n$), $I_h = \{i \in \mathbf{Z}^n : Y_h^i \subseteq \Omega\}$ and 1_A is the characteristic function of a set $A \subseteq \mathbf{R}^n$.

Remark 5.7. This corrector result permits to study the limit behaviour of the bounded solutions u_h to quasi-linear equations of the form

$$-\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du_h)) + \gamma u_h = H(\frac{x}{\varepsilon_h}, u_h, Du_h)\,,$$

where $a \in \mathcal{N}_\sharp$, $\gamma > 0$, and $H = H(x,s,\xi)$ is Y-periodic in x, continuous in the pair $[s,\xi]$ and grows at most like $|\xi|^2$ (for more details see [20]).

Conclusive remarks

Let us conclude this chapter with the statement of some further results on homogenization of nonlinear monotone operators in divergence form.

The case $1 < p < +\infty$ has been studied under analogous hypotheses of uniform strict monotonicity and equicontinuity for a by Fusco and Moscariello in [27] and [28]. Given two positive constants α and β, they prove an homogenization result for

$$\mathcal{A}^h u = -\mathrm{div}(a(\frac{x}{\varepsilon_h}, u, Du))\,,$$

where $a(x,s,\xi)$ verifies the following structure conditions:

a) $a(\cdot,s,\xi)$ is Y-periodic and Lebesgue measurable on $\mathbf{R}^n$;

b) for a.e. $x\in\mathbf{R}^n$, for every s, s_1, $s_2\in\mathbf{R}$, and ξ_1, $\xi_2\in\mathbf{R}^n$

if $p\ge 2$

i) $(a(x,s,\xi_1)-a(x,s,\xi_2),\xi_1-\xi_2)\ \ge\ \alpha|\xi_1-\xi_2|^p$

ii) $|a(x,s_1,\xi_1)-a(x,s_2,\xi_2)|\ \le\ \beta(1+|s_1|+|s_2|+|\xi_1|+|\xi_2|)^{p-2}(|s_1-s_2|+|\xi_1-\xi_2|)$

if $1<p\le 2$

j) $(a(x,s,\xi_1)-a(x,s,\xi_2),\xi_1-\xi_2)\ \ge\ \alpha|\xi_1-\xi_2|^2(|\xi_1|+|\xi_2|)^{p-2}$

jj) $|a(x,s_1,\xi_1)-a(x,s_2,\xi_2)|\ \le\ \beta(|s_1-s_2|+|\xi_1-\xi_2|)^{p-1}$

c) $a(x,0,0)\in L^q(\Omega;\mathbf{R}^n)$ if $p>n$, or $a(x,0,0)\in L^{p'}(\Omega;\mathbf{R}^n)$ with $p'>\frac{n}{p-1}$ if $p\le n$.

The main result is the following:

Theorem 5.8. *Let $a:\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}^n$ satisfying a), b) and c). Assume that $f\in L^{p'}(\Omega)$ with $p'>\frac{n}{p}$. Let (ε_h) be a sequence of positive real numbers converging to 0. Let u_h be the solutions to the Dirichlet boundary value problems*

$$\begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon_h},u_h,Du_h))=f & \text{on } \Omega\,,\\ u_h\in W_0^{1,p}(\Omega)\,. \end{cases}$$

Then

$$u_h\rightharpoonup u_0 \qquad \text{weakly in } W_0^{1,p}(\Omega)\,,$$
$$a(\frac{x}{\varepsilon_h},u_h,Du_h)\rightharpoonup b(u_0,Du_0) \qquad \text{weakly in } L^q(\Omega;\mathbf{R}^n)\,,$$

where u_0 is the unique solution to the problem

$$\begin{cases} -\mathrm{div}(b(u_0,Du_0))=f & \text{on } \Omega\,,\\ u_0\in W_0^{1,p}(\Omega)\,. \end{cases}$$

The homogenized operator $b:\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}^n$ is defined by

$$b(s,\xi)=\int_Y a(y,s,\xi+Dw^\xi(y))\,dy\,,$$

where w^ξ is the unique solution to

$$\begin{cases} \int_Y(a(y,s,\xi+Dw^\xi(y)),Dv(y))\,dy=0 & \text{for every } v\in W_\sharp^{1,p}(Y)\\ w^\xi\in W_\sharp^{1,p}(Y)\,. \end{cases}$$

Finally, in [19] the regularity conditions on a (required until this point) has been weakend and also the general case where a is a possibly multivalued map has been considered. To state the main result let us introduce some notation and definition.

For every open subset U in $\mathbf{R}^n$ we denote by $\mathcal{L}(U)$ the σ-field of all Lebesgue measurable subsets of U, and by $\mathcal{B}(\mathbf{R}^n)$ the σ-field of all Borel subsets of $\mathbf{R}^n$. Let $1<p<+\infty$, and let us fix two constants $m_1\ge 0$, $m_2\ge 0$, and two constants $c_1>0$, $c_2>0$.

Definition 5.9. By $M(\mathbf{R}^n)$ we denote that class of all (possibly) multivalued functions $a : \mathbf{R}^n \to \mathbf{R}^n$ which satisfy the following conditions:

i) a is maximal monotone;

ii) the estimates

$$\begin{aligned} |\eta|^q &\leq m_1 + c_1(\eta,\xi) \\ |\xi|^p &\leq m_2 + c_2(\eta,\xi) \end{aligned}$$

hold for every $\xi \in \mathbf{R}^n$ and $\eta \in a(\xi)$.

For every open subset U of $\mathbf{R}^n$, by $M_U(\mathbf{R}^n)$ we denote the class of all multivalued functions $a : U \times \mathbf{R}^n \to \mathbf{R}^n$ with closed values which satisfy the following conditions:

iii) for a.e. $y \in U$, $a(y,\cdot) \in M(\mathbf{R}^n)$;

iv) a is measurable with respect to $\mathcal{L}(U) \otimes \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^n)$, i.e.,

$$a^{-1}(C) = \{[y,\xi] \in U \times \mathbf{R}^n : a(y,\xi) \cap C \neq \emptyset\} \in \mathcal{L}(U) \otimes \mathcal{B}(\mathbf{R}^n)$$

for every closed set $C \subseteq \mathbf{R}^n$.

Now we can state the homogenization result:

Theorem 5.10. *Let $a \in \mathcal{M}_{\mathbf{R}^n}(\mathbf{R}^n)$ be such that $a(\cdot,\xi)$ is Y-periodic for every $\xi \in \mathbf{R}^n$. Let (ε_h) be a sequence of positive real numbers converging to 0. Let u_h be the solutions and g_h be the momenta to the Dirichlet boundary value problems*

$$\begin{cases} g_h(x) \in a(\frac{x}{\varepsilon_h}, Du_h(x)) & \text{for a.e. } x \in \Omega\,, \\ -\mathrm{div} g_h = f & \text{on } \Omega\,, \\ u_h \in W_0^{1,p}(\Omega)\,. \end{cases}$$

Then, up to a subsequence,

$$\begin{aligned} u_h &\rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega)\,, \\ g_h &\rightharpoonup g \qquad \text{weakly in } L^q(\Omega;\mathbf{R}^n)\,, \end{aligned}$$

where u is a solution and g is a momentum of the homogenized problem

$$\begin{cases} g(x) \in b(Du(x)) & \text{for a.e. } x \in \Omega\,, \\ -\mathrm{div} g = f & \text{on } \Omega\,, \\ u \in W_0^{1,p}(\Omega)\,. \end{cases}$$

For every $\xi \in \mathbf{R}^n$, the set $b(\xi)$ is defined by

$$b(\xi) = \{\nu \in \mathbf{R}^n : \exists w^\xi \in W_\sharp^{1,p}(Y) \exists k \in L^q(Y;\mathbf{R}^n) \text{ satisfying (5.21) and } \nu = \int_Y k(y)dy\}\,,$$

and

$$\begin{cases} w^\xi \in W_\sharp^{1,p}(Y), \qquad k \in L^q(Y;\mathbf{R}^n)\,, \\ k(y) \in a(y, \xi + Dw^\xi(y)) \qquad \text{for a.e. } y \in Y\,, \\ \int_Y (k(y), Dv(y))\, dy = 0 \qquad \text{for every } v \in W_\sharp^{1,p}(Y)\,. \end{cases} \tag{5.21}$$

Note: The main examples of maps of the class $M_{\mathbf{R}^n}(\mathbf{R}^n)$ have the form

$$a(x,\xi) = \partial_\xi \psi(x,\xi)\,, \tag{5.22}$$

where ∂_ξ denotes the subdifferential with respect to ξ and $\psi : \mathbf{R}^n \times \mathbf{R}^n \to [0,+\infty[$ is measurable in (x,ξ), convex in ξ, and satisfies the inequalities

$$c_1|\xi|^p \le \psi(x,\xi) \le c_2(1+|\xi|^p)$$

for suitable constants $0 < c_1 \le c_2$. In this case the operator $-\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du))$ is the subdifferential of the functional

$$\Psi_h(u) = \int_\Omega \psi(\frac{x}{\varepsilon_h}, Du)\,dx\,. \tag{5.23}$$

Note that the homogenization of a family of variational integrals of the form (5.23) has been studied by Marcellini in [32] and by Carbone and Sbordone in [17] using the techniques of Γ-convergence introduced by De Giorgi.

Let us point out that if ψ is not assumed to be differentiable the map a can be multivalued. Moreover, the "multivalued approach" finds also a motivation in the fact that, under the general assumptions on $a \in M_{\mathbf{R}^n}(\mathbf{R}^n)$, the additional hypothesis on a to be single-valued is not enough to ensure the same property for the homogenized operator b (see [19], Section 4).

III. G-convergence; H-convergence

6. Setting of the problem. G-convergence for second order linear (uniformly) elliptic operators. The symmetric case

Let Ω be a bounded open subset of $\mathbf{R}^n$. Let α and β be constants such that $0 < \alpha \le \beta < +\infty$.
Let us denote by $\mathcal{M}(\alpha,\beta)$ the set of all functions $a : \Omega \to M^{n\times n}$ satisfying the following properties:
(6.1) $a_{ij} \in L^\infty(\Omega)$ for $i,j = 1,\dots,n$, and $|a(x)\xi| \le \beta|\xi|$ for a.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$;
(6.2) $(a(x)\xi,\xi) \ge \alpha|\xi|^2$ for a.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$.

Let us consider a sequence (a^h) in $\mathcal{M}(\alpha,\beta)$ and let $f \in H^{-1,2}(\Omega)$ (for the sake of simplicity, without loss of generality, we consider from now on a right hand side term independent of h). Then, for every fixed h, there exists a unique solution u_h to the Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}(a^h(x)Du_h) = f & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,, \end{cases}$$

and

$$\alpha\|u_h\|_{H_0^{1,2}(\Omega)} \le \|f\|_{H^{-1,2}(\Omega)}\,.$$

Hence, there exists a subsequence $(u_{\sigma(h)})$ of (u_h) such that

$$u_{\sigma(h)} \rightharpoonup u_0 \quad \text{weakly in } H_0^{1,2}(\Omega)\,.$$

As in the periodic case, the problem is then the following: what can we say about u_0? Does u_0 satisfy an equation of the same type as u_h?

Remark 6.1. If $(a^h) \in \mathcal{M}(\alpha,\beta)$, and

$$a^h \to a^0 \qquad \text{strongly in } L^\infty(\Omega; \mathbf{R}^{n^2})$$

we can pass to the limit in $a^h Du_h$ and we have

$$a^h Du_h \rightharpoonup a^0 Du_0 \qquad \text{weakly in } L^2(\Omega; \mathbf{R}^n)$$

and hence u_0 is the solution (unique since $a^0 \in \mathcal{M}(\alpha,\beta)$) to

$$\begin{cases} -\mathrm{div}(a^0 Du_0) = f & \text{in } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases}$$

Let us note that the previous result is not true if we do not have the strong convergence of the sequence (a^h).

Indeed, let $\Omega =]x_0, x_1[\subset \mathbf{R}$ and $f \in L^2(\Omega)$. Let us consider the sequence $a^h \in \mathcal{M}(\alpha,\beta) = \{a^h \in L^\infty(\Omega) : \alpha \le a^h(x) \le \beta \text{ for a.e. } x \in \Omega\}$ defined by $a^h(x) = g(hx)$, where $g : \mathbf{R} \to \mathbf{R}$ is a 1-periodic function defined on $]0,1[$ by

$$g(x) = \begin{cases} \alpha & \text{if } 0 < x < \frac{1}{2} \\ \beta & \text{if } \frac{1}{2} \le x < 1\,. \end{cases}$$

Then we get (up to subsequences)

$$\frac{1}{a^h} \overset{*}{\rightharpoonup} \frac{1}{a^0} = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right), \qquad \text{in } L^\infty(\Omega) \text{ weak*}$$

while

$$a^h \overset{*}{\rightharpoonup} b^0 = \frac{1}{2}(\alpha + \beta) \qquad \text{in } L^\infty(\Omega) \text{ weak*}$$

and the sequence of solutions u_h to

$$\begin{cases} -\frac{d}{dx}\left(a^h(x)\frac{du_h}{dx}(x)\right) = f & \text{in } \Omega\,, \\ u_h(x_0) = u_h(x_1) = 0 \end{cases}$$

converge in the weak topology of $H_0^{1,2}(\Omega)$ to the solution of the Dirichlet boundary value problem

$$\begin{cases} -\frac{d}{dx}\left(a^0(x)\frac{du_0(x)}{dx}\right) = f & \text{in } \Omega\,, \\ u_0(x_0) = u_0(x_1) = 0\,. \end{cases}$$

Let us point out that only in dimension $n = 1$ the weak* limit of $(\frac{1}{a^h})$ caracterizes the coefficients of the matrix we are looking for. This is not longer true in dimension $n > 1$ as shown by an example in [33].

In order to answer to the above questions and other related questions for a more general class of problems we follow the approach which uses the theory of G-convergence. A first notion of G-convergence for second order linear elliptic operators was introduced by De Giorgi and S. Spagnolo in [23], [42], [43],[44] as the convergence, in a suitable topology, of the Green's operator associated to the Dirichlet boundary value problems, in the case that $a^h \in \mathcal{M}(\alpha,\beta)$ and $a^h(x)$ is symmetric. Let us recall it briefly here.

Definition 6.2. For every $h \in \mathbf{N}$ let $a^h \in \mathcal{M}(\alpha,\beta)$, $a^h_{ij}(x) = a^h_{ji}(x)$ for a.e. $x \in \Omega$ and for every $i,j = 1,\dots,n$ and let $a^0 \in \mathcal{M}(\alpha,\beta)$, $a^0_{ij}(x) = a^0_{ji}(x)$ for a.e. $x \in \Omega$ and for every $i,j = 1,\dots,n$. We then say that (a^h) G-*converges to* a^0 if for every $f \in H^{-1,2}(\Omega)$ the solutions u_h of the equations

$$(6.3) \qquad \begin{cases} -\operatorname{div}(a^h(x)Du_h) = f & \text{on } \Omega\,, \\ u_h \in H^{1,2}_0(\Omega) \end{cases}$$

satisfy

$$u_h \rightharpoonup u_0 \qquad \text{weakly in } H^{1,2}_0(\Omega)\,,$$

where u_0 is the solution to

$$(6.4) \qquad \begin{cases} -\operatorname{div}(a^0(x)Du_0) = f & \text{on } \Omega\,, \\ u_0 \in H^{1,2}_0(\Omega)\,. \end{cases}$$

The main result (which motivates the definition) is the sequential compactness of the class of symmetric functions belonging to $\mathcal{M}(\alpha,\beta)$ with respect to the G-convergence.

Theorem 6.3. *Given a sequence $(a^h) \subset \mathcal{M}(\alpha,\beta)$, $a^h(x)$ symmetric, then there exist a subsequence $(a^{\sigma(h)})$ of (a^h) and $a^0 \in \mathcal{M}(\alpha,\beta)$, $a^0(x)$ symmetric such that $(a^{\sigma(h)})$ G-converges to a^0.*

Remark 6.4. The above result can be expressed as follows: given a sequence $(a^h) \subset \mathcal{M}(\alpha,\beta)$, $a^h(x)$ symmetric, there exist a matrix $a^0 \in \mathcal{M}(\alpha,\beta)$, $a^0(x)$ symmetric (called the G-limit) and an increasing sequence of integers $(\sigma(h))$, such that for every $f \in H^{-1,2}(\Omega)$ the sequence $(u_{\sigma(h)})$ of the solutions to (6.3) corresponding to $(a^{\sigma(h)})$ converges weakly in $H^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to the solution u_0 to (6.4).

The original proof of Spagnolo is rather technical and uses results of the semigroup theory for linear operators and of the G-convergence of parabolic equations. Many different proofs have been given subsequently (see, for example [46], [41]).

We would like to notice that in [44] also the following localization property is proven.

Theorem 6.5. *Assume that (a^h), (b^h), a^0 and b^0 belong to $\mathcal{M}(\alpha,\beta)$ and are symmetric. If (a^h) G-converges to a^0, (b^h) G-converges to b^0, and $a^h(x) = b^h(x)$ for a.e. x in an open subset Ω' of Ω, then $a^0(x) = b^0(x)$ for a.e. $x \in \Omega'$.*

7. H-convergence for second order linear (uniformly) elliptic operators. The non-symmetric case

The notion of G-convergence has been extended to the non-symmetric case by Murat and Tartar under the name of H-convergence (see [33], [47] and [48]). Let us recall the definition (see [33]). Let α' and β' be constants satisfying $0 < \alpha' \le \beta' < +\infty$.

Definition 7.1. Let $a^h \in \mathcal{M}(\alpha,\beta)$ and let $a^0 \in \mathcal{M}(\alpha',\beta')$. We then say that (a^h) H-*converges to* a^0 if for every $f \in H^{-1,2}(\Omega)$ the solutions u_h to the equations

$$\begin{cases} -\mathrm{div}(a^h(x)Du_h) = f & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega) \end{cases} \tag{7.1}$$

satisfy

$$\begin{cases} u_h \rightharpoonup u_0 & \text{weakly in } H_0^{1,2}(\Omega)\,, \\ a^h Du_h \rightharpoonup a^0 Du_0 & \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,, \end{cases} \tag{7.2}$$

where u_0 is the solution to

$$\begin{cases} -\mathrm{div}(a^0(x)Du_0) = f & \text{on } \Omega \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases} \tag{7.3}$$

Remark 7.2. Let us note that in the non-symmetric case (see also the nonlinear cases) a definition of H-convergence as in the symmetric case would not determine uniquely the H-limit as the following example shows.

Assume $n = 3$, and let $\varphi \in C_0^\infty(\Omega)$. Let us define $a(x) = I$, where I is the identity matrix and let

$$b(x) = I + \begin{pmatrix} 0 & -D_3\varphi(x) & D_2\varphi(x) \\ D_3\varphi(x) & 0 & -D_1\varphi(x) \\ -D_2\varphi(x) & D_1\varphi(x) & 0 \end{pmatrix}\,.$$

It is easy to see that a and b belong to $\mathcal{M}(\alpha,\beta)$ with $\alpha = 1$ and $\beta = (1 + \max_\Omega |D\varphi|)$. Note that $b(x)\xi = \xi + D\varphi \wedge \xi$, where $\wedge$ denotes the external product in $\mathbf{R}^n$ and

$$\int_\Omega ((D\varphi \wedge Du), Dv)\, dx = 0 \qquad \text{for every } u, v \in H^{1,2}(\Omega)\,.$$

It follows that

$$\int_\Omega (a(x)Du, Dv)\, dx = \int_\Omega (b(x)Du, Dv)\, dx \qquad \text{for every } u, v \in H^{1,2}(\Omega)\,.$$

This implies that the operator $\mathcal{A}u = -\mathrm{div}(a(x)Du)$ coincides with the operator $\mathcal{B}u = -\mathrm{div}(b(x)Du)$ in spite of the fact that $a(x) \neq b(x)$. $\square$

Let us show now that the condition (7.2) in the above definition determines uniquely the H-limit a^0.

Proposition 7.3. *Let (a^h) be a sequence of functions of the class $\mathcal{M}(\alpha,\beta)$ and let $a^0 \in \mathcal{M}(\alpha',\beta')$ and $b^0 \in \mathcal{M}(\alpha'',\beta'')$ such that (a^h) H-converges to a^0 and (a^h) H-converges to b^0. Then, $a^0 = b^0$ a.e. on Ω.*

Proof. Let $\omega \subset\subset \Omega$ and let $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ on ω. For every $\lambda \in \mathbf{R}^n$ let us define $f^\lambda = -\mathrm{div}(a^0 D((\lambda,x)\varphi))$. Let us consider for $h = 0,1,\ldots$ the solutions u_h^λ to the equations

$$\begin{cases} -\mathrm{div}(a^h Du_h^\lambda) = f^\lambda & \text{on } \Omega\,, \\ u_h^\lambda \in H_0^{1,2}(\Omega)\,. \end{cases}$$

By the coercivity of a^0 it turns out that

$$u_0^\lambda = (\lambda,x)\varphi \qquad \text{on } \Omega\,,$$

and being a^0 by assumption an H-limit of (a^h) we have

$$\begin{cases} u_h^\lambda \rightharpoonup u_0^\lambda & \text{weakly in } H_0^{1,2}(\Omega)\,, \\ a^h Du_h^\lambda \rightharpoonup a^0 Du_0^\lambda & \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,. \end{cases}$$

Analogously for b^0 we have

$$\begin{cases} u_h^\lambda \rightharpoonup u_0^\lambda & \text{weakly in } H_0^{1,2}(\Omega)\,, \\ a^h Du_h^\lambda \rightharpoonup b^0 Du_0^\lambda & \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,. \end{cases}$$

By the uniqueness of the weak limit in $L^2(\Omega;\mathbf{R}^n)$ we may conclude that $a^0 Du_0^\lambda = b^0 Du_0^\lambda$ a. e. on Ω. Since $Du_0^\lambda = \lambda$ on ω, we get $a^0 = b^0$ a.e. on ω. Thus, $a^0 = b^0$ a.e. on Ω. □

The main result obtained by Tartar and Murat (see [33]) is the sequential compactness of the class $\mathcal{M}(\alpha,\beta)$ with respect to the H-convergence.

Theorem 7.4. *Given a sequence $(a^h) \subset \mathcal{M}(\alpha,\beta)$, then there exist a subsequence $(a^{\sigma(h)})$ of (a^h) and $a^0 \in \mathcal{M}(\alpha,\frac{\beta^2}{\alpha})$ such that $(a^{\sigma(h)})$ H-converges to a^0.*

Note. The above result shows that the class $\mathcal{M}(\alpha,\beta)$ is "stable" with respect to the H-convergence as far as coerciveness is concerned, but unstable with regard to the norm of the matrices (compare with the compactness result for the symmetric case).

The rest of this section is devoted to the study of some properties of the H-convergence and the proof of Theorem 7.4.

The next lemma, together with the compensated compactness lemma (see Lemma A.2.1), will be crucial in the sequel. Given $M \in M^{n\times n}$, we denote by M^T the transpose matrix of M.

Lemma 7.5. *Let $a^h \in \mathcal{M}(\alpha,\beta)$. Let (u_h) and (v_h) be two sequences in $H^{1,2}(\Omega)$ such that the following conditions are satisfied:*

$$\begin{cases} u_h \rightharpoonup u_0 & \text{weakly in } H^{1,2}(\Omega) \\ \xi^h = a^h Du_h \rightharpoonup \xi^0 & \text{weakly in } L^2(\Omega;\mathbf{R}^n) \\ -\mathrm{div}(a^h Du_h) \to -\mathrm{div}\xi^0 & \text{strongly in } H^{-1,2}(\Omega) \end{cases} \tag{7.4}$$

$$\begin{cases} v_h \rightharpoonup v_0 & \text{weakly in } H^{1,2}(\Omega) \\ \eta^h = (a^h)^T Dv_h \rightharpoonup \eta^0 & \text{weakly in } L^2(\Omega;\mathbf{R}^n) \\ -\mathrm{div}((a^h)^T Dv_h) \to -\mathrm{div}\eta^0 & \text{strongly in } H^{-1,2}(\Omega)\,. \end{cases} \tag{7.5}$$

Then

$$(\xi^0, Dv_0) = (Du_0, \eta^0) \qquad a.e. \text{ on } \Omega\,. \tag{7.6}$$

Proof. Let us write

$$(\xi^h, Dv_h) = (a^h Du_h, Dv_h) = (Du_h, (a^h)^T Dv_h) = (Du_h, \eta^h)\,.$$

Hence

$$\int_\Omega (\xi^h, Dv_h)\varphi\, dx = \int_\Omega (Du_h, \eta^h)\varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. By the compensated compactness lemma we may conclude that

$$\int_\Omega (\xi^0, Dv_0)\varphi\, dx = \int_\Omega (Du_0, \eta^0)\varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$, and (7.6) follows immediately. □

Proposition 7.6. *Let (a^h) be a sequence in $\mathcal{M}(\alpha,\beta)$ which H-converges to $a^0 \in \mathcal{M}(\alpha',\beta')$. Then, the sequence $(a^h)^T$ H-converges to $(a^0)^T$.*

Proof. Let $g \in H^{-1,2}(\Omega)$. We have to prove that the solutions v_h to

$$\begin{cases} -\mathrm{div}((a^h)^T Dv_h) = g & \text{on } \Omega\,, \\ v_h \in H_0^{1,2}(\Omega) \end{cases}$$

satisfy

$$\begin{cases} v_h \rightharpoonup v_0 & \text{weakly in } H_0^{1,2}(\Omega) \\ (a^h)^T Dv_h \rightharpoonup (a^0)^T Dv_0 & \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,, \end{cases}$$

where v_0 is the solution to

$$\begin{cases} -\mathrm{div}((a^0)^T Dv_0) = g & \text{on } \Omega\,, \\ v_0 \in H_0^{1,2}(\Omega)\,. \end{cases}$$

Let us note that the sequence (v_h) is uniformly bounded in $H_0^{1,2}(\Omega)$; furthermore, $((a^h)^T Dv_h)$ is uniformly bounded in $L^2(\Omega;\mathbf{R}^n)$. Hence, there exist a subsequence $\sigma(h)$ of h and two functions $v \in H_0^{1,2}(\Omega)$ and $\eta \in L^2(\Omega;\mathbf{R}^n)$ such that

$$\begin{cases} v_{\sigma(h)} \rightharpoonup v & \text{weakly in } H_0^{1,2}(\Omega)\,, \\ (a^{\sigma(h)})^T Dv_{\sigma(h)} \rightharpoonup \eta & \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,. \end{cases}$$

Clearly, $-\mathrm{div}\eta = g$ on Ω. On the other hand, given $f \in H^{-1,2}(\Omega)$ and u_h the solutions to

$$\begin{cases} -\mathrm{div}(a^h Du_h) = f & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,, \end{cases}$$

we have by assumption that

$$\begin{cases} u_h \rightharpoonup u_0 & \text{weakly in } H_0^{1,2}(\Omega)\,, \\ a^h Du_h \rightharpoonup a^0 Du_0 & \text{weakly in } L^2(\Omega;\mathbf{R}^n)\,, \end{cases}$$

where u_0 is the solution to

$$\begin{cases} -\mathrm{div}(a^0 Du_0) = f & \text{on } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases}$$

By Lemma 7.5 we get

$$(a^0 Du_0, Dv) = (Du_0, \eta) \quad \text{a.e. on } \Omega\,. \tag{7.7}$$

Since f can be chosen arbitrarily in $H^{-1,2}(\Omega)$, arguing as in the proof of Proposition 7.3, we can take on $\omega \subset\subset \Omega$, $Du_0 = \lambda$, where $\lambda \in \mathbf{R}^n$ is arbitrary. Then (7.7) becomes

$$(a^0\lambda, Dv) = (\lambda, \eta) \qquad \text{a.e. on } \omega\,.$$

Since this is true for every $\lambda \in \mathbf{R}^n$ we can conclude that $\eta = (a^0)^T Dv$ on Ω. The equality $-\mathrm{div}\eta = g$ implies then $v = v_0$, $\eta = (a^0)^T Dv_0$. By the uniqueness of the limits, we can conclude that the whole sequences (v_h) and $(a^h Dv_h)$ converge to v_0 and $a^0 Dv_0$, respectively. This concludes the proof. □

Proof of Theorem 7.4. The proof of Theorem 7.4 is divided in several steps. The proof of Step 1 is given in the Appendix.

Step 1:

Proposition 7.7. *Let F be a separable Banach space and let G be a reflexive Banach space. Let $\mathcal{L}(F;G)$ be the set of all linear and continuous operators from F into G. Assume that for every $h \in \mathbf{N}$*

(i) $T^h \in \mathcal{L}(F;G)$;
(ii) $\|T^h\|_{\mathcal{L}(F;G)} \leq c \qquad c > 0\,.$

Then there exist a subsequence $(T^{\sigma(h)})$ of (T^h) and an operator $T^0 \in \mathcal{L}(F;G)$ such that for every $f \in F$

$$T^{\sigma(h)} f \rightharpoonup T^0 f \qquad \textit{weakly in } G\,. \tag{7.8}$$

Proposition 7.8. *Let V be a reflexive and separable Banach space. Let α and β be two positive constants and let (T^h) be a sequence of operators such that for every $h \in \mathbf{N}$*

(i) $T^h \in \mathcal{L}(V;V^*)$;
(ii) $\|T^h\|_{\mathcal{L}(V;V^*)} \leq \beta$;
(iii) for every $v \in V$, $\quad \langle T^h v, v\rangle_{V^*,V} \geq \alpha\|v\|_V^2\,.$

Then there exist a subsequence $(T^{\sigma(h)})$ *of* (T^h) *and an operator* $T^0 \in \mathcal{L}(V;V^*)$ *such that*

$$\begin{cases} T^0 \in \mathcal{L}(V;V^*) \\ \|T^0\|_{\mathcal{L}(V;V^*)} \leq \frac{\beta^2}{\alpha} \\ \textit{for every } v \in V, \quad \langle T^0 v, v\rangle_{V^*,V} \geq \alpha\|v\|_V^2 . \end{cases} \tag{7.9}$$

Moreover, for every $f \in V^*$ *we have*

$$(T^{\sigma(h)})^{-1} f \rightharpoonup (T^0)^{-1} f \qquad \textit{weakly in } V . \tag{7.10}$$

Step 2. We construct the test functions which will be used in Lemma 7.5.

Let Ω' be a bounded open subset of $\mathbf{R}^n$ such that $\Omega \subset\subset \Omega'$. Let us consider the sequence (b^h) in $\mathcal{M}(\alpha,\beta,\Omega')$ (note that $\mathcal{M}(\alpha,\beta,\Omega')$ denotes the set $\mathcal{M}(\alpha,\beta)$, where Ω has been replaced by Ω') such that

$$b^h = (a^h)^T \qquad \text{on } \Omega \tag{7.11}$$

(for example take $b^h = \alpha I$ on $\Omega' \setminus \Omega$).

Let us consider the sequence of operators $(\mathcal{B}^h) \subset \mathcal{L}(H_0^{1,2}(\Omega'); H^{-1,2}(\Omega'))$ defined for $h \in \mathbf{N}$ by

$$\mathcal{B}^h u = -\text{div}(b^h Du) .$$

By Proposition 7.8 (it is easy to verify that $\mathcal{B}^h$ satisfies the hypotheses (ii) and (iii) of Proposition 7.8) there exist a subsequence $(\mathcal{B}^{\sigma(h)})$ of $(\mathcal{B}^h)$ and an operator $\mathcal{B}^0 \in \mathcal{L}(H_0^{1,2}(\Omega'); H^{-1,2}(\Omega'))$ such that for every $g \in H^{-1,2}(\Omega')$

$$(\mathcal{B}^{\sigma(h)})^{-1} g \rightharpoonup (\mathcal{B}^0)^{-1} g \qquad \text{weakly in } H_0^{1,2}(\Omega') . \tag{7.12}$$

Given $\varphi \in C_0^\infty(\Omega')$ such that $\varphi = 1$ on Ω, we denote by g_i the function in $H^{-1,2}(\Omega')$ defined by

$$g_i = \mathcal{B}^0((e_i, x)\varphi) . \tag{7.13}$$

For every $i \in \{1,\ldots,n\}$, let us denote by $v_{\sigma(h),i}$ the solutions to

$$\begin{cases} \mathcal{B}^{\sigma(h)} v_{\sigma(h),i} = g_i \qquad \text{on } \Omega' , \\ v_{\sigma(h),i} \in H_0^{1,2}(\Omega') . \end{cases}$$

This definition together with (7.12) and (7.11) implies that for every $i \in \{1,2,\ldots,n\}$ we have

$$\begin{cases} -\text{div}((a^{\sigma(h)})^T Dv_{\sigma(h),i}) = g_i \qquad \text{on } \Omega , \\ v_{\sigma(h),i} \in H^{1,2}(\Omega) . \end{cases}$$

Furthermore, by (7.12)

$$v_{\sigma(h),i} \rightharpoonup (e_i, \cdot) \qquad \text{weakly in } H^{1,2}(\Omega) .$$

By passing to a subsequence of $\sigma(h)$, let us denote it by $\tau(h)$, we have for every $i \in \{1,2,\ldots,n\}$

$$(a^{\tau(h)})^T Dv_{\tau(h),i} \rightharpoonup \eta_i \qquad \text{weakly in } L^2(\Omega;\mathbf{R}^n) .$$

Note that for every $i \in \{1, 2, \ldots, n\}$, the sequence $(v_{\tau(h),i})$ satisfies (7.5).

Let us define $a^0 \in L^2(\Omega; \mathbf{R}^{n^2})$ by

$$(a^0(x))_{i,j} = (\eta_i(x))_j \qquad \text{for a.e. } x \in \Omega, \text{ for every } i, j \in \{1, 2, \ldots, n\}.$$

In the remaining steps we shall prove that $(a^{\tau(h)})$ H-converges to a^0.

Step 3. For the sake of simplicity we shall in the sequel simply write h instead of $\tau(h)$. For every $h \in \mathbf{N}$, let us denote by $\mathcal{A}^h$ the operator in $\mathcal{L}(H_0^{1,2}(\Omega); H^{-1,2}(\Omega))$ defined by

$$\mathcal{A}^h u = -\mathrm{div}(a^h Du).$$

It turns out that $\mathcal{A}^h$ is an isomorphism. Moreover, let us consider the operator $\mathcal{T}^h \in \mathcal{L}(H^{-1,2}(\Omega); L^2(\Omega; \mathbf{R}^n))$ defined by

$$\mathcal{T}^h f = a^h D((\mathcal{A}^h)^{-1} f).$$

We have

$$\|\mathcal{T}^h f\|_{L^2(\Omega;\mathbf{R}^n)} \le \beta \|(\mathcal{A}^h)^{-1} f\|_{H_0^{1,2}(\Omega)} \le \frac{\beta}{\alpha} \|f\|_{H^{-1,2}(\Omega)}$$

for every $f \in H^{-1,2}(\Omega)$. By applying Proposition 7.8 to the operator $\mathcal{A}^h$ and Proposition 7.7 to the operator $\mathcal{T}^h$ we deduce that there exist a subsequence $\rho(h)$ of h (recall that h stands here for the subsequence $\tau(h)$; however, no confusion can occur) and two operators $\mathcal{A}^0 \in \mathcal{L}(H_0^{1,2}(\Omega); H^{-1,2}(\Omega))$ and $\mathcal{T}^0 \in \mathcal{L}(H^{-1,2}(\Omega); L^2(\Omega; \mathbf{R}^n))$ such that for every $f \in H^{-1,2}(\Omega)$ we have

$$\begin{aligned} &(\mathcal{A}^{\rho(h)})^{-1} f \rightharpoonup (\mathcal{A}^0)^{-1} f \qquad \text{weakly in } H_0^{1,2}(\Omega) \\ &\mathcal{T}^{\rho(h)} f \rightharpoonup \mathcal{T}^0 f \qquad \text{weakly in } L^2(\Omega; \mathbf{R}^n). \end{aligned}$$

For $f \in H^{-1,2}(\Omega)$, we set

$$(\mathcal{A}^h)^{-1} f = u_h, \qquad (\mathcal{A}^0)^{-1} f = u_0;$$

(here h stands for $\tau(h)$). We have then

$$\begin{cases} u_{\rho(h)} \rightharpoonup u_0 \qquad \text{weakly in } H_0^{1,2}(\Omega) \\ a^{\rho(h)} Du_{\rho(h)} \rightharpoonup \mathcal{T}^0 f = \xi \qquad \text{weakly in } L^2(\Omega; \mathbf{R}^n) \\ -\mathrm{div}(a^{\rho(h)} Du_{\rho(h)}) = f \qquad \text{on } \Omega. \end{cases}$$

We note know that the sequence $(u_{\rho(h)})$ satisfies the hypothesis (7.4) of Lemma 7.5. Moreover, by taking into account the sequence $(v_{\tau(h)})$ constructed in the previous step and Lemma 7.5, we obtain for every $i \in \{1, 2, \ldots, n\}$

$$(\xi, D(e_i, x)) = (Du_0, \eta_i) \qquad \text{a.e. on } \Omega.$$

By the definition of a^0 this is nothing but

$$\mathcal{T}^0 f = \xi = a^0 Du_0.$$

Step 4. We prove that a^0 belongs to $\mathcal{M}(\alpha, \frac{\beta^2}{\alpha})$.

By definition $a^0 \in L^2(\Omega; \mathbf{R}^{n^2})$. Hence, for every $u_0 \in H_0^{1,2}(\Omega)$ we have $a^0 Du_0 \in L^2(\Omega; \mathbf{R}^n)$. By the compensated compactness lemma we get that

$$(a^{\rho(h)} Du_{\rho(h)}, Du_{\rho(h)}) \to (a^0 Du_0, Du_0) \qquad \text{in } \mathcal{D}'(\Omega). \tag{7.14}$$

By the ellipticity assumption of a^h we have

$$\int_\Omega (a^{\rho(h)} Du_{\rho(h)}, Du_{\rho(h)})\varphi\, dx \geq \alpha \int_\Omega |Du_{\rho(h)}|^2 \varphi\, dx \tag{7.15}$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Then, by (7.14), (7.15) and the weak lower semicontinuity of the norm in $L^2(\Omega;\mathbf{R}^n)$ we get

$$\int_\Omega (a^0 Du_0, Du_0)\varphi\, dx \geq \alpha \int_\Omega |Du_0|^2 \varphi\, dx \tag{7.16}$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Note that (7.16) holds for every $u_0 \in H_0^{1,2}(\Omega)$ (since f ranges all over $H^{-1,2}(\Omega)$ and $\mathcal{A}^0$ is an isomorphism). By taking $u_0 = (\lambda, x)\psi$, where $\psi \in C_0^\infty(\Omega)$ and $\psi = 1$ in a neighbourhood of the support of φ and $\lambda \in \mathbf{R}^n$ arbitrary, from (7.16) we deduce

$$\int_\Omega (a^0(x)\lambda, \lambda)\varphi\, dx \geq \alpha \int_\Omega |\lambda|^2 \varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Hence

$$(a^0(x)\lambda, \lambda) \geq |\lambda|^2$$

for every $\lambda \in \mathbf{R}^n$ and for a.e. $x \in \Omega$.

Let us prove now that $|a^0(x)\lambda| \leq \frac{\beta^2}{\alpha}|\lambda|$ for a.e. $x \in \Omega$ and for every $\lambda \in \mathbf{R}^n$. By the assumptions on a^h it follows that for every $h \in \mathbf{N}$ the following inequality holds

$$((a^h)^{-1}(x)\mu, \mu) \geq \frac{\alpha}{\beta^2}|\mu|^2 \qquad \text{for a.e. } x \in \Omega, \text{ for every } \mu \in \mathbf{R}^n\,.$$

This yields

$$\int_\Omega (Du_h, a^h Du_h)\varphi^2\, dx \geq \frac{\alpha}{\beta^2}\int_\Omega |a^h Du_h|^2 \varphi^2\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$ and for every $h \in \mathbf{N}$; hence, in particular it holds for every $\rho(h)$. By passing to the limit (taking into account the compensated compactness lemma and the weak lower semicontinuity of the norm in $L^2(\Omega;\mathbf{R}^n)$) we obtain

$$\int_\Omega (Du_0, a^0 Du_0)\varphi^2\, dx \geq \frac{\alpha}{\beta^2}\int_\Omega |a^0 Du_0|^2 \varphi^2\, dx\,.$$

Proceeding as above we get for every $\lambda \in \mathbf{R}^n$ and for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (\lambda, a^0\lambda)\varphi^2\, dx \geq \frac{\alpha}{\beta^2}\int_\Omega |a^0\lambda|^2\varphi^2\, dx\,.$$

From this inequality we can deduce

$$\frac{\alpha}{\beta^2}\|a^0\lambda\varphi\|_{L^2(\Omega;\mathbf{R}^n)} \leq \|a^0\lambda\varphi\|_{L^2(\Omega;\mathbf{R}^n)}\|\lambda\varphi\|_{L^2(\Omega;\mathbf{R}^n)}\,.$$

Finally,

$$\|a^0\lambda\varphi\|_{L^2(\Omega;\mathbf{R}^n)} \leq \frac{\beta^2}{\alpha}\|\varphi\|_{L^2(\Omega)}|\lambda|\,,$$

for every $\varphi \in C_0^\infty(\Omega)$ and for every $\lambda \in \mathbf{R}^n$. By the converse of Hölder's inequality (see [26], Proposition 6.14) we obtain that $a^0\lambda \in L^\infty(\Omega;\mathbf{R}^n)$ and

$$\|a^0\lambda\|_{L^\infty(\Omega;\mathbf{R}^n)} \le \frac{\beta^2}{\alpha}|\lambda|$$

for every $\lambda \in \mathbf{R}^n$.

Step 5. In the previous step we have shown that a^0 belongs to $\mathcal{M}(\alpha, \frac{\beta^2}{\alpha})$. The limit u_0 of the sequence $(u_{\rho(h)})$ is defined in a unique manner (independent of the subsequence $\rho(h)$ extract from the sequence $\tau(h)$) by

$$\begin{cases} -\mathrm{div}(a^0 Du_0) = f & \text{in } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases}$$

Moreover, by the uniqueness of the limits, we have that the whole sequences $(u_{\tau(h)})$ and $(a^{\tau(h)}u_{\tau(h)})$ (and not the subsequences determined by $\rho(h)$) converge. We may conclude that $a^{\tau(h)}$ H-converges to a^0; the proof of Theorem 7.4 is so accomplished. □

Remark 7.9. Let us conclude this section by noting that a corrector result for the class $\mathcal{M}(\alpha,\beta)$ has been proved in [33]. Moreover, some properties of the H-convergence for quasi-linear elliptic operators were studied by L. Boccardo, Th. Gallouet and F. Murat in [12], [13] and [14].

8. Some further remarks on G-convergence and H-convergence

The first results in the nonlinear case are due to L. Tartar, who studied (in [47]) the properties of the H-convergence for monotone problems of the type

$$\begin{cases} -\mathrm{div}(a_h(x, Du_h)) = f & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,, \end{cases}$$

assuming that the maps a_h are uniformly strictly monotone and uniformly Lipschitz-continuous on $\mathbf{R}^n$ (note that the vector-valued case is considered in [45] whereas more general classes of uniformly equicontinuous strictly monotone operators on $W^{1,p}(\Omega)$, with $p \ge 2$, are considered by Raitum in [39]).

By $\mathcal{N}(\alpha,\beta)$ we denote the set of all functions $a : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ such that for every $\xi \in \mathbf{R}^n$, $a(\cdot,\xi)$ is Lebesgue measurable and satisfies the following properties:

(8.1) (strict monotonicity) $(a(x,\xi_1) - (x,\xi_2), \xi_1 - \xi_2) \ge \alpha|\xi_1 - \xi_2|^2$

(8.2) (Lipschitz-continuity) $|a(x,\xi_1) - a(x,\xi_2)| \le \beta|\xi_1 - \xi_2|$

for a.e. $x \in \mathbf{R}^n$ and for every ξ_1, $\xi_2 \in \mathbf{R}^n$. Moreover

(8.3) $a(x,0) = 0$ for a.e. $x \in \mathbf{R}^n$.

Let α' and β' be constants satisfying $0 < \alpha' \le \beta' < +\infty$.

Definition 8.1. Let $a_h \in \mathcal{N}(\alpha, \beta)$ and let $a_0 \in \mathcal{N}(\alpha', \beta')$. We say that (a_h) H-*converges to* a_0 if for every $f \in H^{-1,2}(\Omega)$ the solutions u_h to the equations

$$\begin{cases} -\operatorname{div}(a_h(x, Du_h)) = f & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega) \end{cases} \tag{8.4}$$

satisfy

$$\begin{cases} u_h \rightharpoonup u_0 & \text{weakly in } H_0^{1,2}(\Omega)\,, \\ a_h(\cdot, Du_h) \rightharpoonup a_0(\cdot, Du_0) & \text{weakly in } L^2(\Omega; \mathbf{R}^n)\,, \end{cases} \tag{8.5}$$

where u_0 is the solution to

$$\begin{cases} -\operatorname{div}(a_0(x, Du_0)) = f & \text{on } \Omega\,, \\ u_0 \in H_0^{1,2}(\Omega)\,. \end{cases}$$

The following theorem, due to Tartar (see [47] and [50]), justifies the definition (8.1) of H-convergence; its proof is reproduced in [24].

Theorem 8.2. *Given a sequence $(a_h) \subset \mathcal{N}(\alpha, \beta)$, there exist a subsequence $(a_{\sigma(h)})$ of (a_h) and $a_0 \in \mathcal{N}(\alpha, \frac{\beta^2}{\alpha})$ such that $(a_{\sigma(h)})$ H-converges to a_0.*

Remark 8.3. Let us mention that a corrector result for the class $\mathcal{N}(\alpha, \beta)$ has been proved by Murat in [36].

Moreover, in [11] a convergence result for the strongly non linear equations

$$\begin{cases} -\operatorname{div}(a_h(x, Du_h)) + c_h(x) g(u_h) = f & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,, \end{cases}$$

where $a_h \in \mathcal{N}(\alpha, \beta)$, has been proved.

A compactness result (in the sense of H-convergence) for equations of the type

$$\begin{cases} -\operatorname{div}(a_h(x, u_h, Du_h)) = f & \text{on } \Omega\,, \\ u_h \in H_0^{1,2}(\Omega)\,, \end{cases}$$

with $a_h(x, s, \xi) \in \mathcal{N}(\alpha, \beta)$ for every $s \in \mathbf{R}$ is shown in [24].

Finally, a general notion of G-convergence for a sequence of maximal monotone (possibly multivalued) operators of the form $\mathcal{A}_h u = -\operatorname{div}(a_h(x, Du))$ has been introduced in [18]. Let us point out that, in order to include the case (5.22), the authors consider the class $\mathcal{M}_\Omega(\mathbf{R}^n)$ (see Definition 5.9) and do not assume the maps a to be continuous or strictly monotone. The main results of the paper are the local character of the G-convergence and the sequential compactness of $\mathcal{M}_\Omega(\mathbf{R}^n)$ with respect to the G-convergence.

Appendix

A.1. L^p and Sobolev Spaces

We give here only the definitions and main results that we used in the previous chapters. Most of the theorems are standard and their proofs as well as a deeper analysis are available in several textbooks on Functional Analysis.

We start with the abstract definition of the notion of weak convergence (for more details on it we refer to [25], [51] or to [16]) and then apply it to L^p and Sobolev spaces.

A.1.1. Weak convergence

Let us start with the definition.

Definition A.1.1. Let X be a real Banach space, X^* its dual and $\langle \cdot , \cdot \rangle$ the canonical pairing over $X^* \times X$.

i) We say that the sequence (x_h) in X *converges weakly to* $x \in X$ and we denote

$$x_h \rightharpoonup x \text{ in } X$$

if $\langle x^*, x_h \rangle \to \langle x^*, x \rangle$ for every $x^* \in X^*$.

ii) We say that the sequence (x_h^*) in X^* *converges weak * to* $x^* \in X^*$ and we denote

$$x_h^* \overset{*}{\rightharpoonup} x^* \text{ in } X^*$$

if $\langle x_h^*, x \rangle \to \langle x^*, x \rangle$ for every $x \in X$.

Then the following results hold.

Theorem A.1.2. *Let X be a Banach space. Let (x_h) and (x_h^*) be two sequence in X and in X^*, respectively.*

i) Let $x_h \rightharpoonup x$, then there exists a constant $K > 0$ such that $\|x_h\| \leq K$;

furthermore $\|x\| \leq \liminf_{h \to \infty} \|x_h\|$.

ii) Let $x_h^* \overset{*}{\rightharpoonup} x^*$, then there exists a constant $K > 0$ such that $\|x_h^*\|_{X^*} \leq K$;

furthermore $\|x^*\|_{X^*} \leq \liminf_{h \to \infty} \|x_h^*\|_{X^*}$.

iii) If $x_h \to x$ (strongly), then $x_h \rightharpoonup x$ (weakly).

*iv) If $x_h^* \to x^*$ (strongly in X^*), then $x_h^* \overset{*}{\rightharpoonup} x^*$ (weak *).*

v) If $x_h \rightharpoonup x$ (weakly) and $x_h^* \to x^*$ (strongly in X^*), then $\langle x_h^*, x_h \rangle \to \langle x^*, x \rangle$.

Theorem A.1.3. *Let X be a reflexive Banach space. Let (x_h) be a sequence in X and K be a positive constant such that $\|x_h\| \leq K$. Then there exist $x \in X$ and a subsequence $(x_{\sigma(h)})$ of (x_h) such that $x_{\sigma(h)} \rightharpoonup x$ in X.*

Theorem A.1.4. *Let X be a separable Banach space. Let (x_h^*) be a sequence in X^* and K be a positive constant such that $\|x_h^*\|_{X^*} \leq K$. Then there exist $x^* \in X^*$ and a subsequence $(x_{\sigma(h)}^*)$ of (x_h^*) such that $x_{\sigma(h)}^* \overset{*}{\rightharpoonup} x^*$ in X^*.*

A.1.2. L^p spaces

We apply the above results to the L^p spaces which are defined as follows (for more details see [1], [16], [52]).

Definition A.1.5. Let Ω be an open subset of $\mathbf{R}^n$.

i) Let $1 \leq p < +\infty$. We denote by $L^p(\Omega;\mathbf{R}^n)$ the set of all measurable functions $f : \Omega \to \mathbf{R}^n$ such that
$$\|f\|_{L^p(\Omega;\mathbf{R}^n)} \equiv \Big(\int_\Omega |f(x)|^p\,dx\Big)^{1/p} < +\infty\,.$$
It can be shown that $\|\cdot\|_{L^p(\Omega;\mathbf{R}^n)}$ is a norm.

ii) Let $p = +\infty$. A measurable function $f : \Omega \to \mathbf{R}^n$ is said to be in $L^\infty(\Omega;\mathbf{R}^n)$ if
$$\|f\|_{L^\infty(\Omega;\mathbf{R}^n)} \equiv \inf\{\alpha : |f(x)| \leq \alpha \text{ a.e. in } \Omega\} < +\infty\,.$$
One proves that $\|\cdot\|_{L^\infty(\Omega;\mathbf{R}^n)}$ defines a norm.

iii) $L^p_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$ denotes the linear space of measurable functions u such that $u \in L^p(\Omega';\mathbf{R}^n)$ for every $\Omega' \subset\subset \Omega$ (note that $u_h \to u$ in $L^p_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$ if $u_h \to u$ in $L^p(\Omega';\mathbf{R}^n)$ for every $\Omega' \subset\subset \Omega$).

Note: When dealing with scalar functions defined on Ω, we drop the target space $\mathbf{R}^n$ in the notation, and write just $L^p(\Omega)$ or $L^p_{\mathrm{loc}}(\Omega)$.

Remark A.1.6.

a) Let $1 \leq p \leq +\infty$. We denote by q the conjugate exponent of p, i.e., $1/p + 1/q = 1$, where it is understood that if $p = 1$ then $q = +\infty$ and reciprocally.

b) Let $1 \leq p < +\infty$. Then the dual space of $L^p(\Omega;\mathbf{R}^n)$ is $L^q(\Omega;\mathbf{R}^n)$. We point out also that the dual space of $L^\infty(\Omega;\mathbf{R}^n)$ contains strictly $L^1(\Omega;\mathbf{R}^n)$.

c) The notion of weak convergence in $L^p(\Omega;\mathbf{R}^n)$ becomes then as follows: If $1 \leq p < +\infty$, then $f_h \rightharpoonup f$ weakly in $L^p(\Omega;\mathbf{R}^n)$ if
$$\int_\Omega (f_h(x), g(x))\,dx \to \int_\Omega (f(x), g(x))\,dx$$
for every $g \in L^q(\Omega;\mathbf{R}^n)$. For the case $p = +\infty$, $f_h \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega;\mathbf{R}^n)$ weak* if
$$\int_\Omega (f_h(x), g(x))\,dx \to \int_\Omega (f(x), g(x))\,dx$$
for every $g \in L^1(\Omega;\mathbf{R}^n)$.

Theorem A.1.7. *For every $1 \leq p \leq +\infty$, $L^p(\Omega;\mathbf{R}^n)$ is a Banach space. It is separable if $1 \leq p < +\infty$ and reflexive if $1 < p < +\infty$. Moreover, $L^2(\Omega;\mathbf{R}^n)$ turns out to be a Hilbert space with the scalar product defined by $(f,g)_{L^2(\Omega;\mathbf{R}^n)} = \int_\Omega (f(x), g(x))\,dx$.*

A.1.3. Sobolev spaces

We mention here some important results on Sobolev spaces that we have used in the previous chapters.

Let us give first the definition of Sobolev spaces.

Definition A.1.8. Let Ω be an open subset of $\mathbf{R}^n$ and $1 \leq p \leq +\infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : Du \in L^p(\Omega; \mathbf{R}^n)\},$$

where $Du = (D_1 u, D_2 u, \ldots, D_n u) = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$ denotes the first order distributional derivative of the function u.

On $W^{1,p}(\Omega)$ we define the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega;\mathbf{R}^n)}\right)^{1/p}$$

Definition A.1.9. Let $1 \leq p < +\infty$. $W^{1,p}_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$. $W^{-1,q}(\Omega)$ with $1/p + 1/q = 1$ indicates the dual space of $W^{1,p}_0(\Omega)$.

Remark A.1.10. If $p = 2$, the notations $H^{1,2}(\Omega)$ or $H^1(\Omega)$ are very common for $W^{1,2}(\Omega)$. Moreover, $H^{1,2}_0(\Omega)$ or $H^1_0(\Omega)$ stand for $W^{1,2}_0(\Omega)$. The spaces $H^{1,2}(\Omega)$ and $H^{1,2}_0(\Omega)$ are naturally endowed with the scalar product $(u,v)_{H^{1,2}(\Omega)} = (u,v)_{L^2(\Omega)} + \sum_{i=1}^n (D_i u, D_i v)_{L^2(\Omega)}$ which induces the norm $\|u\|_{H^{1,2}(\Omega)}$.

Theorem A.1.11. *The space $W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq +\infty$. $W^{1,p}(\Omega)$ is separable if $1 \leq p < +\infty$ and reflexive if $1 < p < +\infty$.*

Moreover, the space $W^{1,p}_0(\Omega)$ endowed with the norm induced by $W^{1,p}(\Omega)$ is a separable Banach space; it is reflexive if $1 < p < +\infty$.

The spaces $H^{1,2}(\Omega)$ and $H^{1,2}_0(\Omega)$ are separable Hilbert spaces.

We now quote the Sobolev and Rellich-Kondrachov imbedding theorems.

Theorem A.1.12. *Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary.*

i) If $1 \leq p < n$, then

$$W^{1,p}(\Omega) \subset L^q(\Omega) \quad \textit{for every } 1 \leq q \leq np/(n-p)$$

and the imbedding is compact for every $1 \leq q < np/(n-p)$.

ii) If $p = n$, then

$$W^{1,p}(\Omega) \subset L^q(\Omega) \quad \textit{for every } 1 \leq q < +\infty$$

and the imbedding is compact.

iii) If $p > n$, then

$$W^{1,p}(\Omega) \subset C(\overline{\Omega})$$

and the imbedding is compact.

Remark A.1.13.

a) The regularity of the boundary $\partial\Omega$ in the theorem can be weakened (see, for example, [1]). Note that if the space $W^{1,p}(\Omega)$ is replaced by $W^{1,p}_0(\Omega)$, then no regularity of the boundary is required.

b) The compact imbedding can be read in the following way. Let

$$u_h \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega).$$

Case I : If $1 \leq p < n$, then $u_h \to u$ strongly in $L^q(\Omega)$, $\quad 1 \leq q < np/(n-p)$;
Case II : If $p = n$, then $u_h \to u$ strongly in $L^q(\Omega)$, $\quad 1 \leq q < +\infty$;
Case III : If $p > n$, then $u_h \to u$ strongly in $L^\infty(\Omega)$.

Let us state two important inequalities.

Theorem 1.14.

i) *(Poincaré inequality) Let* Ω *be a bounded open set and let* $1 \leq p < +\infty$*. Then there exists a constant* $K > 0$ *such that*

$$\|u\|_{L^p(\Omega)} \leq K \|Du\|_{L^p(\Omega;\mathbf{R}^n)}$$

for every $u \in W_0^{1,p}(\Omega)$*.*

ii) *(Poincaré-Wirtinger inequality) Let* Ω *be a bounded open convex set and let* $1 \leq p < +\infty$*. Then there exists a constant* $K > 0$ *such that*

$$\|u - \mathcal{M}_\Omega(u)\|_{L^p(\Omega)} \leq K \|Du\|_{L^p(\Omega;\mathbf{R}^n)}$$

for every $u \in W^{1,p}(\Omega)$*.*

Remark A.1.15. From the previous theorem it follows that $\|Du\|_{L^p(\Omega;\mathbf{R}^n)}$ defines a norm on $W_0^{1,p}(\Omega)$, denoted by $\|u\|_{W_0^{1,p}(\Omega)}$, which is equivalent to the norm $\|u\|_{W^{1,p}(\Omega)}$.

A.1.4. Extension and convergence lemmas for periodic functions

Let us start with the extension properties of periodic functions (see [45] Annexe 2). Let $Y =]0,1[^n$ be the unit cube in $\mathbf{R}^n$ and let $1 < p < +\infty$. By $W_\sharp^{1,p}(Y)$ we denote the subset of $W^{1,p}(Y)$ of all the functions u with mean value zero which have the same trace on the opposite faces of Y. In the case $p = 2$ we use the notation $H_\sharp^{1,2}(Y)$.

Lemma A.1.16. *Let* $f \in W_\sharp^{1,p}(Y)$*. Then* f *can be extended by periodicity to an element of* $W_{\rm loc}^{1,p}(\mathbf{R}^n)$*.*

Lemma A.1.17. *Let* $g \in L^q(Y;\mathbf{R}^n)$ *such that* $\int_Y (g, Dv) = 0$ *for every* $v \in W_\sharp^{1,p}(Y)$*. Then* g *can be extended by periodicity to an element of* $L_{\rm loc}^q(\mathbf{R}^n;\mathbf{R}^n)$*, still denoted by* g *such that* $-{\rm div}\, g = 0$ *in* $\mathcal{D}'(\mathbf{R}^n)$*.*

Let us conclude this section with a result for the weak convergence on L^p spaces which has been used frequently in the previous chapters. For a proof of it we refer to [45] Annexe 2, [21] Chapter 2, Theorem 1.5.

Theorem A.1.18. *Let* $f \in L^p(Y)$*. Then* f *can be extended by periodicity to a function (still denoted by* f*) belonging to* $L_{\rm loc}^p(\mathbf{R}^n)$*. Moreover, if* (ε_h) *is a sequence of positive real numbers converging to* 0 *and* $f_h(x) = f(\frac{x}{\varepsilon_h})$*, then*

$$f_h \rightharpoonup \mathcal{M}(f) = \frac{1}{|Y|}\int_Y f(y)\,dy \quad \textit{weakly in } L_{\rm loc}^p(\mathbf{R}^n)$$

if $1 \leq p < +\infty$*, and*

$$f_h \overset{*}{\rightharpoonup} \mathcal{M}(f) \quad \textit{in } L^\infty(\mathbf{R}^n) \textit{ weak*}$$

if $p = +\infty$*.*

It is clear that the above results still hold for Y not necessarily the unit cube in $\mathbf{R}^n$ but a parallelogram of the type described in Section 1.

Remark A.1.19. Let us point out some features of the weak convergence. To this aim, let us consider $Y =]0, 2\pi[$ and $f(x) = \sin x$. Let (ε_h) be a sequence of positive numbers converging to 0. By Theorem A.1.18 we have that $f_h(x) = f(\frac{x}{\varepsilon_h})$ converges to 0 in $L^\infty(Y)$ weak* (hence weakly in $L^2(Y)$). In particular,

$$\int_0^{2\pi} f_h(x)\, dx \to \frac{1}{2\pi} \int_0^{2\pi} \sin y\, dy = 0\,,$$

i.e., the mean values of f_h converges to 0. On the other hand, we have that (f_h) does not converge a.e. on Y. Furthermore,

$$(A.1.1) \qquad \|f_h - 0\|^2_{L^2(Y)} = \int_0^{2\pi} \sin^2(\frac{x}{\varepsilon_h}) dx \to (\frac{1}{\pi} \int_0^{\pi} \sin^2 y\, dy) 2\pi = \pi \neq 0\,,$$

which shows that we do not have convergence of (f_h) to f in the strong topology of $L^2(Y)$.

This example shows also another mathematical difficulty one meets by handling with weak convergent sequences. More precisely, if two sequences and their product converge in the weak topology, the limit of the product ist not equal, in general, to the product of the limits. Indeed, (A.1.1) proves that $f_h^2 = f_h \times f_h$ does not converge weakly in $L^2(Y)$ to 0.

A.2. A Compensated Compactness Lemma

noindent The next lemma, which has been used frequently in the previous chapters, helps to overcome the difficulties present by passing to the limit in products of only weakly convergent sequences.

Lemma A.2.1. *Let $1 < p < +\infty$. Let (u_h) be a sequence converging to u weakly in $W^{1,p}(\Omega)$, and let (g_h) be a sequence in $L^q(\Omega; \mathbf{R}^n)$ converging weakly to g in $L^q(\Omega; \mathbf{R}^n)$. Moreover assume that $(-\mathrm{div}\, g_h)$ converges to $-\mathrm{div}\, g$ strongly in $W^{-1,q}(\Omega)$. Then*

$$\int_\Omega (g_h, Du_h)\varphi\, dx \to \int_\Omega (g, Du)\varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Proof. The lemma is a simple case of compensated compactness (see ([34], [35], [49]). It can be proved by observing that

$$\int_\Omega (g_h, Du_h)\varphi\, dx = < -\mathrm{div}\, g_h, u_h\varphi > - \int_\Omega u_h (g_h, D\varphi)\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. □

Note that (g_h, Du_h) is the product of two sequences which converge only in the weak topology, and that by passing to the limit we get the product of the limits. This fact is known as the phenomenon of "compensated compactness".

A.3. Abstract existence theorems

A.3.1. Lax-Milgram Lemma

Let H be a Hilbert space. A bilinear form a on H is called *continuous* (or *bounded*) if there exists a positive constant K such that

$$|a(u, v)| \leq K\|x\|\|v\| \qquad \text{for every } u,\ v \in H$$

and *coercive* if there exists a positive constant α such that

$$a(u, u) \geq \alpha\|u\|^2 \qquad \text{for every } u \in H\,.$$

A particular example of continuous, coercive bilinear form is the scalar product of H itself.

Lemma A.3.1. *Let a be a continuous, coercive bilinear form on a Hilbert space H. Then for every bounded linear functional f in H^*, there exists a unique element $u \in H$ such that*

$$a(u,v) = \langle f, v\rangle \qquad \text{for every } v \in H\,.$$

For a proof of this classical lemma we refer to [16], [29].

A.3.2. Maximal monotone operators

Let X be a Banach space and X^* its dual space. Let A be a single-valued operator from $D(A)$ to X^*, where $D(A)$ is a linear subspace of X and is called the domain of A. The range $R(A)$ of A is the set of all points f of X^* such that there exists $x \in D(A)$ with $Ax = f$. Then

a) A is said to be *monotone* if

$$\langle Ax_1 - Ax_2, x_1 - x_2\rangle \geq 0 \qquad \text{for every } x_1\,,\ x_2 \in D(A)\,.$$

b) A is said to be *strictly monotone* if for every x_1, $x_2 \in D(A)$

$$\langle Ax_1 - Ax_2, x_1 - x_2\rangle = 0 \qquad \text{implies } x_1 = x_2$$

c) A is said to be *maximal monotone* if for every pair $[x,y] \in X \times X^*$ such that

$$\langle y - A\xi, x - \xi\rangle \geq 0 \qquad \text{for every } \xi \in D(A)$$

it follows that $y = Ax$.

d) A is said to be *hemicontinuous* if

$$\lim_{t\to 0} A(x+ty) = Ax \qquad \text{weakly in } X^*$$

for any $x \in D(A)$ and $y \in X$ such that $x + ty \in D(A)$ for $0 \leq t \leq 1$.

Theorem A.3.2. *Let X be a Banach space and let $A : X \to X^*$ be everywhere defined (i.e., $D(A) = X$), monotone and hemicontinuous. Then A is maximal monotone. In addition, if X is reflexive and A is coercive, i.e.,*

$$\lim_{\|x\|\to\infty} \frac{\langle Ax, x\rangle}{\|x\|} = +\infty\,,$$

then $R(A) = X^*$.

Proof. If X is a Hilbert space the proof of the previous theorem can be found in [15]. For the general case see [38] Chapter III, Corollary 2.3 and Theorem 2.10, or [30] Chapter 2, Theorem 2.1. □

A.4. Proof of Proposition 7.7 and of Proposition 7.8

Proof of Proposition 7.7. Since F is separable, there exists a countable dense subset X of F. By the assumptions on T^h and G and by using a diagonalization argument there exists a subsequence $(T^{\sigma(h)})$ of (T^h) such that for every $x \in X$, $(T^{\sigma(h)}x)$ converges weakly to a limit in G. Let us denote this limit by $T^0 x$.

Now, given $f \in F$ and $g^* \in G^*$, by approximating f by $x \in X$ one proves easily that the sequence $(\langle T^{\sigma(h)} f, g^*\rangle_{G,G^*})$ is a Cauchy sequence in $\mathbf{R}$. Let us denote by $\langle T^0 f, g^*\rangle$ its limit. The linearity of T^0 is immediate; by taking into account the weak lower semicontinuity of the norm and assumption (ii) we get

$$\|T^0 f\|_G = \|\lim_{h\to\infty} T^{\sigma(h)} f\|_G \leq \liminf_{h\to\infty} \|T^{\sigma(h)} f\|_G \leq c\|f\|_G\,.$$

Hence $T^0 \in \mathcal{L}(F;G)$. The proof of Proposition 7.7 is then accomplished. □

Proof of Proposition 7.8. Since (i) holds, we can define the bilinear form $a_h : V \times V \to \mathbf{R}$ by

$$a_h(u,v) = \langle T^h u, v\rangle$$

for every $u,\ v \in V$. By the hypotheses (ii) and (iii) it follows immediately that a_h is continuous and coercive. Hence, by the Lax-Milgram lemma for every $f \in V^*$ there exists a unique function $u \in V$ such that

$$a_h(u,v) = \langle f, v\rangle \qquad \text{for every } v \in V\,.$$

It turns out that the operators T^h are invertible and

$$\|(T^h)^{-1} f\|_V = \|u\|_V \le \frac{1}{\alpha}\|f\|_{V^*}$$

for every $f \in V^*$; thus $\|(T^h)^{-1}\|_{\mathcal{L}(V;V^*)} \le \frac{1}{\alpha}$. By Proposition 7.7 there exist a subsequence $\sigma(h)$ of h and an operator $S \in \mathcal{L}(V^*;V)$ such that for every $f \in V^*$

$$(T^{\sigma(h)})^{-1} f \rightharpoonup Sf \qquad \text{weakly in } V\,.$$

We get

$$\begin{aligned}\langle (T^{\sigma(h)})^{-1} f, f\rangle_{V,V^*} &= \langle (T^{\sigma(h)})^{-1} f, T^{\sigma(h)}(T^{\sigma(h)})^{-1} f\rangle_{V,V^*} \ \ge \\ &\ge \alpha\|(T^{\sigma(h)})^{-1} f\|_V^2 \ \ge \frac{\alpha}{\beta^2}\|f\|_{V^*}^2\,.\end{aligned}$$

Hence, for every $f \in V^*$

$$\langle Sf, f\rangle_{V,V^*} \ \ge \frac{\alpha}{\beta^2}\|f\|_{V^*}^2\,.$$

This proves that S is coercive. This fact together with the property that $S \in \mathcal{L}(V^*;V)$ ensures that S is invertible. Let us denote by $T^0 \in \mathcal{L}(V;V^*)$ its inverse. Note that for every $v \in V$ we have

$$\frac{\alpha}{\beta^2}\|T^0 v\|_{V^*}^2 \ \le\ \langle ST^0 v, T^0 v\rangle_{V,V^*} \ \le\ \langle v, T^0 v\rangle_{V,V^*} \le \|v\|_V\|T^0 v\|_{V^*}\,.$$

We conclude that

$$\|T^0\|_{\mathcal{L}(V;V^*)} \le \frac{\beta^2}{\alpha}\,.$$

On the other hand, we have for every $f \in V^*$

$$\alpha\|(T^{\sigma(h)})^{-1} f\|_V^2 \ \le \langle T^{\sigma(h)}(T^{\sigma(h)})^{-1} f, (T^{\sigma(h)})^{-1} f\rangle_{V^*,V} = \langle f, (T^{\sigma(h)})^{-1} f\rangle_{V^*,V}\,;$$

by taking the weak lower semicontinuity of the norm in V into account we obtain for every $f \in V^*$

$$\alpha\|Sf\|_V^2 \ \le \langle f, Sf\rangle_{V^*,V}\,.$$

By taking in particular $f = T^0 v$ we conclude that

$$\alpha\|v\|_V^2 \ \le \langle T^0 v, v\rangle_{V^*,V}$$

for every $v \in V$, which concludes the proof of Proposition 7.8. ☐

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