LOCAL RINGS WITH ZERO-DIMENSIONAL FORMAL FIBERS

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Abstract. We study Noetherian local rings whose all formal fibers are of dimension zero. Universal catenarity and going-up property of the canonical map to the completion are considered. We present several characterizations of these rings, including a characterization of Weierstrass preparation type. A characterization of local rings with going up property by a strong form of Lichtenbaum-Hartshorne Theorem is obtained. As an application, we give an upper bound for dimension of formal fibers of a large class of algebras over these rings.

0. Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. In [11], Matsumura proposes to study the maximum of the dimensions of all formal fibers of R. Denote this number by $\alpha(R)$. In an attempt to relate this invariant with other invariants of R, Matsumura gives several estimates of $\alpha(R)$ in terms of the dimension of R and computes it for some concrete examples. In general one has $0 \leq \alpha(R) \leq \dim(R)-1$. Matsumura constructs examples of local rings with $\alpha(R) = 0, \dim(R)-1, \dim(R)-2$ (cf. [11]). Later, in an attempt to answer a question of Matsumura, Rotthaus [16] constructs examples with $\alpha(R) = i$ for any $i \in \{0, 1, \ldots, \dim(R)-1\}$. However, the interesting question how to compute $\alpha(R)$ by means of known numerical invariants of R is not yet answered. To find an answer to this question, beginning with understanding the class of local rings whose formal fibers have dimension zero seems to be natural. These local rings will be called local rings with trivial formal fibers. They are the object of investigation in this note.

Examples of local rings with trivial formal fibers consist of complete local rings and one-dimensional local rings. Higher dimensional and non-complete examples are harder to construct. A well-known example is due to Nagata [13, Example E3.1, Appendix]. There is also a construction of local rings with trivial formal fibers due to Rotthaus [16]. There are interesting examples coming from arithmetics and geometry. For instance, let K be a p-adic field, that is, a finite extension of $\mathbb{Q}_p$, and let X be a projective curve over the ring of integers $\mathcal{O}_K$. If x is a point of the special fiber of X, then the local ring $\mathcal{O}_{X,x}$ of germs of regular functions at x has all formal fibers of dimension zero (see Example 1.1). This is in fact very special class of local rings with trivial formal fibers which is a motivation for our present work. Further examples with applications in algebraic number theory could be found in [6], [3].

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Local rings with trivial formal fibers have been studied by several authors. In [7], Heinzer and Rotthaus consider excellent Henselian local rings with trivial formal fibers and show that they satisfy the Noetherian intermediate rings property. Recently, Zöschinger [17] gave several characterizations of local rings whose canonical map to the completion satisfies the going-up property. These are in fact local rings with trivial formal fibers.

The aim of this note is to study several basic properties of local rings with trivial formal fibers and some applications to higher dimensional case.

In Section 1 we first present the example of local rings of regular functions on a curve over the ring of p-adic integers, these rings have trivial formal fibers. The main result of this section is Theorem 1.4 and its consequence where we consider over a local ring with trivial formal fibers certain class of algebras, including the polynomial algebras, and give an upper bound for the dimension of their formal fibers.

Universally catenary local rings with trivial formal fibers are considered in Section 2. By using a result of Charters and Loepp, we obtain an example of a local ring with trivial formal fibers which is not universally catenary. This distinguishes the subclass to the whole class of local rings considered in the previous section. We then present several characterizations of local rings in this subclass including a characterization of Weierstrass preparation type (Theorem 2.4 and its consequence).

In the last Section 3 we consider local rings whose canonical map $R \hookrightarrow \hat{R}$ satisfies the going-up theorem. An easy argument shows that R is universally catenary and $\alpha(R) = 0$. The converse does not hold (see Example 3.1). Recently Zöschinger [17] studied such rings with going-up property and gave several characterizations. We will give another characterization by means of a strong form of Lichtenbaum-Hartshorne Vanishing Theorem. The famous theorem of Lichtenbaum and Hartshorne characterizes the vanishing of the top local cohomology module of a local ring supported on an ideal by some conditions on the completion. Several authors have attempted to replace the conditions on $\hat{R}$ by similar conditions on R. It is of course impossible in general. However, we will show that such desired conditions are possible for local rings with going-up property, it even characterizes such local rings among universally catenary local rings with trivial formal fibers (see Theorem 3.2).

Throughout this note all rings will be commutative and Noetherian with a unit, local rings mean commutative rings with a unique maximal ideal.

1. Local rings with trivial formal fibers

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\hat{R}$ be its $\mathfrak{m}$-adic completion. For each prime $\mathfrak{p} \in \operatorname{Spec} R$, denote by $\alpha(R, \mathfrak{p})$ the Krull dimension of the fiber ring $\hat{R} \otimes_R k(\mathfrak{p})$, where $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, and $\alpha(R) := \max\{\alpha(R, \mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} R\}$. We identify R with a subring of $\hat{R}$ via the canonical map $R \to \hat{R}$. The restriction morphism on spectra is denoted by $\tau : \operatorname{Spec} \hat{R} \to \operatorname{Spec} R$, $P \mapsto P \cap R$. Since the completion is flat, if $\mathfrak{p} \subseteq \mathfrak{q}$ are two primes of R then $\alpha(R, \mathfrak{p}) \geq \alpha(R, \mathfrak{q})$ by [10, Theorem 15.3], hence $\alpha(R) = \max\{\alpha(R, \mathfrak{p}) : \mathfrak{p} \in \min \operatorname{Spec} R\}$, here we denote by $\min \operatorname{Spec}(R)$ the set of all minimal primes of R. In particular, all formal fibers of a ring have dimension zero if and only if the dimensions of the generic formal fibers are zero. Such a local ring is called a local ring with trivial formal fibers.

Typical examples of non-complete local rings with trivial formal fibers are 1-dimensional local rings. There are examples coming naturally from arithmetics and geometry (see, for example, Harbater, Hartmann and Krashen [6] or Cuong [3]). To motivate the present work, we have an example.

Example 1.1. Let K be a field of p-adic numbers and $\mathcal{O}_K$ be its ring of integers. Let X be a projective curve over $\mathcal{O}_K$. Let x be a point of the special fiber of X. Then all the formal fibers of the local domain $\mathcal{O}_{X,x}$ of germs of regular functions at x have dimension zero.

Using the triviality of the formal fibers of the ring $\mathcal{O}_{X,x}$ in the example and combining with some other properties of the ring, one can prove that the extension $\mathcal{O}_{X,x} \subset \widehat{\mathcal{O}}_{X,x}$ satisfies the Weierstrass Preparation Theorem (see [3]). The later theorem was used effectively in algebraic number theory to estimate the u-invariant of quadratic forms over the fractional field of $\mathcal{O}_{X,x}$. This leads to interesting computation of u-invariant of function fields over p-adic numbers (see [6]). The proof of the conclusion in the example will be postponed till the end of this section.

The flatness of the completion implies that the restriction morphism τ satisfies the going-down theorem, in particular it is surjective. So if P is a prime of $\hat{R}$, then $\operatorname{height}(P) \geq \operatorname{height}(P \cap R)$, and consequently, $\tau(\min \operatorname{Spec}(\hat{R})) = \min \operatorname{Spec} R$. Using this we have the following characterization for $\alpha(R) = 0$.

Proposition 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. The following statements are equivalent.*

(a) $\alpha(R) = 0$.
(b) $\min \operatorname{Spec}(\hat{R}) = \tau^{-1}(\min \operatorname{Spec}(R))$.
(c) For any prime P of $\hat{R}$, $\operatorname{height}_{\hat{R}}(P) > 0$ *if and only if* $\operatorname{height}_R(P \cap R) > 0$.

Moreover, any of the above equivalent conditions implies that $\operatorname{Ass}(\hat{R}) = \tau^{-1}(\operatorname{Ass}(R))$. *So via the restriction morphism τ, the associated primes of the completion could be obtained from the associated primes of R and vice versa.*

Proof. $(a) \Rightarrow (c)$: The "if" part is straightforward by the going-down theorem. For the "only if" part, suppose $\alpha(R) = 0$ and take $P \in \operatorname{Spec}(\hat{R})$ with $\operatorname{height}_{\hat{R}}(P) > 0$. Let $Q \subset P$, $Q \neq P$, be a prime ideal of $\hat{R}$. Since $\alpha(R) = 0$, $Q \cap R \neq P \cap R$. So $\operatorname{height}_R(P \cap R) > 0$.
$(c) \Rightarrow (b)$: We have $\min \operatorname{Spec}(\hat{R}) \subseteq \tau^{-1}(\min \operatorname{Spec} R)$. Let $P \in \operatorname{Spec}(\hat{R})$ such that $P \cap R \in \min \operatorname{Spec}(R)$. Hence $\operatorname{height}_R(P \cap R) = 0$ and then $\operatorname{height}_{\hat{R}}(P) = 0$ by the assumption. So P is a minimal prime of the completion $\hat{R}$ and $\min \operatorname{Spec}(\hat{R}) = \tau^{-1}(\min \operatorname{Spec} R)$.
$(b) \Rightarrow (a)$: We have $\alpha(R) = \dim \tau^{-1}(\min \operatorname{Spec}(R))$ (see, for example, [10, Theorem 15.3]). So $\alpha(R) = \dim(\min \operatorname{Spec}(\hat{R})) = 0$.

In order to prove the last conclusion, we now assume that $\alpha(R) = 0$. We will need the following fact.

Claim: For a Noetherian local ring S, $\tau(\operatorname{Ass}(\hat{S})) = \operatorname{Ass}(S)$.

The conclusion of the claim in fact holds true for any faithfully flat ring extension. A proof could be found in [14, Lemma 3.4].

Since $\alpha(R) = 0$, for each associated prime $\mathfrak{p} \in \operatorname{Ass}(R)$, we have $\alpha(R/\mathfrak{p}) = 0$. So by the claim and (c), we have

$$\operatorname{Ass}(\hat{R}/\mathfrak{p}\hat{R}) \subseteq \tau^{-1}(\{\mathfrak{p}\}) = \min\operatorname{Spec}(\hat{R}/\mathfrak{p}\hat{R}) \subseteq \operatorname{Ass}(\hat{R}/\mathfrak{p}\hat{R}).$$

Thus $\operatorname{Ass}(\hat{R}/\mathfrak{p}\hat{R}) = \tau^{-1}(\{\mathfrak{p}\})$. On the other hand, there is an inclusion $\hat{R}/\mathfrak{p}\hat{R} \hookrightarrow \hat{R}$ which is induced from an inclusion $R/\mathfrak{p} \hookrightarrow R$ as $\mathfrak{p}$ is an associated prime of R and the faithful flatness of the completion. Then there is an inclusion of sets

$$\operatorname{Ass}(\hat{R}) \supseteq \bigcup_{\mathfrak{p}\in\operatorname{Ass}(R)} \operatorname{Ass}(\hat{R}/\mathfrak{p}\hat{R}) = \tau^{-1}(\operatorname{Ass}(R)).$$

Therefore $\operatorname{Ass}(\hat{R}) = \tau^{-1}(\operatorname{Ass}(R))$ by using again the claim. □

The next proposition gives a characterization of Weierstrass preparation type for local rings with trivial formal fibers. The idea was in fact used in several places for constructing examples of such local rings (cf. [16]).

Proposition 1.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. All formal fibers of R have dimension zero, that is, $\alpha(R) = 0$, if and only if for any element $\hat{a} \in \hat{R}$, there is $\hat{b} \in \hat{R}$ such that $\hat{a}\hat{b} \in R$ and* $\operatorname{height}_R(\hat{a}\hat{b}.R) = \operatorname{height}_{\hat{R}}(\hat{a}\hat{R})$.

Proof. For the sufficient condition, let $P \in \operatorname{Spec}\hat{R}$ with $\operatorname{height}_{\hat{R}}(P) \geq 1$. Take an element $\hat{a} \in P$ such that $\operatorname{height}_{\hat{R}}(\hat{a}\hat{R}) = 1$. By the assumption, there is $\hat{b} \in \hat{R}$ such that $\hat{a}\hat{b} \in R$ and $\operatorname{height}_R(\hat{a}\hat{b}R) = 1$. In particular, $\operatorname{height}_R(P \cap R) \geq 1$. This is actually an equivalence since τ satisfies the going-down theorem. So $\alpha(R) = 0$ following Proposition 1.2.

Conversely, let $\min\operatorname{Spec}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and $\min\operatorname{Spec}(\hat{R}) = \{P_1, \ldots, P_r\}$. We have $\tau^{-1}(\mathfrak{p}_1, \ldots, \mathfrak{p}_s) = \{P_1, \ldots, P_r\}$ by Proposition 1.2, since $\alpha(R) = 0$. Put $S = R\backslash(\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s)$ which is a multiplicative set. The fiber ring of τ at $\min\operatorname{Spec}(R)$ is $S^{-1}\hat{R}$. Note that $\dim S^{-1}\hat{R} = 0$ since $\alpha(R) = 0$, so $S^{-1}\hat{R}$ is Artinian. Its maximal ideals are $S^{-1}P_1, \ldots, S^{-1}P_r$. Take an element $\hat{a} \in \hat{R}$. It suffices to prove the assertion for the case $\operatorname{height}_{\hat{R}}(\hat{a}\hat{R}) = 1$. In other words, $\hat{a} \notin P_1 \cup \ldots \cup P_r$ and thus $\hat{a} \in S^{-1}\hat{R}$ is invertible. Write $\hat{a}^{-1} = \frac{\hat{b}}{x} \in S^{-1}\hat{R}$ for some $\hat{b} \in \hat{R}$, $x \in S$. Then there is $y \in S$ such that $\hat{a}\hat{b}y = xy \in R$ with $\operatorname{height}_R(xyR) = 1$. □

Now we are going to estimate the dimension of formal fibers of certain algebras over a local ring R provided $\alpha(R) = 0$. For a Noetherian local ring A, the bounds $0 \leq \alpha(A) \leq \dim(A) - 1$ are obvious. The highest possibility $\dim(A) - 1$ of $\alpha(A)$ occurs rather popularly in nature, as Matsumura [11] points out that $\alpha(A) = \dim(A) - 1$ if A is essentially of finite type over a field. In the following we will present a class of local rings A with $\alpha(A) \leq \dim(A) - 2$. This is an interesting way to produce local rings with higher (but not highest) dimensional formal fibers from those with trivial formal fibers.

We first denote the least dimension of irreducible components of $\operatorname{Spec}(R)$ by $\operatorname{sdim}(R)$. So $\operatorname{sdim}(R) = \min\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in \min\operatorname{Spec}(R)\}$.

Theorem 1.4. *Let $(R, \mathfrak{m}_R, k)$ be a Noetherian local ring. Let $R \subseteq A$ be a ring extension where $(A, \mathfrak{m}_A)$ is a Noetherian local ring such that $\dim(A) > 1$, $\mathfrak{m}_A \cap R = \mathfrak{m}_R$ and the natural inclusion $R/\mathfrak{m}_R \hookrightarrow A/\mathfrak{m}_A$ is a finite field extension. We assume further that for each element $x \in R$, if $\operatorname{height}_R(xR) = 1$ then $\operatorname{height}_A(xA) = 1$ in the ring A. Then $\alpha(A) \leq \dim(A) - 2$ if either*

(a) R is complete and of positive dimension; or,
(b) $\alpha(R)=0$ and $\operatorname{sdim}(\hat{R})\geq 2$.

Proof. (a) In the first case we assume $(R,\mathfrak{m}_R)$ is a complete local ring of positive dimension. The present proof extends Matsumura's ideas in [11].

We assume the contrary that $\alpha(A)=\dim(A)-1$. Then there is a prime P of $\widehat{A}$ such that $\dim\widehat{A}/P=1$ and $\dim(A/P\cap A)=\dim(A)$. Denote $\mathfrak{p}=P\cap A$. By the assumption, the ideal $\mathfrak{m}_RA\subseteq A$ is of positive height, thus $\mathfrak{m}_RA$ is not contained in $\mathfrak{p}$. Then $\mathfrak{m}_R\widehat{A}+P$ is an $\mathfrak{m}_A\widehat{A}$-primary ideal in $\widehat{A}$. Let $n>0$ be an integer such that $\mathfrak{m}_A^{n+1}\widehat{A}\subseteq\mathfrak{m}_R\widehat{A}+P$. We choose a set of generators $f_1,\dots,f_r\in A$ of $\mathfrak{m}_A$. Since $\widehat{A}/\mathfrak{m}_A\widehat{A}\simeq A/\mathfrak{m}_A$ is a finite extension of $k=R/\mathfrak{m}_R$, we can write $\widehat{A}/\mathfrak{m}_A\widehat{A}=kw_1+\ldots+kw_t$ for some $w_1,\dots,w_t\in A$. We have

$$\begin{aligned}\widehat{A}&=Rw_1+\ldots+Rw_t+\mathfrak{m}_A\widehat{A}\\&=(Rw_1+\ldots+Rw_t)+(f_1\widehat{A}+\ldots+f_r\widehat{A})\\&=\ldots=\sum_{\substack{1\leq i\leq t\\ \alpha_1+\ldots+\alpha_r\leq n}}w_if_1^{\alpha_1}\ldots f_r^{\alpha_r}R+\mathfrak{m}_A^{n+1}\widehat{A}\\&=\sum_{\substack{1\leq i\leq t\\ \alpha_1+\ldots+\alpha_r\leq n}}w_if_1^{\alpha_1}\ldots f_r^{\alpha_r}R+P+\mathfrak{m}_R\widehat{A}.\end{aligned}$$

Denote $M=\widehat{A}/P$, then

$$M=R[\bar{w}_1,\dots,\bar{w}_t][\bar{f}_1,\dots,\bar{f}_r]_{\leq n}+\mathfrak{m}_RM,$$

where $\bar{w}_1,\dots,\bar{w}_t,\bar{f}_1,\dots,\bar{f}_r\in A/\mathfrak{p}$ and the subindex $\leq n$ means the subset of polynomials of degree bounded above by n. So $M/\mathfrak{m}_RM$ is finitely generated over k. Note that M is separated in the $\mathfrak{m}_R\widehat{A}$-adic topology, Theorem 30.6 of [13] concludes that $\widehat{A}/P$ is an R-module generated by the finite set $\{w_if_1^{\alpha_1}\ldots f_r^{\alpha_r}:1\leq i\leq t,\alpha_1+\ldots+\alpha_r\leq n\}$. Combining this with the inclusions $R/(\mathfrak{p}\cap R)\subseteq A/\mathfrak{p}\subseteq\widehat{A}/P$, we obtain

$$\dim(R/(\mathfrak{p}\cap R))=\dim(A/\mathfrak{p})=\dim\widehat{A}/P=1.$$

This contradicts the assumption $\dim(A/\mathfrak{p})=\dim(A)\geq 2$. Therefore $\alpha(A)\leq\dim(A)-2$.

(b) We now consider the general case where R is not necessarily complete. We have a commutative diagram of Cartesian product

$$\begin{array}{ccc}\widehat{R}&\longrightarrow&\widehat{R}\otimes_R A\\ \uparrow&&\uparrow\\ R&\longrightarrow&A.\end{array}$$

Denote the $\mathfrak{m}_RA$-adic completion and the $\mathfrak{m}_A$-adic completion of A respectively by A' and $\widehat{A}$. We have

$$A'\simeq\varprojlim_s A/\mathfrak{m}_R^sA\simeq\varprojlim_s(R/\mathfrak{m}_R^s\otimes_R(\widehat{R}\otimes_R A)).$$

So A' is isomorphic to the $\mathfrak{m}_R(\widehat{R}\otimes_R A)$-adic completion $(\widehat{R}\otimes_R A)'$ of $\widehat{R}\otimes_R A$. We then get a commutative diagram

$$\begin{tikzcd} \widehat{R} \arrow[r] & \widehat{R}\otimes_R A \arrow[r] & (\widehat{R}\otimes_R A)' \arrow[d, "\simeq"] & \\ R \arrow[u] \arrow[r] & A \arrow[u] \arrow[r] & A' \arrow[r] & \widehat{A}. \end{tikzcd}$$

We denote the composition $\widehat{R}\to\widehat{R}\otimes_R A\to A'$ by φ and $R_0 := \mathrm{Im}(\varphi)$. Consider the commutative triangle

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & \widehat{R} \arrow[dl, "\varphi"] \\ & R_0 & \end{tikzcd}$$

where the map $R\to R_0$ is induced from the inclusions $R\hookrightarrow A\hookrightarrow A'$. In particular, $\mathrm{Ker}(\varphi)\cap R=0$. The assumption $\alpha(R)=0$ then implies that $\mathrm{Ker}(\varphi)$ is included in some minimal primes of $\hat{R}$. In particular, $\dim(R_0)\geq \mathrm{sdim}(\hat{R})\geq 2$.

We have obtained a complete local ring R_0 with positive dimension and maximal ideal $\mathfrak{m}_R R_0$ such that $R_0\subseteq A'$ is a subring satisfying

(i) $\mathfrak{m}_A A'\cap R_0=\mathfrak{m}_R R_0$ is the maximal ideal of R_0;

(ii) $A'/\mathfrak{m}_A A'\simeq A/\mathfrak{m}_A$ is a finite extension of $R_0/\mathfrak{m}_R R_0\simeq R/\mathfrak{m}_R$;

(iii) For each $x\in\hat{R}$, if $\mathrm{height}_{R_0}(xR_0)=1$ then $\mathrm{height}_{A'}(xA')=1$. Indeed, we have $\mathrm{height}_{\hat{R}}(x\hat{R})=1$ as well. Proposition 1.2 gives us $\mathrm{height}_R(R\cap x\hat{R})=1$. Taking an element $a\in R\cap x\hat{R}$ such that $\mathrm{height}_R(aR)=1$. By the assumption, we have $\mathrm{height}_{A'}(aA')=\mathrm{height}_A(aA)=1$. Since $aA'\subseteq xA'$, it shows that $\mathrm{height}_{A'}(xA')=1$.

This is exactly the situation in the first case. So if P is a prime of $\widehat{A}$ with $\dim(\widehat{A}/P)=1$, from the the proof of Case (a), either $\mathfrak{m}_R\subseteq P$ or $\dim(R_0/P\cap R_0)=\dim(\hat{A}/P)=1$.

We will show that $P\cap A$ is not a minimal prime of A. This is obvious if $\mathfrak{m}_R\subseteq P$. So assume that $\mathfrak{m}_R\not\subseteq P$ and $\dim(R_0/P\cap R_0)=\dim(\hat{A}/P)=1$. Let $\mathfrak{q}=\varphi^{-1}(P\cap R_0)$ be the prime of $\hat{R}$ corresponding to $P\cap R_0$ via the projection φ. The dimension of the quotient $\hat{R}/\mathfrak{q}$ is also one. As $\mathrm{sdim}(\hat{R})\geq 2$, $\mathfrak{q}$ is not a minimal prime of the completion $\hat{R}$. The assumption $\alpha(R)=0$ implies that the ideal $P\cap R=\mathfrak{q}\cap R$ is not a minimal prime of R (cf. Proposition 1.2). In particular, the ideal $P\cap A$ is not a minimal prime of A. We therefore obtain $\alpha(A)\leq\dim A-2$. □

Remark 1.5. 1. In Theorem 1.4, it is required that if x is an element of R and $\mathrm{height}_R(xR)=1$, then the ideal xA is of height 1 in the algebra A. Equivalently, minimal primes of A restrict to minimal primes of R. This requirement is fulfilled in the following important cases

(i) A is a domain; or

(ii) The extension $R\subseteq A$ satisfies the going-down theorem (for example, A is flat over R).

2. The assumption $\mathrm{sdim}(\hat{R})\geq 2$ in the second case is important. For instance, consider $R=k[X]_{(X)}$ where k is a field, and $A=k[X,Y]_{(X,Y)}$. We have $\dim(R)=\dim(\hat{R})=\mathrm{sdim}(\hat{R})=1$ and $\alpha(R)=0$. However, $\alpha(A)=1=\dim(A)-1$.

Theorem 1.4 gives an interesting relation between local rings with trivial formal fibers and those with higher dimensional formal fibers. The following consequence of the theorem provides us a rich class of local algebras which satisfy the conditions in the theorem. It is worth noting that Example 1.1 is a special case of this corollary.

Corollary 1.6. *Keep the assumption on $(R, \mathfrak{m}_R, k)$ in Theorem 1.4. Let X be a projective scheme over* $\operatorname{Spec}(R)$ *and suppose that X dominates* $\operatorname{Spec}(R)$*. Let x be a closed point in the special fiber of X over $k = R/\mathfrak{m}$. Then we have*

$$\alpha(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{X,x}) - 2.$$

Proof. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$. Since x is a closed point of the special fiber of X, we have $\mathfrak{m}_x \cap R = \mathfrak{m}_R$. Furthermore, by the geometric formulation of Hilbert's Nullstellensatz (see Mumford [12, Proposition 3, page 99]), $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is a finite extension of the residue field k. So $\mathcal{O}_{X,x}$ satisfies all conditions in Theorem 1.4, it leads to $\alpha(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{X,x}) - 2$. □

2. Universal catenarity

Local rings with trivial formal fibers seems to be quite close to complete local rings. In fact, this is not a right intuition, there are some fundamental properties of complete local rings like universal catenarity which are not satisfied generally by local rings with trivial formal fibers.

There are examples of local rings with trivial formal fibers and the rings are not universally catenary. In order to give an example of such a local ring, we first recall a result of Charters and Loepp [2, Theorem 3.1 and Lemma 2.8] which is very helpful in seeking for examples of local rings with prescribed generic formal fibers.

Proposition 2.1 (Charters and Loepp)**.** *Let $(T, \mathfrak{m}_T)$ be a complete local ring. Let $W \subset \operatorname{Spec}(T)$ be a non-empty subset of primes which satisfies:*
(a) W has finitely many maximal elements;
(b) $\mathfrak{m}_T \notin W$ and $\operatorname{Ass}(T) \subseteq W$;
(c) If $P \in \operatorname{Spec}(T)$, $Q \in W$ and $P \subseteq Q$ then $P \in W$;
(d) For any $P \in W$, $P \cap \mathbb{Z}.1_T = 0$.
Then there is a Noetherian local domain $(R, \mathfrak{m})$ such that T is isomorphic to the $\mathfrak{m}$-adic completion of R and the generic formal fiber of R, $\tau^{-1}(\{0\}) = W$. Moreover, if $\mathfrak{p}$ is a prime of R, $\mathfrak{p} \neq 0$, then $T \otimes_R k(\mathfrak{p}) \simeq k(\mathfrak{p})$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Using this result, we are able to give examples of local rings with trivial formal fibers which are fail to be universally catenary, or even catenary.

Example 2.2. Let k be a field and $S = k[[X, Y, Z]]/(X, Y) \cap (Z)$. We have $\min \operatorname{Spec}(S) = \{(X, Y), (Z)\}$. Proposition 2.1 shows that there is a local domain R with $\hat{R} = S$ and its generic formal fiber is $\tau^{-1}(\{0\}) = \{(X, Y), (Z)\}$. Hence $\alpha(R) = 0$. On the other hand, R is obviously not formally equidimensional. Therefore it is not universally catenary due to a well-known theorem of Ratliff [15, Theorem 2.6].

It is remarkable that though the local domain R is not universally catenary, it is catenary and all its formal fibers are regular. The catenarity is immediate since $\dim R = 2$. The regularity of the formal fibers of R follows from Proposition 2.1. By the same idea, we could also obtain a local domain R with $\alpha(R) = 0$ and R is not even catenary.

Remark 2.3. Let S be the local ring in Example 2.2. Denote the Henselization of S by S^h. It is well-known that S and S^h are analytically isomorphic and the canonical homomorphism $S \to S^h$ is regular with zero-dimensional fibers (see [5, Théorème 18.6.9]). Hence $\alpha(S^h) = \alpha(S) = 0$.

It is worth noting that the formal fibers of S are geometrically normal. Then S^h is universally catenary due to Heinzer-Rotthaus-Wiegand [8, Proposition 2.2]. So the completion extension $S \to \hat{S}$ factors through a "small" universally catenary local ring with trivial formal fibers, namely, through S^h.

The next theorem is a characterization of Weierstrass Preparation type for universally catenary local rings with trivial formal fibers.

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. The following statements are equivalent.*

(a) R is universally catenary and $\alpha(R) = 0$.
(b) For any prime $P \in \operatorname{Spec}(\hat{R})$, $\dim \hat{R}/P = \dim R/P \cap R$.
(c) For any ideal $I \subset \hat{R}$, $\dim \hat{R}/I = \dim R/I \cap R$.

If in addition R is equidimensional, then any of the equivalent statements above is equivalent to

(d) For any parameter element $x \in \hat{R}$, there is $y \in \hat{R}$ such that $xy \in R$ and xy is a parameter element of R.

Proof.
$(a) \Leftrightarrow (b)$: The famous theorem of Ratliff [15, Theorem 2.6] (see also [10, Theorem 31.7]) tells us that R is universally catenary if and only if it is formally equidimensional. Or equivalently,

$$\dim \hat{R}/P + \operatorname{height}(P/\mathfrak{p}\hat{R}) = \dim R/\mathfrak{p},$$

for any prime $P \in \operatorname{Spec} \hat{R}$, where $\mathfrak{p} = P \cap R$. On the other hand, $\alpha(R) = 0$ is equivalent to $\operatorname{height}(P/\mathfrak{p}\hat{R}) = 0$ for any $P \in \operatorname{Spec} \hat{R}$. So R is universally catenary with $\alpha(R) = 0$ if and only if $\dim \hat{R}/P = \dim R/P \cap R$ for all $P \in \operatorname{Spec} \hat{R}$, and $\mathfrak{p} = P \cap R$. This proves the equivalence of (a) and (b).
$(b) \Leftrightarrow (c)$: Let $I \subset \hat{R}$ be a proper ideal. We have $\sqrt{I} \cap R = \sqrt{I \cap R}$ and $\dim(\hat{R}/I) = \dim(\hat{R}/\sqrt{I})$, $\dim(R/I \cap R) = \dim(R/\sqrt{I \cap R})$, so replacing I by its radical $\sqrt{I}$ we can assume that I and $I \cap R$ are radical ideals. Write $I = \cap_{i=1}^{r} P_i$ where $P_i \in \operatorname{Spec}(\hat{R})$'s are primes of I. Take a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ of $R \cap I$ such that $\dim(R/\mathfrak{p}) = \dim(R/I \cap R)$. We have

$$\mathfrak{p} \supseteq I \cap R = \bigcap_{i=1}^{r} (P_i \cap R),$$

which induces that $\mathfrak{p} \supseteq P_i \cap R$ for some P_i by the Prime Avoidance Theorem. Using the assumption we obtain $\dim(R/\mathfrak{p}) \leq \dim(R/P_i \cap R) = \dim(\hat{R}/P_i) \leq \dim(\hat{R}/I)$. So $\dim \hat{R}/I = \dim R/I \cap R$. The converse is trivial.

To prove the last assertion, we assume that R is equidimensional.
$(a) \Rightarrow (d)$: Assume $\alpha(R) = 0$ and R is universally catenary. Since R is equidimensional, $\hat{R}$ is also equidimensional by the theorem of Ratliff. So x is a parameter element of $\hat{R}$ if and only if $\operatorname{height}_{\hat{R}}(x\hat{R}) = 1$. Proposition 1.3 guarantees the existence of an element $y \in \hat{R}$ such that $xy \in R$ and $\operatorname{height}_R(xyR) = \operatorname{height}_{\hat{R}}(x\hat{R}) = 1$. In particular, xy is a parameter element of R.

$(d) \Rightarrow (a)$: We prove first $\alpha(R) = 0$ by using Proposition 1.3. Indeed, let $x \in \hat{R}$. If $\operatorname{height}_{\hat{R}}(x\hat{R}) = 0$ then take $y = 0$ we get $xy = 0 \in R$ and $\operatorname{height}_R(xyR) = 0 = \operatorname{height}_{\hat{R}}(x\hat{R})$. If $\operatorname{height}_{\hat{R}}(x\hat{R}) = 1$ then x is a parameter element of $\hat{R}$. By the assumption, let $y \in \hat{R}$ be such that $xy \in R$ and xy is a parameter element of R. So $\operatorname{height}_R(xyR) = 1 = \operatorname{height}_{\hat{R}}(x\hat{R})$ since R is equidimensional. Proposition 1.3 applies to imply that $\alpha(R) = 0$.

In order to prove the universal catenarity, again by the theorem of Ratliff it suffices to show that $\hat{R}$ is equidimensional. It is worth noting that a local ring is equidimensional if and only if every parameter element generates a principal ideal of height one. So let x be a parameter element of $\hat{R}$. Let $y \in \hat{R}$ be such that $xy \in R$ and xy is a parameter element of R. Since R is equidimensional, $\operatorname{height}_R(xyR) = 1$. The argument before Proposition 1.2 gives us

$$1 = \operatorname{height}_R(xyR) \leq \operatorname{height}_R(xy\hat{R} \cap R) \leq \operatorname{height}_R(x\hat{R} \cap R) \leq \operatorname{height}_{\hat{R}}(x\hat{R}).$$

So $\operatorname{height}_{\hat{R}}(x\hat{R}) = 1$ and $\hat{R}$ is equidimensional. This completes the proof. □

Theorem 2.4 leads to the following consequence which is another characterization of universally catenary rings with trivial formal fibers. For a subset $X \subseteq \operatorname{Spec} R$ and an integer $i \in \mathbb{Z}$, we denote $X_i := \{\mathfrak{p} \in X : \dim R/\mathfrak{p} = i\}$.

Corollary 2.5. *Let R be a Noetherian local ring. The following statements are equivalent.*

(a) R is universally catenary and $\alpha(R) = 0$.

(b) $\operatorname{Ass}(\hat{R})_i = \tau^{-1}(\operatorname{Ass}(R)_i)$, for each $i = 1, \ldots, \dim R$.

(c) $\min \operatorname{Spec}(\hat{R})_i = \tau^{-1}(\min \operatorname{Spec}(R)_i)$, for each $i = 1, \ldots, \dim R$.

In particular, if R is equidimensional then R is universally catenary with trivial formal fibers if and only if $\operatorname{Spec}(\hat{R})_d = \tau^{-1}(\operatorname{Spec}(R)_d)$, where $d = \dim R$.

Proof.

$(a) \Rightarrow (b)$: We have proved in Proposition 1.2 that if $\alpha(R) = 0$ then $\operatorname{Ass}(\hat{R}) = \tau^{-1}(\operatorname{Ass} R)$. Then Theorem 2.4(b) induces that (a) implies (b).

$(b) \Rightarrow (c)$: is obvious.

$(c) \Rightarrow (a)$: We have

$$\min \operatorname{Spec} \hat{R} = \bigcup_{i>0} \min \operatorname{Spec}(\hat{R})_i = \bigcup_{i>0} \tau^{-1}(\min \operatorname{Spec}(R)_i) = \tau^{-1}(\min \operatorname{Spec} R).$$

Hence $\alpha(R) = 0$ by Proposition 1.2. Moreover, for each minimal prime $\mathfrak{p} \in \min \operatorname{Spec} R$, let $i = \dim R/\mathfrak{p}$, we then have

$$\operatorname{Ass}(\hat{R}/\mathfrak{p}\hat{R}) \subseteq \tau^{-1}(\mathfrak{p}) \subseteq \tau^{-1}(\min \operatorname{Spec}(R)_i) = \min \operatorname{Spec}(\hat{R})_i.$$

Particularly, $\hat{R}/\mathfrak{p}\hat{R}$ is equidimensional. Therefore R is universally catenary by Ratliff's theorem. This completes the proof. □

In the rest of this section, we consider some characterizations of universal catenarity among local rings with trivial formal fibers. Recall that a local ring R is called quasi-unmixed if $\operatorname{Ass}(R) = \operatorname{Ass}(R)_{\dim R}$ and it is unmixed if its completion is quasi-unmixed.

Proposition 2.6. *Let R be a Noetherian local ring with $\alpha(R) = 0$. The following statements are equivalent.*

(a) R is universally catenary.
(b) For any prime $\mathfrak{p} \in \operatorname{Spec}(R)$, $R/\mathfrak{p}$ is unmixed.
(c) R is a homomorphic image of a Cohen-Macaulay ring.

Proof. The equivalence of (a) and (b) is a direct consequence of Corollary 2.5.
$(a) \Leftrightarrow (c)$: Note that all the fiber rings of the extension $R \subseteq \hat{R}$ are Cohen-Macaulay since $\alpha(R) = 0$. The conclusion then follows from [9, Corollary 1.2]. $\square$

Proposition 2.6 leads to the following characterization in terms of local cohomology of the ring.

Corollary 2.7. *Let R be a Noetherian local ring. Suppose R is quasi-unmixed and $\alpha(R) = 0$. Then R is universally catenary if and only if $\dim R/\mathfrak{a}_i(R) < i$ for $i = 1, 2, \ldots, \dim(R) - 1$, where $\mathfrak{a}_i(R) = \operatorname{Ann}_R(H^i_{\mathfrak{m}}(R))$ is the annihilator ideal of the i-th local cohomology module $H^i_{\mathfrak{m}}(R)$.*

Proof. To prove the corollary, we need to use the following result about local cohomology of local rings (for a proof, see [4, Proposition 3.6]).
Claim: Let S be a homomorphic image of a Cohen-Macaulay ring. Fix an integer $0 \leq i \leq \dim S$. For a prime $\mathfrak{p}$ of S, $\mathfrak{p} \supseteq \mathfrak{a}_0(S) \ldots \mathfrak{a}_i(S)$ if and only if $\operatorname{depth}(S_{\mathfrak{p}}) + \dim S/\mathfrak{p} \leq i$.
Necessary condition: Assume R is universally catenary. Then it is an image of a Cohen-Macaulay local ring by Proposition 2.6. Let $\mathfrak{p}$ be a prime of R. Using the claim, if $\mathfrak{p} \supseteq \mathfrak{a}_i(R)$ then $\operatorname{depth}(R_{\mathfrak{p}}) + \dim R/\mathfrak{p} \leq i$. But $\operatorname{depth}(R_{\mathfrak{p}}) > 0$ as R is quasi-unmixed, then $\dim(R/\mathfrak{p}) < i$. So $\dim R/\mathfrak{a}_i(R) < i$.
Sufficient condition: We assume the contrary that R is not universally catenary. Since R is quasi-unmixed and $\alpha(R) = 0$, Corollary 2.5 shows that there is a minimal prime P of $\hat{R}$ such that $\dim(\hat{R}/P) < \dim R$. Put $i = \dim(\hat{R}/P)$. As R is a homomorphic image of a regular local ring, thank to Cohen Theorem on structure of complete local rings, the claim applies to shows that $\dim(\hat{R}/\mathfrak{a}_i(\hat{R})) = i$.

On the other hand, the flat base change theorem for local cohomology gives rise to an isomorphism $H^i_{\mathfrak{m}}(R) \otimes_R \hat{R} \simeq H^i_{\mathfrak{m}\hat{R}}(\hat{R})$. It induces $\mathfrak{a}_i(\hat{R}) \cap R = \mathfrak{a}_i(R)$. Hence $\dim(R/\mathfrak{a}_i(R)) \geq \dim(\hat{R}/\mathfrak{a}_i(\hat{R})) = i$. This contradicts to the assumption. Therefore R is universally catenary. $\square$

3. Local rings with going-up property

We say that the extension $R \subseteq \hat{R}$ satisfies the going-up theorem if for any prime $\mathfrak{p} \subseteq \mathfrak{q}$ of R, for any prime P of $\hat{R}$ with $P \cap R = \mathfrak{p}$, there is a prime Q of $\hat{R}$ such that $Q \cap R = \mathfrak{q}$ and $P \subseteq Q$. We will also say a local ring with going-up property to indicate such a ring R.

A local ring of dimension one clearly satisfies the going-up theorem. It could be shown that for the local rings $\mathcal{O}_{X,x}$ in Example 1.1, if $\mathfrak{p}$ is any prime of $\mathcal{O}_{X,x}$ then $\mathfrak{p}\widehat{\mathcal{O}}_{X,x}$ is a prime of the completion $\widehat{\mathcal{O}}_{X,x}$ (see [3]). So $\mathcal{O}_{X,x}$ satisfies the going-up theorem.

If the completion $\hat{R}$ is integral over R then $\hat{R}/R$ satisfies the going-up theorem by [10, Theorem 9.4]. Non-trivial examples of this kind of local rings seem to be very rare. One can find in [13, Example E3.1, Appendix] for such an example.

We consider four classes of Noetherian local rings

$\mathcal{R}^0 := \{$Noetherian local rings with trivial formal fibers$\}$;

$\mathcal{R}^0_{uc} := \{R \in \mathcal{R}^0 : R \text{ is universally catenary}\};$
$\mathcal{R}^0_{gu} := \{\text{Noetherian local rings with going-up property}\};$
$\mathcal{R}^0_{int} := \{\text{Noetherian local rings } R \text{ such that } \hat{R} \text{ is integral over } R\}.$

Let R be a Noetherian local ring with going up property. Since the inclusion $R \subseteq \hat{R}$ satisfies both the going-up and going-down theorems, it could be shown without difficulty that R is universally catenary and $\alpha(R) = 0$. So we have the inclusions

$$\mathcal{R}^0_{int} \subset \mathcal{R}^0_{gu} \subset \mathcal{R}^0_{uc} \subset \mathcal{R}^0.$$

The last inclusion is trict as we have seen in Example 2.2. The other two inclusions are also strict as we can see in the following examples.

Example 3.1. We will show by examples that the three classes $\mathcal{R}^0_{int}$, $\mathcal{R}^0_{gu}$, $\mathcal{R}^0_{uc}$ are distinct.

1. In the first example we present a local ring R with going-up property but $\hat{R}$ is not integral over R, so $\mathcal{R}^0_{int} \neq \mathcal{R}^0_{gu}$. We set $R = \mathbb{Q}[X]_{(X)}$. Then $\hat{R} = \mathbb{Q}[[X]]$. The extension $\mathbb{Q}[X]_{(X)} \subset \mathbb{Q}[[X]]$ clearly satisfies the going-up theorem. The Laurent series of the exponent function e^X is

$$e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \ldots + \frac{X^n}{n!} + \ldots$$

So $e^X \in \mathbb{Q}[[X]]$. Since e is transcendental over $\mathbb{Q}$, e^X is not integral over $\mathbb{Q}[X]_{(X)}$ and $\hat{R}/R$ is not integral.

2. Let k be a field. We consider the complete local ring $k[[X, Y, Z]]/(Y) \cap (Z)$. Using Proposition 2.1, there is a local domain R such that

(a) $\hat{R} = k[[X, Y, Z]]/(Y) \cap (Z)$;
(b) The generic fiber of the completion extension is $\tau^{-1}(\{0\}) = \{(Y), (Z)\} = \operatorname{Ass}(\hat{R})$;
(c) If P is a prime of $\hat{R}$ not in the generic fiber, then $\{P\} = \tau^{-1}(P \cap R)$.

By Corollary 2.5, the conditions (a) and (b) imply that $\alpha(R) = 0$ and R is universally catenary. The ideal $Q := (X, Y)$ is a prime of $\hat{R}$ which is obviously not in the generic fiber of τ. Let $\mathfrak{q} = Q \cap R$, then $\mathfrak{q} \neq 0$. Since $\{Q\} = \tau^{-1}(\{\mathfrak{q}\})$ by (c) and $Q \not\supseteq (Z)$, the extension $R \subseteq \hat{R}$ does not satisfy the going-up theorem.

In [17], Zöschinger proves that the extension $R \subseteq \hat{R}$ satisfies the going-up theorem if and only if

(a) $\alpha(R) = 0$ and R is universally catenary;
(b) For all primes $\mathfrak{p} \subseteq \mathfrak{q} \neq \mathfrak{m}$ of R, if $P \in \operatorname{Ass}(\hat{R}/\mathfrak{p}\hat{R})$ with $\dim(\hat{R}/P) = \dim(R/\mathfrak{p})$, then $\dim \hat{R}/(\mathfrak{q}\hat{R} + P) > 0$.

Example 3.1.2. shows that the condition (b) in Zöschinger's theorem could not be removed.

The well-known Lichtenbaum-Hartshorne Vanishing Theorem provides a sufficient and necessary condition for the vanishing of certain top local cohomology modules by means of some conditions on the completion of the ring. For local rings with going-up property, we will show that it is possible to check an analogous condition on the ring itself. So, in this senses, local rings with going-up property are quite closed to complete local rings. It is also worth noting that recently several authors have tried to extend the Vanishing Theorem in this direction, for examples, in the case the completion is integral over the ground ring. The next theorem gives a strong form of Lichtenbaum-Hartshorne Vanishing Theorem in a more general context with a simpler proof. It also provide another characterization of local rings with going-up property.

Recall that we denote $\operatorname{Assh}(R) := \operatorname{Ass}(R)_{\dim R} = \{\mathfrak{p} \in \operatorname{Spec}(R) : \dim R/\mathfrak{p} = \dim R\}$.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a universally catenary Noetherian local ring with $\alpha(R) = 0$. The following statements are equivalent:*

(a) The extension $R \subseteq \hat{R}$ satisfies the going-up theorem.
(b) For any ideal $I \subset R$ and any quotient S of R, $H_I^{\dim(S)}(S) = 0$ if and only if $\dim R/(\mathfrak{p} + I) > 0$ for any prime ideal $\mathfrak{p} \in \operatorname{Assh}(S)$.

Proof. $(a) \Rightarrow (b)$: Assume the extension $R \subseteq \widehat{R}$ satisfies the going-up theorem. Then so does the induced map $S \to \hat{S}$, and it suffices to prove the conclusion for the case $S = R$. The Lichtenbaum-Hartshorne Vanishing Theorem says that $H_I^{\dim(R)}(R) = 0$ if and only if $\dim \hat{R}/(P + I\hat{R}) > 0$ for any prime $P \in \operatorname{Assh}(\hat{R})$. We will show that the later condition is equivalent to $\dim(R/\mathfrak{p} + I) > 0$ for any prime $\mathfrak{p} \in \operatorname{Assh}(R)$.

For one direction, let $\mathfrak{p} \in \operatorname{Assh}(R)$ and let $P \in \operatorname{Assh}(\hat{R})$ such that P restricts to $\mathfrak{p}$. We have $\dim R/\mathfrak{p} + I = \dim \hat{R}/\mathfrak{p}\hat{R} + I\hat{R} \geq \dim \hat{R}/P + I\hat{R} > 0$. Conversely, assume $\dim R/\mathfrak{p} + I > 0$ for any $\mathfrak{p} \in \operatorname{Assh}(R)$. Let $P \in \operatorname{Assh}(\hat{R})$. Set $\mathfrak{p} = P \cap R$. Let $\mathfrak{q} \supseteq \mathfrak{p} + I$ be a prime ideal such that $\dim R/\mathfrak{q} > 0$. The going-up theorem applies to R and implies the existence of a prime ideal Q of $\hat{R}$ such that $P \subseteq Q$ and $Q \cap R = \mathfrak{q}$. Hence $Q \supseteq P + I\hat{R}$. Moreover, $\dim \hat{R}/Q > 0$ since $\mathfrak{q} \neq \mathfrak{m}$. Therefore, $\dim \hat{R}/P + I\hat{R} > 0$.

$(b) \Rightarrow (a)$: Let $\mathfrak{p} \subset \mathfrak{q} \neq \mathfrak{m}$ be prime ideals of R and let $P \in \operatorname{Spec}(\hat{R})$ be an associated prime of $\hat{R}/\mathfrak{p}\hat{R}$. As $\alpha(R) = 0$, this means that $P \cap R = \mathfrak{p}$ following Proposition 1.2. Denote $S = R/\mathfrak{p}$. Since R is universally catenary with trivial formal fibers, so is the quotient S. Corollary 2.5 implies that $P\hat{S} \in \operatorname{Assh}(\hat{S})$. On the other hand, the fact $\dim S/\mathfrak{q} > 0$ induces $H_{\mathfrak{q}}^{\dim(S)}(S) = 0$. Hence $\dim \hat{S}/\mathfrak{q}\hat{S} + P\hat{S} > 0$ by Lichtenbaum-Hartshorne Theorem, or equivalently, $\dim \hat{R}/\mathfrak{q}\hat{R} + P > 0$. Combining this with the assumption $\alpha(R) = 0$ and R is universally catenary, we obtain a going-up theorem for the extension $R \subseteq \hat{R}$ by the result of Zöschinger [17, Satz 1]. □

Theorem 3.2 has an immediate consequence.

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a local domain with going-up property. Then for any ideal $I \subset R$ which is not $\mathfrak{m}$-primary, $H_I^{\dim R}(R) = 0$.*

One of the most important applications of Lichtenbaum-Hartshorne Vanishing Theorem is to study the connectedness of algebraic sets. The idea comes back to Hartshorne and has been developed further by others. The crucial step is to use the Vanishing Theorem together with the Mayer-Vietoris Sequence to prove connectedness results for the spectrum of a complete local ring. By the same idea and by the usage of Theorem 3.2, we are able to state similar results for local rings with going-up property. This extension has its own geometric meaning since the local rings could come from geometry as we have seen in Example 1.1.

We will state one of the main results about connectedness, namely, the Connectedness Bound for local rings with going-up property. This result is central in the theory as it induces most of other results. It is stated and proved for complete local rings in [1, 19.2.9]. Here the proof for our case is similar to those for complete case (see [1, 19.2.7, 19.2.8, 19.2.9]) except Theorem 3.2 being used in the place of the Local Lichtenbaum-Hartshorne Theorem.

Recall that for a local ring $(R, \mathfrak{m})$ we denote

$c(R) := \min\{n :$ there is an ideal $\mathfrak{b} \subset R$ such that $\dim R/\mathfrak{b} = n$ and $\mathrm{Spec}(R) \setminus \mathrm{V}(\mathfrak{b})$ is disconnected$\}$;

$\mathrm{sdim}(R) := \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in \min \mathrm{Spec}(R)\}$;

The arithmetic rank of an ideal $\mathfrak{a} \subset R$ is defined as the least number of elements in $\mathfrak{m}$ which generate an ideal of the same radical as $\mathfrak{a}$ and it is denoted by $\mathrm{ara}(\mathfrak{a})$. We have

Corollary 3.4 (Connectedness bound for local rings with going-up property)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with going-up property. Let $\mathfrak{a}$ be a proper ideal. Then*
$$c(R/\mathfrak{a}) + \mathrm{ara}(\mathfrak{a}) \geq \min\{c(R), \mathrm{sdim}(R) - 1\}.$$
Consequently, $c(R/\mathfrak{a}) + \mathrm{ara}(\mathfrak{a}) \geq c(R) - 1$.

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