A Hybrid Subgradient Algorithm for Nonexpansive Mappings and Equilibrium Problems *

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Abstract. We propose a strongly convergent algorithm for finding a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings in a real Hilbert space. The proposed algorithm uses only one projection and does not require any Lipschitz condition for the bifunctions.

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1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its reduced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of $\mathcal{H}$. We recall that a mapping $T : C \to C$ is said to be a *contraction* on C with a constant $\delta \in (0, 1)$ iff

$$\|T(x) - T(y)\| \leq \delta \|x - y\|, \quad \forall x, y \in C.$$

If $\delta = 1$, then T is called *nonexpansive* on C. We denote by $Fix(T)$ the set of all fixed points of T. It is well known that T is a closed convex set. Suppose that Δ is an open convex set containing C and $f : \Delta \times \Delta \to \mathcal{R}$ is a bifunction such that $f(x, x) = 0$ for all $x \in C$. Such a bifunction is called an *equilibrium bifunction*. We consider the equilibrium problem defined as

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0, \quad \text{for all } y \in C. \qquad EP(C, f)$$

This problem is also often called the Ky Fan inequality due to his contribution to this field. It is well known (see e.g. [7, 14]) that some important problems such as convex programs, variational inequalities, the Kakutani fixed point, minimax problems and Nash equilibrium models can be formulated as an equilibrium problem of the form $EP(C, f)$. We denote the set of solutions of $EP(C, f)$ by $Sol(C, f)$. Recall [7] that the bifunction f is

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(i) strongly monotone on C with modulus $\beta > 0$, shortly β-strongly monotone on C, iff

$$f(x,y) + f(y,x) \leq -\beta\|y-x\|^2, \quad \forall x, y \in C;$$

(ii) monotone on C iff

$$f(x,y) + f(y,x) \leq 0, \quad \forall x, y \in C;$$

(iii) pseudomonotone on a set $A \subseteq C$ with respect to x iff

$$f(x,y) \geq 0 \ \text{ implies } f(y,x) \leq 0, \quad \forall y \in A;$$

We say that f is pseudomonotone on A if it is pseudomonotone on A with respect to every $x \in A$.

Clearly,

$$(i) \Rightarrow (ii) \Rightarrow (iii).$$

The bifunction f is said to be *Lipschitz-type continuous* on C with constants $c_1 > 0$ and $c_2 > 0$ (see e.g. [13]) iff

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \quad \forall x, y, z \in C. \tag{1.1}$$

A basis solution method for solving equilibrium problems is the projection method. It is well known [9] that the projection method is not convergent for monotone variational inequality, which is a special case of monotone equilibrium problems. In order to obtain convergence of the projection method for equilibrium problems, the extragradient method introduced by Korpelevics in [8] is extended to pseusomonotone equilibrium problems [20]. However the extragradient method requires two projections onto the constrained set C, which is computationally expensive except when C has special structure. Efforts for avoiding and/or reducing computational costs in computing the projection have been made by using penalty function methods [2, 6, 14, 15] and relaxing the constrained convex set by polyhedral convex ones [3, 10]. Another idea is to considering conditions on the bifunctions involved that enables replacing two projections by only one [16]. In [16] an algorithm has been proposed for solving a wide class of equilibrium problems that requires only one projection rather than two ones as in the extragradient method. Computational results reported in [16] show efficiency of this algorithm in finite dimensional Euclidean spaces.

The problem $P(C,f,T)$ of finding a common point in the solution set of Problem $EP(C,f)$ and the set of fixed points of a nonexpansive mapping T recently becomes an attractive subject, and various methods have been developed for solving this problem (see e.g. [1, 2, 18, 19, 21, 23] and the references therein). Most of the existing algorithms for this problem are based on the proximal point method applying to equilibrium problem $EP(C,f)$ combining with a Mann's iteration to the problem of finding a fixed point of T.

Tada and Takahashi in [19] proposed an algorithm, where at each iteration k the iterate x^{k+1} is defined as follows

$$z^k \in C \text{ such that } \ f(z^k, y) + \frac{1}{\lambda_k}\langle y - z^k, z^k - x^k\rangle \geq 0, \quad \forall y \in C, \tag{1.2}$$

$$\begin{cases} w^k = \alpha_k x^k + (1-\alpha_k)T(z^k), \\ C_k = \{z \in \mathcal{H} : \ \|w^k - z\| \leq \|x^k - z\|\}, \\ D_k = \{z \in \mathcal{H} : \ \langle x^k - z, x^0 - x^k\rangle \geq 0\}, x^{k+1} = P_{C_k \cap D_k}(x^0), \end{cases}$$

where $\lambda_k > 0$ is the regularization parameter at iteration k, $x^0 \in C$ and P_C is the metric projection onto C. Under the main assumption that the bifunction f is monotone on C, the sequence $\{x^k\}$ strongly converges to the projection of the starting point onto the solution set of Problem $P(C, f, T)$, provided the sequences $\{\lambda_k\}, \{\alpha_k\}$ satisfy some properties. In this algorithm and some other ones using scheme (1.2), the bifunctions involved are assumed to be monotone on C. Recently, Anh in [1] proposed to use the extragradient-type iteration instead of the proximal point iteration (1.2) for solving Problem $P(C, f, T)$. More precisely, given $x^k \in C$, the proximal point iteration (1.2) is replaced by the two following mathematical programs, which seems numerically easier than (1.2)

$$\begin{cases} y^k = \text{argmin}\{f(x^k, y) + \frac{1}{2\lambda_k}\|y - x^k\|^2 : \ y \in C\}, \\ z^k = \text{argmin}\{f(y^k, z) + \frac{1}{2\lambda_k}\|z - x^k\|^2 : \ z \in C\}. \end{cases} \tag{1.3}$$

It was proved that if f is pseudomonotone and satisfies the Lipschitz-type condition (1.1), then the sequence $\{x^k\}$ strongly converges to a solution of Problem $P(C, f, T)$.

It should be emphasized that the Lipschitz-type condition (1.1), in general is not satisfied, and if yes, finding the constants c_1 and c_2 is not an easy task. Furthermore solving the strongly convex programs (1.3) is expensive excepts special cases when C has a simple structure.

The purpose of this paper is to propose a strongly convergent algorithm for solving Problem $P(C, f, T)$, which is a combination of the well-known Mann iterative scheme for fixed point [12] and the projection method for equilibrium problems. The proposed algorithm can be considered as an extension of the one in [16] to problem $P(C, f, T)$ in real Hilbert spaces.

The paper is organized as follows. In the next section we describe the algorithm and state some lemmas which will be used in the proof for the convergence of the proposed algorithm. The convergence analysis of the algorithm is presented in the third section.

2 Preliminaries

Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. We write $x^n \rightharpoonup x$ to indicate that the sequence $\{x^n\}$ converges weakly to x as $n \to \infty$, and $x^n \to x$ means that $\{x^n\}$ converges strongly to x. Since C is closed, convex, for any $x \in \mathcal{H}$, there exists an uniquely point in C, denoted by $P_C(x)$ satisfying

$$\|x - P_C(x)\| \le \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of $\mathcal{H}$ to C. It is well known that P_C satisfies the following properties:

$$\langle x - y, P_C(x) - P_C(y)\rangle \ge \|P_C(x) - P_C(y)\|^2, \quad \forall x, y \in \mathcal{H}, \tag{2.1}$$

$$\langle x - P_C(x), P_C(x) - y\rangle \ge 0, \quad \forall x \in \mathcal{H}, y \in C, \tag{2.2}$$

$$\|x - y\|^2 \ge \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \quad \forall x \in \mathcal{H}, y \in C. \tag{2.3}$$

Let us assume that the bifunction $f : \Delta \times \Delta \to \mathcal{R}$ and the nonexpansive mapping $T : C \to C$ satisfy the following conditions:

A_1. For each $x \in C$, $f(x, x) = 0$ and $f(x, \cdot)$ is convex on C;

A_2. $\partial_\epsilon f(x,\cdot)(x)$ is nonempty for each $\epsilon > 0$ and $x \in C$, where $\partial_\epsilon f(x,\cdot)(x)$ stands for ϵ-subdifferential of the convex function $f(x,.)$ at x;

A_3. f is pseudomonotone on C with respect to every solution of Problem $EP(C,f)$ and satisfies the following condition, called paramonotonicity property

$$x \in Sol(C,f), y \in C, f(y,x) = f(x,y) = 0 \ \Rightarrow y \in Sol(C,f); \qquad (2.4)$$

A_4. For each $x \in C$, $f(\cdot,x)$ is weakly upper semicontinuous on the open set Δ;

A_5. The solution set S of Problem $P(C,f,T)$ is nonempty.

Suppose that the sequences $\{\lambda_k\}, \{\beta_k\}, \{\epsilon_k\}$, $\{\delta_k\}$ of nonnegative numbers satisfy the following conditions

$$\begin{cases} 0 < \underline{\lambda} < \lambda_k < \overline{\lambda},\ 0 < a < \delta_k < b < 1,\ \delta_k \to 1/2, \\ \beta_k > 0, \sum_{k=0}^{\infty} \beta_k = +\infty, \sum_{k=0}^{\infty} \beta_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \beta_k \epsilon_k < +\infty. \end{cases} \qquad (2.5)$$

We note that bifunctions satisfying (2.4) are used in [16], and in [11] for variational inequality. Clearly, in the optimization problem, where $f(x,y) = \varphi(y) - \varphi(x)$, the condition (2.4) is automatically satisfied.

Now the iteration scheme for finding a common point in the set of solutions of Problem $EP(C,f)$ and the set of fixed points of the nonexpansive mapping T can be written as follows:

Pick $x^0 \in C$. At each iteration $k = 0, 1, ...$ do the followings:

$$\begin{cases} \text{Compute } w^k \in \partial_{\epsilon_k} f(x^k,\cdot)(x^k); \\ \text{Take } \gamma_k := \max\{\lambda_k, \|w^k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ \text{Compute } y^k = P_C(x^k - \alpha_k w^k) \text{ and let } x^{k+1} := \delta_k x^k + (1-\delta_k) T(y^k). \end{cases} \qquad (2.6)$$

To investigate the convergence of this scheme, we recall the following technical lemmas which will be used in the sequel.

Lemma 2.1 *(see [22]) Suppose that $\{a_k\}$ and $\{\beta_k\}$ are two sequences of nonnegative real numbers such that*

$$a_{k+1} \le a_k + \beta_k, \quad k \ge 0,$$

where $\sum_{k=0}^{\infty} \beta_k < \infty$. Then the sequence $\{a_k\}$ is convergent.

Lemma 2.2 *(see [17]) Let $\mathcal{H}$ be a real Hilbert space, $\{\delta_k\}$ be a sequence of real numbers such that $0 < a \le \delta_k \le b < 1$ for all $k = 0, 1, \cdots$, and let $\{v^k\}$, $\{w^k\}$ be sequences of $\mathcal{H}$ such that*

$$\limsup_{k\to\infty} \|v^k\| \le c, \limsup_{k\to\infty} \|w^k\| \le c,$$

and

$$\lim_{k\to\infty} \|\delta_k v^k + (1-\delta_k) w^k\| = c, \quad \text{for some} \quad c > 0.$$

Then, $\lim_{k\to\infty} \|v^k - w^k\| = 0.$

3 Convergence Analysis

Now, we state and prove the main convergence theorem for the proposed iteration scheme (2.6).

Theorem 3.1 *Suppose that Assumptions A_1 - A_5 are satisfied, the parameters δ, λ and the sequences $\{\lambda_k\}, \{\beta_k\}, \{\epsilon_k\}$, $\{\delta_k\}$ satisfy restrictions (2.5). Then the sequences $\{x^k\}$, $\{y^k\}$ and $\{P_S(x^k)\}$ generated by (2.6) strongly converge to the same point $\bar{x}$ and $\bar{x} = \lim_{k\to\infty} P_S(x^k)$.*

The theorem is proved through several claims.

Claim 1. For every $x^* \in S$ and every k, one has

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k f(x^k, x^*) + 2(1-\delta_k)\alpha_k\epsilon_k + 2(1-\delta_k)\beta_k^2, \quad (3.1)$$

and there exists the limit $\lim\limits_{k\to\infty} \|x^k - x^*\| := c$.

Proof of Claim 1. It follows from $x^{k+1} = \delta_k x^k + (1-\delta_k)T(y^k)$ and $x^* \in Fix(T)$ that

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 =& \|\delta_k(x^k - x^*) + (1-\delta_k)\big(T(y^k) - T(x^*)\big)\|^2 \\
\le& \delta_k\|x^k - x^*\|^2 + (1-\delta_k)\|T(y^k) - T(x^*)\|^2 \\
\le& \delta_k\|x^k - x^*\|^2 + (1-\delta_k)\|y^k - x^*\|^2 \\
=& \delta_k\|x^k - x^*\|^2 + (1-\delta_k)\big(\|x^k - x^*\|^2 - \|y^k - x^k\|^2 + 2\langle x^k - y^k, x^* - y^k\rangle\big) \\
\le& \|x^k - x^*\|^2 + 2(1-\delta_k)\langle x^k - y^k, x^* - y^k\rangle. \quad (3.2)
\end{aligned}$$

Since $y^k = P_C(x^k - \alpha_k w^k)$ and $x^* \in C$, one has

$$\langle x^k - y^k, x^* - y^k\rangle \le \alpha_k\langle w^k, x^* - y^k\rangle.$$

Combining this inequality with (2.6) and (3.2) yields

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 \le& \|x^k - x^*\|^2 + 2(1-\delta_k)\langle x^k - y^k, x^* - y^k\rangle \\
\le& \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k\langle w^k, x^* - y^k\rangle \\
=& \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k\langle w^k, x^* - x^k\rangle + 2(1-\delta_k)\alpha_k\langle w^k, x^k - y^k\rangle \\
\le& \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k\langle w^k, x^* - x^k\rangle + 2(1-\delta_k)\alpha_k\|w^k\|\|x^k - y^k\| \\
=& \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k\langle w^k, x^* - x^k\rangle \\
&+ 2(1-\delta_k)\frac{\beta_k}{\max\{\lambda_k, \|w^k\|\}}\|w^k\|\|x^k - y^k\| \\
\le& \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k\langle w^k, x^* - x^k\rangle + 2(1-\delta_k)\beta_k\|x^k - y^k\|. \quad (3.3)
\end{aligned}$$

Using again $y^k = P_C(x^k - \alpha_k w^k)$, $x^k \in C$, it follows from (2.6) that

$$\begin{aligned}
\|x^k - y^k\|^2 \le& \alpha_k\langle w^k, x^k - y^k\rangle \\
\le& \alpha_k\|w^k\|\|x^k - y^k\| \\
=& \frac{\beta_k}{\max\{\lambda_k, \|w^k\|\}}\|w^k\|\|x^k - y^k\| \\
\le& \beta_k\|x^k - y^k\|,
\end{aligned}$$

which implies that $\|x^k - y^k\| \leq \beta_k$. Using this fact and (3.3), we get

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k\langle w^k, x^* - x^k\rangle + 2(1-\delta_k)\beta_k^2. \tag{3.4}$$

Since $w^k \in \partial_{\epsilon_k} f(x^k, \cdot)(x^k)$, $x^* \in C$ and $f(x,x) = 0$ for all $x \in C$, we have

$$\begin{aligned}\langle w^k, x^* - x^k\rangle &\leq f(x^k, x^*) - f(x^k, x^k) + \epsilon_k \\ &\leq f(x^k, x^*) + \epsilon_k.\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we obtain the inequality (3.1). On the other hand, since $x^* \in Sol(C, f)$, i.e., $f(x^*, x) \geq 0$ for all $x \in C$, by pseudomonotonicity of f, we have $f(x, x^*) \leq 0$ for all $x \in C$. Replacing x by $x^k \in C$, we get $f(x^k, x^*) \leq 0$. Then from (3.1) and (3.4), it follows that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k\epsilon_k + 2(1-\delta_k)\beta_k^2. \tag{3.6}$$

Now applying Lemma 2.1 to (3.6), by Assumption (2.5), we obtain the existence of $c := \lim_{k\to\infty} \|x^k - x^*\|$.

Claim 2. $\limsup_{k\to\infty} f(x^k, x^*) = 0$ for every $x^* \in S$.

Proof of Claim 2. Since f is pseudomonotone on C and $f(x^*, x^k) \geq 0$, we have $-f(x^k, x^*) \geq 0$. Then, by Claim 1, for every k, one has

$$\begin{aligned}2(1-\delta_k)\alpha_k[-f(x^k, x^*)] &\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \\ &\quad + 2(1-\delta_k)\alpha_k\epsilon_k + 2(1-\delta_k)\beta_k^2 \\ &\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2\alpha_k\epsilon_k + 2\beta_k^2.\end{aligned} \tag{3.7}$$

Summing up the above inequalities for every k, we obtain

$$\begin{aligned}0 &\leq 2\sum_{k=0}^{\infty}(1-\delta_k)\alpha_k[-f(x^k, x^*)] \\ &\leq \|x^0 - x^*\|^2 + 2\sum_{k=0}^{\infty}\alpha_k\epsilon_k + 2\sum_{k=0}^{\infty}\beta_k^2 < +\infty.\end{aligned} \tag{3.8}$$

It follows from the algorithm and Assumption (2.5) that

$$\alpha_k = \frac{\beta_k}{\gamma_k} = \frac{\beta_k}{\max\{\lambda_k, \|w^k\|\}} \geq \frac{\beta_k}{\lambda_k} \geq \frac{\beta_k}{\overline{\lambda}},$$

which together with $0 < a < \delta_k < b < 1$ and (3.8), implies

$$\begin{aligned}0 &\leq 2(1-b)\sum_{k=0}^{\infty}\frac{\beta_k}{\overline{\lambda}}[-f(x^k, x^*)] \\ &\leq 2\sum_{k=0}^{\infty}(1-\delta_k)\alpha_k[-f(x^k, x^*)] < +\infty.\end{aligned}$$

Thus

$$\sum_{k=0}^{\infty}\beta_k[-f(x^k, x^*)] < +\infty.$$

Then, by $\sum_{k=0}^{\infty} \beta_k = \infty$ and $-f(x^k, x^*) \geq 0$, we can deduce that $\limsup_{k\to\infty} f(x^k, x^*) = 0$ as desired.

Claim 3. For any $x^* \in S$, suppose that $\{x^{k_i}\}$ is the subsequence of of $\{x^k\}$ such that

$$\limsup_{k\to\infty} f(x^k, x^*) = \lim_{i\to\infty} f(x^{k_i}, x^*), \tag{3.9}$$

and $\bar{x}$ is a weakly limit point of $\{x^{k_i}\}$. Then $\bar{x}$ solves $EP(C, f)$.

Proof of Claim 3. For simplicity of notation, without lost of generality, we may assume that x^{k_i} weakly converges to $\bar{x}$ as $i \to \infty$. Since $f(., x^*)$ is weakly upper semicontinuous, (3.9), we have

$$\begin{aligned} f(\bar{x}, x^*) \geq& \limsup_{i\to\infty} f(x^{k_i}, x^*) = \lim_{i\to\infty} f(x^{k_i}, x^*) \\ =& \limsup_{k\to\infty} f(x^k, x^*) = 0. \end{aligned} \tag{3.10}$$

On the other hand, since f is pseudomonotone and $f(x^*, \bar{x}) \geq 0$, we have $f(\bar{x}, x^*) \leq 0$. Thus, $f(\bar{x}, x^*) = 0$, which by pseudomonotonicity implies $f(x^*, \bar{x}) \leq 0$. Hence $f(x^*, \bar{x}) = 0$ Then, from Assumption (A_4), it follows that $\bar{x}$ is a solution of EP(f, C) as well.

Claim 4. Any weakly cluster point of the sequence $\{x^k\}$ is a fixed point of T, in particular, $\bar{x} \in S$.

Proof of Claim 4. Let $\bar{y}$ be a weakly cluster point of $\{x^k\}$ and $\{x^{k_j}\}$ be a subsequence of $\{x^k\} \subset C$ weakly converging to $\bar{y}$. By convexity, C is weakly closed. Hence $\bar{y} \in C$. For each $x^* \in S$, by nonexpansiveness of T, we can write

$$\|T(y^k) - x^*\| \leq \|y^k - x^*\| \leq \|x^k - x^*\| + \|y^k - x^k\| \leq \|x^k - x^*\| + \beta_k,$$

which implies

$$\limsup_{k\to\infty} \|T(y^k) - x^*\| \leq \lim_{k\to\infty} (\|x^k - x^*\| + \beta_k) = c.$$

On other hand, since $\delta_k(x^k - x^*) + (1 - \delta_k)(T(y^k) - x^*) = \|x^{k+1} - x^*\|$, we have

$$\lim_{k\to\infty} \|\delta_k(x^k - x^*) + (1 - \delta_k)(T(y^k) - x^*)\| = \lim_{k\to\infty} \|x^{k+1} - x^*\| = c.$$

Then, applying Lemma 2.2 with $v^k := x^k - x^*$, $w^k := T(y^k) - x^*$, it results

$$\lim_{k\to\infty} \|T(y^k) - x^k\| = 0. \tag{3.11}$$

At the same time, we have

$$\begin{aligned} \|T(x^k) - x^k\| \leq& \|T(x^k) - T(y^k)\| + \|x^k - T(y^k)\| \\ \leq& \|x^k - y^k\| + \|x^k - T(y^k)\| \leq \beta_k + \|x^k - T(y^k)\|. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), it follows that

$$\lim_{k\to\infty} \|T(x^k) - x^k\| = 0.$$

Suppose in contrary that $\bar{y} \neq T(\bar{y})$. Then, since $x^{k_j} \rightharpoonup \bar{y}$, by Opial's Theorem and (3.11) one can write

$$\begin{aligned}\liminf_{j\to\infty}\|x^{k_j} - \bar{y}\| &< \liminf_{j\to\infty}\|x^{k_j} - T(\bar{y})\| \\ &= \liminf_{j\to\infty}\Big[\|x^{k_j} - T(x^{k_j})\| + \|T(x^{k_j}) - T(\bar{y})\|\Big] \\ &\leq \lim_{j\to\infty}\|x^{k_j} - T(x^{k_j})\| + \liminf_{j\to\infty}\|T(x^{k_j}) - T(\bar{y})\| \\ &\leq \liminf_{j\to\infty}\|x^{k_j} - \bar{y}\|, \end{aligned} \tag{3.13}$$

which is a contradiction. Hence $\bar{y} = T(\bar{y})$. Apply Claim 3 with $\bar{y} = \bar{x}$ we obtain $\bar{x} \in S$.

Claim 5.

$$\lim_{k\to\infty} x^k = \lim_{k\to\infty} y^k = \lim_{k\to\infty} P_S(x^k) = \bar{x}.$$

Proof of Claim 5. Since f is pseudomonotone and T is nonexpansive on C, the solution set S is convex. It follows from (3.6) that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \eta_k, \tag{3.14}$$

where $\eta_k := 2(1-\delta_k)\alpha_k\epsilon_k + 2(1-\delta_k)\beta_k^2 > 0$ for all $k \geq 0$ and $\sum_{k=0}^{\infty} \eta_k < +\infty$. Then, by definition of x^{k+1}, we have

$$\begin{aligned}\|x^{k+1} - P_S(x^{k+1})\|^2 &\leq \|\delta_k(x^k - P_S(x^k)) + (1-\delta_k)(T(y^k) - P_S(x^k))\|^2 \\ &\leq \delta_k\|x^k - P_S(x^k)\|^2 + (1-\delta_k)\|T(y^k) - P_S(x^k)\|^2. \end{aligned} \tag{3.15}$$

Now using the property(2.3) of the metric projection we have

$$\|T(y^k) - P_S(x^k)\|^2 \leq \|T(y^k) - x^k\|^2 - \|x^k - P_S(x^k)\|^2,$$

which, by (3.15), implies that

$$\begin{aligned}\|x^{k+1} - P_S(x^{k+1})\|^2 &\leq \delta_k\|x^k - P_S(x^k)\|^2 + (1-\delta_k)\|T(y^k) - x^k\|^2 \\ &\quad - (1-\delta_k)\|x^k - P_S(x^k)\|^2 \\ &= (2\delta_k - 1)\|x^k - P_S(x^k)\|^2 \\ &\quad + (1-\delta_k)\|T(y^k) - x^k\|^2. \end{aligned} \tag{3.16}$$

Note that, by (3.11), $\|T(y^k) - x^k\|^2 \to 0$. Using this fact, by boundedness of the sequence $\{x^k - P_S(x^k)\}$, letting $\delta_k \to 1/2$ we obtain from the latter inequality that

$$\lim_{k\to\infty}\|x^{k+1} - P_S(x^{k+1})\| = 0 \tag{3.17}$$

For simplicity of notation, let $z^k := P_S(x^k)$. Then, for all $m > k$, since S is convex, we have $\frac{1}{2}(z^m + z^k) \in S$, and therefore

$$\begin{aligned}\|z^m - z^k\|^2 =&2\|x^m - z^m\|^2 + 2\|x^m - z^k\|^2 - 4\|x^m - \frac{1}{2}(z^m + z^k)\|^2 \\ \leq&2\|x^m - z^m\|^2 + 2\|x^m - z^k\|^2 - 4\|x^m - z^m\|^2 \\ =&2\|x^m - z^k\|^2 - 2\|x^m - z^m\|^2. \end{aligned} \tag{3.18}$$

Since $z^k \in S$, it follows from ((3.14)) with $x^* = z^k$ that

$$\begin{aligned}\|x^m - z^k\|^2 &\le \|x^{m-1} - z^k\|^2 + \eta_{m-1}\\ &\le \|x^{m-2} - z^k\|^2 + \eta_{m-1} + \eta_{m-2}\\ &\le \cdots\\ &\le \|x^k - z^k\|^2 + \sum_{j=k}^{m-1} \eta_j.\end{aligned}$$

Combining this inequality with (3.18), we have

$$\|z^m - z^k\|^2 \le 2\|x^k - z^k\|^2 + 2\sum_{j=k}^{m-1} \eta_j - 2\|x^m - z^m\|^2,$$

which, together with $\sum\limits_{k=0}^{\infty} \eta_k < \infty$ and (3.17), implies that $\{z^k\}$ is a Cauchy sequence. Hence, $\{z^k\}$ strongly converges to some point $\bar{z} \in S$. However, since $z^{k_i} := P_S(x^{k_i})$, letting $i \to \infty$, we obtain in the limit that

$$\bar{z} = \lim_i P_S(x^{k_i}) = P_S(\bar{x}) = \bar{x} \in S,$$

which implies that $\bar{x} = \bar{z}$, and therefore, $z^k := P_S(x^k) \to \bar{z} = \bar{x} \in S$. Then, from (3.16) and $z^k \to \bar{x}$, $\|x^k - T(y^k)\| \to 0$, we can conclude that $x^k \to \bar{x}$, Finally, since $\lim_{k\to\infty} \|x^k - y^k\| = 0$, we have $\lim_{k\to\infty} y^k = \bar{x}$. □

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