# ON THE PETERSON HIT PROBLEM

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Abstract. Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in $k$ variables $x_1, x_2, \ldots, x_k$, each of degree 1. We study the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for $P_k$ as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we study a minimal set of generators for $\mathcal{A}$-module $P_k$ in some so-call generic degrees and apply these results to explicitly determine the hit problem for $k = 4$.

*Dedicated to Prof. N. H. V. Hưng on the occasion of his sixtieth birthday*

## 1. Introduction and statement of results

Let $V_k$ be an elementary abelian 2-group of rank $k$. Denote by $BV_k$ the classifying space of $V_k$. It may be thought of as the product of $k$ copies of the real projective space $\mathbb{RP}^\infty$. Then

$$P_k := H^*(BV_k) \cong \mathbb{F}_2[x_1, x_2, \ldots, x_k],$$

a polynomial algebra in $k$ variables $x_1, x_2, \ldots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

Being the cohomology of a space, $P_k$ is a module over the mod 2 Steenrod algebra $\mathcal{A}$. The action of $\mathcal{A}$ on $P_k$ can explicitly be given by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0,\\ x_j^2, & i = 1,\\ 0, & \text{otherwise,}\end{cases}$$

and subject to the Cartan formula

$$Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f)Sq^{n-i}(g),$$

for $f, g \in P_k$ (see Steenrod and Epstein [29]).

A polynomial $f$ in $P_k$ is called *hit* if it can be written as a finite sum $f = \sum_{i>0} Sq^i(f_i)$ for some polynomials $f_i$. That means $f$ belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$. We are interested in the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra $P_k$ as a module over the Steenrod algebra. In other words, we want to find a basis of the $\mathbb{F}_2$-vector space $QP_k := P_k/\mathcal{A}^+.P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

The hit problem was first studied by Peterson [21, 22], Wood [36], Singer [27], and Priddy [23], who showed its relationship to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence for

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the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. The vector space $QP_k$ was explicitly calculated by Peterson [21] for $k = 1, 2$, by Kameko [14] for $k = 3$. The case $k = 4$ has been treated by Kameko [16] and by us [30].

Several aspects of the hit problem were then investigated by many authors. (See Boardman [1], Bruner, Hà and Hưng [2], Carlisle and Wood [3], Crabb and Hubbuck [4], Giambalvo and Peterson [5], Hà [6], Hưng [7], Hưng and Nam [8, 9], Hưng and Peterson [10, 11], Janfada and Wood [12, 13], Kameko [14, 15], Minami [17], Mothebe [18], Nam [19, 20], Repka and Selick [24], Singer [28], Silverman [25], Walker and Wood [33, 34, 35], Wood [37, 38] and others.)

The $\mu$-function is one of the numerical functions that have much been used in the context of the hit problem. For a positive integer $n$, by $\mu(n)$ one means the smallest number $r$ for which it is possible to write $n = \sum_{1\leqslant i\leqslant r}(2^{d_i} - 1)$, where $d_i > 0$. A routine computation shows that $\mu(n) = s$ if and only if there exists uniquely a sequence of integers $d_1 > d_2 > \ldots > d_{s-1} \geqslant d_s > 0$ such that

$$n = 2^{d_1} + 2^{d_2} + \ldots + 2^{d_{s-1}} + 2^{d_s} - s. \tag{1.1}$$

From this it implies $n - s$ is even and $\mu(\frac{n-s}{2}) \leqslant s$.

Denote by $(QP_k)_n$ the subspace of $QP_k$ consisting of all the classes represented by homogeneous polynomials of degree $n$ in $P_k$.

Peterson [21] made the following conjecture, which was subsequently proved by Wood [36].

**Theorem 1.1** (Wood [36])**.** *If $\mu(n) > k$, then $(QP_k)_n = 0$.*

One of the main tools in the study of the hit problem is Kameko's homomorphism $\widetilde{Sq}^0_* : QP_k \to QP_k$. This homomorphism is induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\ldots x_ky^2,\\ 0, & \text{otherwise,}\end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_*Sq^{2t} = Sq^t\widetilde{Sq}^0_*$, and $\widetilde{Sq}^0_*Sq^{2t+1} = 0$ for any non-negative integer $t$.

**Theorem 1.2** (Kameko [14])**.** *Let $m$ be a positive integer. If $\mu(2m+k) = k$, then $\widetilde{Sq}^0_* : (QP_k)_{2m+k} \to (QP_k)_m$ is an isomorphism of $GL_k$-modules.*

Based on Theorems 1.1 and 1.2, the hit problem is reduced to the case of degree $n$ with $\mu(n) = s < k$.

The hit problem in the case of degree $n$ of the form (1.1) with $s = k - 1$, $d_{i-1} - d_i > 1$ for $2 \leqslant i < k$ and $d_{k-1} > 1$ was studied by Crabb and Hubbuck [4], Nam [19] and Repka and Selick [24].

In this paper, we explicitly determine the hit problem for the case $k = 4$. First, we study the hit problem for the cases of degree $n$ of the form (1.1) for $s = k - 1$. The following theorem gives an inductive formula for the dimension of $(QP_k)_n$ in this case.

**Theorem 1.3.** *Let* $n = \sum_{1\leqslant i\leqslant k-1}(2^{d_i}-1)$ *with* $d_i$ *positive integers such that* $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1}$, *and let* $m = \sum_{1\leqslant i\leqslant k-2}(2^{d_i-d_{k-1}}-1)$. *If* $d_{k-1} \geqslant k-1 \geqslant 1$, *then*

$$\dim(QP_k)_n = (2^k-1)\dim(QP_{k-1})_m.$$

For $d_{k-1} \geqslant k$, the theorem follows from the results in Nam [19] and the present author [32]. However, for $d_{k-1} = k-1$, the theorem is new.

Based on Theorem 1.3, we explicitly compute $QP_4$.

**Theorem 1.4.** *Let* $n$ *be an arbitrary positive integer with* $\mu(n) < 4$. *The dimension of the* $\mathbb{F}_2$*-vector space* $(QP_4)_n$ *is given by the following table:*

| $n$ | $s=1$ | $s=2$ | $s=3$ | $s=4$ | $s\geqslant 5$ |
|---|---|---|---|---|---|
| $2^{s+1}-3$ | 4 | 15 | 35 | 45 | 45 |
| $2^{s+1}-2$ | 6 | 24 | 50 | 70 | 80 |
| $2^{s+1}-1$ | 14 | 35 | 75 | 89 | 85 |
| $2^{s+2}+2^{s+1}-3$ | 46 | 94 | 105 | 105 | 105 |
| $2^{s+3}+2^{s+1}-3$ | 87 | 135 | 150 | 150 | 150 |
| $2^{s+4}+2^{s+1}-3$ | 136 | 180 | 195 | 195 | 195 |
| $2^{s+t+1}+2^{s+1}-3, t\geqslant 4$ | 150 | 195 | 210 | 210 | 210 |
| $2^{s+1}+2^{s}-2$ | 21 | 70 | 116 | 164 | 175 |
| $2^{s+2}+2^{s}-2$ | 55 | 126 | 192 | 240 | 255 |
| $2^{s+3}+2^{s}-2$ | 73 | 165 | 241 | 285 | 300 |
| $2^{s+4}+2^{s}-2$ | 95 | 179 | 255 | 300 | 315 |
| $2^{s+5}+2^{s}-2$ | 115 | 175 | 255 | 300 | 315 |
| $2^{s+t}+2^{s}-2, t\geqslant 6$ | 125 | 175 | 255 | 300 | 315 |
| $2^{s+2}+2^{s+1}+2^{s}-3$ | 64 | 120 | 120 | 120 | 120 |
| $2^{s+3}+2^{s+2}+2^{s}-3$ | 155 | 210 | 210 | 210 | 210 |
| $2^{s+t+1}+2^{s+t}+2^{s}-3, t\geqslant 3$ | 140 | 210 | 210 | 210 | 210 |
| $2^{s+3}+2^{s+1}+2^{s}-3$ | 140 | 225 | 225 | 225 | 225 |
| $2^{s+u+1}+2^{s+1}+2^{s}-3, u\geqslant 3$ | 120 | 210 | 210 | 210 | 210 |
| $2^{s+u+2}+2^{s+2}+2^{s}-3, u\geqslant 2$ | 225 | 315 | 315 | 315 | 315 |
| $2^{s+t+u}+2^{s+t}+2^{s}-3, u\geqslant 2, t\geqslant 3$ | 210 | 315 | 315 | 315 | 315. |

The space $QP_4$ was also computed in Kameko [16] by using computer calculation. However the manuscript is unpublished at the time of the writing.

Carlisle and Wood showed in [3] that the dimension of the vector space $(QP_k)_m$ is uniformly bounded by a number depended only on $k$. In 1990, Kameko made the following conjecture in his Johns Hopkins University PhD thesis [14].

**Conjecture 1.5** (Kameko [14]). *For every nonnegative integer* $m$,

$$\dim(QP_k)_m \leqslant \prod_{1\leqslant i\leqslant k}(2^i-1).$$

The conjecture was shown by Kameko himself for $k \leqslant 3$ in [14]. From Theorem 1.4, we see that the conjecture is also true for $k = 4$.

By induction on $k$, using Theorem 1.3, we obtain the following.

**Corollary 1.6.** *Let* $n = \sum_{1\leqslant i\leqslant k-1}(2^{d_i}-1)$ *with* $d_i$ *positive integers. If* $d_1 - d_2 \geqslant 2, d_{i-1} - d_i \geqslant i-1, 3 \leqslant i \leqslant k-1, d_{k-1} \geqslant k-1$*, then*

$$\dim(QP_k)_n = \prod_{1\leqslant i\leqslant k}(2^i - 1).$$

For the case $d_{i-1} - d_i \geqslant i, 2 \leqslant i \leqslant k-1$, and $d_{k-1} \geqslant k$, this result is due to Nam [19]. This corollary also shows that Kameko's conjecture is true for the degree $n$ as given in the corollary.

By induction on $k$, using Theorems 1.3, 1.4 and the fact that the dual of the Kameko squaring is an epimorphism, one gets the following.

**Corollary 1.7.** *Let* $n = \sum_{1\leqslant i\leqslant k-2}(2^{d_i}-1)$ *with* $d_i$ *positive integers and let* $d_{k-1} = 1$, $n_r = \sum_{1\leqslant i\leqslant r-2}(2^{d_i - d_{r-1}}-1)-1$ *with* $r = 5, 6, \ldots, k$*. If* $d_1 - d_2 \geqslant 4$, $d_{i-2} - d_{i-1} \geqslant i$*, for* $4 \leqslant i \leqslant k$ *and* $k \geqslant 5$*, then*

$$\dim(QP_k)_n = \prod_{1\leqslant i\leqslant k}(2^i - 1) + \sum_{5\leqslant r\leqslant k}\Big(\prod_{r+1\leqslant i\leqslant k}(2^i - 1)\Big)\dim \mathrm{Ker}(\widetilde{Sq}^0_*)_{n_r},$$

*where* $(\widetilde{Sq}^0_*)_{n_r} : (QP_r)_{2n_r+r} \to (QP_r)_{n_r}$ *denotes the squaring operation* $\widetilde{Sq}^0_*$ *in degree* $2n_r + r$*. Here, by convention,* $\prod_{r+1\leqslant i\leqslant k}(2^i-1) = 1$ *for* $r = k$*.*

This corollary has been proved in [32] for the case $d_{i-2} - d_{i-1} > i+1$ with $3 \leqslant i \leqslant k$.

Obviously $2n_r + r = \sum_{1\leqslant i\leqslant r-2}(2^{e_i}-1)$, where $e_i = d_i - d_{r-1} + 1$ for $1 \leqslant i \leqslant r-2$. So, in degree $2n_r + r$ of $P_r$, there is a so-called spike $x = x_1^{2^{e_1}-1}x_2^{2^{e_2}-1}\ldots x_{r-2}^{2^{e_{r-2}}-1}$, i.e. a monomial whose exponents are all of the form $2^e - 1$ for some $e$. Since the class $[x]$ in $(QP_k)_{2n_r+r}$ represented by the spike $x$ is nonzero and $\widetilde{Sq}^0_*([x]) = 0$, we have $\mathrm{Ker}(\widetilde{Sq}^0_*)_{n_r} \neq 0$, for any $5 \leqslant r \leqslant k$. Therefore, by Corollary 1.7, Kameko's conjecture is not true in degree $n = 2n_k + k$ for any $k \geqslant 5$, where $n_k = 2^{d_1-1} + 2^{d_2-1} + \ldots + 2^{d_{k-2}-1} - k + 1$.

This paper is organized as follows. In Section 2, we recall some needed information on the admissible monomials in $P_k$ and Singer's criterion on the hit monomials. We prove Theorem 1.3 in Section 3 by describing a basis of $(QP_k)_n$ in terms of a given basis of $(QP_{k-1})_m$. In Section 4, we recall the results on the hit problem for $k \leqslant 3$. Theorem 1.4 will be proved in Section 5 by explicitly determining all of the admissible monomials in $P_4$.

The first formulation of this paper was given in a 240-page preprint in 2007 [30], which was then publicized to a remarkable number of colleagues. One year latter, we found the negative answer to Kameko's conjecture on the hit problem [31, 32]. Being led by the insight of this new study, we have remarkably reduced the length of the paper.

## 2. Preliminaries

In this section, we recall some results in Kameko [14] and Singer [28] which will be used in the next sections.

**Notation 2.1.** Throughout the paper, we use the following notations.

$$\begin{aligned}\mathbb{N}_k &= \{1,2,\ldots,k\},\\ X_I &= X_{i_1,i_2,\ldots,i_r} = x_1\ldots\hat{x}_{i_1}\ldots\hat{x}_{i_r}\ldots x_k\\ &= \prod_{i\in\mathbb{N}_k\setminus I} x_i, \quad I=\{i_1,i_2,\ldots,x_{i_r}\}\subset\mathbb{N}_k,\end{aligned}$$

In particular, we have

$$\begin{aligned}&X_{\mathbb{N}_k}=1,\\ &X_\emptyset = x_1x_2\ldots x_k,\\ &X_i = x_1\ldots\hat{x}_i\ldots x_k,\ 1\leqslant i\leqslant k.\end{aligned}$$

Let $\alpha_i(a)$ denote the $i$-th coefficient in dyadic expansion of a nonnegative integer $a$. That means $a=\alpha_0(a)2^0+\alpha_1(a)2^1+\alpha_2(a)2^2+\ldots$, for $\alpha_i(a)=0$ or 1 and $i\geqslant 0$. Denote by $\alpha(a)$ the number of one in dyadic expansion of $a$.

Let $x=x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}\in P_k$. Denote by $\nu_j(x)=a_j, 1\leqslant j\leqslant k$. Set

$$I_i(x)=\{j\in\mathbb{N}_k:\alpha_i(\nu_j(x))=0\},$$

for $i\geqslant 0$. Then we have

$$x=\prod_{i\geqslant 0} X_{I_i(x)}^{2^i}.$$

For a polynomial $f$ in $P_k$, we denote by $[f]$ the class in $QP_k$ represented by $f$. For a subset $S\subset P_k$, we denote

$$[S]=\{[f]:f\in S\}\subset QP_k.$$

**Definition 2.2.** For a monomial $x$, define two sequences associated with $x$ by

$$\begin{aligned}\omega(x)&=(\omega_1(x),\omega_2(x),\ldots,\omega_i(x),\ldots),\\ \sigma(x)&=(a_1,a_2,\ldots,a_k),\end{aligned}$$

where $\omega_i(x)=\sum_{1\leqslant j\leqslant k}\alpha_{i-1}(\nu_j(x))=\deg X_{I_{i-1}(x)},\ i\geqslant 1$.

The sequence $\omega(x)$ is called the weight vector of $x$ (see Wood [37]). The weight vectors and the sigma vectors can be ordered by the left lexicographical order.

Let $\omega=(\omega_1,\omega_2,\ldots,\omega_i,\ldots)$ be a sequence of nonnegative integers such that $\omega_i=0$ for $i\gg 0$. Define $\deg\omega=\sum_{i>0}2^{i-1}\omega_i$. Denote by $P_k(\omega)$ the subspace of $P_k$ spanned by all monomials $y$ such that $\deg y=\deg\omega$, $\omega(y)\leqslant\omega$ and $P_k^-(\omega)$ the subspace of $P_k$ spanned by all monomials $y\in P_k(\omega)$ such that $\omega(y)<\omega$. Denote by $\mathcal{A}_s^+$ the subspace of $\mathcal{A}$ spanned by all $Sq^j$ with $1\leqslant j<2^s$.

**Definition 2.3.** Let $\omega$ be a sequence of nonnegative integers and $f,g$ two homogeneous polynomials of the same degree in $P_k$.

i) $f\equiv g$ if and only if $f-g\in\mathcal{A}^+P_k$.

ii) $f\simeq_{(s,\omega)} g$ if and only if $f-g\in\mathcal{A}_s^+P_k+P_k^-(\omega)$.

Since $\mathcal{A}_0^+P_k=0$, $f\simeq_{(0,\omega)} g$ if and only if $f-g\in P_k^-(\omega)$. If $x$ is a monomial in $P_k$ and $\omega=\omega(x)$, then we denote $x\simeq_s g$ if and only if $x\simeq_{(s,\omega(x))} g$.

Obviously, the relations $\equiv$ and $\simeq_{(s,\omega)}$ are equivalence relations.

We recall some relations on the action of the Steenrod squares on $P_k$.

**Proposition 2.4.** *Let $f$ be a homogeneous polynomial in $P_k$.*

i) *If $i>\deg f$, then $Sq^i(f)=0$. If $i=\deg f$, then $Sq^i(f)=f^2$.*

ii) *If $i$ is not divisible by $2^s$, then $Sq^i(f^{2^s})=0$ while $Sq^{r2^s}(f^{2^s})=(Sq^r(f))^{2^s}$.*

**Proposition 2.5.** *Let $x, y$ be monomials and $f, g$ homogeneous polynomials in $P_k$ such that* $\deg x = \deg f$, $\deg y = \deg g$.

i) *If $\omega_i(x) \leqslant 1$ for $i > s$ and $x \simeq_s f$, then $xy^{2^s} \simeq_s fy^{2^s}$.*

ii) *If $\omega_i(x) = 0$ for $i > s$, $x \simeq_s f$ and $y \simeq_r g$, then $xy^{2^s} \simeq_{s+r} fg^{2^s}$.*

*Proof.* Suppose that

$$x + f + \sum_{1\leqslant i<2^s} Sq^i(z_i) = h \in P_k^-(\omega(x))$$

where $z_i \in P_k$. From this and Proposition 2.4, we have $Sq^i(z_i)y^{2^s} = Sq^i(z_iy^{2^s})$. Observe that $\omega_i(xy^{2^s}) = \omega_i(x)$ for $i = 1, 2, \ldots, s$. If $z$ is a monomial and $z \in P_k^-(\omega(x))$, then there exists an index $i \geqslant 1$ such that $\omega_j(z) = \omega_j(x)$, $j = 1, 2, \ldots, i-1$ and $\omega_i(z) < \omega_i(x)$. If $i > s$, then $\omega_i(x) = 1, \omega_i(z) = 0$. Then we have

$$\alpha_{i-1}\left(\deg x - \sum_{j=1}^{i-1} 2^{j-1}\omega_j(x)\right) = \alpha_{i-1}\left(2^{i-1} + \sum_{j>i} 2^{j-1}\omega_j(x))\right) = 1.$$

On the other hand, since $\deg x = \deg z$, $\omega_i(z) = 0$ and $\omega_j(z) = \omega_j(x)$, $j = 1, 2, \ldots, i-1$, one gets

$$\alpha_{i-1}\left(\deg x - \sum_{j=1}^{i-1} 2^{j-1}\omega_j(x)\right) = \alpha_{i-1}\left(\deg z - \sum_{j=1}^{i-1} 2^{j-1}\omega_j(z)\right)$$
$$= \alpha_{i-1}\left(\sum_{j>i} 2^{j-1}\omega_j(z)\right) = 0.$$

This is a contradiction. Hence $1 \leqslant i \leqslant s$.

From these about equalities and the fact that $h \in P_k^-(\omega(x))$, one gets

$$xy^{2^s} + fy^{2^s} + \sum_{1\leqslant i<2^s} Sq^i(z_iy^{2^s}) = hy^{2^s} \in P_k^-(\omega(xy^{2^s})).$$

The first part of the proposition is proved.

Suppose that $y + g + \sum_{1\leqslant j<2^r} Sq^j(u_j) = h_1 \in P_k^-(\omega(y))$, where $u_j \in P_k$. Then

$$xy^{2^s} = xg^{2^s} + xh_1^{2^s} + \sum_{1\leqslant j<2^r} xSq^{j2^s}(u_j^{2^s}).$$

Since $\omega_i(x) = 0$ for $i > s$ and $h_1 \in P_k^-(\omega(y))$, we get $xh_1^{2^s} \in P_k^-(\omega(xy^{2^s}))$. Using the Cartan formula and Proposition 2.4, we obtain

$$xSq^{j2^s}(u_j^{2^s}) = Sq^{j2^s}(xu_j^{2^s}) + \sum_{0<b\leqslant j} Sq^{b2^s}(x)(Sq^{j-b}(u_j))^{2^s}.$$

Since $\omega_i(x) = 0$ for $i > s$, we have $x = \prod_{0\leqslant i<s} X_{I_i(x)}^{2^i}$. Using the Cartan formula and Proposition 2.4, we see that $Sq^{b2^s}(x)$ is a sum of polynomials of the form

$$\prod_{0\leqslant i<s} (Sq^{b_i}(X_{I_i(x)}))^{2^i},$$

where $\sum_{0\leqslant i<s} b_i 2^i = b2^s$ and $0 \leqslant b_i \leqslant \deg X_{I_i(x)}$. Let $\ell$ be the smallest index such that $b_\ell > 0$ with $0 \leqslant \ell < s$. Suppose that a monomial $z$ appears as a term of the polynomial $\left(\prod_{0\leqslant i<s}(Sq^{b_i}(X_{I_i(x)}))^{2^i}\right)(Sq^{j-b}(u_j))^{2^s}$. Then $\omega_t(z) = \deg X_{I_{t-1}}(x) =$

$\omega_t(x) = \omega_t(xy^{2^s})$ for $t \leqslant \ell$, and $\omega_{\ell+1}(z) = \deg X_{I_\ell(x)} - b_\ell < \deg X_{I_\ell(x)} = \omega_{\ell+1}(x) = \omega_{\ell+1}(xy^{2^s})$. Hence

$$\Big(\prod_{0\leqslant i<s}(Sq^{b_i}(X_{I_i(x)}))^{2^i}\Big)(Sq^{j-b}(u_j))^{2^s} \in P_k^-(\omega(xy^{2^s})).$$

This implies $Sq^{b2^s}(x)(Sq^{j-b}(u_j))^{2^s} \in P_k^-(\omega(xy^{2^s}))$ for $0 < b \leqslant j$. So one gets

$$xy^{2^s} + xg^{2^s} + \sum_{1\leqslant j<2^r} Sq^{j2^s}(xu_j^{2^s}) \in P_k^-(\omega(xy^{2^s})).$$

Since $h \in P_k^-(\omega(x))$, we have $hg^{2^s} \in P_k^-(\omega(xy^{2^s}))$. Using Proposition 2.4 and the Cartan formula, we get

$$xg^{2^s} + fg^{2^s} + \sum_{1\leqslant i<2^s} Sq^i(z_ig^{2^s}) = hg^{2^s} \in P_k^-(\omega(xy^{2^s})).$$

Note that $1 \leqslant j2^s < 2^{r+s}$ for $1 \leqslant j < 2^r$. Combining the above equalities gives $xy^{2^s} - fg^{2^s} \in \mathcal{A}_{r+s}P_k + P_k^-(\omega(xy^{2^s}))$. This implies $xy^{2^s} \simeq_{r+s} xg^{2^s} \simeq_{r+s} fg^{2^s}$. The proposition is proved. □

**Definition 2.6.** Let $x, y$ be monomials of the same degree in $P_k$. We say that $x < y$ if and only if one of the following holds

i) $\omega(x) < \omega(y)$;
ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

**Definition 2.7.** A monomial $x$ is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$ for $j = 1, 2, \ldots, t$ and $x - \sum_{j=1}^t y_j \in \mathcal{A}^+P_k$.

A monomial $x$ is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree $n$ in $P_k$ is a minimal set of $\mathcal{A}$-generators for $P_k$ in degree $n$.

**Definition 2.8.** A monomial $x$ is said to be strictly inadmissible if and only if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$, for $j = 1, 2, \ldots, t$ and $x - \sum_{j=1}^t y_j \in \mathcal{A}_s^+P_k$ with $s = \max\{i\ ; \omega_i(x) > 0\}$.

It is easy to see that if $x$ is strictly inadmissible, then it is inadmissible. The following theorem is a modification of a result in [14].

**Theorem 2.9** (Kameko [14], Sum [32])**.** *Let $x, y, w$ be monomials in $P_k$ such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*

i) *If $w$ is inadmissible, then $xw^{2^r}$ is also inadmissible.*
ii) *If $w$ is strictly inadmissible, then $xw^{2^r}y^{2^{r+s}}$ is inadmissible.*

**Proposition 2.10** ([32])**.** *Let $x$ be an admissible monomial in $P_k$. Then we have*

i) *If there is an index $i_0$ such that $\omega_{i_0}(x) = 0$, then $\omega_i(x) = 0$ for all $i > i_0$.*
ii) *If there is an index $i_0$ such that $\omega_{i_0}(x) < k$, then $\omega_i(x) < k$ for all $i > i_0$.*

Now, we recall a result of Singer [28] on the hit monomials in $P_k$.

**Definition 2.11.** A monomial $z$ in $P_k$ is called a spike if $\nu_j(z) = 2^{s_j} - 1$ for $s_j$ a nonnegative integer and $j = 1, 2, \ldots, k$. If $z$ is a spike with $s_1 > s_2 > \ldots > s_{r-1} \geqslant s_r > 0$ and $s_j = 0$ for $j > r$, then it is called a minimal spike.

The following is a criterion for the hit monomials in $P_k$.

**Theorem 2.12** (Singer [28]). *Suppose $x \in P_k$ is a monomial of degree $n$, where $\mu(n) \leqslant k$. Let $z$ be the minimal spike of degree $n$. If $\omega(x) < \omega(z)$, then $x$ is hit.*

From this theorem, we see that if $z$ is a minimal spike, then $P_k(\omega(z)) \subset \mathcal{A}^+P_k$. The following lemmas were proved in [32].

**Lemma 2.13** ([32]). *Let $n = \sum_{1\leqslant i\leqslant k-1}(2^{d_i}-1)$ with $d_i$ positive integers such that $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1} > 0$, and $x$ a monomial of degree $n$ in $P_k$. If $[x] \neq 0$, then $\omega_i(x) = k-1$ for $1 \leqslant i \leqslant d_{k-1}$.*

**Lemma 2.14** ([32]). *Let $n = \sum_{1\leqslant i\leqslant k-1}(2^{d_i}-1)$ with $d_i$ positive integers such that $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1} > 0$, and $x$ a monomial in $P_k$ such that $\omega_i(x) = k-1$, for $i = 1, 2, \ldots, s \leqslant d_{k-1}$ and $\omega_i(x) = 0$ for $i > s$. Suppose $y$, $f$ and $g$ are polynomials in $P_k$ with $\deg f = \deg x$ and $\deg y = \deg g = (n - \deg x)/2^s = 2^{d_1-s} + \ldots + 2^{d_{k-2}-s} + 2^{d_{k-1}-s} - k + 1$.*
i) *If $x \simeq_s f$, then $xg^{2^s} \equiv fg^{2^s}$.*
ii) *If $y \equiv g$, then $xy^{2^s} \equiv xg^{2^s}$.*

For latter use, we set

$$P_k^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}\ ;\ a_1a_2\ldots a_k = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}\ ;\ a_1a_2\ldots a_k > 0\}\rangle.$$

It is easy to see that $P_k^0$ and $P_k^+$ are the $\mathcal{A}$-submodules of $P_k$. Furthermore, we have the following.

**Proposition 2.15.** *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces*

$$QP_k = QP_k^0 \oplus QP_k^+.$$

*Here $QP_k^0 = P_k^0/\mathcal{A}^+.P_k^0$ and $QP_k^+ = P_k^+/\mathcal{A}^+.P_k^+$.*

## 3. Proof of Theorem 1.3

We denote

$$\mathcal{N}_k = \{(i;I); I = (i_1, i_2, \ldots, i_r), 1 \leqslant i < i_1 < \ldots < i_r \leqslant k,\ 0 \leqslant r < k\}.$$

Let $(i;I) \in \mathcal{N}_k$ and $j \in \mathbb{N}_k$. Denote by $r = \ell(I)$ the length of $I$, and

$$I \cup j = \begin{cases} I, & \text{if } j \in I,\\ (i_1, \ldots, i_{t-1}, j, i_t, \ldots, i_r), & \text{if } i_{t-1} < j < i_t,\ 1 \leqslant t \leqslant r+1. \end{cases}$$

Here $i_0 = 0$ and $i_{r+1} = k+1$.

For $2 \leqslant h < k$, we set $\mathcal{N}_{h-1} \cup h = \{(i; I\cup h); (i;I) \in \mathcal{N}_{h-1}\}$. Then we have

$$\mathcal{N}_k = (\mathcal{N}_1 \cup 2) \cup \ldots \cup (\mathcal{N}_{k-1} \cup k) \cup \{(1;\emptyset), \ldots, (k;\emptyset)\}. \tag{3.1}$$

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i = f_{k;i} : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i,\\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

**Definition 3.1.** Let $(i;I) \in \mathbb{N}_k$, let $r = \ell(I)$, and let $u$ be an integer with $1 \leqslant u \leqslant r$. A monomial $x \in P_{k-1}$ is said to be $u$-compatible with $(i;I)$ if all of the following hold:

i) $\nu_{i_1-1}(x) = \nu_{i_2-1}(x) = \ldots = \nu_{i_{(u-1)}-1}(x) = 2^r - 1$,
ii) $\nu_{i_u-1}(x) > 2^r - 1$,
iii) $\alpha_{r-t}(\nu_{i_u-1}(x)) = 1,\ \forall t,\ 1 \leqslant t \leqslant u$,
iv) $\alpha_{r-t}(\nu_{i_t-1}(x)) = 1,\ \forall t,\ u < t \leqslant r$.

Clearly, a monomial $x$ can be $u$-compatible with a given $(i;I) \in \mathcal{N}_k, r = \ell(I) > 0$, for at most one value of $u$. By convention, $x$ is 1-compatible with $(i;\emptyset)$.

**Definition 3.2.** Let $(i;I) \in \mathcal{N}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1}+\ldots+2^{r-u}} \prod_{u<t\leqslant r} x_{i_t}^{2^{r-t}}$ for $1 \leqslant u \leqslant r = \ell(I)$, $x_{(\emptyset,1)} = 1$. For a monomial $x$ in $P_{k-1}$, we define the monomial $\phi_{(i;I)}(x)$ in $P_k$ by setting

$$\phi_{(i;I)}(x) = \begin{cases} (x_i^{2^r-1} f_i(x))/x_{(I,u)}, & \text{if there exists } u \text{ such that} \\ & \ x \text{ is } u\text{-compatible with } (i,I), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have an $\mathbb{F}_2$-linear map $\phi_{(i;I)} : P_{k-1} \to P_k$. In particular, $\phi_{(i;\emptyset)} = f_i$.

Let $x = X^{2^d-1} y^{2^d}$, with $y$ a monomial in $P_{k-1}$ and $X = x_1 x_2 \ldots, x_{k-1} \in P_{k-1}$.

If $r < d$, then $x$ is 1-compatible with $(i;I)$ and

$$\phi_{(i;I)}(x) = \phi_{(i;I)}(X^{2^d-1}) f_i(y)^{2^d} = x_i^{2^r-1} \prod_{1\leqslant t\leqslant r} x_{i_t}^{2^d-2^{r-t}-1} X_{i,i_1,\ldots,i_r}^{2^d-1} f_i(y)^{2^d}. \quad (3.2)$$

If $d = r$, $\nu_{j-1}(y) = 0, j = i_1, i_2, \ldots, i_{u-1}$ and $\nu_{i_u-1}(y) > 0$, then $x$ is $u$-compatible with $(i;I)$ and

$$\phi_{(i;I)}(x) = \phi_{(i_u;J_u)}(X^{2^d-1}) f_i(y)^{2^d}, \quad (3.3)$$

where $J_u = (i_{u+1}, \ldots, i_r)$.

Let $B$ be a finite subset of $P_{k-1}$ consisting of some homogeneous polynomials in degree $n$. We set

$$\Phi^0(B) = \bigcup_{1\leqslant i\leqslant k} \phi_{(i;\emptyset)}(B) = \bigcup_{1\leqslant i\leqslant k} f_i(B).$$

$$\Phi^+(B) = \bigcup_{(i;I)\in\mathcal{N}_k, 0<\ell(I)\leqslant k-1} \phi_{(i;I)}(B) \setminus P_k^0.$$

$$\Phi(B) = \Phi^0(B) \bigcup \Phi^+(B).$$

It is easy to see that if $B_{k-1}(n)$ is a minimal set of generators for $P_{k-1}$ in degree $n$, then $\Phi^0(B_{k-1}(n))$ is a minimal set of generators for $\mathcal{A}$-module $P_k^0$ in degree $n$ and $\Phi^+(B_{k-1}(n)) \subset P_k^+$.

**Proposition 3.3.** *Let $n = \sum_{1\leqslant i\leqslant k-1}(2^{d_i} - 1)$ with $d_i$ positive integers such that $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1} \geqslant k-1 \geqslant 1$. If $B_{k-1}(n)$ is a minimal set of generators for $\mathcal{A}$-module $P_{k-1}$ in degree $n$, then $B_k(n) = \Phi(B_{k-1}(n))$ is also a minimal set of generators for $\mathcal{A}$-module $P_k$ in degree $n$.*

For $d_{k-1} \geqslant k$, this proposition is a modification of a result in Nam [19]. For $d_{k-2} = d_{k-1} > k$, it has been proved in [32].

We prepare some lemmas for the proof of this proposition.

**Lemma 3.4.** *Let $j_0, j_1, \ldots, j_{d-1} \in \mathbb{N}_k$. Then there is $(i;I) \in \mathcal{N}_k$ such that*

$$x = \prod_{0 \leqslant t < d} X_{j_t}^{2^t} \simeq_{d-1} \phi_{(i;I)}(X^{2^d-1}),$$

*where $i = \min\{j_0, j_1, \ldots, j_{d-1}\}$.*

**Lemma 3.5.** *Let $n = \sum_{1 \leqslant i \leqslant k-1}(2^{d_i} - 1)$ with $d_i$ positive integers such that $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1} > 0$, and let $y_0$ be a monomial in $(P_k)_{m-1}$, $y_i = y_0 x_i$ for $1 \leqslant i \leqslant k$, and $(i;I) \in \mathcal{N}_k$.*

i) *If $0 < r = \ell(I) < d = d_{k-1}$, then*

$$\phi_{(i;I)}(X^{2^d-1})y_i^{2^d} \equiv \sum_{1 \leqslant j < i} \phi_{(j;I)}(X^{2^d-1})y_j^{2^d} + \sum_{i < j \leqslant k} \phi_{(i_j;I_j)}(X^{2^d-1})y_j^{2^d},$$

*where $i_j = \min(j, I), I_j = I$ for $j < \min I$, and $I_j = (I \cup j) \setminus \{i_j\}$ for $j \geqslant \min I$.*

ii) *If $r + 1 < d$, then*

$$\phi_{(i;I)}(X^{2^d-1})y_i^{2^d} \equiv \sum_{1 \leqslant j < i} \phi_{(j;I\cup i)}(X^{2^d-1})y_j^{2^d} + \sum_{i < j \leqslant k} \phi_{(i;I\cup j)}(X^{2^d-1})y_j^{2^d}.$$

Denote by $I_t = (t+1, t+2, \ldots, k)$ for $1 \leqslant t \leqslant k$. Set

$$Y_t = \sum_{r=t}^{k} \phi_{(t;I_t)}(X^{2^d-1})x_r^{2^d},\ d > k+1-t.$$

**Lemma 3.6.** *For $1 < t \leqslant k$,*

$$Y_t \simeq_{(k,\omega)} \sum_{(j;J)} \phi_{(j;J)}(X^{2^d-1})x_j^{2^d},$$

*where the sum runs over some $(j;J) \in \mathcal{N}_k$ with $1 \leqslant j < t$, $J \subset I_{t-1}$, $J \neq I_{t-1}$ and $\omega = \omega(X_1^{2^d-1}x_1^{2^d})$.*

We assume that all elements of $B_{k-1}(n)$ are monomials. Denote by $\mathcal{B} = B_{k-1}(n)$. We set

$$\begin{aligned}\mathcal{C} &= \{z \in \mathcal{B} : \nu_1(z) > 2^{k-1} - 1\},\\ \mathcal{D} &= \{z \in \mathcal{B} : \nu_1(z) = 2^{k-1} - 1, \nu_2(z) > 2^{k-1} - 1\},\\ \mathcal{E} &= \{z \in \mathcal{B} : \nu_1(z) = \nu_2(z) = 2^{k-1} - 1\}.\end{aligned}$$

Since $\omega_k(z) \geqslant k-3$ for all $z \in \mathcal{B}$, we have $\mathcal{B} = \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$. If $d = d_{k-1} > k-1$, then $\mathcal{D} = \mathcal{E} = \emptyset$. If $d_{k-2} > d_{k-1} = k-1$, then $\mathcal{E} = \emptyset$. We set $\bar{\mathcal{B}} = \{\bar{z}; X^{2^d-1}\bar{z}^{2^d} \in \mathcal{B}\}$. If either $d \geqslant k$ or $I \neq I_1$, then $\phi_{(i;I)}(z) = \phi_{(i;I)}(X^{2^d-1})f_i(\bar{z})^{2^d}$. If $d = d_{k-1} = k-1$, then

$$\phi_{(1;I_1)}(z) = \begin{cases}\phi_{(2;I_2)}(X^{2^d-1})f_1(\bar{z})^{2^d}, & \text{if } z \in \mathcal{C},\\ \phi_{(3;I_3)}(X^{2^d-1})f_2(\bar{z})^{2^d}, & \text{if } z \in \mathcal{D},\\ \phi_{(4;I_4)}(X^{2^d-1})f_3(\bar{z})^{2^d}, & \text{if } z \in \mathcal{E}.\end{cases} \tag{3.4}$$

For any $(i;I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases}x_j, & \text{if } 1 \leqslant j < i,\\ \sum_{s \in I} x_{s-1}, & \text{if } j = i,\\ x_{j-1}, & \text{if } i < j \leqslant k.\end{cases}$$

Then $p_{(i;I)}$ is a homomorphism of $\mathcal{A}$-modules. In particular, for $I=\emptyset$, we have $p_{(i;\emptyset)}(x_i)=0$.

**Lemma 3.7.** *Let $z\in\mathcal{B}$, $(i;I),(j;J)\in\mathcal{N}_k$ and $\ell(J)\leqslant\ell(I)$.*

i) *If either $d\geqslant k$ or $d=k-1$ and $I\neq I_1$, then*

$$p_{(j;J)}(\phi_{(i;I)}(z))\equiv\begin{cases}z, & \text{if } (j;J)=(i;I),\\ 0, & \text{if } (j;J)\neq(i;I).\end{cases}$$

ii) *If $z\in\mathcal{C}$ and $d=k-1$, then*

$$p_{(i;I)}(\phi_{(1;I_1)}(z))\equiv\begin{cases}z, & \text{if } (i;I)=(1;I_1),\\ 0 \bmod\langle\mathcal{D}\cup\mathcal{E}\rangle, & \text{if } (i;I)=(2;I_2)\\ 0, & \text{otherwise}\ .\end{cases}$$

iii) *If $z\in\mathcal{D}$, then*

$$p_{(i;I)}(\phi_{(1;I_1)}(z))\equiv\begin{cases}z, & \text{if } (i;I)=(1;I_1),(1;I_2),(2;I_2),\\ 0 \bmod\langle\mathcal{E}\rangle, & \text{if } (i;I)=(3;I_3),\\ 0, & \text{otherwise}\ .\end{cases}$$

iv) *If $z\in\mathcal{E}$, then*

$$p_{(i;I)}(\phi_{(1;I_1)}(z))\equiv\begin{cases}z & \text{if } I_3\subset I,\\ 0, & \text{otherwise.}\end{cases}$$

The above lemmas will be proved in the end of the section.

We recall the following.

**Lemma 3.8** (Nam [19])**.** *Let $x$ be a monomial in $P_k$. Then $x\equiv\sum\bar{x}$, where $\bar{x}$ are monomials with $\nu_1(\bar{x})=2^t-1$ and $t=\alpha(\nu_1(x))$.*

*Proof of Proposition 3.3.* Denote by $\mathcal{P}(n)$ the subspace of $(P_k)_n$ spanned by all elements of the set $B_k(n)$.

Let $x$ be a monomial of degree $n$ in $P_k$ and $[x]\neq0$. By Lemma 2.13, we have $\omega_i(x)=k-1$ for $1\leqslant i\leqslant d_{k-1}=d$. Hence we obtain $x=\Big(\prod_{0\leqslant t<d}X_{j_t}^{2^t}\Big)\bar{y}^{2^d}$, for suitable monomial $\bar{y}\in(P_k)_m$, with $m=\sum_{1\leqslant i\leqslant k-2}(2^{d_i-d}-1)$.

According to Lemmas 3.4 and 2.14, there is $(i;I)\in\mathcal{N}_k$ such that

$$x=\Big(\prod_{0\leqslant t<d}X_{j_t}^{2^t}\Big)\bar{y}^{2^d}\equiv\phi_{(i;I)}(X^{2^d-1})\bar{y}^{2^d}, \tag{3.5}$$

where $r=\ell(I)<d$.

Set $h_u=2^{d_1-u}+\ldots+2^{d_{k-2}-u}+2^{d_{k-1}-u}-k+1$, for $0\leqslant u\leqslant d$. We have $h_0=n,\ h_d=m,\ 2h_u+k-1=h_{u-1}$ and $\mu(2h_u+k-1)=k-1$ for $1\leqslant u\leqslant d$. By Theorem 1.2, the squaring operation $(\widetilde{Sq}^0_*)_{h_u}:(QP_{k-1})_{h_{u-1}}\to(QP_{k-1})_{h_u}$ is an isomorphism of $\mathbb{F}_2$-vector spaces. So the iterated squaring operation

$$(\widetilde{Sq}^0_*)^d=(\widetilde{Sq}^0_*)_{h_d}\ldots(\widetilde{Sq}^0_*)_{h_1}:(QP_{k-1})_n\to(QP_{k-1})_m$$

is also an isomorphism of $\mathbb{F}_2$-vector spaces. Hence

$$\bar{B}_{k-1}(m)=(\widetilde{Sq}^0_*)^d(B_{k-1}(n))=\{\bar{z}\in(P_{k-1})_m : X^{2^d-1}\bar{z}^{2^d}\in B_{k-1}(n)\}$$

is a minimal set of $\mathcal{A}$-generators for $P_{k-1}$ in degree $m$.

Now, we prove $[x] \in [\mathcal{P}(n)]$. The proof is divided into many cases.

**Case 3.5.1.** *$\bar{y} = f_i(y)$ with $y \in (P_{k-1})_m$.*

Since $y \in (P_{k-1})_m$, we have $y \equiv \bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_s$ with $\bar{z}_t$ monomials in $\bar{B}_{k-1}(m)$. Using Lemma 2.14, we get

$$x \equiv \phi_{(i;I)}(X^{2^d-1})f_i(y)^{2^d} \equiv \sum_{1 \leqslant t \leqslant s} \phi_{(i;I)}(X^{2^d-1})f_i(\bar{z}_t)^{2^d}.$$

Since $\phi_{(i;I)}(X^{2^d-1})f_i(\bar{z}_t)^{2^d} = \phi_{(i;I)}(X^{2^d-1}\bar{z}_t^{2^d})$ and $X^{2^d-1}\bar{z}_t^{2^d} \in B_{k-1}(n)$, we get $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.2.** *$d \geqslant k$, $\bar{y} = x_i^a f_i(y)$ with $y \in (P_{k-1})_{m-a}$.*

If $i = 1$ and either $I \neq I_1$ or $d > k$, then $d - r - 1 \geqslant 1$. Applying Lemma 3.5(ii) with $y_0 = x_1^{a-1}f_1(y)$, we get

$$x \equiv \sum_{2 \leqslant j \leqslant k} \phi_{(1;I\cup j)}(X^{2^d-1})(x_1^{a-1}f_1(x_{j-1}y))^{2^d}.$$

From this and the inductive hypothesis, we obtain $[x] \in [\mathcal{P}(n)]$.

If $I = I_1$ and $d = k$, then $r = d - 1$. Using Lemma 3.5(i) with $y_0 = x_1^{a-1}f_1(y)$ and Lemma 3.6, we get

$$\begin{aligned} x &\equiv \sum_{j=2}^{k} \phi_{(2;I_2)}(X^{2^k-1})(x_j y_0)^{2^k} = Y_2 y_0^{2^k} \\ &\equiv \sum_{J \neq I_1} \phi_{(1;J)}(X^{2^k-1})(x_1^a f_1(y))^{2^k}. \end{aligned}$$

Since $J \neq I_1$, one gets $[x] \in [\mathcal{P}(n)]$.

Suppose $i > 1$. Then $r+1 < k \leqslant d$. Applying Lemma 3.5(ii) with $y_0 = x_i^{a-1}f_i(y)$, we obtain

$$x \equiv \sum_{1 \leqslant j < i} \phi_{(j;I\cup i)}(X^{2^d-1})y_j^{2^d} + \sum_{i < j \leqslant k} \phi_{(i;I\cup j)}(X^{2^d-1})y_j^{2^d},$$

where $y_j = x_j y_0 = x_i^{a-1}f_i(x_{j-1}y)$ for $j > i$. Using the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$. So the proposition is proved for $d \geqslant k$.

In the remaining part of the proof, we assume that $d = k - 1$.

**Case 3.5.3.** *$(i;I) = (2;I_2)$ and $\bar{y} = f_1(y)$ with $y \in (P_{k-1})_m$, $\nu_1(y) > 0$.*

Since $y \in (P_{k-1})_m$, we have $y \equiv \bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_s$ with $\bar{z}_t$ monomials in $\bar{B}_{k-1}(m)$. Using Lemma 2.14, we get

$$x \equiv \phi_{(2;I_2)}(X^{2^d-1})f_1(y)^{2^d} \equiv \sum_{1 \leqslant t \leqslant s} \phi_{(2;I_2)}(X^{2^d-1})f_1(\bar{z}_t)^{2^d}.$$

If $\nu_1(\bar{z}_t) > 0$, then $\phi_{(2;I_2)}(X^{2^d-1})f_1(\bar{z}_t)^{2^d} = \phi_{(1;I_1)}(X^{2^d-1}\bar{z}_t^{2^d})$. If $\nu_1(\bar{z}_t) = 0$, then $f_1(\bar{z}_t) = f_2(\bar{z}_t)$ and $\phi_{(2;I_2)}(X^{2^d-1})f_1(\bar{z}_t)^{2^d} = \phi_{(2;I_2)}(X^{2^d-1}\bar{z}_t^{2^d})$. Hence $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.4.** *$(i;I) = (3;I_3)$ and $\bar{y} = f_2(y)$ with $y \in (P_{k-1})_m$, $\nu_1(y) = 0, \nu_2(y) > 0$.*

Since $y \in (P_{k-1})_m$ and $\nu_1(y) = 0$, we have $y \equiv \bar z_1 + \bar z_2 + \ldots + \bar z_s$ with $\bar z_t$ polynomials in $B_{k-1}(m)$ and $\nu_1(\bar z_t) = 0$. Using Lemma 2.14, we get

$$x \equiv \phi_{(3;I_3)}(X^{2^d-1})f_2(y)^{2^d} \equiv \sum_{1\leqslant t\leqslant s}\phi_{(3;I_3)}(X^{2^d-1})f_2(\bar z_t)^{2^d}.$$

If $\nu_2(\bar z_t) > 0$, then $\phi_{(3;I_3)}(X^{2^d-1})f_2(\bar z_t)^{2^d} = \phi_{(1;I_1)}(X^{2^d-1}\bar z_t^{2^d})$. If $\nu_2(\bar z_t) = 0$, then $f_2(\bar z_t) = f_3(\bar z_t)$ and $\phi_{(3;I_3)}(X^{2^d-1})f_2(\bar z_t)^{2^d} = \phi_{(3;I_3)}(X^{2^d-1}\bar z_t^{2^d})$. Hence $[x] \in [\mathcal P(n)]$.

**Case 3.5.5.** *$(i;I) = (4;I_4)$ and $\bar y = f_3(y)$ with $y \in (P_{k-1})_m,\ \nu_1(y) = \nu_2(y) = 0$.*

Since $y \in (P_{k-1})_m$ and $\nu_1(y) = \nu_2(y) = 0$, we have $y \equiv \bar z_1 + \bar z_2 + \ldots + \bar z_s$ with $\bar z_t$ polynomials in $B_{k-1}(m)$ and $\nu_1(\bar z_t) = \nu_2(\bar z_t) = 0$. Using Lemma 2.14, we get

$$x \equiv \phi_{(4;I_4)}(X^{2^d-1})(f_3(y))^{2^d} \equiv \sum_{1\leqslant t\leqslant s}\phi_{(4;I_4)}(X^{2^d-1})f_3(\bar z_t)^{2^d}.$$

If $\nu_3(\bar z_t) > 0$, then $\phi_{(4;I_4)}(X^{2^d-1})f_3(\bar z_t)^{2^d} = \phi_{(1;I_1)}(X^{2^d-1}\bar z_t^{2^d})$. If $\nu_3(\bar z_t) = 0$, then $f_3(\bar z_t) = f_4(\bar z_t)$ and $\phi_{(4;I_4)}(X^{2^d-1})f_3(\bar z_t)^{2^d} = \phi_{(4;I_4)}(X^{2^d-1}\bar z_t^{2^d})$. Hence $[x] \in [\mathcal P(n)]$.

**Case 3.5.6.** *$\bar y = x_1^{2^s}f_1(y)$ with $y \in (P_{k-1})_{m-2^s}$, $i = 1$ and $\ell(I) < k-2$.*

According to Lemma 3.8, $x_1^{2^s}f_1(y)^{2^d} \equiv x_1f_1(g)$, for some polynomial $g$. So we assume $s = 0$. Using Lemma 3.5(ii) with $y_0 = f_1(y)$, we have

$$x \equiv \sum_{r=2}^{k}\phi_{(1;I\cup r)}(X^{2^d-1})(f_1(x_{r-1}y))^{2^d}.$$

Hence by Case 3.5.1, $[x] \in [\mathcal P(n)]$.

**Case 3.5.7.** *$\bar y = x_2^{2^s}f_2(y)$ with $y \in (P_{k-1})_{m-2^s}, \nu_1(y) = 0,\ i = 2$ and $\ell(I) < k-3$.*

Using Lemma 3.8, we need only to prove $[x] \in [\mathcal P(n)]$ for $s = 0$. Using Lemma 3.5(ii) with $y_0 = f_2(y)$, one gets

$$x \equiv \phi_{(1;I\cup 2)}(X^{2^d-1})(x_1f_2(y))^{2^d} + \sum_{r=3}^{k}\phi_{(2;I\cup r)}(X^{2^d-1})(f_2(x_{r-1}y))^{2^d}.$$

Since $\nu_1(y) = 0$, $f_2(y) = f_1(y)$, from this equalities, Cases 3.5.1 and 3.5.6, we get $[x] \in [\mathcal P(n)]$.

**Case 3.5.8.** *$\bar y = x_3^{2^s}f_3(y)$, with $y \in (P_{k-1})_{m-2^s}$, $\nu_1(y) = \nu_2(y) = 0$ and $i = 3$.*

We need only to prove $[x] \in [\mathcal P(n)]$ for $s = 0$. Note that since $\nu_1(y) = \nu_2(y) = 0$, we have $f_1(y) = f_2(y) = f_3(y)$. If $I = I_3$, then by Case 3.5.4, $[x] \in [\mathcal P(n)]$. If $\ell(I) < k-4$, then using Lemma 3.5(ii) with $y_0 = f_3(y)$, we get

$$x \equiv \phi_{(1;I\cup 3)}(X^{2^d-1})(x_1f_1(y))^{2^d} + \phi_{(2;I\cup 3)}(X^{2^d-1})(f_2(x_1y))^{2^d} + \sum_{r=4}^{k}\phi_{(3;I\cup r)}(X^{2^d-1})(f_3(x_{r-1}y))^{2^d}.$$

From this equalities and Cases 3.5.1, 3.5.6, 3.5.7, we get $[x] \in [\mathcal P(n)]$.

If $d_{k-2} > d_{k-1}$ and $I \neq I_3$, then $\omega_k(x) = \omega_1(y) + 1 = k-2$. Hence $\alpha_0(\nu_j(y)) = 1$ for $j = 3, \ldots, k-1$. Applying Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2f_2(y))^{2^d}.$$

Hence by Cases 3.5.6 and 3.5.7, we get $[x] \in [\mathcal{P}(n)]$.

Suppose $d_{k-2} = d_{k-1}$ and $\ell(I) = k-4$. Then $I = I_{3,u} = (4, \ldots, \hat{u}, \ldots, k)$ with $4 \leqslant u \leqslant k$. Since $\omega_k(x) = \omega_1(y) + 1 = k-3$, we have $\omega_1(y) = k-4$. Hence there exists uniquely $3 \leqslant t < k$ such that $\alpha_0(\nu_t(y)) = 0$.

If $t = u-1$, then using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2f_2(y))^{2^d} + \phi_{(4;I_4)}(X^{2^d-1})(f_3(x_ty))^{2^d}.$$

By Cases 3.5.5, 3.5.6 and 3.5.7, we get $[x] \in [\mathcal{P}(n)]$.

If $u = 4 < t+1$, then using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2f_2(y))^{2^d} + \phi_{(5;I_5)}(X^{2^d-1})(f_3(x_ty))^{2^d}.$$

Applying Lemma 3.5(i) with $y_0 = f_3(x_ty/x_4)$ and Theorem 2.12, we have

$$\phi_{(5;I_5)}(X^{2^d-1})(f_3(x_ty))^{2^d} \equiv \sum_{1\leqslant i\leqslant 3} \phi_{(i;I_5)}(X^{2^d-1})(x_if_i(x_ty/x_4))^{2^d}.$$

Since $\ell(I_5) = k-5 < k-4$, using Cases 3.5.6, 3.5.7 and the above equalities, we get $[x] \in [\mathcal{P}(n)]$.

Suppose that $4 < u \neq t+1$. Using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12, we obtain

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2f_2(y))^{2^d} + \phi_{(4;I\setminus 4)}(X^{2^d-1})(f_3(x_ty))^{2^d}.$$

Applying Lemma 3.5(i) with $y_0 = f_3(x_ty/x_3)$ and Theorem 2.12, we have

$$\phi_{(4;I\setminus 4)}(X^{2^d-1})(f_3(x_ty))^{2^d} \equiv \sum_{1\leqslant i\leqslant 3} \phi_{(i;I\setminus 4)}(X^{2^d-1})(x_if_i(x_ty/x_3))^{2^d}.$$

Since $\ell(I\setminus 4) = k-5 < k-4$, using Cases 3.5.6, 3.5.7 and the above equalities, we get $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.9.** *$\bar{y} = x_3^bx_4^cf_4(y)$ for $y \in (P_{k-1})_{m-b-c}$ with $\nu_j(y) = 0, j = 1, 2, 3$ and $i = 4$.*

Using Lemmas 3.8 and 2.14, we assume that $b = 2^s - 1$. We prove $[x] \in [\mathcal{P}(n)]$ by double induction on $(\ell(I), c)$. If $c = 0$, then by Case 3.5.1, $[x] \in [\mathcal{P}(n)]$. If $I \neq I_4$, then applying Lemma 3.5(ii) with $y_0 = x_3^bx_4^{c-1}f_4(y)$, we have

$$x \equiv \phi_{(1;I\cup 4)}(X^{2^d-1})(x_1f_1(x_2^bx_3^{c-1}y))^{2^d} + \phi_{(2;I\cup 4)}(X^{2^d-1})(x_2f_2(x_2^bx_3^{c-1}y))^{2^d}$$
$$+ \phi_{(3;I\cup 4)}(X^{2^d-1})(x_3^{2^s}f_3(x_3^{c-1}y))^{2^d} + \sum_{r=5}^{k} \phi_{(4;I\cup r)}(X^{2^d-1})(x_3^bx_4^{c-1}f_4(x_{r-1}y))^{2^d}.$$

From this equalities, Cases 3.5.6, 3.5.7, 3.5.8 and the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$.

If $I = I_4$, then applying Lemma 3.5(i) with $y_0 = x_3^b x_4^{c-1} f_4(y)$, we obtain

$$x \equiv \phi_{(1;I_4)}(X^{2^d-1})(x_1 f_1(x_2^b x_3^{c-1} y))^{2^d} + \phi_{(2;I_4)}(X^{2^d-1})(x_2 f_2(x_2^b x_3^{c-1} y))^{2^d} + \phi_{(3;I_4)}(X^{2^d-1})(x_3^{2^s} f_3(x_3^{c-1} y))^{2^d} + Y_5 y_0^{2^d}.$$

By Lemma 3.6 and Lemma 2.14,

$$Y_5 y_0^{2^d} \equiv \sum \phi_{(j;J)}(X^{2^d-1}) y_j^{2^d},$$

where $1 \leqslant j < 5, J \subset I_4$ and $J \neq I_4$. From the above equalities, Cases 3.5.6, 3.5.7, 3.5.8 and the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.10.** *$\bar{y} = x_3^b f_3(y)$ for $y \in (P_{k-1})_{m-b}$ with $\nu_1(y) = \nu_2(y) = 0$ and $i = 3$.*

We prove $[x] \in [\mathcal{P}(n)]$ by double induction on $(\ell(I), b)$. If $b = 0$, then by Case 3.5.1, $[x] \in [\mathcal{P}(n)]$. If $I = I_3$, then by Case 3.5.4, $[x] \in [\mathcal{P}(n)]$.

Suppose $b > 0$. If $\ell(I) < k-4$, then applying Lemma 3.5(ii) with $y_0 = x_3^{b-1} f_3(y)$, we obtain

$$x \equiv \phi_{(1;I\cup 3)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I\cup 3)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d} + \sum_{r=4}^{k} \phi_{(3;I\cup r)}(X^{2^d-1})(x_3^{b-1} f_3(x_{r-1} y))^{2^d}.$$

Using Cases 3.5.6, 3.5.7 and the inductive hypothesis, we obtain $[x] \in [\mathcal{P}(n)]$.

Suppose that $\ell(I) = k-4$, and $I = I_{3,u} = (4, \ldots, \hat{u}, \ldots, k)$, $3 < u \leqslant k$. If $d_{k-2} > d_{k-1}$, then $\omega_k(x) = \omega_1(y) + 1 = k-2$. Hence $\alpha_0(\nu_j(y)) = 1$ for $j = 3, \ldots, k-1$. Applying Lemma 3.5(i) with $y_0 = x_3^{b-1} f_3(y)$ and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d}.$$

Hence by Cases 3.5.6 and 3.5.7, we get $[x] \in [\mathcal{P}(n)]$.

Suppose $d_{k-2} = d_{k-1}$. Since $\omega_k(x) = \omega_1(y) + 1 = k-3$, we have $\omega_1(y) = k-4$. Hence there exists uniquely $3 \leqslant t \leqslant k-1$ such that $\alpha_0(\nu_t(y)) = 0$.

If $t = u-1$, then using Lemma 3.5(i) with $y_0 = x_3^{b-1} f_3(y)$ and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d} + \phi_{(4;I_4)}(X^{2^d-1})(x_3^{b-1} f_3(x_t y))^{2^d}.$$

From this equalities, Cases 3.5.6, 3.5.7 and 3.5.9, we get $[x] \in [\mathcal{P}(n)]$.

If $u = 4 < t+1$, then using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d} + \phi_{(5;I_5)}(X^{2^d-1})(x_3^{b-1} f_3(x_t y))^{2^d}.$$

Applying Lemma 3.5(i) with $y_0 = x_3^{b-1} f_3(x_t y/x_4)$ and Theorem 2.12, we have

$$\phi_{(5;I_5)}(X^{2^d-1})(x_3^{b-1} f_3(x_t y))^{2^d} \equiv \sum_{1 \leqslant i \leqslant 3} \phi_{(i;I_5)}(X^{2^d-1})(x_3^{b-1} x_i f_3(x_t y/x_4))^{2^d}.$$

Since $\ell(I_5) = k-5 < k-4$, using the above equalities, Cases 3.5.6, 3.5.7 and the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$.

Suppose that $4 < u \neq t+1$. Using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12, we obtain

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1f_1(x_2^{b-1}y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2f_2(x_2^{b-1}y))^{2^d} + \phi_{(4;I\setminus 4)}(X^{2^d-1})(x_3^{b-1}f_3(x_ty))^{2^d}.$$

From the above equalities, Cases 3.5.6, 3.5.7 and 3.5.9, we get $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.11.** *$\bar{y} = x_2^{2^s}f_2(y)$ for $y \in (P_{k-1})_{m-2^s}$ with $\nu_1(y) = 0$ and $i = 2$.*

It suffices to prove $[x] \in [\mathcal{P}(n)]$ for $s = 0$. If $\ell(I) < k-3$, then $[x] \in [\mathcal{P}(n)]$ by Case 3.5.7. If $I = I_2$, then by Case 3.5.3, $[x] \in [\mathcal{P}(n)]$.

Suppose $\ell(I) = k-3$. Then $I = I_{2,u} = (3,\ldots,\hat{u},\ldots,k)$. If $u = 3$, then using Lemma 3.5(i) with $y_0 = f_2(y)$, we get

$$x \equiv \phi_{(1;I_3)}(X^{2^d-1})(x_1f_1(y))^{2^d} + \phi_{(3;I_3)}(X^{2^d-1})(f_2(x_2y))^{2^d} + \sum_{i=4}^{k}\phi_{(4;I_4)}(X^{2^d-1})(f_2(x_{r-1}y))^{2^d}.$$

Using Cases 3.5.4, 3.5.6, 3.5.9, and the above equalities, we obtain $[x] \in [\mathcal{P}(n)]$.

If $u > 3$, then using Lemma 3.5(i) with $y_0 = f_2(y)$, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1f_1(y))^{2^d} + \phi_{(3;I_3)}(X^{2^d-1})(f_2(x_{u-1}y))^{2^d} + \sum_{4\leqslant r\leqslant k, r\neq u}\phi_{(3;I\setminus 3)}(X^{2^d-1})(f_2(x_{r-1}y))^{2^d}.$$

Using Cases 3.5.4, 3.5.6, 3.5.10, and the above equalities, we obtain $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.12.** *$\bar{y} = x_2^ax_3^bf_3(y)$ for $y \in (P_{k-1})_{m-a-b}$ with $\nu_1(y) = \nu_2(y) = 0$ and $i = 3$.*

According to Lemma 3.8, we assume $a = 2^s-1$. We prove $[x] \in [\mathcal{P}(n)]$ by double induction on $(\ell(I), b)$.

If $b = 0$, then by Case 3.5.1, $[x] \in P[n]$. If $I \neq I_3$, then using Lemma 3.5(ii) with $y_0 = x_2^ax_3^{b-1}f_3(y)$, we get

$$x \equiv \phi_{(1;I\cup 3)}(X^{2^d-1})(x_1f_1(x_1^ax_2^{b-1}y))^{2^d} + \phi_{(2;I\cup 3)}(X^{2^d-1})(x_2^{2^s}(f_2(x_2^{b-1}y))^{2^d} + \sum_{r=4}^{k}\phi_{(3;I\cup r)}(X^{2^d-1})(x_2^ax_3^{b-1}f_3(x_{r-1}y))^{2^d}.$$

From this, Cases 3.5.6, 3.5.7 and the inductive hypothesis we obtain $[x] \in [\mathcal{P}(n)]$.

If $I = I_3$, then using Lemma 3.5(i) with $y_0 = x_2^ax_3^{b-1}f_3(y)$, we get

$$x \equiv \phi_{(1;I_3)}(X^{2^d-1})(x_1f_1(x_1^ax_2^{b-1}y))^{2^d} + \phi_{(2;I_3)}(X^{2^d-1})(x_2^{2^s}(f_2(x_2^{b-1}y))^{2^d} + Y_4y_0^{2^d}.$$

By Lemma 3.6 and Lemma 2.14, we have

$$Y_4y_0^{2^d} \equiv \sum_{(j;J)}\phi_{(j;J)}(X^{2^d-1})y_j^{2^d},$$

where $1 \leqslant j < 4$ and $J \subset I_3$ and $J \neq I_3$. Using Cases 3.5.6, 3.5.11, the above equalities and the induction hypothesis, we obtain $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.13.** *$\bar{y} = x_2^a f_2(y)$ for $y \in (P_{k-1})_{m-a}$ with $\nu_1(y) = 0$ and $i = 2$.*

We prove $[x] \in [\mathcal{P}(n)]$ by double induction on $(\ell(I), a)$. If $a = 0$, then by Case 3.5.1, $[x] \in [\mathcal{P}(n)]$. If $I = I_2$, then by Case 3.5.3, $[x] \in [\mathcal{P}(n)]$. Suppose $a > 0$ and $\ell(I) < k - 3$. Applying Lemma 3.5(ii) with $y_0 = x_2^{a-1} f_2(y)$, we get

$$x \equiv \phi_{(1;I\cup 2)}(X^{2^d-1})(x_1 f_1(x_1^{a-1}y))^{2^d} + \sum_{r=3}^{k} \phi_{(2;I\cup r)}(X^{2^d-1})(x_2^{a-1} f_2(x_{r-1}y))^{2^d}.$$

Using Case 3.5.6 and the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$.

Suppose that $I = I_{2,u} = (3, \ldots, \hat{u}, \ldots, k),\ 3 \leqslant u \leqslant k$.

If $u = 3$, then $I = I_3$. Applying Lemma 3.5(i) with $y_0 = x_2^{a-1} f_2(y)$, we get

$$\begin{aligned} x \equiv \phi_{(1;I_3)}(X^{2^d-1})(x_1 f_1(x_1^{a-1}y))^{2^d} &+ \phi_{(3;I_3)}(X^{2^d-1})(x_2^{a-1} f_2(x_2 y))^{2^d} \\ &+ \sum_{r=4}^{k} \phi_{(4;I_4)}(X^{2^d-1})(x_r x_2^{a-1} f_2(y))^{2^d}. \end{aligned}$$

Applying Lemma 3.6 and Lemma 2.14, one gets

$$\begin{aligned} \sum_{r=4}^{k} \phi_{(4;I_4)}(X^{2^d-1})(x_r x_2^{a-1} f_2(y))^{2^d} &= Y_4 y_0^{2^d} \\ &\equiv \sum_{(j;J)} \phi_{(j;J)}(X^{2^d-1})(x_j x_2^{a-1} f_2(y))^{2^d}, \end{aligned}$$

where the last sum runs over some $(j;J)$ with $1 \leqslant j < 4$, $J \subset I_3$ and $J \neq I_3$. Since $\ell(J) < \ell(I_3) = k - 3$, from the above equalities, Cases 3.5.4, 3.5.6, 3.5.12 and the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$.

If $u > 3$, applying Lemma 3.5(i) with $y_0 = x_2^{a-1} f_2(y)$, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_1^{a-1}y))^{2^d} + \sum_{r=3}^{k} \phi_{(3;I\cup r)}(X^{2^d-1})(x_2^{a-1} f_2(x_{r-1}y))^{2^d}.$$

From the last equalities, Cases 3.5.6 and 3.5.12, we have $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.14.** *$\bar{y} = x_1^{2^s} f_1(y)$ with $y \in (P_{k-1})_{m-2^s}$ and $i = 1$.*

By Lemma 3.8, we need only to prove $[x] \in [\mathcal{P}(n)]$ for $s = 0$. If $\ell(I) < k-2$, then $[x] \in [\mathcal{P}(n)]$ by Case 3.5.6. Suppose $\ell(I) = k - 2$ and $I = I_{1,u} = (2, \ldots, \hat{u}, \ldots, k)$. If $u = 2$, then $I = I_2$. Applying Lemma 3.5(i) with $y_0 = f_1(y)$, one gets

$$x \equiv \phi_{(2;I_2)}(X^{2^d-1})(f_1(x_1 y))^{2^d} + \sum_{r=3}^{k} \phi_{(3;I_3)}(X^{2^d-1})(f_1(x_r y))^{2^d}.$$

From the last equalities and Cases 3.5.3, 3.5.12, we have $[x] \in [\mathcal{P}(n)]$.

If $u > 2$, then applying Lemma 3.5(i) with $y_0 = f_1(y)$, one obtain

$$x \equiv \phi_{(2;I_2)}(X^{2^d-1})(f_1(x_{u-1}y))^{2^d} + \sum_{2\leqslant r\leqslant k, r\neq u} \phi_{(2;I\cup r)}(X^{2^d-1})(f_1(x_{r-1}y))^{2^d}.$$

From the above equalities and Cases 3.5.3, 3.5.13, we have $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.15.** *$\bar{y} = x_1^a x_2^b f_2(y)$ for $y \in (P_{k-1})_{m-a-b}$ with $\nu_1(y) = 0$ and $i = 2$.*

We prove $[x] \in [\mathcal{P}(n)]$ by double induction on $(\ell(I), b)$. By Lemma 3.8, we assume that $a = 2^s - 1$.

If $b = 0$, then $[x] \in [\mathcal{P}(n)]$ by Case 3.5.1. Suppose that $b > 0$.

If $I \neq I_2$, then applying Lemma 3.5(ii) with $y_0 = x_1^a x_2^{b-1} f_2(y)$, we get

$$\begin{aligned} x \equiv \ & \phi_{(1;I\cup 2)}(X^{2^d-1})(x_1^{2^s} x_2^{b-1} f_2(y))^{2^d} \\ & + \sum_{3\leqslant r\leqslant k} \phi_{(2;I\cup r)}(X^{2^d-1})(x_1^a x_2^{b-1} f_2(x_{r-1}y))^{2^d}. \end{aligned}$$

From the last equalities, Case 3.5.14, and the inductive hypothesis, we have $[x] \in [\mathcal{P}(n)]$.

If $I = I_2$, then applying Lemma 3.5(i) with $y_0 = x_1^a x_2^{b-1} f_2(y)$, we get

$$x \equiv \phi_{(1;I_2)}(X^{2^d-1})(x_1^{2^s} f_2(x_1^{b-1}y))^{2^d} + \sum_{3\leqslant r\leqslant k} \phi_{(3;I_3)}(X^{2^d-1})(x_r x_1^a x_2^{b-1} f_2(y))^{2^d}.$$

By Lemma 3.6 and Lemma 2.14, we have

$$\begin{aligned} \sum_{3\leqslant r\leqslant k} \phi_{(3;I_3)}(X^{2^d-1})(x_r x_1^a x_2^{b-1} f_2(y))^{2^d} &= Y_3 y_0^{2^d} \\ &\equiv \sum_{(j;J)} \phi_{(j;J)}(X^{2^d-1})(x_j x_1^a x_2^{b-1} f_2(y))^{2^d}, \end{aligned}$$

where the last sum runs over some $(j;J)$ with $j = 1, 2$, $J \subset I_2$ and $J \neq I_2$.

From the above equalities, Case 3.5.14, and the inductive hypothesis, we have $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.16.** *$\bar{y} = x_1^a f_1(y)$ for $y \in (P_{k-1})_{m-a}$ and $i = 1$.*

If $a = 0$, then by Case 3.5.1, $[x] \in [\mathcal{P}(n)]$. Suppose that $a > 0$. If $\ell(I) < k - 2$, then applying Lemma 3.5(ii) with $y_0 = x_1^a x_2^{b-1} f_2(y)$, we get

$$x \equiv \sum_{r=2}^{k} \phi_{(1;I\cup r)}(X^{2^d-1})(x_1^{a-1} f_1(x_{r-1}y))^{2^d}.$$

Hence by the inductive hypothesis, we have $[x] \in [\mathcal{P}(n)]$.

Suppose that $\ell(I) = k - 2$. Then $I = I_{1,u} = (2, \ldots, \hat{u}, \ldots, k)$. If $u = 2$, then applying Lemma 3.5(i) with $y_0 = x_1^{a-1} f_1(y)$ and Lemma 2.14, we get

$$x \equiv \phi_{(2;I_2)}(X^{2^d-1})(x_1^{a-1} f_1(x_1 y))^{2^d} + \sum_{r=3}^{k} \phi_{(3;I_3)}(X^{2^d-1})(x_r x_1^{a-1} f_1(y))^{2^d}.$$

By Lemma 3.6 and Lemma 2.14, we have

$$\begin{aligned} \sum_{r=3}^{k} \phi_{(3;I_3)}(X^{2^d-1})(x_r x_1^{a-1} f_1(y))^{2^d} &= Y_3 y_0^{2^d} \\ &\equiv \sum_{(j;J)} \phi_{(j;J)}(X^{2^d-1})(x_j x_1^{a-1} f_1(y))^{2^d}, \end{aligned}$$

where the last sum runs over some $(j;J)$ with $j = 1, 2$, $J \subset I_2$ and $J \neq I_2$.

From the above equalities, Case 3.5.15, and the inductive hypothesis, we have $[x] \in [\mathcal{P}(n)]$.

If $u > 2$, then applying Lemma 3.5(i) with $y_0 = x_1^{a-1}f_1(y)$, we get

$$x \equiv \phi_{(2;I_2)}(X^{2^d-1})(x_{u-1}^{a-1}f_1(x_1y))^{2^d} + \sum_{3\leqslant u\leqslant k, r\neq u}\phi_{(2;I\setminus 2)}(X^{2^d-1})(x_1^{a-1}f_1(x_{r-1}y))^{2^d}.$$

From the above equalities, Case 3.5.15, and the inductive hypothesis, we have $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.17.** $\bar{y} = x_i^a f_i(y)$ *for* $y \in (P_{k-1})_{m-a}$.

If $a = 0$, then by Case 3.5.1, $[x] \in [\mathcal{P}(n)]$. If $a > 0$ and $i = 1, 2$, then by Cases 3.5.15 and 3.5.16, $[x] \in [\mathcal{P}(n)]$. If $a > 0$ and $i > 2$, then applying Lemma 3.5(ii) with $y_0 = x_i^{a-1}f_i(y)$, we get

$$x \equiv \sum_{1\leqslant j<i}\phi_{(j;I\cup i)}(X^{2^d-1})y_j^{2^d} + \sum_{i<j\leqslant k}\phi_{(i;I\cup j)}(X^{2^d-1})y_j^{2^d},$$

where $y_j = x_i^{a-1}f_i(x_{j-1}y)$ for $j > i$. Hence using the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$. So we have proved $[x] \in [\mathcal{P}(n)]$ for all $x \in (P_k)_n$ .

Now we prove that $[B_k(n)]$ is linearly independent in $QP_k$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{((i;I),z)\in\mathcal{N}_k\times B_{k-1}(n)}\gamma_{(i;I),z}\phi_{(i;I)}(z) \equiv 0, \tag{3.6}$$

where $\gamma_{(i;I),z} \in \mathbb{F}_2$.

If $d \geqslant k$, then by induction on $\ell(I)$, we can show that $\gamma_{(i;I),z} = 0$, for all $(i;I) \in \mathcal{N}_k$ and $z \in B_{k-1}(n)$ (see [32] for the case $d > k$).

Suppose that $d = k-1$. By Lemma 3.7, the homomorphism $p_j = p_{(j;\emptyset)}$ sends the relation (3.6) to $\sum_{z\in B_{k-1}(n)}\gamma_{(j;\emptyset),z}z \equiv 0$. This relation implies $\gamma_{(j;\emptyset),z} = 0$ for any $1 \leqslant j \leqslant k$ and $z \in B_{k-1}(n)$.

Suppose $0 < \ell(J) < k-3$ and $\gamma_{(i;I),z} = 0$ for all $(i;I) \in \mathcal{N}_k$ with $\ell(I) < \ell(J)$, $1 \leqslant i \leqslant k$ and $z \in B_{k-1}(n)$. Then using Lemma 3.7 and the relation (3.4), we see that the homomorphism $p_{(j,J)}$ sends the relation (3.6) to $\sum_{z\in B_{k-1}(n)}\gamma_{(j;J),z}z \equiv 0$. Hence we get $\gamma_{(j;J),z} = 0$ for all $z \in B_{k-1}(n)$.

Now, let $(j;J) \in \mathcal{N}_k$ with $\ell(J) = k-3$. If $J \neq I_3$, then using Lemma 3.7, we have $p_{(j;J)}(\phi_{(i;I)}(z)) \equiv 0$ for all $z \in B_{k-1}(n)$ and $(i;I) \in \mathcal{N}_k$ with $(i;I) \neq (j;J)$. So we get

$$p_{(j;J)}(\mathcal{S}) \equiv \sum_{z\in B_{k-1}(n)}\gamma_{(j;J),z}z \equiv 0.$$

Hence $\gamma_{(j;J),z} = 0$, for all $z \in B_{k-1}(n)$.

According to Lemma 3.7, $p_{(j;I_3)}(\phi_{(1;I_1)}(z)) \equiv 0$ for $z \in \mathcal{C}$ and $p_{(j;I_3)}(\phi_{(1;I_1)}(z)) \in \langle\mathcal{E}\rangle$ for $z \in \mathcal{D}\cup\mathcal{E}$. Hence we obtain

$$p_{(j;I_3)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{C}\cup\mathcal{D}}\gamma_{(j;I_3),z}z \equiv 0 \mod \langle\mathcal{E}\rangle.$$

So we get $\gamma_{(j;I_3),z} = 0$ for all $z \in \mathcal{C}\cup\mathcal{D}$.

Now, let $(j;J) \in \mathcal{N}_k$ with $\ell(J) = k-2$. Suppose that $I_3 \not\subset J$. Then using Lemma 3.7, we have $p_{(j;J)}(\phi_{(1;I_1)}(z)) \equiv 0$ for all $z \in \mathcal{B}$. Hence we get

$$p_{(j;J)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{B}}\gamma_{(j;J),z}z \equiv 0.$$

From this, we obtain $\gamma_{(j;J),z} = 0$ for all $z \in \mathcal{B}$.

Suppose that $I_3 \subset J$. Then either $J = I_2, j = 1,2$ or $J = I_3 \cup 2, j = 1$. According to Lemma 3.7, $p_{(j;I_2)}(\phi_{(1;I_1)}(z)) \in \langle \mathcal{D} \cup \mathcal{E} \rangle$ for all $z \in \mathcal{B}$, $p_{(j;I_3\cup 2)}(\phi_{(1;I_1)}(z)) \equiv 0$ for $z \in \mathcal{C} \cup \mathcal{D}$ and $p_{(1;I_3\cup 2)}(\phi_{(1;I_1)}(z)) \in \langle \mathcal{E} \rangle$ for $z \in \ \mathcal{E}$. Hence we obtain

$$p_{(j;I_2)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{C}} \gamma_{(j;I_2),z} z \equiv 0 \mod \langle \mathcal{D} \cup \mathcal{E} \rangle,$$
$$p_{(1;I_3\cup 2)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{C}\cup\mathcal{D}} \gamma_{(1;I_3\cup 2),z} z \equiv 0 \mod \langle \mathcal{E} \rangle.$$

So $\gamma_{(j;I_2),z} = 0$ for $z \in \mathcal{C}$ and $\gamma_{(1;I_3\cup 2),z} = 0$ for $z \in \mathcal{C} \cup \mathcal{D}$. Since $\gamma_{(i;I),z} = 0$, for all $z \in \mathcal{C}$ and $I \neq I_1$, applying Lemma 3.7, we have

$$p_{(1;I_1)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{C}} \gamma_{(1;I_1),z} z \equiv 0 \mod \langle \mathcal{D} \cup \mathcal{E} \rangle.$$

Hence $\gamma_{(1;I_1),z} = 0$ for all $z \in \mathcal{C}$. So the relation (3.6) becomes

$$\begin{aligned}\mathcal{S} = \sum_{1\leqslant i\leqslant 3, z\in\mathcal{E}} \gamma_{(i;I_3),z}\phi_{(i;I_3)}(z) + \sum_{z\in\mathcal{E}} \gamma_{(1;I_3\cup 2),z}\phi_{(1;I_3\cup 2)}(z)\\ + \sum_{1\leqslant i\leqslant 2, z\in\mathcal{D}\cup\mathcal{E}} \gamma_{(i;I_2),z}\phi_{(i;I_2)}(z) + \sum_{z\in\mathcal{D}\cup\mathcal{E}} \gamma_{(1;I_1),z}\phi_{(1;I_1)}(z) \equiv 0. \quad (3.7)\end{aligned}$$

Using the relation (3.7) and Lemma 3.7,

$$p_{(i;I_2)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{D}} (\gamma_{(i;I_2),z} + \gamma_{(1;I_1),z}) z \equiv 0 \mod \langle \mathcal{E} \rangle, \ i = 1,2.$$

This relation implies $\gamma_{(1;I_2),z} = \gamma_{(2;I_2),z} = \gamma_{(1;I_1),z}$ for all $z \in \mathcal{D}$. On the other hand, using the relation (3.7) and Lemma 3.7, one gets

$$p_{(1;I_1)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{D}} (\gamma_{(1;I_2),z} + \gamma_{(2;I_2),z} + \gamma_{(1;I_1),z}) z \equiv 0 \mod \langle \mathcal{E} \rangle.$$

So $\gamma_{(1;I_2),z} + \gamma_{(2;I_2),z} + \gamma_{(1;I_1),z} = 0$. Hence $\gamma_{(1;I_2),z} = \gamma_{(2;I_2),z} = \gamma_{(1;I_1),z} = 0$, for all $z \in \mathcal{D}$. Now, the relation (3.7) becomes

$$\begin{aligned}\mathcal{S} = \sum_{1\leqslant i\leqslant 3, z\in\mathcal{E}} \gamma_{(i;I_3),z}\phi_{(i;I_3)}(z) + \sum_{z\in\mathcal{E}} \gamma_{(1;I_3\cup 2),z}\phi_{(1;I_3\cup 2)}(z)\\ + \sum_{1\leqslant i\leqslant 2, z\in\mathcal{E}} \gamma_{(i;I_2),z}\phi_{(i;I_2)}(f_i(z)) + \sum_{z\in\cup\mathcal{E}} \gamma_{(1;I_1),z}\phi_{(1;I_1)}(z) \equiv 0. \quad (3.8)\end{aligned}$$

Using the relation (3.8) and Lemma 3.7, one gets

$$p_{(i;I_3)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{E}} (\gamma_{(i;I_3),z} + \gamma_{(1;I_1),z}) z \equiv 0, \ i = 1,2,3,$$
$$p_{(1;I_3\cup 2)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{E}} (\gamma_{(1;I_3),z} + \gamma_{(2;I_3),z} + \gamma_{(1;I_3\cup 2),z} + \gamma_{(1;I_1),z}) z \equiv 0,$$
$$p_{(1;I_2)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{E}} (\gamma_{(1;I_3),z} + \gamma_{(3;I_3),z} + \gamma_{(1;I_2),z} + \gamma_{(1;I_1),z}) z \equiv 0,$$
$$p_{(2;I_2)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{E}} (\gamma_{(2;I_3),z} + \gamma_{(3;I_3),z} + \gamma_{(2;I_2),z} + \gamma_{(1;I_1),z}) z \equiv 0,$$

$$p_{(1;I_1)}(\mathcal{S}) \equiv \sum_{z\in\mathcal{E}}(\gamma_{(1;I_3),z} + \gamma_{(2;I_3),z} + \gamma_{(3;I_3),z} \\ + \gamma_{(1;I_2),z} + \gamma_{(2;I_2),z} + \gamma_{(1;I_3\cup 2),z} + \gamma_{(1;I_1),z})z \equiv 0.$$

From the above relations, we get

$$\gamma_{(i;I_3),z} = \gamma_{(j;I_2),z} = \gamma_{(1;I_3\cup 2),z} = \gamma_{(1;I_1),z} = 0$$

for all $z \in \mathcal{E}$, $i = 1, 2, 3$, $j = 1, 2$. The proposition is proved. □

*Proof of Theorem 1.3.* Denote by $|S|$ the cardinal of a set $S$. It is easy to check that $|\mathcal{N}_k| = 2^k - 1$. Let $(i;I), (j;J) \in \mathcal{N}_k$ with $\ell(J) \leqslant \ell(I)$ and $y, z \in B_{k-1}(n)$. Suppose that $\phi_{(j;J)}(y) = \phi_{(i;I)}(z)$. Using Lemma 3.7, we have $y \equiv p_{(j;J)}(\phi_{(i;I)}(z)) \not\equiv 0$. This implies $(i;I) = (j;J)$ and $y = z$. Hence

$$\phi_{(i;I)}(B_{k-1}(n)) \cap \phi_{(j;J)}(B_{k-1}(n)) = \emptyset.$$

for $(i;I) \neq (j;J)$ and $|\phi_{(i;I)}(B_{k-1}(n))| = |B_{k-1}(n)|$. From Proposition 3.3, we have

$$\begin{aligned}\dim(QP_k)_n = |B_k(n)| &= \sum_{(i;I)\in\mathcal{N}_k} |B_{k-1}(n)| \\ &= |\mathcal{N}_k| \dim(QP_{k-1})_n \\ &= (2^k - 1)\dim(QP_{k-1})_n.\end{aligned}$$

The iterated squaring operation $(\widetilde{Sq}^0_*)^d : (QP_{k-1})_n \to (QP_{k-1})_m$ is an isomorphism of $\mathbb{F}_2$-vector spaces. So we get $\dim(QP_{k-1})_n = \dim(QP_{k-1})_m$. The theorem is proved. □

**Remark 3.9.** Let $n = \sum_{1\leqslant i\leqslant k-1}(2^{d_i} - 1)$ with $d_i$ positive integers such that $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1} > 0$, and let $m = \sum_{1\leqslant i\leqslant k-2}(2^{d_i - d_{k-1}} - 1)$. Set $q = \min\{k, d_{k-1}\}$ and $\mathcal{N}_{k,q} = \{(i;I) \in \mathcal{N}_k : \ell(I) < q\}$. Then we have $|\mathcal{N}_{k,q}| = \sum_{1\leqslant j\leqslant q}\binom{k}{j}$. From the proof of Theorem 1.3 we see that the set

$$\Big[\bigcup_{(i;I)\in\mathcal{N}_{k,q}} \phi_{(i;I)}(B_{k-1}(n))\Big]$$

is linearly independent in $QP_k$. So, one gets the following formula in Mothebe [18]:

$$\dim(QP_k)_n \geqslant \sum_{1\leqslant j\leqslant q}\binom{k}{j}\dim(QP_{k-1})_m.$$

In the remaining part of the section, we prove Lemmas 3.4−3.7. We need the following for the proof of Lemma 3.4.

**Lemma 3.10.** *Let $i, j$ be positive integers such that $0 < i < j \leqslant k$, and $a, b > 0$ with $a + b = 2^d - 1$. Then*

$$X_i^a X_j^b \simeq_2 X_i^{2^d-2}X_j = \phi_{(i;j)}(X^{2^d-1}).$$

*Proof.* We prove the lemma by induction on $b$. If $b = 1$, then

$$X_i^a X_j^b = X_i^{2^d-2}X_j.$$

So the lemma holds. Suppose that $b > 1$. Note that $X_i^aX_j^b = x_i^bx_j^aX_{i,j}^{2^d-1}$. If $\alpha_0(b) = 0$, then

$$x \simeq_0 Sq^1(x_i^{b-1}x_j^aX_{i,j}^{2^d-1}) + x_i^{b-1}x_j^{a+1}X_{i,j}^{2^d-1} \simeq_1 X_i^{a+1}X_j^{b-1} \simeq_2 X_i^{2^d-2}X_j.$$

If $\alpha_0(b) = 1, \alpha_1(b) = 0$, then

$$\begin{aligned} x &\simeq_0 Sq^1(x_i^{b-2}x_j^{a+1}X_{i,j}^{2^d-1}) + Sq^2(x_i^{b-2}x_j^aX_{i,j}^{2^d-1}) + x_i^{b-1}x_j^{a+1}X_{i,j}^{2^d-1}\\ &\simeq_2 x_i^{b-1}x_j^{a+1}X_{i,j}^{2^d-1} = X_i^{a+1}X_j^{b-1} \simeq_2 X_i^{2^d-2}X_j. \end{aligned}$$

If $\alpha_0(b) = \alpha_1(b) = 1$, then

$$\begin{aligned} x &\simeq_0 Sq^1(x_i^bx_j^{a-1}X_{i,j}^{2^d-1}) + Sq^2(x_i^{b-1}x_j^{a-1}X_{i,j}^{2^d-1}) + x_i^{b-1}x_j^{a+1}X_{i,j}^{2^d-1}\\ &\simeq_2 x_i^{b-1}x_j^{a+1}X_{i,j}^{2^d-1} = X_i^{a+1}X_j^{b-1} \simeq_2 X_i^{2^d-2}X_j. \end{aligned}$$

The lemma is proved. □

*Proof of Lemma 3.4.* We prove the lemma by induction on $d$. Suppose $d = 2$. If $j_0 = j_1 = i$, then $x = \phi_{(i,\emptyset)}(X^3)$. If $j = j_0 > j_1 = i$, then $x = X_i^2X_j = \phi_{(i,j)}(X^3)$. If $i = j_0 < j_1 = j$, then $x = X_iX_j^2 \simeq_0 Sq^1(X_\emptyset X_{i,j}^2) + X_i^2X_j \simeq_1 X_i^2X_j = \phi_{(i,j)}(X^3)$. So the lemma holds for $d = 2$.

Suppose $d > 2$. By the inductive hypothesis, there is $(i_1; I') \in \mathcal{N}_k$ such that $\prod_{0\leqslant t<d-1} X_{j_t}^{2^t} \simeq_{d-2} \phi_{((i_1;I')}(X^{2^{d-1}-1})$, where $i_1 = \min\{j_0, j_1, \ldots, j_{d-2}\}$. If $j_{d-1} = i_1$, then the lemma holds with $(i; I) = (i_1; I')$. Suppose that $j_{d-1} \neq i_1$.

If $I' = \emptyset$, then using Lemma 3.10, we have

$$x \simeq_{d-2} X_{i_1}^{2^{d-1}-1}X_{j_{d-1}}^{2^{d-1}} \simeq_2 \phi_{(i;j)}(X^{2^d-1}),$$

where $i = \min\{i_1, j_{d-1}\} = \min\{j_0, j_1, \ldots, j_{d-1}\}$. The lemma holds. Suppose $I' = (i_1', i_2', \ldots, i_r'),\ 0 < r < d-1$ and $I_* = (i_2', \ldots, i_r')$, then

$$\phi_{(i_1;I')}(X^{2^{d-1}-1}) = \phi_{(i_1';I_*)}(X^{2^r-1})X_{i_1}^{2^{d-1}-2^r}.$$

If $i_1 < j_{d-1}$ and $r = d-2$, then $X_{i_1}X_{j_{d-1}}^2 \simeq_1 X_{i_1}^2X_{j_{d-1}}$. Hence using Proposition 2.5(ii), one gets

$$\begin{aligned} x &\simeq_{d-2} \phi_{(i_1';I_*)}(X^{2^{d-2}-1})(X_{i_1}X_{j_{d-1}}^2)^{2^{d-2}}\\ &\simeq_{d-1} \phi_{(i_1';I_*)}(X^{2^{d-2}-1})(X_{i_1}^2X_{j_{d-1}})^{2^{d-2}}\\ &= \phi_{(i_1';I_*)}(X^{2^r-1})X_{j_{d-1}}^{2^r}X_{i_1}^{2^d-2^{r+1}}. \end{aligned}$$

If $i_1 < j_{d-1}$ and $r < d-2$, then using Lemma 3.10, we have $X_{i_1}^{2^{d-r-1}-1}X_{j_{d-1}}^{2^{d-r-1}} \simeq_2 X_{i_1}^{2^{d-r}-2}X_{j_{d-1}}$. Hence by Proposition 2.5(ii),

$$\begin{aligned} x &\simeq_{d-2} \phi_{(i_1';I_*)}(X^{2^r-1})(X_{i_1}^{2^{d-r-1}-1}X_{j_{d-1}}^{2^{d-r-1}})^{2^r}\\ &\simeq_{r+2} \phi_{(i_1';I_*)}(X^{2^r-1})(X_{i_1}^{2^{d-r}-2}X_{j_{d-1}})^{2^r}\\ &\simeq_{d-1} \phi_{(i_1';I_*)}(X^{2^r-1})X_{j_{d-1}}^{2^r}X_{i_1}^{2^d-2^{r+1}} \quad \text{(since } r+2 < d). \end{aligned}$$

By the inductive hypothesis, there is $(j; I) \in \mathcal{N}_k$ such that

$$\phi_{(i_1';I_*)}(X^{2^r-1})X_{j_{d-1}}^{2^r} \simeq_r \phi_{(j;I)}(X^{2^{r+1}-1}),$$

for $0 < r \leqslant d-2$. So, from the above equalities and Proposition 2.5(ii), we get $x \simeq_{d-2} \phi_{(j;I)}(X^{2^{r+1}-1})X_{i_1}^{2^d-2^{r+1}} = \phi_{(i_1;I\cup j)}(X^{2^d-1})$. The lemma holds.

If $i_1 > j_{d-1}$ and $r = d-2$, then

$$x \simeq_{d-2} \phi_{(i_1';I_*)}(X^{2^{d-2}-1})(X_{i_1}X_{j_{d-1}}^2)^{2^{d-2}} = \phi_{(j_{d-1};I\cup i_1)}(X^{2^d-1}).$$

If $i_1 > j_{d-1}$ and $r < d-2$, then using Lemma 3.10, we have $X_{i_1}^{2^{d-r-1}-1}X_{j_{d-1}}^{2^{d-r-1}} \simeq_2 X_{j_{d-1}}^{2^{d-r}-2}X_{i_1}$. Hence by Proposition 2.5(ii),

$$\begin{aligned} x &\simeq_{d-2} \phi_{(i_1';I_*)}(X^{2^r-1})(X_{i_1}^{2^{d-r-1}-1}X_{j_{d-1}}^{2^{d-r-1}})^{2^r} \\ &\simeq_{r+2} \phi_{(i_1';I_*)}(X^{2^r-1})(X_{j_{d-1}}^{2^{d-r}-2}X_{i_1})^{2^r} = \phi_{(j_{d-1};I\cup i_1)}(X^{2^d-1}). \end{aligned}$$

Since $r+2 < d$, the lemma is proved. □

From the proof of Lemma 3.4, we easily obtain the following.

**Corollary 3.11.** *Let $(i;I) \in \mathcal{N}_k$, $j \in \mathbb{N}_k$ and a polynomial $y$ in $(P_k)_m$. If $j > i$ and $d > r+1$, then*

i) $\phi_{(i;I)}(X^{2^{r+1}-1})X_j^{2^d-2^{r+1}} \simeq_{d-1} \phi_{(i;I\cup j)}(X^{2^d-1})$.

ii) $X_j^{2^{d-r-1}-1}(\phi_{(i;I)}(X^{2^{r+1}-1}))^{2^{d-r-1}} \simeq_{d-1} \phi_{(i;I\cup j)}(X^{2^d-1})$.

*Proof of Lemma 3.5.* Applying the Cartan formula, we have

$$Sq^1(X_\emptyset^{2^c-1}y_0^{2^c}) = \sum_{1\leqslant j\leqslant k} X_j^{2^c-1}y_j^{2^c},$$

where $c$ is a positive integer. From this, we obtain

$$X_i^{2^c-1}y_i^{2^c} \equiv \sum_{1\leqslant j<i} X_j^{2^c-1}y_j^{2^c} + \sum_{i<j\leqslant k} X_j^{2^c-1}y_j^{2^c}.$$

If $d > r$, then $\phi_{(i;I)}(X^{2^d-1})y_i^{2^d} = \phi_{(i_1;I^+)}(X^{2^r-1})(X_i^{2^c-1}y_i^{2^c})^{2^r}$, with $c = d-r$ and $I^+ = (i_2, i_3, \ldots, i_r)$. Hence using Lemma 2.14, we get

$$\begin{aligned} \phi_{(i;I)}(X^{2^d-1})y_i^{2^d} \equiv &\sum_{1\leqslant j<i} \phi_{(i_1;I^+)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} \\ &+ \sum_{i<j\leqslant k} \phi_{(i_1;I^+)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r}. \end{aligned}$$

Applying Corollary 3.11 and Lemma 2.14, we have

$$\begin{aligned} &\phi_{(i_1;I^+)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} = \phi_{(j;I)}(X^{2^d-1})y_j^{2^d}, \text{ for } j < i, \\ &\phi_{(i_1;I^+)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} \equiv \phi_{(i_j;I_j)}(X^{2^d-1})y_j^{2^d}, \text{ for } j > i. \end{aligned}$$

Hence the first part of the lemma follows.

If $d > r+1$, then $\phi_{(i;I)}(X^{2^d-1})y_i^{2^d} = \phi_{(i;I)}(X^{2^{r+1}-1})(X_i^{2^c-1}y_i^{2^c})^{2^{r+1}}$, with $c = d-r-1$. Hence using Lemma 2.14, we get

$$\begin{aligned} \phi_{(i;I)}(X^{2^d-1})y_i^{2^d} \equiv &\sum_{1\leqslant j<i} \phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} \\ &+ \sum_{i<j\leqslant k} \phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}}. \end{aligned}$$

According to Corollary 3.11 and Lemma 2.14,

$$\begin{aligned}\phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} &= \phi_{(j;I\cup i)}(X^{2^d-1})y_j^{2^d}, \text{ for } j<i,\\ \phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} &\equiv \phi_{(i;I\cup j)}(X^{2^d-1})y_j^{2^d}, \text{ for } j>i.\end{aligned}$$

So the second part of the lemma is proved. □

We need the following lemmas for the proof of Lemma 3.6.

**Lemma 3.12.** *For any integer $0<\ell\leqslant k$,*

$$X_\ell^{2^\ell-1}x_\ell^{2^\ell} \simeq_\ell \sum_{r=\ell}^{k}\sum_{(i;I)\in\mathcal{N}_{\ell-1}}\phi_{(i;I\cup r)}(X^{2^\ell-1})x_r^{2^\ell} + \sum_{r=\ell+1}^{k}X_r^{2^\ell-1}x_r^{2^\ell}.$$

*Proof.* We prove the lemma by induction on $\ell$. For $\ell=1$, the lemma is trivial. Suppose that $\ell\geqslant 1$ and the lemma is true for $\ell$. Using the Cartan formula we have

$$\begin{aligned}X_{\ell+1}^{2^{\ell+1}-1}x_{\ell+1}^{2^{\ell+1}} &= \sum_{r=1}^{\ell}X_r^{2^{\ell+1}-1}x_r^{2^{\ell+1}} + \sum_{r=\ell+2}^{k}X_r^{2^{\ell+1}-1}x_r^{2^{\ell+1}} + Sq^1(X_\emptyset^{2^{\ell+1}-1})\\ &\simeq_1 \sum_{r=1}^{\ell}X_r^{2^{\ell+1-r}-1}(X_r^{2^r-1}x_r^{2^r})^{2^{\ell+1-r}} + \sum_{r=\ell+2}^{k}X_r^{2^{\ell+1}-1}x_r^{2^{\ell+1}}.\end{aligned}$$

Using the inductive hypothesis and Proposition 2.5, we have

$$\begin{aligned}X_r^{2^{\ell+1-r}-1}(X_r^{2^r-1}x_r^{2^r})^{2^{\ell+1-r}} \simeq_{\ell+1} X_r^{2^{\ell+1-r}-1}\Big(&\sum_{m=r+1}^{k}X_m^{2^r-1}x_m^{2^r}\\ &+\sum_{m=r}^{k}\sum_{(i;I)\in\mathcal{N}_{r-1}}\phi_{(i;I\cup m)}(X^{2^r-1})x_m^{2^r}\Big)^{2^{\ell+1-r}}.\end{aligned}$$

According to Corollary 3.11,

$$\begin{aligned}X_r^{2^{\ell+1-r}-1}(X_m^{2^r-1}x_m^{2^r})^{2^{\ell+1-r}} &\simeq_{\ell+1} \phi_{(r;m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}},\\ X_r^{2^{\ell+1-r}-1}(\phi_{(i;I\cup m)}(X^{2^r-1})x_m^{2^r})^{2^{\ell+1-r}} &\simeq_{\ell+1} \phi_{(i;I\cup\{r,m\})}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}.\end{aligned}$$

From the above equalities, we get

$$\begin{aligned}X_r^{2^{\ell+1-r}-1}(X_r^{2^r-1}x_r^{2^r})^{2^{\ell+1-r}} &\simeq_{\ell+1} \sum_{(i;I)\in\mathcal{N}_{r-1}}\phi_{(i;I\cup r)}(X^{2^{\ell+1}-1})x_r^{2^{\ell+1}}\\ &+\sum_{m=r+1}^{k}\Big(\sum_{(i;I)\in\mathcal{N}_{r-1}}\phi_{(i;I\cup\{r,m\})}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}} + \phi_{(r;m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}\Big).\end{aligned}$$

By a direct computation from the above equalities, using the relation (3.1), we have

$$\begin{aligned}\sum_{r=1}^{\ell} X_r^{2^{\ell+1-r}-1}(X_r^{2^r-1}x_r^{2^r})^{2^{\ell+1-r}} &\simeq_{\ell+1} \sum_{r=1}^{\ell}\sum_{(i;I)\in\mathcal{N}_{r-1}} \phi_{(i;I\cup r)}(X^{2^{\ell+1}-1})x_r^{2^{\ell+1}}\\
&+\sum_{m=2}^{\ell}\sum_{r=1}^{m-1}\Big(\sum_{(i;I)\in\mathcal{N}_{r-1}\cup r}\phi_{(i;I\cup m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}+\phi_{(r;m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}\Big)\\
&+\sum_{m=\ell+1}^{k}\sum_{r=1}^{\ell}\Big(\sum_{(i;I)\in\mathcal{N}_{r-1}\cup r}\phi_{(i;I\cup m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}+\phi_{(r;m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}\Big)\\
&=\sum_{r=1}^{\ell}\sum_{(i;I)\in\mathcal{N}_{r-1}}\phi_{(i;I\cup r)}(X^{2^{\ell+1}-1})x_r^{2^{\ell+1}}+\sum_{m=2}^{\ell}\sum_{(i;I)\in\mathcal{N}_{m-1}}\phi_{(i;I\cup m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}\\
&\qquad+\sum_{m=\ell+1}^{k}\sum_{(i;I)\in\mathcal{N}_{\ell-1}}\phi_{(i;I\cup m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}\\
&=\sum_{m=\ell+1}^{k}\sum_{(i;I)\in\mathcal{N}_{\ell}}\phi_{(i;I\cup m)}(X^{2^{\ell+1}-1})x_m^{2^{\ell+1}}.\end{aligned}$$

Combining the above equalities, we get

$$X_{\ell+1}^{2^{\ell+1}-1}x_{\ell+1}^{2^{\ell+1}} \simeq_{\ell+1} \sum_{r=\ell+1}^{k}\sum_{(i;I)\in\mathcal{N}_{\ell}}\phi_{(i;I\cup r)}(X^{2^{\ell+1}-1})x_r^{2^{\ell+1}}+\sum_{r=\ell+2}^{k}X_r^{2^{\ell+1}-1}x_r^{2^{\ell+1}}.$$

The lemma is proved. □

From the proof of this lemma, we obtain

**Corollary 3.13.** *For $2\leqslant d\leqslant k$, we have*

$$\sum_{r=1}^{d-1}X_r^{2^d-1}x_r^{2^d}\simeq_d\sum_{r=d}^{k}\Big(\sum_{(i;I)\in\mathcal{N}_{d-1}}\phi_{(i;I\cup r)}(X^{2^d-1})x_r^{2^d}\Big).$$

**Lemma 3.14.** *For any integer $d>k$, $0\leqslant r\leqslant d-k$ and $0<m<h\leqslant k$,*

$$Z:=\phi_{(m;I_m)}(X^{2^{d-r}-1})X_h^{2^d-2^{d-r}}\simeq_{k-m+1}\phi_{(m;I_m)}(X^{2^d-1}).$$

*Proof.* We prove the lemma by double induction on $(m,r)$. If $m=k-1$, then $h=k$. By Lemma 3.10, we have

$$\phi_{(k-1;k)}(X^{2^{d-r}-1})X_k^{2^d-2^{d-r}}=X_{k-1}^{2^{d-r}-2}X_k^{2^d-2^{d-r}+1}\simeq_2\phi_{(k-1;k)}(X^{2^d-1}).$$

So, the lemma holds. Suppose that $0<m<k-1$. If $h=m+1$, we have

$$Z=\phi_{(m+2;I_{m+2})}(X^{2^{k-m-1}-1})(X_m^{2^{d-k+m-r+1}-2}X_{m+1}^{2^{d-k+m+1}-2^{d-k+m-r+1}+1})^{2^{k-m-1}}.$$

According to Lemma 3.10,

$$X_m^{2^{d-k+m-r+1}-2}X_{m+1}^{2^{d-k+m+1}-2^{d-k+m-r+1}+1}\simeq_2 X_m^{2^{d-k+m+1}-2}X_{m+1}.$$

Hence using Proposition 2.5, we obtain

$$\begin{aligned}Z&\simeq_{k-m+1}\phi_{(m+2;I_{m+2})}(X^{2^{k-m-1}-1})(X_m^{2^{d-k+m+1}-2}X_{m+1})^{2^{k-m-1}}\\&=\phi_{(m;I_m)}(X^{2^d-1}).\end{aligned}$$

The lemma holds. Suppose that $h > m+1$ and $r = 1$. We have

$$Z = \phi_{(m+1;I_{m+1})}(X^{2^{k-m}-1})(X_m^{2^{d-k+m-1}-1}X_h^{2^{d-k+m-1}})^{2^{k-m}}.$$

Since $X_m^{2^{d-k+m-1}-1}X_h^{2^{d-k+m-1}} \simeq_1 X_m^{2^{d-k+m-1}}X_h^{2^{d-k+m-1}-1}$, applying Proposition 2.5 and the inductive hypothesis, we have

$$\begin{aligned}
Z &\simeq_{k-m+1} \phi_{(m+1;I_{m+1})}(X^{2^{k-m}-1})(X_m^{2^{d-k+m-1}}X_h^{2^{d-k+m-1}-1})^{2^{k-m}}\\
&= \phi_{(m+1;I_{m+1})}(X^{2^{k-m}-1})X_h^{2^{d-1}-2^{k-m}}X_m^{2^{d-1}}\\
&\simeq_{k-m} \phi_{(m+1;I_{m+1})}(X^{2^{d-1}-1})X_m^{2^{d-1}}\\
&= \phi_{(m+2;I_{m+2})}(X^{2^{k-m-1}-1})(X_{m+1}^{2^{d-k+m}-1}X_m^{2^{d-k+m}})^{2^{k-m-1}}.
\end{aligned}$$

According to Lemma 3.10,

$$X_{m+1}^{2^{d-k+m}-1}X_m^{2^{d-k+m}} \simeq_2 X_m^{2^{d-k+m+1}-2}X_{m+1}.$$

Hence using Proposition 2.5, one gets

$$\begin{aligned}
Z &\simeq_{k-m+1} \phi_{(m+2;I_{m+2})}(X^{2^{k-m-1}-1})(X_m^{2^{d-k+m+1}-2}X_{m+1})^{2^{k-m-1}}\\
&= \phi_{(m;I_m)}(X^{2^d-1}).
\end{aligned}$$

Now, suppose that $h > m+1$ and $r > 1$. Applying Proposition 2.5 and the inductive hypothesis, one gets

$$\begin{aligned}
Z &= \phi_{(m;I_m)}(X^{2^{d-r}-1})X_h^{2^{d-r}}X_h^{2^d-2^{d-r+1}}\\
&\simeq_{k-m+1} \phi_{(m;I_m)}(X^{2^{d-r+1}-1})X_h^{2^d-2^{d-r+1}}\\
&\simeq_{k-m+1} \phi_{(m;I_m)}(X^{2^d-1}).
\end{aligned}$$

The lemma is proved. □

**Lemma 3.15.** *For any integer $d \geqslant k$,*

$$X_k^{2^d-1}x_k^{2^d} \simeq_k \sum_{(i;I)\in\mathcal{N}_{k-1}} \phi_{(i;I\cup k)}(X^{2^d-1})x_k^{2^d}.$$

*Proof.* By Lemma 3.12, we have

$$X_k^{2^k-1}x_k^{2^k} \simeq_k \sum_{(i;I)\in\mathcal{N}_{k-1}} \phi_{(i;I\cup k)}(X^{2^k-1})x_k^{2^k}.$$

Hence using Proposition 2.5, we get

$$X_k^{2^d-1}x_k^{2^d} = X_k^{2^k-1}x_k^{2^k}X_\emptyset^{2^d-2^k} \simeq_k \sum_{(i;I)\in\mathcal{N}_{k-1}} \phi_{(i;I\cup k)}(X^{2^k-1})X_k^{2^d-2^k}x_k^{2^d}.$$

Let $(i;I) \in \mathcal{N}_{k-1}$. If $I = \emptyset$, then using Lemma 3.10, we have

$$\begin{aligned}
\phi_{(i;I\cup k)}(X^{2^k-1})X_k^{2^d-2^k}x_k^{2^d} &= \phi_{(i;k)}(X^{2^k-1})X_k^{2^d-2^k}x_k^{2^d}\\
&= X_i^{2^k-2}X_k^{2^d-2^k+1}x_k^{2^d}\\
&\simeq_2 X_i^{2^d-2}X_kx_k^{2^d}\\
&= \phi_{(i;k)}(X^{2^d-1})x_k^{2^d}.
\end{aligned}$$

If $I=(i_1,\ldots,i_r), r>0$, then $s=k-\ell(I\cup k)>0$. Hence

$$\begin{aligned}Y:&=\phi_{(i;I\cup k)}(X^{2^k-1})X_k^{2^d-2^k}x_k^{2^d}\\&=\phi_{(i_1;I^+\cup k)}(X^{2^{k-s}-1})(X_i^{2^s-1}X_k^{2^{d-k+s}-2^s})^{2^{k-s}}x_k^{2^d}.\end{aligned}$$

where $I^+=(i_2,\ldots,i_r)$. By Lemma 3.10,

$$X_i^{2^s-1}X_k^{2^{d-k+s}-2^s}\simeq_2 X_i^{2^{d-k+s}-2}X_k.$$

If $(i;I\cup k)\neq(1;I_1)$, then $s\geqslant 2$. Using Proposition 2.5 and Lemma 3.4, one gets

$$\begin{aligned}Y&\simeq_{k-s+2}\phi_{(i_1;I^+\cup k)}(X^{2^{k-s}-1})(X_kX_i^{2^{d-k+s}-2})^{2^{k-s}}x_k^{2^d}\\&=\phi_{(i_1;I^+\cup k)}(X^{2^{k-s}-1})(X_kX_i^2)^{2^{k-s}}X_i^{2^d-2^{k-s+2}}x_k^{2^d}\\&\simeq_k\phi_{(i;I\cup k)}(X^{2^{k-s+2}-1})X_i^{2^d-2^{k-s+2}}x_k^{2^d}\\&=\phi_{(i;I\cup k)}(X^{2^d-1})x_k^{2^d}.\end{aligned}$$

Suppose that $(i;I\cup k)=(1;I_1)$. Then using Lemma 3.14 and Proposition 2.5, we have

$$\phi_{(1;I_1)}(X^{2^k-1})X_k^{2^d-2^k}x_k^{2^d}\simeq_k\phi_{(1;I_1)}(X^{2^d-1})x_k^{2^d}.$$

The lemma is proved. □

**Lemma 3.16.** $Y_1\simeq_{(k,\omega)}0$ *with* $\omega=\omega(X_1^{2^d-1}x_1^{2^d})$. *More precisely,*

$$Y_1=\sum_{0\leqslant i<k}Sq^{2^i}(y_i)+h,$$

*with* $y_i$ *polynomials in* $P_k$, *and* $h\in P_k^-(\omega)$.

*Proof.* First we prove the following by induction on $m$

$$Y_1\simeq_{(k,\omega)}Y_m+\sum_{r=m}^{k}\sum_{(i;I)\in\mathcal{N}_{m-1}}\phi_{(i;I\cup I_{m-1})}(X^{2^d-1})x_r^{2^d}. \tag{3.9}$$

Note that

$$\phi_{(m;I_m)}(X^{2^d-1})x_m^{2^d}=\phi_{(m+1;I_{m+1})}(X^{2^{k-m}-1})(X_m^{2^m-1}x_m^{2^m}X_\emptyset^{2^{d-k+m}-2^m})^{2^{k-m}}.$$

Applying Lemma 3.12 and Proposition 2.5, we have

$$\begin{aligned}X_m^{2^m-1}x_m^{2^m}X_\emptyset^{2^{d-k+m}-2^m}\simeq_m&\sum_{r=m+1}^{k}X_r^{2^{d-k+m}-1}x_r^{2^{d-k+m}}\\&+\sum_{r=m}^{k}\sum_{(i;I)\in\mathcal{N}_{m-1}}\phi_{(i;I\cup r)}(X^{2^m-1})X_r^{2^{d-k+m}-2^m}x_r^{2^{d-k+m}}.\end{aligned}$$

Using Lemma 3.14 and Proposition 2.5, we have

$$\phi_{(m+1;I_{m+1})}(X^{2^{k-m}-1})X_r^{2^d-2^{k-m}}x_r^{2^d}\simeq_{k-m}\phi_{(m+1;I_{m+1})}(X^{2^d-1})x_r^{2^d}.$$

From the above equalities, Proposition 2.5 and Lemma 3.4, one gets

$$\phi_{(m;I_m)}(X^{2^d-1})x_m^{2^d}\simeq_k Y_{m+1}+\sum_{r=m}^{k}\sum_{(i;I)\in\mathcal{N}_{m-1}}\phi_{(i;I\cup I_m\cup r)}(X^{2^k-1})X_r^{2^d-2^k}x_r^{2^d}.$$

If either $r > m$ or $I \neq (2, \dots, m-1)$, then $(i; I \cup I_m \cup r) \neq (1; I_1)$. From the proof of Lemma 3.14, we have

$$\phi_{(i;I\cup I_m\cup r)}(X^{2^k-1})X_r^{2^d-2^k}x_r^{2^d} \simeq_k \phi_{(i;I\cup I_m\cup r)}(X^{2^d-1})x_r^{2^d}.$$

If $r = m$ and $I = (2, \dots, m-1)$, then $(i; I \cup I_m \cup m) = (1; I_1)$. By Lemma 3.14, we have

$$\phi_{(1;I_1)}(X^{2^k-1})X_m^{2^d-2^k}x_m^{2^d} \simeq_k \phi_{(1;I_1)}(X^{2^d-1})x_m^{2^d}.$$

Combining the above equalities, we get

$$\phi_{(m;I_m)}(X_m^{2^d-1})x_m^{2^d} \simeq_k Y_{m+1} + \sum_{r=m}^{k} \sum_{(i;I)\in\mathcal{N}_{m-1}} \phi_{(i;I\cup I_m\cup r)}(X^{2^d-1})x_r^{2^d}.$$

Using the above equalities and the inductive hypothesis, we get

$$\begin{aligned}
Y_1 \simeq_{(k,\omega)} & Y_{m+1} + \sum_{r=m}^{k}\Big(\sum_{(i;I)\in\mathcal{N}_{m-1}} \phi_{(i;I\cup I_{m-1})}(X^{2^d-1})x_r^{2^d}\Big)\\
&+ \sum_{r=m+1}^{k} \phi_{(m;I_m)}(X^{2^d-1})x_r^{2^d} + \sum_{r=m}^{k}\Big(\sum_{(i;I)\in\mathcal{N}_{m-1}} \phi_{(i;I\cup I_m\cup r)}(X^{2^d-1})x_r^{2^d}\Big)\\
&= Y_{m+1} + \sum_{r=m+1}^{k}\Big(\sum_{(i;I)\in\mathcal{N}_{m-1}\cup m} \phi_{(i;I\cup I_m)}(X^{2^d-1})x_r^{2^d}\Big)\\
&+ \sum_{r=m+1}^{k}\Big(\sum_{(i;I)\in\mathcal{N}_{m-1}} \phi_{(i;I\cup I_m)}(X^{2^d-1})x_r^{2^d}\Big) + \sum_{r=m+1}^{k} \phi_{(m;I_m)}(X^{2^d-1})x_r^{2^d}\\
&\quad (\text{since } m\cup I_m = I_{m-1} \text{ and } I_m \cup r = I_m \text{ for } r > m)\\
&= Y_{m+1} + \sum_{r=m+1}^{k}\Big(\sum_{(i;I)\in\mathcal{N}_m} \phi_{(i;I\cup I_m)}(X^{2^d-1})x_r^{2^d}\Big)\\
&\quad (\text{since } \mathcal{N}_m = \mathcal{N}_{m-1}\cup(\mathcal{N}_{m-1}\cup m)\cup\{(m;\emptyset)\}).
\end{aligned}$$

The relation (3.9) is proved.

Since $Y_k = X_k^{2^d-1}x_k^{2^d}$, using the relation (3.9) with $m = k$ and Lemma 3.12, one gets

$$Y_1 \simeq_{(k,\omega)} X_k^{2^d-1}x_k^{2^d} + \sum_{(i;I)\in\mathcal{N}_{k-1}} \phi_{(i;I\cup k)}(X^{2^d-1})x_k^{2^d} \simeq_{(k,\omega)} 0.$$

The lemma is proved. □

*Proof of Lemma 3.6.* We have $Y_m = Z^{2^d-1}Y_1(x_m, \dots, x_k)$ with $Z = x_1x_2\dots x_{m-1}$. By Lemma 3.16, $Y_m$ is a sum of polynomials of the form $f = Z^{2^d-1}(Sq^{2^i}(y) + h)$ with $0 \leqslant i \leqslant k-m$, $y$ a monomial in $P_{k-m+1} = P_{k-m+1}(x_m, \dots, x_k)$ and $h \in P_{k-m+1}^-(\omega^*), \omega^* = \omega((x_{m+1}\dots x_k)^{2^d-1}x_m^{2^d})$. Then $Z^{2^d-1}h \in P_k^-(\omega)$ with $\omega = \omega(X_1^{2^d-1}x_1^{2^d})$. Using the Cartan formula, we have

$$f \simeq_{(0,\omega)} Sq^{2^i}(Z^{2^d-1}y) + \sum_{1\leqslant t\leqslant 2^i} Sq^t(Z^{2^d-1})Sq^{2^i-t}(y).$$

By a direct computation using the Cartan formula, we can show that if $0 < t < 2^i$, then $\omega_u(Sq^t(Z^{2^d-1})Sq^{2^i-t}(y)) < k-1$ for some $u \leqslant d$. Hence one gets

$$f \simeq_{(k,\omega)} Sq^{2^i}(Z^{2^d-1})y \simeq_{(k,\omega)} \sum_{0<j<m} Z^{2^d-1}x_j^{2^i}y.$$

Since $\omega_u(Z^{2^d-1}x_j^{2^i}) = m-2$ for $i < u \leqslant k$, if $Z^{2^d-1}x_j^{2^i}y \notin P_k^-(\omega)$, then $\omega_u(y) = k-m$ for $i < u \leqslant k$. According to Lemma 3.4, there is $(j;J) \in \mathcal{N}_k$ such that $Z^{2^d-1}x_j^{2^i}y \simeq_i \phi_{(j;J)}(X^{2^d-1})x_j^{2^d}$. Here $J \subset I_{m-1}$. Since $0 \leqslant \ell(J) = i \leqslant k-m < \ell(I_{m-1}) = k-m+1$, we have $J \neq I_{m-1}$. The lemma is proved. □

The following will be used in the proof of Lemma 3.7.

**Lemma 3.17.** *Let $(j;J),(i;I) \in \mathcal{N}_k$ with $\ell(I) < d$. Then*

$$p_{(j;J)}\phi_{(i;I)}(X^{2^d-1}) \simeq_0 \begin{cases} X^{2^d-1}, & (i;I) \subset (j;J), \\ 0, & (i;I) \not\subset (j;J). \end{cases}$$

*Proof.* Suppose that $(i;I) \not\subset (j;J)$. If $i \notin (j;J)$, then from (3.2), we see that $p_{(j;J)}(\phi_{(i;I)}(X^{2^d-1}))$ is a sum of monomials of the form

$$w = x_{i'}^{2^r-1}f_{k-1;i'}(z),$$

for suitable monomial $z$ in $P_{k-2}$. Here $i' = i$ if $j > i$ and $i' = i-1$ if $j < i$. In this case, we have $\alpha_r(2^r-1) = 0$ and $\omega_{r+1}(w) < k-1$. Hence $w \in P_{k-1}^-(\omega^{(d)})$, where $\omega^{(d)} = \omega(X^{2^d-1})$. Suppose that $i \in (j;J)$. Since $(i;I) \not\subset (j;J)$, there is $1 \leqslant t \leqslant r$, such that $i_t \notin (j;J)$, then from (3.2), we see that $p_{(j;J)}(\phi_{(i;I)}(X^{2^d-1}))$ is a sum of monomials of the form

$$w = x_{i_t-1}^{2^r-2^{r-t}-1}f_{k-1;i_t-1}(z),$$

for some monomial $z$ in $P_{k-2}$. It is easy to see that $\alpha_{r-t}(2^r-2^{r-t}-1) = 0$ and $\omega_{r-t+1}(w) < k-1$. Hence $w \in P_{k-1}^-(\omega^{(d)})$.

Suppose that $(i;I) \subset (j;J)$. If $i = j$, then from (3.2), we see that the polynomial $p_{(j;J)}(\phi_{(i;I)}(X^{2^d-1}))$ is a sum of monomials of the form

$$w = \Big(\prod_{1\leqslant t\leqslant r} x_{i_t-1}^{2^r-2^{r-t}-1+b_t}\Big)\Big(\prod_{j+1\in J\setminus I} x_j^{2^d-1+c_j}\Big)\Big(\prod_{j+1\notin J} x_j^{2^d-1}\Big),$$

where $b_1+b_2+\ldots+b_r+\sum_{j+1\in J\setminus I}c_j = 2^r-1$. If $c_j > 0$, then $\alpha_{u_j}(2^d-1+c_j) = 0$ with $u_j$ the smallest index such that $\alpha_{u_j}(c_j) = 1$. Hence $w \in P_{k-1}^-(\omega^{(d)})$. If $b_t = 0$ for suitable $1 \leqslant t \leqslant r$, then $\alpha_{r-t}(2^r-2^{r-t}-1) = 0$ and $\omega_{r-t+1}(w) < k-1$. Hence $w \in P_{k-1}^-(\omega^{(d)})$. Suppose that $b_t > 0$ for any $t$. Let $v_t$ be the smallest index such that $\alpha_{v_t}(b_t) = 1$. If $v_t \neq r-t$, then $\alpha_{v_t}(2^r-2^{r-t}-1+b_t) = 0$ and $w \in P_{k-1}^-(\omega^{(d)})$. So $u_t = r-t$ and $b_t = 2^{r-t}+b_t'$ with $b_t' \geqslant 0$. If $b_t' > 0$, then $\alpha_{v_t'}(2^r-2^{r-t}-1+b_t) = \alpha_{v_t'}(2^r-1+b_t') = 0$ with $v_t'$ the smallest index such that $\alpha_{v_t'}(b_t') = 1$. Hence $w \in P_{k-1}^-(\omega^{(d)})$. This implies $b_t' = 0$ for $1 \leqslant t \leqslant r$ and $w = g$.

If $i \in J$, then from (3.2), we see that the polynomial $p_{(j;J)}(\phi_{(i;I)}(X^{2^d-1}))$ is a sum of monomials of the form

$$w = x_{i-1}^{2^r-1+b_0}\Big(\prod_{1\leqslant t\leqslant r} x_{i_t-1}^{2^r-2^{r-t}-1+b_t}\Big)\Big(\prod_{j+1\in J\setminus(i;I)} x_j^{2^d-1+c_j}\Big)\Big(\prod_{j+1\notin J} x_j^{2^d-1}\Big),$$

where $b_0 + b_1 + b_2 + \ldots + b_r + \sum_{j+1\in J\setminus(i;I)} c_j = 2^d - 1$. By a same argument as above, we see that $w \in P^-_{k-1}(\omega^{(d)})$ if either $c_j > 0$ or $b_t \neq 2^{r-t}$ for some $j, t$ with $t > 0$. Suppose $c_j = 0$ and $b_t = 2^{r-t}$ with all $j$ and $t > 0$. Then $2^d - 1 = b_0 + b_1 + b_2 + \ldots + b_r + \sum_{j+1\in J\setminus(i;I)} c_j = b_0 + 2^r - 1$ and $w = X^{2^d-1}$. The lemma is proved. □

The following is easily be proved by a direct computation.

**Lemma 3.18.** *The following diagram is commutative:*

$$\begin{array}{ccc} P_{k-1} & \xrightarrow{f_i} & P_k \\ \Big\downarrow{\scriptstyle p_{(i;I_i)}} & & \Big\downarrow{\scriptstyle p_{(i+1;I_{i+1})}} \\ P_{k-2} & \xrightarrow{f_i} & P_{k-1}. \end{array}$$

*Proof of Lemma 3.7.* i) Suppose that either $d \geqslant k$ or $d = k-1$ and $I \neq I_1$, then $\phi_{(i;I)}(z) = \phi_{(i;I)}(X^{2^d-1})f_i(\bar z)^{2^d}$. Hence the first part of the lemma follows from Lemma 3.17.

ii) According to (3.4), $\phi_{(1;I_1)}(z) = \phi_{(2;I_2)}(X^{2^d-1})f_1(\bar z)^{2^d}$. Hence from Lemmas 3.17 and 3.18, we have

$$\begin{aligned} p_{(i;I)}(\phi_{(1;I_1)}(z)) &\equiv p_{(i;I)}(\phi_{(2;I_2)}(X^{2^d-1}))p_{(i;I)}(f_1(\bar z)^{2^d}) \\ &\equiv \begin{cases} z & \text{if } (i;I) = (1;I_1), \\ X^{2^d-1}f_1p_{(1;I_1)}(\bar z^{2^d}) \in \langle \mathcal{D}\cup\mathcal{E}\rangle, & \text{if } (i;I) = (2;I_2), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

iii) Let $z \in \mathcal{D}$. Using the relation (3.4), Lemma 3.17 and Lemma 3.18, one has

$$\begin{aligned} p_{(i;I)}(\phi_{(1;I_1)}(z)) &\equiv p_{(i;I)}(\phi_{(3;I_3)}(X^{2^d-1}))p_{(i;I)}(f_2(\bar z)^{2^d}) \\ &\equiv \begin{cases} z & \text{if } I_2 \subset I, \\ X^{2^d-1}f_2p_{(2;I_2)}(\bar z)^{2^d} \in \langle \mathcal{E}\rangle, & \text{if } (i;I) = (3;I_3), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

iv) Let $z \in \mathcal{E}$. Using the relation (3.4), Lemma 3.17 and Lemma 3.18, one gets

$$\begin{aligned} p_{(4;I_4)}(\phi_{(4;I_4)}(z)) &= p_{(4;I_4)}(\phi_{(4;I_4)}(X^{2^d-1}))p_{(4;I_4)}(f_3(\bar z)^{2^d}) \\ &\equiv X^{2^d-1}f_3p_{(3;I_3)}((\bar z)^{2^d}). \end{aligned}$$

If a monomial $y$ is a term of $f_3p_{(3;I_3)}((\bar z)^{2^d})$, then $\omega_1(y) < k-3$. According to Theorem 2.12, $y \equiv 0$. Hence $X^{2^d-1}f_3p_{(3;I_3)}(\bar z)^{2^d} \equiv 0$. So using Lemma 3.18 one gets

$$p_{(i;I)}(\phi_{(1;I_1)}(z)) \equiv p_{(i;I)}(\phi_{(4;I_4)}(X^{2^d-1}))p_{(i;I)}(f_3(\bar z)^{2^d}) \equiv \begin{cases} z & \text{if } I_3 \subset I, \\ 0, & \text{otherwise .} \end{cases}$$

The lemma is completely proved. □

## 4. The cases $k \leqslant 3$

In this section and the next sections, we denote by $B_k(n)$ the set of all admissible monomials of degree $n$ in $P_k$, $B_k^0(n) = B_k(n) \cap P_k^0$, $B_k^+(n) = B_k(n) \cap P_k^+$. For an $\omega$-vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ of degree $n$, we set $B_k(\omega) = B_k(n) \cap P_k(\omega)$, $B_k^+(\omega) = B_k^+(n) \cap P_k(\omega)$. Then $[B_k(\omega)]$ and $[B_k^+(\omega)]$, respectively are the basses of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega)$.

If there is $i_0 = 0, i_1, i_2, \ldots, i_r > 0$ such that $i_1 + i_2 + \ldots + i_r = m$ and $\omega_{i_1+\ldots+i_{s-1}+t} = a_s, 1 \leqslant t \leqslant i_s, 1 \leqslant s \leqslant r$, then we denote $\omega = (a_1^{(i_1)}, a_2^{(i_2)}, \ldots, a_r^{(i_r)})$. If $i_u = 1$, then we denote $a_u^{(1)} = a_u$.

Using Lemma 5.3.3(i) in Subsection 5.3 and Theorem 2.9, we easily obtain the following.

**Proposition 4.1.** *For any $s \geqslant 1$,*

$$B_k(1^{(s)}) = \big\{x_{i_1}x_{i_2}^2 \ldots x_{i_{m-1}}^{2^{m-2}}x_{i_m}^{2^s-2^{m-1}};\ 1 \leqslant i_1 < \ldots < i_m \leqslant k, 1 \leqslant m \leqslant \min\{s,k\}\big\}.$$

It is well known that if $n \neq 2^u - 1$ then $B_1(n) = \emptyset$. If $n = 2^u - 1$ for $u \geqslant 0$, then $B_1(n) = B_1(1^{(u)}) = \{x^{2^u-1}\}$. It is easy to see that $\Phi(B_1(0)) = \{1\} = B_2(0)$, $\Phi(B_1(1)) = \{x_1, x_2\} = B_2(1)$. According to Proposition 3.3, for $u > 1$, we have

$$B_2(2^u-1) = \Phi(B_1(2^u-1)) = \{x_1^{2^u-1}, x_2^{2^u-1}, x_1x_2^{2^u-2}\},$$

By Theorem 1.1, $B_2(n) = \emptyset$ if $n \neq 2^{t+u} + 2^t - 2$ for all nonnegative integers $t, u$. We define the $\mathbb{F}_2$-linear map $\psi : (P_k)_m \to (P_k)_{2m+k}$ by $\psi(y) = X_\emptyset y^2$ for any monomial $y \in (P_k)_m$. From Theorem 1.2 and Theorem 1.3, we have

**Theorem 4.2** (Peterson [21])**.** *If $n = 2^{t+u} + 2^t - 2$, with $t, u$ positive integers, then*

$$\begin{aligned} B_2(n) &= \psi^t(\Phi(B_1(2^u-1))) \\ &= \begin{cases} \{(x_1x_2)^{2^t-1}\}, & u = 0,\\ \{x_1^{2^{t+1}-1}x_2^{2^t-1},\ x_1^{2^t-1}x_2^{2^{t+1}-1}\}, & u = 1,\\ \{x_1^{2^{t+u}-1}x_2^{2^t-1}, x_1^{2^t-1}x_2^{2^{t+u}-1}, x_1^{2^{t+1}-1}x_2^{2^{t+u}-2^t-1}\}, & u > 1.\end{cases}\end{aligned}$$

By Theorems 1.1 and 1.2, for $k = 3$, we need only to consider the cases of degree $n = 2^s - 2,\ n = 2^s - 1$ and $n = 2^{s+t} + 2^s - 2$ with $s, t$ positive integers. A direct computation using Theorem 1.3 we have

**Theorem 4.3** (Kameko [14])**.**

i) *If $n = 2^s - 2$, then $B_3(2^s-2) = \Phi(B_2(2^s-2))$.*

ii) *If $n = 2^s - 1$, then $B_3(2^s-1) = B_3(1^{(s)}) \cup \psi(\Phi(B_2(2^{s-1}-2)))$.*

iii) *If $n = 2^{s+t} + 2^s - 2$, then*

$$B_3(n) = \begin{cases}\Phi(B_2(8)) \cup \{x_1^3x_2^4x_3\}, & \text{if } s = 1, t = 2,\\ \Phi(B_2(2^{s+t}+2^s-2)), & \text{otherwise.}\end{cases}$$

## 5. Proof of Theorem 1.4

For $1 \leqslant i \leqslant k$, define $\varphi_i : QP_k \to QP_k$, the homomorphism induced by the $\mathcal{A}$-homomorphism $\overline{\varphi}_i : P_k \to P_k$, which is determined by $\overline{\varphi}_1(x_1) = x_1 + x_2, \overline{\varphi}_1(x_j) = x_j$ for $j > 1$, and $\overline{\varphi}_i(x_i) = x_{i-1}, \overline{\varphi}_i(x_{i-1}) = x_i$, $\overline{\varphi}_i(x_j) = x_j$ for $j \neq i, i-1,\ 1 < i \leqslant k$. Note that the general linear group $GL_k$ is generated by $\overline{\varphi}_i,\ 0 < i \leqslant k$ and the symmetric group $\Sigma_k$ is generated by $\overline{\varphi}_i,\ 1 < i \leqslant k$.

Let $B$ be a finite subset of $P_k$ consisting of some monomials of degree $n$. To prove the set $[B]$ is linearly independent in $QP_k$, we order the set $B$ by the order as in Definition 2.6 and denote the elements of $B$ by $d_i = d_{n,i}, 0 < i \leqslant b = |B|$ in such away that $d_{n,i} < d_{n,j}$ if and only if $i < j$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1\leqslant j\leqslant b} \gamma_j d_{n,j} \equiv 0,$$

with $\gamma_j \in \mathbb{F}_2$. For $(i;I) \in \mathcal{N}_k$, we explicitly compute $p_{(i;I)}(\mathcal{S})$ in terms of a minimal set of $\mathcal{A}$-generators in $P_{k-1}$. Computing from some relations $p_{(i;I)}(\mathcal{S}) \equiv 0$ with $(i;I) \in \mathcal{N}_k$ and $\overline{\varphi}_i(\mathcal{S}) \equiv 0$, we will obtain $\gamma_j = 0$ for all $j$.

5.1. **The case of degree $n = 2^{s+1} - 3$.**

In this subsection we prove the following.

**Proposition 5.1.1.** *For any $s \geqslant 1$, $\Phi(B_3(n))$ is a minimal set of generators for $\mathcal{A}$-module $P_4$ in degree $n = 2^{s+1} - 3$.*

We need the following lemma for the proof of the proposition.

**Lemma 5.1.2.** *If $x$ is an admissible monomial of degree $2^{s+1} - 3$ in $P_4$, then $\omega(x) = (3^{(s-1)}, 1)$.*

*Proof.* It is easy to see that the lemma holds for $s = 1$. Suppose $s \geqslant 2$. Obviously, $z = x_1^{2^s-1}x_2^{2^{s-1}-1}x_3^{2^{s-1}-1}$ is the minimal spike of degree $2^{s+1} - 3$ in $P_4$ and $\omega(z) = (3^{(s-1)}, 1)$. Since $2^{s+1}-3$ is odd, we get either $\omega_1(x) = 1$ or $\omega_1(x) = 3$. If $\omega_1(x) = 1$, then $\omega(x) < \omega(z)$. By Theorem 2.12, $x$ is hit. This contradicts the fact that $x$ is admissible. Hence we have $\omega_1(x) = 3$. Using Proposition 2.10 and Theorem 2.12, we obtain $\omega_i(x) = 3,\ i = 1, 2, \ldots, s-1$. From this, it implies

$$2^{s+1} - 3 = \deg x = \sum_{i\geqslant 1} 2^{i-1}\omega_i(x) = 3(2^{s-1} - 1) + \sum_{i\geqslant s} 2^{i-1}\omega_i(x).$$

The last equality implies $\omega_s(x) = 1$ and $\omega_i(x) = 0$ for $i > s$. The lemma is proved. □

From Lemma 3.10, we have the following.

**Lemma 5.1.3.** *The following monomials are strictly inadmissible:*

$$X_1x_1^2, X_iX_j^2,\ 1 \leqslant i < j \leqslant 4.$$

*Proof of Proposition 5.1.1.* We have $n = 2^{s+1} - 3 = 2^s + 2^{s-1} + 2^{s-1} - 3$. Hence the proposition follows from Theorem 1.3 for $s \geqslant 4$. According to Kameko [14],

$$B_3(n) = \{v_1 = X^{2^{s-1}-1}x_3^{2^{s-1}}, v_2 = X^{2^{s-1}-1}x_2^{2^{s-1}}, v_3 = X^{2^{s-1}-1}x_1^{2^{s-1}}\},$$

where $X = x_1x_2x_3$.

It is easy to see that $\Phi(B_3(1)) = \{x_1, x_2, x_3, x_4\}$. Hence the proposition holds for $s = 1$. For $s = 2$, using Lemma 5.1.3, we see that

$$\Phi^+(B_3(5)) = \{x_1x_2x_3x_4^2,\ x_1x_2x_3^2x_4,\ x_1x_2^2x_3x_4\}$$

is a minimal set of generators for $(P_4^+)_5$. A direct computation using Lemmas 5.1.2 and 5.1.3 shows that for $s = 3$, $\Phi^+(B_3(13))$ is the set of 23 following monomials:

$$X_i^2X_jx_m^4,\ 1 \leqslant i < j \leqslant 4,\ m \neq i, X_i^2X_jx_i^4,\ 2 \leqslant i < j \leqslant 4,\ X_3^3x_3^4,\ X_4^3x_4^4.$$

Using Lemmas 5.1.2, 5.1.3 and Theorem 2.12, we see that if $x$ is an admissible monomial of degree 13 in $P_4^+$, then $x \in \Phi^+(B_3(13))$. Hence $(QP_4^+)_{13}$ is generated by $[\Phi^+(B_3(13))]$. Now we prove that the set $[\Phi^+(B_3(13))]$ is linearly independent.

Suppose there is a linear relation

$$\sum_{j=1}^{23} \gamma_j d_j \equiv 0, \tag{5.1.3.1}$$

where $\gamma_j \in \mathbb{F}_2, 1 \leqslant j \leqslant 23$.

Consider the homomorphisms $p_{(1;i)} : P_4 \to P_3, i = 2,3,4$. By a direct computation from (5.1.3.1), we have

$$\begin{aligned}
p_{(1;2)}(\mathcal{S}) &\equiv \gamma_1 v_1 + \gamma_2 v_2 + \gamma_7 v_3 \equiv 0,\\
p_{(1;3)}(\mathcal{S}) &\equiv \gamma_3 v_1 + (\gamma_5 + \gamma_{16}) v_2 + \gamma_8 v_3 \equiv 0,\\
p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_4 + \gamma_{15}) v_1 + \gamma_6 v_2 + \gamma_9 v_3 \equiv 0.
\end{aligned}$$

From the above equalities it implies

$$\begin{cases} \gamma_j = 0,\ j = 1,2,3,6,7,8,9,\\ \gamma_5 = \gamma_{16},\ \gamma_4 = \gamma_{16}. \end{cases} \tag{5.1.3.2}$$

Substituting (5.1.3.2) into the relation (5.1.3.1), we have

$$\mathcal{S} = \gamma_4 d_4 + \gamma_5 d_5 + \sum_{10 \leqslant j \leqslant 23} \gamma_j d_j \equiv 0. \tag{5.1.3.3}$$

Applying the homomorphisms $p_{(2;3)}, p_{(2;4)}, p_{(3;4)} : P_4 \to P_3$ to (5.1.3.3), we get

$$\begin{aligned}
p_{(2;3)}(\mathcal{S}) &\equiv \gamma_{10} v_1 + (\gamma_{12} + \gamma_{16} + \gamma_{18}) v_2 + \gamma_{21} v_3 \equiv 0,\\
p_{(2;4)}(\mathcal{S}) &\equiv (\gamma_{11} + \gamma_{15} + \gamma_{19}) v_1 + \gamma_{13} v_2 + \gamma_{22} v_3 \equiv 0,\\
p_{(3;4)}(\mathcal{S}) &\equiv (\gamma_{14} + \gamma_{15} + \gamma_{16} + \gamma_{17}) v_1 + \gamma_{20} v_2 + \gamma_{23} v_3 \equiv 0.
\end{aligned}$$

Hence we get

$$\begin{cases} \gamma_j = 0,\ j = 10,13,20,21,22,23,\\ \gamma_{12} + \gamma_{16} + \gamma_{18} = \gamma_{11} + \gamma_{15} + \gamma_{19} = 0,\\ \gamma_{14} + \gamma_{15} + \gamma_{16} + \gamma_{17} = 0. \end{cases} \tag{5.1.3.4}$$

Substituting (5.1.3.4) into the relation (5.1.3.3) we get

$$\mathcal{S} = \gamma_4 d_4 + \gamma_5 d_5 + \gamma_{11} d_{11} + \gamma_{12} d_{12} + \sum_{14 \leqslant j \leqslant 19} \gamma_j d_j \equiv 0. \tag{5.1.3.5}$$

The homomorphisms $p_{(1;(2,3))}, p_{(1;(2,4))}, p_{(1;(3,4))} : P_4 \to P_3$, send (5.1.3.5) respectively to

$$\begin{aligned}
p_{(1;(2,3))}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{12} + \gamma_{16}) v_2 + \gamma_{18} v_3 \equiv 0,\\
p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_4 + \gamma_{11} + \gamma_{15}) v_1 + \gamma_{19} v_3 \equiv 0\\
p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_4 + \gamma_{14} + \gamma_{15}) v_1 + (\gamma_5 + \gamma_{16} + \gamma_{17}) v_2 \equiv 0.
\end{aligned}$$

From this we obtain

$$\begin{cases} \gamma_{18} = \gamma_{19} = \gamma_5 + \gamma_{12} + \gamma_{16} = 0,\\ \gamma_4 + \gamma_{11} + \gamma_{15} = \gamma_4 + \gamma_{14} + \gamma_{15} = \gamma_5 + \gamma_{16} + \gamma_{17} = 0. \end{cases} \tag{5.1.3.6}$$

Combining (5.1.3.2), (5.1.3.4) and (5.1.3.6), we obtain $\gamma_j = 0,\ j = 1, 2, \ldots, 23$. The proposition is proved. $\square$

5.2. **The case of degree $n = 2^{s+1} - 2$.**

It is well-known that, Kameko's homomorphism

$$\widetilde{Sq}^0_* : (QP_k)_{2m+k} \to (QP_k)_m$$

is an epimorphism. Hence we have

$$(QP_k)_{2m+k} \cong (QP_k)_m \oplus (QP_k^0)_{2m+k} \oplus (\mathrm{Ker}\widetilde{Sq}^0_* \cap (QP_k^+)_{2m+k}),$$

and $(QP_k)_m \cong \langle[\psi(B_k(m))]\rangle \subset (QP_k)_{2m+k}$.

For $k = 4$, from Theorem 4.3, it is easy to see that

$$\Phi(B_3(2)) = \Phi^0(B_3(2)) = \{x_ix_j \mid 1 \leqslant i < j \leqslant 4\}.$$

For $m = 2^s - 3,\ s \geqslant 2$, we have

$$|\Phi^0(B_3(6))| = 18,\ |\Phi^0(B_3(2^{s+1} - 2))| = 22,\ \text{for } s \geqslant 3,$$

$$|\psi(B_4(1))| = 4,\ \mathrm{Ker}\widetilde{Sq}^0_* \cap [B_4^+(6)] = \{[x_1x_2^2x_3x_4^2], [x_1x_2x_3^2x_4^2]\}.$$

Hence $\dim(QP_4)_2 = 6$, $\dim(QP_4)_6 = 24$.

The main result of this subsection is:

**Proposition 5.2.1.** *For any $s \geqslant 3$, $(QP_4^+)_{2^{s+1}-2} \cap \mathrm{Ker}\widetilde{Sq}^0_*$ is an $\mathbb{F}_2$-vector space of dimension 13 with a basis consisting of all the classes represented by the following admissible monomials:*

$d_1 = x_1x_2x_3^{2^s-2}x_4^{2^s-2}$, $d_2 = x_1x_2^2x_3^{2^s-4}x_4^{2^s-1}$, $d_3 = x_1x_2^2x_3^{2^s-3}x_4^{2^s-2}$,
$d_4 = x_1x_2^2x_3^{2^s-1}x_4^{2^s-4}$, $d_5 = x_1x_2^3x_3^{2^s-4}x_4^{2^s-2}$, $d_6 = x_1x_2^3x_3^{2^s-2}x_4^{2^s-4}$,
$d_7 = x_1x_2^{2^s-2}x_3x_4^{2^s-2}$, $d_8 = x_1x_2^{2^s-1}x_3^2x_4^{2^s-4}$, $d_9 = x_1^3x_2x_3^{2^s-4}x_4^{2^s-2}$,
$d_{10} = x_1^3x_2x_3^{2^s-2}x_4^{2^s-4}$, $d_{11} = x_1^3x_2^3x_3^4x_4^4,\ s = 3$, $d_{11} = x_1^3x_2^5x_3^{2^s-6}x_4^{2^s-4},\ s > 3$,
$d_{12} = x_1^3x_2^{2^s-3}x_3^2x_4^{2^s-4}$, $d_{13} = x_1^{2^s-1}x_2x_3^2x_4^{2^s-4}$,

The proof of this theorem is based on some lemmas.

**Lemma 5.2.2.** *If $x$ is an admissible monomial of degree $2^{s+1} - 2$ in $P_4$ and $[x] \in \mathrm{Ker}\widetilde{Sq}^0_*$, then $\omega(x) = (2^{(s)})$.*

*Proof.* We prove the lemma by induction on $s$. Obviously, the lemma holds for $s = 1$. Observe that $z = (x_1x_2)^{2^s-1}$ is the minimal spike of degree $2^{s+1} - 2$ in $P_4$ and $\omega(z) = (2^{(s)})$. Since $2^{s+1} - 2$ is even, using Theorem 2.12 and the fact that $[x] \in \mathrm{Ker}\widetilde{Sq}^0_*$, we obtain $\omega_1(x) = 2$. Hence $x = x_ix_jy^2$, where $y$ is a monomial of degree $2^s - 2$ and $1 \leqslant i < j \leqslant 4$. Since $x$ is admissible, by Theorem 2.9, $y$ is also admissible. Now, the lemma follows from the inductive hypothesis. $\square$

The following lemma is proved by a direct computation.

**Lemma 5.2.3.** *The following monomials are strictly inadmissible:*

i) $x_i^2x_jx_k^3,\ x_i^3x_j^4x_k^7,\ i < j, k \neq i, j,\ x_1^2x_2^2x_3x_4, x_1^2x_2x_3^2x_4, x_1^2x_2x_3x_4^2, x_1x_2^2x_3^2x_4$.

ii) $x_1x_2^6x_3^3x_4^4, x_1^3x_2^4x_3x_4^6, x_1^3x_2^4x_3^3x_4^4$.

iii) $x_1x_2^7x_3^{10}x_4^{12}, x_1^7x_2x_3^{10}x_4^{12}, x_1^3x_2^3x_3^{12}x_4^{12}, x_1^3x_2^5x_3^8x_4^{14}, x_1^3x_2^5x_3^{14}x_4^8, x_1^7x_2^7x_3^8x_4^8$.

*Proof of Proposition 5.2.1.* Let $x$ be an admissible monomial in $P_4$ and $[x] \in \mathrm{Ker}\widetilde{Sq}^0_*$. By Lemma 5.2.2, $\omega_i(x) = 2$, for $1 \leqslant i \leqslant s$. By induction on $s$, we see that if $x \neq d_i$, for $i = 1, 2, \ldots, 13$, then there is a monomial $w$, which is given in Lemma 5.2.3 such that $x = wy^{2^u}$ for some monomial $y$ and positive integer $u$. By Theorem 2.9, $x$ is inadmissible. Hence $\mathrm{Ker}\widetilde{Sq}^0_* \cap (QP_4^+)$ is spanned by the classes $[d_i]$ with $i = 1, 2, \ldots, 13$. Now, we prove that the classes $[d_i]$ with $i = 1, 2, \ldots, 13$, are linearly independent.

Suppose there is a linear relation

$$\sum_{1 \leqslant i \leqslant 13} \gamma_i d_i \equiv 0, \tag{5.2.3.1}$$

with $\gamma_i \in \mathbb{F}_2$.

According to Kameko [14], for $s \geqslant 3$, $B_3(n) \cap (P_3^+)_n$ is the set consisting of 4 monomials:

$$w_1 = x_1x_2^{2^s-2}x_3^{2^s-1}, \quad w_2 = x_1x_2^{2^s-1}x_3^{2^s-2},$$
$$w_3 = x_1^3x_2^{2^s-3}x_3^{2^s-2}, \quad w_4 = x_1^{2^s-1}x_2x_3^{2^s-2}.$$

Apply the homomorphisms $p_{(1;2)}, p_{(3;4)} : P_4 \to P_3$ to the relation (5.2.3.1) and we obtain

$$\gamma_2 w_1 + \gamma_4 w_2 + \gamma_3 w_3 + \gamma_7 w_4 \equiv 0.$$
$$\gamma_7 w_1 + \gamma_8 w_2 + \gamma_{12} w_3 + \gamma_{13} w_4 \equiv 0.$$

From these relations, we get $\gamma_i = 0,\ i = 2, 3, 4, 7, 8, 12, 13$. Then the relation 5.2.3.1 becomes

$$\gamma_1 d_1 + \gamma_5 d_5 + \gamma_6 d_6 + \gamma_9 d_9 + \gamma_{10} d_{10} + \gamma_{11} d_{11} \equiv 0. \tag{5.2.3.2}$$

Apply the homomorphisms $p_{(1;4)}, p_{(2;3)} : P_4 \to P_3$ to the relation (5.2.3.2) and we get

$$(\gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11})w_1 + \gamma_6 w_3 \equiv 0,$$
$$(\gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11})w_2 + \gamma_9 w_3 \equiv 0.$$

These equalities imply $\gamma_6 = \gamma_9 = \gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11} = 0$. Hence we obtain

$$\gamma_1 d_1 + \gamma_5 d_5 + \gamma_{10} d_{10} + \gamma_{11} d_{11} \equiv 0. \tag{5.2.3.3}$$

For $s > 3$, apply the homomorphisms $p_{(1;3)}, p_{(2;4)} : P_4 \to P_3$ to (5.2.3.3), we get

$$\gamma_1 w_2 + \gamma_5 w_3 \equiv 0,$$
$$\gamma_1 w_1 + \gamma_{10} w_3 \equiv 0.$$

From the above equalities, we get $\gamma_i = 0, i = 1, 2, \ldots, 13$.

For $s = 3$, apply the homomorphisms $p_{(1;3)}, p_{(2;4)} : P_4 \to P_3$ to (5.2.3.3), we get

$$(\gamma_1 + \gamma_{11})w_2 + \gamma_8 w_3 \equiv 0,$$
$$(\gamma_1 + \gamma_{11})w_1 + \gamma_{10} w_3 \equiv 0.$$

From the above equalities, we get $\gamma_i = 0, i = 2, \ldots, 10, 12, 13$ and $\gamma_1 = \gamma_{11}$. So the relation (5.2.3.3) becomes

$$\gamma_1(d_1 + d_{11}) \equiv 0.$$

Now, we prove that $[d_1 + d_{11}] \neq 0$. Suppose the contrary, that the polynomial $d_1 + d_{11} = x_1x_2x_3^6x_4^6 + x_1^3x_2^3x_3^4x_4^4$ is hit. Then by the unstable property of the action of $\mathcal{A}$ on the polynomial algebra, we have

$$x_1x_2x_3^6x_4^6 + x_1^3x_2^3x_3^4x_4^4 = Sq^1(A) + Sq^2(B) + Sq^4(C),$$

for some polynomials $A \in (P_4^+)_{13}, B \in (P_4^+)_{12}, C \in (P_4^+)_{10}$. Let $(Sq^2)^3$ acts on the both sides of the above equality. Since $(Sq^2)^3Sq^1 = 0$ and $(Sq^2)^3Sq^2 = 0$, we get

$$(Sq^2)^3(x_1x_2x_3^6x_4^6 + x_1^3x_2^3x_3^4x_4^4) = (Sq^2)^3Sq^4(C).$$

On the other hand, by a direct computation, it is not difficult to check that

$$(Sq^2)^3(x_1x_2x_3^6x_4^6 + x_1^3x_2^3x_3^4x_4^4) \neq (Sq^2)^3Sq^4(C),$$

for all $C \in (P_4^+)_{10}$. This is a contradiction. Hence $[d_1 + d_{11}] \neq 0$ and $\gamma_1 = \gamma_{11} = 0$. The proposition is proved. □

5.3. **The case of degree $n = 2^{s+1} - 1$.**

First, we determine the $\omega$-vector of an admissible monomial of degree $2^{s+1} - 1$ in $P_4$.

**Lemma 5.3.1.** *If $x$ is an admissible monomial of degree $2^{s+1} - 1$ in $P_4$ then either $\omega(x) = (1^{(s+1)})$ or $\omega(x) = (3, 2^{(s-1)})$ or $\omega(x) = (1, 3)$ for $s = 2$.*

*Proof.* Obviously, the lemma holds for $s = 1$. Suppose $s \geqslant 2$. By a direct computation we see that if $w$ is a monomial in $P_4$ such that $\omega(w) = (1, 3, 2)$ or $\omega(w) = (1, 1, 3)$, then $w$ is strictly inadmissible.

Since $2^{s+1} - 1$ is odd, we have either $\omega_1(x) = 1$ or $\omega_1(x) = 3$. If $\omega_1(x) = 1$, then $x = x_iy^2$, where $y$ is a monomial of degree $2^s - 1$. Hence either $\omega_1(y) = 1$ or $\omega_1(y) = 3$. So the lemma holds for $s = 2$. Suppose that $s \geqslant 3$. If $\omega_1(y) = 3$, then $y = X_iy_1^2$, where $y_1$ is a monomial of degree $2^{s-1} - 2$. Since $y_1$ is admissible, using Proposition 2.10, one gets $\omega_1(y_1) = 2$. Hence $x$ is inadmissible. If $\omega_1(y) = 1$, then $y = x_jy_1^2$, where $y_1$ is an admissible monomial of degree $2^{s-1} - 1$. By the inductive hypothesis $\omega(y_1) = (1^{(s-1)})$. So we get $\omega(x) = (1^{(s+1)})$.

Suppose that $\omega_1(x) = 3$. Then $x = X_iy^2$, where $y$ is an admissible monomial of degree $2^s - 2$. Since $x$ is admissible, by Lemma 5.2.3, $\omega(y) = (2^{(s-1)})$. The lemma is proved. □

For $s = 1$, we have $(QP_4)_3 = (QP_4^0)_3$. Hence $B_4(3) = \Phi^0(B_3(3))$. Using Proposition 4.1 and Theorem 4.3, we have

$$|\Phi^0(B_3(3))| = 14,\ |\Phi^0(B_3(7))| = 26,\ |\Phi^0(B_3(15))| = 38,$$
$$|\Phi^0(B_3(2^{s+1} - 1))| = 42,\ \text{for } s \geqslant 4.$$

For $s = 2$, $B_4(7) = B_4(1^{(3)}) \cup B_4(1, 3) \cup B_4(3, 2)$. By a direct computation, we have $B_4(1, 3) = \{x_1X_1^2\},\ B_4(3, 2) = \Phi(B_3(7))$.

Recall that

$$B_3(2^{s+1} - 1) = B_3(1^{(s+1)}) \cup \psi(\Phi(B_2(2^s - 2))),$$

where $B_2(2^s - 2) = \{x_1^{2^{s-1}-1}x_2^{2^{s-1}-1}\}$. Hence $B_3(3, 2^{(s-1)}) = \psi(\Phi(B_2(2^s - 2)))$.

**Proposition 5.3.2.** *For any $s \geqslant 3$, $B_4(3, 2^{(s-1)}) = (\Phi(B_3(3, 2^{(s-1)})) \cup A(s)$, where $A(s)$ is determined as follows:*

$$A(3) = \{x_1^3x_2^4x_3x_4^7, x_1^3x_2^4x_3^7x_4, x_1^3x_2^7x_3^4x_4, x_1^7x_2^3x_3^4x_4, x_1^3x_2^4x_3^3x_4^5\},$$
$$A(4) = \{x_1^3x_2^4x_3^{11}x_4^{13}, x_1^3x_2^7x_3^8x_4^{13}, x_1^7x_2^3x_3^8x_4^{13}, x_1^7x_2^7x_3^8x_4^9, x_1^7x_2^7x_3^9x_4^8, \},$$
$$A(s) = \{x_1^3x_2^4x_3^{2^s-5}x_4^{2^s-3}\}, s \geqslant 5.$$

Combining Lemma 5.3.1 and Propositions 4.1, 5.3.2, we have

$$B_4(2^{s+1}-1) = B_4(1^{(s+1)}) \cup \Phi(B_3(3, 2^{(s-1)})) \cup A(s).$$

The following can easily be proved by a direct computation.

**Lemma 5.3.3.** *The following monomials are strictly inadmissible:*

i) $x_i^2x_j, x_i^3x_j^4,\ 1 \leqslant i < j \leqslant 4$.
ii) $X_2x_1^2x_2^2,\ X_1x_1^2x_i^2,\ i = 2, 3, 4$.
iii) $x_i^3x_j^{12}x_kx_\ell^{15},\ x_i^3x_j^4x_k^9x_\ell^{15}, x_i^3x_j^5x_k^8x_\ell^{15},\ i < j < k,\ \ell \neq i,\ j,\ k$.
iv) $x_1^7x_2^{11}x_3^{12}x_4,\ x_1^3x_2^{12}x_3^3x_4^{13},\ X_jx_1^2x_2^4x_3^8x_4^8x_j^6,\ x_1^7x_2^{11}x_3^4x_4^8x_j,$
$x_1^3x_2^3x_3^{12}x_4^8x_i^4x_j,\ x_1^3x_2^3x_3^{24}x_4^{29}x_i^4,\ i = 1, 2,\ j = 3, 4$.

*Proof of Proposition 5.3.2.* By a direct computation using Lemma 5.3.1, Lemma 5.3.3 and Theorem 2.9 we see that if $x$ is a monomial of degree $2^{s+1}-1$ in $P_4$ and $x \notin \Phi(B_3(3, 2^{(s-1)})) \cup A(s)$, then there is a monomial $w$ which is given in Lemma 5.3.3 such that $x = wy^{2^u}$ for some monomial $y$ and integer $u > 1$. Hence $x$ is inadmissible.

Now we prove that the set $[B_4(3, 2^{(s-1)})]$ is linearly independent in $QP_4^+$. For $s = 3$, we have $|B_4(3, 2, 2)| = 36$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leqslant i \leqslant 36} \gamma_i d_i \equiv 0, \tag{5.3.3.1}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{15,i}$.

A simple computation, we see that $B_3(3; 2, 2) = \psi(\Phi(B_2(6)))$ is the set consisting of 6 monomials:

$$v_1 = x_1x_2^7x_3^7,\ v_2 = x_1^3x_2^5x_3^7, v_3 = x_1^3x_2^7x_3^5, v_4 = x_1^7x_2x_3^7, v_5 = x_1^7x_2^3x_3^5, v_6 = x_1^7x_2^7x_3.$$

By a direct computation, we have

$$\begin{aligned}
p_{(1;2)}(\mathcal{S}) &\equiv \gamma_3v_2 + \gamma_4v_3 + (\gamma_9 + \gamma_{22})v_4 + (\gamma_{10} + \gamma_{23})v_5 + (\gamma_{11} + \gamma_{24})v_6 \equiv 0,\\
p_{(1;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_{16})v_1 + \gamma_5v_2 + (\gamma_7 + \gamma_{20})v_3 + \gamma_{13}v_5 + (\gamma_{15} + \gamma_{30})v_6 \equiv 0,\\
p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_2 + \gamma_{19})v_1 + (\gamma_6 + \gamma_{21} + \gamma_{27})v_2 + \gamma_8v_3 + (\gamma_{12} + \gamma_{29})v_4 + \gamma_{14}v_5 \equiv 0,\\
p_{(2;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_3 + \gamma_5 + \gamma_9)v_1 + (\gamma_{16} + \gamma_{22})v_2\\
&\quad + (\gamma_{18} + \gamma_{20} + \gamma_{23} + \gamma_{26})v_3 + \gamma_{32}v_5 + (\gamma_{34} + \gamma_{36})v_6 \equiv 0,\\
p_{(2;4)}(\mathcal{S}) &\equiv (\gamma_2 + \gamma_4 + \gamma_8 + \gamma_{11})v_1\\
&\quad + (\gamma_{17} + \gamma_{21})v_2 + (\gamma_{19} + \gamma_{24}v_3 + \gamma_{31} + \gamma_{35})v_4 + \gamma_{33}v_5 \equiv 0,\\
p_{(3;4)}(\mathcal{S}) &\equiv (\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{15})v_1 + (\gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28})v_2\\
&\quad + (\gamma_{29} + \gamma_{30})v_3 + (\gamma_{31} + \gamma_{32} + \gamma_{33} + \gamma_{34})v_4 + (\gamma_{35} + \gamma_{36})v_5 \equiv 0.
\end{aligned}$$

From these equalities, we obtain

$$\begin{cases}\gamma_j = 0,\ j = 3, 4, 5, 8, 13, 14, 32, 33,\\ \gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22},\ \gamma_2 = \gamma_{11} = \gamma_{19} = \gamma_{24},\ \gamma_7 = \gamma_{20},\\ \gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22},\quad \gamma_{10} = \gamma_{23},\ \gamma_{17} = \gamma_{21},\\ \gamma_{12} = \gamma_{15} = \gamma_{29} = \gamma_{30},\quad \gamma_{31} = \gamma_{34} = \gamma_{35} = \gamma_{36},\\ \gamma_6 + \gamma_{21} + \gamma_{27} = \gamma_7 + \gamma_{10} + \gamma_{18} + \gamma_{26} = \gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28} = 0.\end{cases} \tag{5.3.3.2}$$

A direct computation using 5.3.3.2 and Theorem 2.12, we get

$$\begin{aligned} p_{(1;(2,3))}(\mathcal{S}) &\equiv \gamma_{18}w_3 + \gamma_{26}w_5 + \gamma_{28}w_6 \equiv 0,\\ p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_6 + \gamma_{10} + \gamma_{27})w_2 + \gamma_{25}w_4 + \gamma_{27}w_5 \equiv 0,\\ p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_{17} + \gamma_{18})w_1\\ &\quad + (\gamma_6 + \gamma_7 + \gamma_{17} + \gamma_{25} + \gamma_{26} + \gamma_{27})w_2 + (\gamma_{17} + \gamma_{28})w_3 \equiv 0.\end{aligned}$$

Combining the above equalities and (5.3.3.2), one gets $\gamma_j = 0$ for $j \neq 1$, 2 , 9, 11, 12, 15, 16, 19, 22, 24 , 29, 30, 31 and $\gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}$, $\gamma_2 = \gamma_{11} = \gamma_{19} = \gamma_{24}$, $\gamma_{12} = \gamma_{15} = \gamma_{29} = \gamma_{30}$, $\gamma_{31} = \gamma_{34} = \gamma_{35} = \gamma_{36}$. Hence the relation (5.3.3.1) becomes

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_3 + \gamma_{31}\theta_4 \equiv 0, \tag{5.3.3.3}$$

where

$$\begin{aligned}\theta_1 &= d_1 + d_9 + d_{16} + d_{22},\quad \theta_2 = d_2 + d_{11} + d_{19} + d_{24},\\ \theta_3 &= d_{12} + d_{15} + d_{29} + d_{30},\quad \theta_4 = d_{31} + d_{34} + d_{35} + d_{36}.\end{aligned}$$

Now, we prove that $\gamma_1 = \gamma_2 = \gamma_{12} = \gamma_{31} = 0$.

The proof is divided into 4 steps.

*Step 1.* Under the homomorphism $\varphi_1$, the image of (5.3.3.3) is

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_3 + \gamma_{31}(\theta_4 + \theta_3) \equiv 0. \tag{5.3.3.4}$$

Combining (5.3.3.3) and (5.3.3.4), we get

$$\gamma_{31}\theta_3 \equiv 0. \tag{5.3.3.5}$$

If the polynomial $\theta_3$ is hit, then we have

$$\theta_3 = Sq^1(A) + Sq^2(B) + Sq^4(C),$$

for some polynomials $A \in (P_4^+)_{14}, B \in (P_4^+)_{13}, C \in (P_4^+)_{11}$. Let $(Sq^2)^3$ act on the both sides of this equality. We get

$$(Sq^2)^3(\theta_3) = (Sq^2)^3Sq^4(C),$$

By a direct calculation, we see that the monomial $x = x_1^8x_2^7x_3^4x_4^2$ is a term of $(Sq^2)^3(\theta_3)$. If this monomial is a term of $(Sq^2)^3Sq^4(y)$ for a monomial $y \in (P_4^+)_{11}$, then $y = x_2^7f_2(z)$ with $z \in P_3$ and $\deg z = 4$. Using the Cartan formula, we see that $x$ is a term of $x_2^7(Sq^2)^3Sq^4(z) = x_2^7(Sq^2)^3(z^2) = 0$. Hence

$$(Sq^2)^3(\theta_3) \neq (Sq^2)^3Sq^4(C),$$

for all $C \in (P_4^+)_{11}$ and we have a contradiction. So $[\theta_3] \neq 0$ and $\gamma_{31} = 0$.

*Step 2.* Since $\gamma_{31} = 0$, the homomorphism $\varphi_2$ sends (5.3.3.3) to

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_4 \equiv 0. \tag{5.3.3.6}$$

Using the relation (5.3.3.6) and by the same argument as given in Step 1, we get $\gamma_{12} = 0$.

*Step 3.* Since $\gamma_{31} = \gamma_{12} = 0$, the homomorphism $\varphi_3$ sends (5.3.3.3) to

$$\gamma_1[\theta_1] + \gamma_2[\theta_3] = 0. \tag{5.3.3.7}$$

Using the relation (5.3.3.7) and by the same argument as given in Step 2, we obtain $\gamma_3 = 0$.

*Step 4.* Since $\gamma_{31} = \gamma_{12} = \gamma_2 = 0$, the homomorphism $\varphi_4$ sends (5.3.3.3) to

$$\gamma_1\theta_2 = 0.$$

Using this relation and by the same argument as given in Step 3, we obtain $\gamma_1 = 0$.

For $s \geqslant 4$, $B_3(3, 2^{(s-1)})) = \psi(\Phi(B_2(2^{s-1} - 2)))$ is the set consisting of 7 monomials:

$$v_1 = x_1x_2^{2^s-1}x_3^{2^s-1},\ v_2 = x_1^3x_2^{2^s-3}x_3^{2^s-1},\ v_3 = x_1^3x_2^{2^s-1}x_3^{2^s-3},\ v_4 = x_1^7x_2^{2^s-5}x_3^{2^s-3},$$
$$v_5 = x_1^{2^s-1}x_2x_3^{2^s-1},\ v_6 = x_1^{2^s-1}x_2^3x_3^{2^s-3},\ v_7 = x_1^{2^s-1}x_2^{2^s-1}x_3.$$

Suppose that $s = 4$. Then we have $|B_4((3, 2^{(3)}))| = 46$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1\leqslant j\leqslant 46}\gamma_jd_j = 0, \tag{5.3.3.8}$$

with $\gamma_j \in \mathbb{F}_2$ and $d_i = d_{31,i}$.

By a direct computation using Theorem 2.12, we have

$$\begin{aligned}
p_{(1;2)}(\mathcal{S}) &\equiv \gamma_3w_2 + \gamma_4w_3 + (\gamma_9 + \gamma_{25})w_4 + \gamma_{12}w_5 + \gamma_{13}w_6 + \gamma_{14}w_7 \equiv 0,\\
p_{(1;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_{19})w_1 + \gamma_5w_2 + (\gamma_7 + \gamma_{23} + \gamma_{37} + \gamma_{39})w_3\\
&\quad + (\gamma_{10} + \gamma_{28})w_4 + \gamma_{16}w_6 + \gamma_{18}w_7 \equiv 0,\\
p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_2 + \gamma_{22})w_1 + (\gamma_6 + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{40})w_2\\
&\quad + \gamma_8w_3 + \gamma_{11}w_4 + (\gamma_{15} + \gamma_{34})w_5 + \gamma_{17}w_6 \equiv 0.
\end{aligned}$$

From these equalities, we get

$$\begin{cases}\gamma_j = 0,\ j = 3, 4, 5, 8, 11, 12, 13, 14, 16, 17, 18,\\
\gamma_9 = \gamma_{25},\ \gamma_1 = \gamma_{19},\ \gamma_7 + \gamma_{23} + \gamma_{37} + \gamma_{39} = 0,\ \gamma_{10} = \gamma_{28},\\
\gamma_2 = \gamma_{22},\ \gamma_6 + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{40} = 0,\ \gamma_{15} + \gamma_{34} = 0.\end{cases} \tag{5.3.3.9}$$

Using the relations (5.3.3.9), and Theorem 2.12, we obtain

$$\begin{aligned}
p_{(2;3)}(\mathcal{S}) &\equiv \gamma_1w_1 + \gamma_1w_2 + (\gamma_9 + \gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{26} + \gamma_{31} + \gamma_{39})w_3\\
&\quad + (\gamma_{35} + \gamma_{37})w_4 + \gamma_{43}w_6 \equiv 0,\\
p_{(2;4)}(\mathcal{S}) &\equiv \gamma_2w_1 + \gamma_{45}w_7 + (\gamma_{20} + \gamma_{24} + \gamma_{38} + \gamma_{40})w_2 + \gamma_2w_3 + \gamma_{36}w_4\\
&\quad + (\gamma_{42} + \gamma_{46})w_5 + \gamma_{44}w_6 \equiv 0,\\
p_{(3;4)}(\mathcal{S}) &\equiv \gamma_{15}w_1 + (\gamma_{30} + \gamma_{31} + \gamma_{32} + \gamma_{33})w_2\\
&\quad + \gamma_{15}w_3 + \gamma_{41}w_4 + (\gamma_{42} + \gamma_{43} + \gamma_{44} + \gamma_{45})w_5 + \gamma_{42}w_6 \equiv 0.
\end{aligned}$$

From these equalities, we get

$$\begin{cases}\gamma_j = 0,\ j = 1, 2, 15, 36, 41, 42, 43, 44, 45, 46,\\
\gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{26} + \gamma_{31} + \gamma_{39} = 0,\\
\gamma_{35} = \gamma_{37},\ \gamma_{20} + \gamma_{24} + \gamma_{38} + \gamma_{40} = 0,\\
\gamma_{30} + \gamma_{31} + \gamma_{32} + \gamma_{33} = 0.\end{cases} \tag{5.3.3.10}$$

By a direct computation using (5.3.3.9), (5.3.3.10) and Theorem 2.12, we have

$$\begin{aligned}
p_{(1;(2,3))}(\mathcal{S}) &\equiv (\gamma_7 + \gamma_{21} + \gamma_{23} + \gamma_{39})w_3 + \gamma_{26}w_4 + \gamma_{31}w_6 + \gamma_{33}w_7 \equiv 0,\\
p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_6 + \gamma_9 + \gamma_{20} + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{38} + \gamma_{40})w_2\\
&\quad + \gamma_{27}w_4 + \gamma_{30}w_5 + \gamma_{32}w_6 \equiv 0,\\
p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_6 + \gamma_{10} + \gamma_{23} + \gamma_{24} + \gamma_{26} + \gamma_{27} + \gamma_{29} + \gamma_{30} + \gamma_{31} + \gamma_{32})w_2\\
&\quad + (\gamma_7 + \gamma_{23} + \gamma_{24} + \gamma_{33} + \gamma_{35} + \gamma_{38} + \gamma_{39} + \gamma_{40})w_3\\
&\quad + (\gamma_{20} + \gamma_{21} + \gamma_{35})w_1 + \gamma_{29}w_4 \equiv 0,\\
p_{(2;(3,4))}(\mathcal{S}) &\equiv (\gamma_{10} + \gamma_{20} + \gamma_{23} + \gamma_{24} + \gamma_{29} + \gamma_{30} + \gamma_{35} + \gamma_{38} + \gamma_{39} + \gamma_{40})w_2\\
&\quad + (\gamma_9 + \gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{24} + \gamma_{26} + \gamma_{27} + \gamma_{29} + \gamma_{31} + \gamma_{32})w_3\\
&\quad + (\gamma_6 + \gamma_7 + \gamma_9 + \gamma_{10})w_1 + \gamma_{38}w_4 \equiv 0.
\end{aligned}$$

Combining the above equalities, 5.3.3.9 and 5.3.3.10, we get

$$\begin{cases}\gamma_j = 0,\ j \ne 7, 10, 21, 23, 24, 28, 35, 37, 39, 40,\\ \gamma_7 = \gamma_{10} = \gamma_{28},\ \gamma_{21} = \gamma_{35} = \gamma_{37},\\ \gamma_7 + \gamma_{21} + \gamma_{23} + \gamma_{39} = 0.\end{cases} \tag{5.3.3.11}$$

Hence we obtain

$$\gamma_7\theta_1 + \gamma_{21}\theta_2 + \gamma_{39}\theta_3 + \gamma_{24}\theta_4 \equiv 0, \tag{5.3.3.12}$$

where

$$\begin{aligned}
\theta_1 &= d_7 + d_{10} + d_{23} + d_{28},\\
\theta_2 &= d_{21} + d_{23} + d_{35} + d_{37},\\
\theta_3 &= d_{23} + d_{39}, \quad \theta_4 = d_{24} + d_{40}.
\end{aligned}$$

Now, we prove $\gamma_7 = \gamma_{21} = \gamma_{24} = \gamma_{39} = 0$. The proof is divided into 4 steps.

*Step 1.* The homomorphism $\varphi_1$ sends (5.3.3.12) to

$$\gamma_7\theta_1 + \gamma_{21}(\theta_2 + \theta_1) + \gamma_{24}\theta_3 + \gamma_{39}\theta_4 \equiv 0. \tag{5.3.3.13}$$

Combining (5.3.3.12) and (5.3.3.13) gives

$$\gamma_{25}\theta_1 \equiv 0. \tag{5.3.3.14}$$

By an analogous argument as given in the proof of the proposition for the case $s = 3$, $[\theta_1] \ne 0$. So we get $\gamma_{21} = 0$.

*Step 2.* Applying the homomorphism $\varphi_2$ to (5.3.3.8), we obtain

$$\gamma_7\theta_2 + \gamma_{24}\theta_3 + \gamma_{39}\theta_4 = 0. \tag{5.3.3.15}$$

Using (5.3.3.15) and by a same argument as given in Step 1, we get $\gamma_7 = 0$.

*Step 3.* Under the homomorphism $\varphi_3$, the image of (5.3.3.8) is

$$\gamma_{24}[\theta_2] + \gamma_{39}[\theta_4] = 0. \tag{5.3.3.16}$$

Using (5.3.3.16) and by a same argument as given in Step 3, we obtain $\gamma_{24} = 0$.

*Step 4.* Since $\gamma_7 = \gamma_{22} = \gamma_{24} = 0$, the homomorphism $\varphi_3$ sends (5.3.3.8) to

$$\gamma_{39}[\theta_3] = 0.$$

From this equality and by a same argument as given in Step 3, we get $\gamma_{39} = 0$.

For $s \geqslant 5$, $|B_4(3, 2^{(s-1)})| = 43$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{1 \leqslant j \leqslant 43} \gamma_j d_j \equiv 0, \qquad (5.3.3.17)$$

with $\gamma_j \in \mathbb{F}_2$.

Using the relations $p_{(j;J)}(\mathcal{S}) \equiv 0$, for $(j;J) \in \mathcal{N}_4$ and the admissible monomials $v_i, i = 1, 2, \ldots, 7$, we obtain $\gamma_j = 0$ for any $j$. The proposition is proved. □

5.4. **The case of degree** $2^{s+t+1} + 2^{s+1} - 3$**.**

First of all, we determine the $\omega$-vector of an admissible monomial of degree $n = 2^{s+t+1} + 2^{s+1} - 3$ for any positive integers $s, t$.

**Lemma 5.4.1.** *Let $x$ be a monomial of degree $2^{s+t+1} + 2^{s+1} - 3$ in $P_4$ with $s, t$ are the positive integers. If $x$ is admissible, then either $\omega(x) = (3^{(s)}, 1^{(t+1)})$ or $\omega(x) = (3^{(s+1)}, 2^{(t-1)})$.*

*Proof.* Observe that the monomial $z = x_1^{2^{s+t+1}-1} x_2^{2^s-1} x_3^{2^s-1}$ is the minimal spike of degree $2^{s+t+1} + 2^{s+1} - 3$ in $P_4$ and $\omega(z) = (3^{(s)}, 1^{(t+1)})$. Since $x$ is admissible and $2^{s+t+1} + 2^{s+1} - 3$ is odd, using Theorem 2.12, we obtain $\omega_1(x) = 3$. Using Theorem 2.12 and Proposition 2.10, we get $\omega_i(x) = 3$ for $i = 1, 2, \ldots, s$.

Let $x' = \prod_{i \geqslant 1} X_{I_{i+s-1}(x)}^{2^{i-1}}$. Then $\omega_i(x') = \omega_{i+s}(x), i \geqslant 1$ and $\deg(x') = 2^{t+1} - 1$. Since $x$ is admissible, using Theorem 2.9, we see that $x'$ is also admissible. By Lemmas 5.3.1, either $\omega(x') = (1^{(t+1)})$ or $\omega(x') = (3, 2^{(t-1)})$ or $\omega(x') = (1, 3)$ for $t = 2$. By a direct computation we see that if $\omega(x') = (1, 3)$, then $x$ is inadmissible. So, the lemma is proved. □

Using Theorem 1.3, we easily obtain the following.

**Proposition 5.4.2.** *For any positive integers $s, t$ with $s \geqslant 3$, $\Phi(B_3(n))$ is a minimal set of generators for $\mathcal{A}$-module $P_4$ in degree $n = 2^{s+t+1} + 2^{s+1} - 3$.*

Hence it suffices to consider the cases $s = 1$ and $s = 2$.

5.4.1. **The subcase** $s = 1$**.**

For $s = 1$, $n = 2^{t+2} + 1 = (2^{t+2} - 1) + (2 - 1) + (2 - 1)$. Hence $\mu(2^{t+2} + 1) = 3$ and Kameko's homomorphism

$$\widetilde{Sq}^0_* : (QP_3)_{2^{t+2}+1} \to (QP_3)_{2^{t+1}-1}$$

is an isomorphism. So, we get

$$B_3(n) = \psi(B_3(2^{t+1} - 1)) = \psi(B_3(1^{(t+1)})) \cup \psi(B_3(3, 2^{(t-1)})).$$

**Proposition 5.4.3.** *For any positive integer $t$, $\Phi(B_3(n)) \cup B(t)$ is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n = 2^{t+2} + 1$, where the set $B(t)$ is determined as follows:*

$$B(1) = \{x_1^3x_2^4x_3x_4\}, \quad B(2) = \{x_1^3x_2^5x_3^8x_4\},$$
$$B(3) = \{x_1^3x_2^7x_3^{11}x_4^{12}, x_1^7x_2^3x_3^{11}x_4^{12}, x_1^7x_2^{11}x_3^3x_4^{12}, x_1^7x_2^7x_3^8x_4^{11}, x_1^7x_2^7x_3^{11}x_4^8\},$$
$$B(t) = \{x_1^3x_2^7x_3^{2^{t+1}-5}x_4^{2^{t+1}-4}, x_1^7x_2^3x_3^{2^{t+1}-5}x_4^{2^{t+1}-4},$$
$$x_1^7x_2^{2^{t+1}-5}x_3^3x_4^{2^{t+1}-4}, x_1^7x_2^7x_3^{2^{t+1}-8}x_4^{2^{t+1}-5}\}, \text{ for } t > 3.$$

The following lemma is proved by a direct computation.

**Lemma 5.4.4.** *The following monomials are strictly inadmissible:*

i) $X_2x_1^2x_2^{12},\ X_3^3x_3^4x_i^4,\ i=1,2,\ X_jx_1^2x_2^4x_j^8,\ X_2^3x_2^4x_j^4,\ j=3,4.$
ii) $X_3x_1^2x_2^4x_3^{24},\ X_3x_1^2x_2^4x_j^8x_4^{16}, j=3,4.$
iii) $X_3X_2^2x_1^4x_2^8x_4^{12},\ X_4X_2^2x_1^4x_2^8x_3^{12},\ X_4X_3^2x_1^4x_2^{12}x_3^8,\ X_4X_3^2x_1^{12}x_2^4x_3^8$
iv) $X_j^3x_i^4x_j^8x_m^{12},\ 1\leqslant i<j\leqslant 4,\ m\neq i,j.$
v) $X_jX_2^2x_1^4x_3^4x_2^8x_4^8, j=3,4,\ X_j^32x_1^4x_3^4x_2^8x_4^8,\ j=2,4.$
vi) $X_3^3x_1^4x_2^4x_3^{24}x_4^{24},\ X_3^3x_1^4x_2^4x_i^8x_4^8x_3^{16}x_4^{16},\ X_4X_2^2x_1^4x_2^4x_i^8x_4^8x_3^{16}x_4^{16},\ i=1,2,$
$X_j^3x_1^{12}x_2^{12}x_3^{16}x_4^{16},\ j=3,4,\ X_4X_3^2x_1^{12}x_2^{12}x_3^{16}x_4^{16}.$

*Proof of Proposition 5.4.2.* Let $x$ be an admissible monomial of degree $n=2^{t+2}+1$. According to Lemma 5.4.1, $x=X_iy^2$ with $y$ a monomial of degree $2^{t+1}-1$. Since $x$ is admissible, by Theorem 2.12, $y$ is admissible. By a direct computation, we see that if $y\in B_4(2^{t+1}-1)$ and $X_iy^2\notin\Phi(B_3(n))\cup B(t)$, then there is a monomial $w$ which is given in one of Lemma 5.1.3, 5.3.3, 5.4.4 such that $X_iy^2=wz^{2^u}$ with some positive integer $u$ and monomial $z$. By Theorem 2.9, $x$ is inadmissible.

For $t=1$, we have $|C_4^+(9)|=18$. Suppose there is a linear relation

$$\mathcal{S}=\sum_{i=1}^{18}\gamma_id_i\equiv 0, \tag{5.4.4.1}$$

with $\gamma_i\in\mathbb{F}_2$. A direct computation from the relations $p_{(r;j)}(\mathcal{S})\equiv 0$, for $1\leqslant r<j\leqslant 4$, we obtain $\gamma_i=0$ for $i\neq 1,4,9,10,11,12$ and $\gamma_1=\gamma_2=\gamma_3=\gamma_{10}=\gamma_{11}=\gamma_{12}$. Hence the relation (5.4.4.1) becomes $\gamma_1\theta\equiv 0$ where $\theta=d_1+d_4+d_9+d_{10}+d_{11}+d_{12}$.

We prove $\gamma_1=0$. Suppose $\theta$ is hit. Then we get

$$\theta=Sq^1(A)+Sq^2(B)+Sq^4(C),$$

for some polynomials $A\in(P_4^+)_8, B\in(P_4^+)_7, C\in(P_4^+)_5$. Let $(Sq^2)^3$ act on the both sides of this equality. It is easy to check that $(Sq^2)^3Sq^4(C)=0$ for all $C\in(P_4^+)_5$. Since $(Sq^2)^3$ annihilates $Sq^1$ and $Sq^2$, the right hand side is sent to zero. On the other hand, a direct computation shows

$$(Sq^2)^3(\theta)=(1,2,4,8)+\text{symmetries}\neq 0.$$

Hence we have a contradiction. So we obtain $\gamma_1=0$.

For $t=2$, $|B_4^+(17)|=47$. Suppose there is a linear relation

$$\mathcal{S}=\sum_{i=1}^{47}\gamma_id_i\equiv 0, \tag{5.4.4.2}$$

with $\gamma_i\in\mathbb{F}_2$ and $d_i=d_{17,i}$. A direct computation from the relations $p_{(j;J)}(\mathcal{S})\equiv 0$, for $(j;J)\in\mathcal{N}_4$, we obtain $\gamma_i=0$ for $i\neq 1,4,8,9,10,11,17,18$ and $\gamma_1=\gamma_2=\gamma_8=\gamma_9=\gamma_{10}=\gamma_{11}=\gamma_{17}=\gamma_{18}$. Hence the relation (5.4.4.2) becomes $\gamma_1\theta\equiv 0$ where $\theta=d_1+d_4+d_8+d_{11}+d_{13}+d_{16}+d_{17}+d_{18}$.

By a same argument as given in the proof of the proposition for $t=1$, we see that $[\theta]\neq 0$. Hence $\gamma_1=0$.

For $t=3$, we have $|B_4^+(33)|=84$, and $|B_4^+(2^{t+2}+1)|=94$ for $t\geqslant 4$. Suppose there is a linear relation

$$\mathcal{S}=\sum_{i=1}^{84}\gamma_id_i\equiv 0, \tag{5.4.4.3}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{33,i}$. A direct computation from the relations $p_{(j;J)}(\mathcal{S}) \equiv 0$, for $(j;J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i \notin E$ with $E = \{1, 3, 8, 9, 13, 14, 17, 24, 25, 42, 43, 59, 60, 65, 66, 67\}$ and $\gamma_i = \gamma_1$ for all $i \in E$. Hence the relation 5.4.4.3 become $\gamma_1\theta \equiv 0$ with $\theta = \sum_{i\in E} d_i$.

By a same argument as given in the proof of the proposition for $t = 1$, we see that $[\theta] \neq 0$. Therefore $\gamma_1 = 0$.

Now, we prove the set $B_4^+(n)$ is linearly independent for $t > 3$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{94} \gamma_i d_i \equiv 0, \tag{5.4.4.4}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n,i}$. A direct computation from the relations $p_{(j;J)}(\mathcal{S}) \equiv 0$, for $(j;J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i$. $\square$

5.4.2. **The subcase $s = 2$.**

For $s = 2$, we have $n = 2^{t+3}+5$. According to Theorem 1.2, the iterated Kameko homomorphism

$$(\widetilde{Sq}^0_*)^2 : (QP_3)_{2^{t+3}+5} \to (QP_3)_{2^{t+1}-1}$$

is an isomorphism. So we get

$$B_3(n) = \psi^2(B_3(2^{t+1}-1)) = \psi^2(B_3(1^{(t+1)})) \cup \psi^2(\Phi(B_3(3, 2^{(t-1)})).$$

**Proposition 5.4.5.**
i) $B_4(n) = \Phi(B_3(21)) \cup \{x_1^7x_2^9x_3^2x_4^3,\ x_1^7x_2^9x_3^3x_4^2,\ x_1^3x_2^7x_3^8x_4^3,\ x_1^7x_2^3x_3^8x_4^3\}$ *is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree* 21.
ii) *For any integer $t > 1$, $\Phi(B_3(n))$ is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n = 2^{t+3} + 5$.*

The following lemma is proved by a direct computation.

**Lemma 5.4.6.** *The following monomials are strictly inadmissible:*

i) $X_2^3x_3^4,\ X_i^4X_j^3,\ 1 \leqslant i < j \leqslant 4,\ X_2^3x_1^4x_2^8$.
ii) $X_3^3x_i^4x_3^{24},\ X_3^3x_i^4x_3^8x_4^{16},\ X_4^3x_i^4x_3^8x_4^{16},\ X_4^7x_i^8x_4^8, i = 1, 2$.
iii) $x_1^7x_2^{11}x_3^{17}x_4^2,\ X_j^3x_2^8x_j^{16}, X_j^7x_3^8x_4^8, j = 3, 4$
iv) $x_1^{15}x_2^{15}x_3^{16}x_4^{23},\ x_1^{15}x_2^{15}x_3^{23}x_4^{16},\ x_1^{15}x_2^{15}x_3^{17}x_4^{22}$.

*Proof of Proposition 5.4.5.* Let $x$ be an admissible monomial of degree $n = 2^{t+3}+5$. According to Lemma 5.4.1, $x = X_iy^2$ with $y$ a monomial of degree $2^{t+2} + 1$. Since $x$ is admissible, by Theorem 2.12, $y$ is admissible.

By a direct computation, we see that if $y \in B_4(2^{t+2} + 1)$ and $X_iy^2$ is not in the set given in Proposition 5.4.5, then there is a monomial $w$ which is given in one of Lemmas 5.1.3, 5.3.3, 5.4.6 such that $X_iy^2 = wz^{2^u}$ with some positive integer $u$ and monomial $z$.

By Theorem 2.9, $x$ is inadmissible. Hence $QP_4(n)$ is generated by the set given in the proposition.

For $t = 1$, we have $|B_4^+(21)| = 66$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{66} \gamma_i d_i \equiv 0, \tag{5.4.6.1}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{21,i}$.

By a simple computation, we see that $B_3(21)$ is the set consisting of 7 monomials:

$$v_1 = x_1^3x_2^3x_3^{15},\ v_2 = x_1^3x_2^7x_3^{11},\ v_3 = x_1^3x_2^{15}x_3^3,\ v_4 = x_1^7x_2^3x_3^{11},$$
$$v_5 = x_1^7x_2^{11}x_3^3,\ v_6 = x_1^{15}x_2^3x_3^3,\ v_7 = x_1^7x_2^7x_3^7.$$

A direct computation, we have

$$\begin{aligned}
p_{(1;2)}(\mathcal{S}) &\equiv \gamma_1v_1 + \gamma_2v_2 + \gamma_3v_3 + \gamma_{10}v_4 + \gamma_{11}v_5 + \gamma_{16}v_6 + \gamma_{57}v_7 \equiv 0,\\
p_{(1;3)}(\mathcal{S}) &\equiv \gamma_4v_1 + \gamma_6 + \gamma_{27}v_2 + (\gamma_8 + \gamma_{30} + \gamma_{49})v_3 + \gamma_{12}v_4\\
&\quad + (\gamma_{14} + \gamma_{38}v_5 + \gamma_{17})v_6 + \gamma_{58}v_7 \equiv 0,\\
p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{26} + \gamma_{48})v_1 + (\gamma_7 + \gamma_{29}v_2 + \gamma_9)v_3 + (\gamma_{13} + \gamma_{37})v_4\\
&\quad + \gamma_{15}v_5 + \gamma_{18}v_6 + \gamma_{59}v_7 \equiv 0,\\
p_{(2;3)}(\mathcal{S}) &\equiv \gamma_{19}v_1 + (\gamma_{21} + \gamma_{27} + \gamma_{32} + \gamma_{60})v_2 + (\gamma_{23} + \gamma_{30} + \gamma_{34} + \gamma_{38} + \gamma_{40})v_3\\
&\quad + \gamma_{43}v_4 + (\gamma_{45} + \gamma_{49} + \gamma_{51})v_5 + \gamma_{54}v_6 + \gamma_{63}v_7 \equiv 0,\\
p_{(2;4)}(\mathcal{S}) &\equiv (\gamma_{20} + \gamma_{26} + \gamma_{33} + \gamma_{37} + \gamma_{41})v_1 + \gamma_{22} + \gamma_{29} + \gamma_{35} + \gamma_{61})v_2\\
&\quad + \gamma_{24}v_3 + (\gamma_{44} + \gamma_{48} + \gamma_{52})v_4 + \gamma_{46}v_5 + \gamma_{55}v_6 + \gamma_{64}v_7 \equiv 0,\\
p_{(3;4)}(\mathcal{S}) &\equiv (\gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28} + \gamma_{29} + \gamma_{30} + \gamma_{31})v_1\\
&\quad + (\gamma_{36} + \gamma_{37} + \gamma_{38} + \gamma_{39} + \gamma_{62})v_2 + \gamma_{42}v_3\\
&\quad + (\gamma_{47} + \gamma_{48} + \gamma_{49} + \gamma_{50} + \gamma_{65})v_4 + \gamma_{53}v_5 + \gamma_{56}v_6 + \gamma_{66}v_7 \equiv 0.
\end{aligned}$$

From the above equalities, we get $\gamma_i = 0$, for $i = 1, 2, 3, 4, 9, 10, 11, 12, 15, 16, 17, 18, 19, 24, 42, 43, 46, 53, 54, 55, 56, 57, 58, 59, 63, 64, 66$ and $\gamma_6 = \gamma_{27}.\ \gamma_8+\gamma_{30}+\gamma_{49} = 0,\ \gamma_{14} = \gamma_{38},\ \gamma_5+\gamma_{26}+\gamma_{48} = 0,\ \gamma_7 = \gamma_{29},\ \gamma_{13} = \gamma_{37},\ \gamma_6+\gamma_{21}+\gamma_{32}+\gamma_{60} = 0,\ \gamma_{14}+\gamma_{23}+\gamma_{30}+\gamma_{34}+\gamma_{40}\gamma_{45}+\gamma_{49}+\gamma_{51} = 0,\ \gamma_{20}+\gamma_{26}+\gamma_{33}+\gamma_{37}+\gamma_{41} = 0,\ \gamma_7+\gamma_{22}+\gamma_{35}+\gamma_{61} = 0,\ \gamma_{44}+\gamma_{48}+\gamma_{52} = 0,\ \gamma_6+\gamma_7+\gamma_{25}+\gamma_{26}+\gamma_{28}+\gamma_{30}+\gamma_{31} = 0,\ \gamma_{14}+\gamma_{36}+\gamma_{37}+\gamma_{39}+\gamma_{62} = 0,\ \gamma_{47}+\gamma_{48}+\gamma_{49}+\gamma_{50}+\gamma_{65} = 0.$

With the aid of the above equalities have

$$\begin{aligned}
p_{(1;(2,3))}(\mathcal{S}) &\equiv \gamma_{21}v_2 + (\gamma_8 + \gamma_{23} + \gamma_{30} + \gamma_{45} + \gamma_{49})v_3 + \gamma_{32}v_4\\
&\quad + (\gamma_{34} + \gamma_{45} + \gamma_{49} + \gamma_{51})v_5 + (\gamma_{40} + \gamma_{51})v_6 + \gamma_{60}v_7 \equiv 0,\\
p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{20} + \gamma_{26} + \gamma_{44} + \gamma_{48})v_1 + \gamma_{22}v_2\\
&\quad + (\gamma_{33} + \gamma_{44} + \gamma_{48} + \gamma_{52})v_4 + \gamma_{35}v_5 + (\gamma_{41} + \gamma_{52})v_6 + \gamma_{61}v_7 \equiv 0.
\end{aligned}$$

From this, we obtain $\gamma_i = 0$, for $i = 21, 22, 32, 35, 60, 61$ and $\gamma_8+\gamma_{23}+\gamma_{30}+\gamma_{45}+\gamma_{49} = 0,\ \gamma_{34}+\gamma_{45}+\gamma_{49}+\gamma_{51} = 0,\ \gamma_{40} = \gamma_{51},\ \gamma_5+\gamma_{20}+\gamma_{26}+\gamma_{44}+\gamma_{48} = 0,\ \gamma_{33}+\gamma_{44}+\gamma_{48}+\gamma_{52} = 0,\ \gamma_{41} = \gamma_{52}$. By a direct computation using the above equalities, one gets

$$\begin{aligned}
p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{25} + \gamma_{26} + \gamma_{47} + \gamma_{48})v_1 + (\gamma_{28} + \gamma_{47} + \gamma_{48} + \gamma_{49} + \gamma_{50})v_2\\
&\quad + (\gamma_8 + \gamma_{30} + \gamma_{31} + \gamma_{49} + \gamma_{50})v_3 + \gamma_{36}v_4 + \gamma_{39}v_5 + \gamma_{62}v_7 \equiv 0,\\
p_{(2;(3,4))}(\mathcal{S}) &\equiv (\gamma_{13} + \gamma_{20} + \gamma_{25} + \gamma_{26} + \gamma_{33} + \gamma_{36} + \gamma_{40} + \gamma_{41})v_1 + (\gamma_6 + \gamma_7\\
&\quad + \gamma_{13} + \gamma_{14} + \gamma_{28} + \gamma_{33} + \gamma_{34} + \gamma_{36} + \gamma_{39})v_2 + (\gamma_{14} + \gamma_{23} + \gamma_{30} + \gamma_{31}\\
&\quad + \gamma_{34} + \gamma_{39} + \gamma_{40} + \gamma_{41})v_3 + (\gamma_{44} + \gamma_{47} + \gamma_{48} + \gamma_{51} + \gamma_{52})v_4\\
&\quad + (\gamma_{45} + \gamma_{49} + \gamma_{50} + \gamma_{51} + \gamma_{52})v_5 + \gamma_{65}v_7 \equiv 0.
\end{aligned}$$

So we obtain $\gamma_{36} = \gamma_{39} = \gamma_{62} = \gamma_{65} = 0,\ \gamma_5+\gamma_{25}+\gamma_{26}+\gamma_{47}+\gamma_{48} = 0,\ \gamma_{28}+\gamma_{47}+\gamma_{48}+\gamma_{49}+\gamma_{50} = 0,\ \gamma_8+\gamma_{30}+\gamma_{31}+\gamma_{49}+\gamma_{50} = 0,\ \gamma_{13}+\gamma_{20}+\gamma_{25}+\gamma_{26}+\gamma_{33}+\gamma_{40}+\gamma_{41} =$

$0,\ \gamma_6+\gamma_7+\gamma_{13}+\gamma_{14}+\gamma_{28}+\gamma_{33}+\gamma_{34}=0,\ \gamma_{14}+\gamma_{23}+\gamma_{30}+\gamma_{31}+\gamma_{34}+\gamma_{40}+\gamma_{41}=0,\ \gamma_{44}+\gamma_{47}+\gamma_{48}+\gamma_{51}+\gamma_{52}=0,\ \gamma_{45}+\gamma_{49}+\gamma_{50}+\gamma_{51}+\gamma_{52}=0.$

Combining the above equalities, one gets $\gamma_i=0$ for $i\neq 5$, 8, 13, 14, 20, 23, 25, 26, 30, 31, 37, 38, 40, 41, 44, 45, 47, 48, 49, 50, 51, $\gamma_i=\gamma_5$ for $i=8,13,14,37,38$, $\gamma_i=\gamma_{20}$ for $i=23,44,45$, $\gamma_i=\gamma_{25}$ for $i=40,47,51$, $\gamma_i=\gamma_{31}$ for $i=41,50,52$, $\gamma_{20}+\gamma_{25}+\gamma_{49}=0$, $\gamma_5+\gamma_{20}+\gamma_{26}+\gamma_{31}=0$, $\gamma_{20}+\gamma_{31}+\gamma_{48}=0$, $\gamma_5+\gamma_{20}+\gamma_{25}+\gamma_{30}=0$.

Substituting the above equalities into the relation (5.4.6.1), we have

$$\gamma_{25}[\theta_1]+\gamma_{31}[\theta_2]+\gamma_5[\theta_3]+\gamma_{20}[\theta_4]=0, \tag{5.4.6.2}$$

where

$$\begin{aligned}
\theta_1 &= d_{25}+d_{30}+d_{40}+d_{47}+d_{49}+d_{51},\\
\theta_2 &= d_{26}+d_{31}+d_{41}+d_{48}+d_{50}+d_{52},\\
\theta_3 &= d_5+d_8+d_{13}+d_{14}+d_{26}+d_{30}+d_{37}+d_{38},\\
\theta_4 &= d_{20}+d_{23}+d_{26}+d_{30}+d_{44}+d_{45}+d_{48}+d_{49}.
\end{aligned}$$

We need to show that $\gamma_5=\gamma_{20}=\gamma_{25}=\gamma_{31}=0$. The proof is divided into 4 steps.

*Step 1.* The homomorphism $\varphi_1$ sends (5.4.6.2) to

$$\gamma_{25}[\theta_1]+\gamma_{31}[\theta_2]+(\gamma_5+\gamma_{20})[\theta_3]+\gamma_{20}[\theta_4]=0. \tag{5.4.6.3}$$

Combining (5.4.6.2) and (5.4.6.3) gives

$$\gamma_{20}[\theta_3]=0.$$

We prove $[\theta_3]\neq 0$. We have $\varphi_2\varphi_3([\theta_1])=[\theta_3]$. So we need only to prove that $[\theta_1]\neq 0$. Suppose $[\theta_1]=0$. Then the polynomial $\theta_1$ is hit and we have

$$\theta_1 = Sq^1(A)+Sq^2(B)+Sq^4(C)+Sq^8(D),$$

for some polynomials $A\in(P_4^+)_{20}, B\in(P_4^+)_{19}, C\in(P_4^+)_{17}, D\in(P_4^+)_{13}$.

Let $(Sq^2)^3$ act on the both sides of this equality. Since $(Sq^2)^3Sq^1=0$ and $(Sq^2)^3Sq^2=0$, we get

$$(Sq^2)^3(\theta_3)=(Sq^2)^3Sq^4(C)+(Sq^2)^3Sq^8(D).$$

By a direct computation, we see that the monomial $x=x_1^7x_2^{12}x_3^2x_4^6$ is a term of $(Sq^2)^3(\theta_1)$. If this monomial is a term of $(Sq^2)^3Sq^8(y)$, then $y=x_1^7f_1(z)$ with $z$ a monomial of degree 6 in $P_3$ and $x$ is a term of $x_1^7(Sq^2)^3Sq^8(f_1(z))=0$. So the monomial $x$ is not a term of $(Sq^2)^3Sq^8(D)$ for all $D\in(P_4^+)_{13}$.

If this monomial is a term of $(Sq^2)^3Sq^4(y)$, where the monomial $y$ is a term of $C$, then $y=x_1^7f_1(z)$ with $z$ a monomial of degree 10 in $P_3$ and $x$ is a term of $x_1^7(Sq^2)^3Sq^4(f_1(z))=0$. By a direct computation, we see that either $x_1^7x_2^6x_3x_4^3$ or $x_1^7x_2^5x_3^2x_4^3$ is a term of $C$.

If $x_1^7x_2^6x_3x_4^3$ is a term of $C$ then

$$(Sq^2)^3(\theta_1+Sq^4(x_1^7x_2^6x_3x_4^3))=(Sq^2)^3(Sq^4(C')+Sq^8(D)),$$

where $C'=C+x_1^7x_2^6x_3x_4^3$. The monomial $x'=x_1^{16}x_2^6x_3^2x_4^3$ is a term of the polynomial $(Sq^2)^3(\theta_1+Sq^4(x_1^7x_2^6x_3x_4^3))$. If $x'$ is a term of the polynomial $(Sq^2)^3Sq^8(y')$, with $y'$ a monomial in $(P_4^+)_{13}$. Then $y'=x_1^ax_2^bx_3^cx_4^3$ with $a\geqslant 7, b\geqslant 3, c>0$. This contradicts with the fact that $\deg y'=13$. So $x'$ is not a term of $(Sq^2)^3Sq^8(D)$ for all $D\in(P_4^+)_{13}$. Hence $x'$ is a term of $(Sq^2)^3(Sq^4(C')$. By a direct computation, we see that either $x_1^7x_2^6x_3x_4^3$ or $x_1^7x_2^5x_3^2x_4^3$ is a term of $C'$. Since $x_1^7x_2^6x_3x_4^3$ is not a term of $C'$, the monomial $x_1^7x_2^5x_3^2x_4^3$ is a term of $C'$. Then we have

$$(Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_3^2x_4^3)) = (Sq^2)^3(Sq^4(C'') + Sq^8(D)),$$

where $C'' = C' + x_1^7x_2^5x_3^2x_4^3 = C + x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_3^2x_4^3$. Now the monomial $x = x_1^7x_2^{12}x_3^2x_4^6$ is a term of

$$(Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_3^2x_4^3)).$$

Hence either $x_1^7x_2^6x_3x_4^3$ or $x_1^7x_2^5x_3^2x_4^3$ is a term of $C$ is a term of $C''$. On the other hand, the two monomials $x_1^7x_2^6x_3x_4^3$ and $x_1^7x_2^5x_3^2x_4^3$ are not the terms of $C''$. We have a contradiction. Hence one gets $\gamma_{20} = 0$.

*Step 2.* Since $\gamma_{20} = 0$, the homomorphism $\varphi_2$ sends (5.4.6.3) to

$$\gamma_{25}[\theta_1] + \gamma_{31}[\theta_2] + \gamma_5[\theta_3] = 0. \tag{5.4.6.4}$$

Using (5.4.6.4) and the result in Step 1, we get $\gamma_5 = 0$.

*Step 3.* The homomorphism $\varphi_3$ sends (5.4.6.3) to

$$\gamma_{25}[\theta_4] + \gamma_{31}[\theta_2] = 0. \tag{5.4.6.5}$$

Using the relation (5.4.6.5) and the result in Step 2, we obtain $\gamma_{25} = 0$.

*Step 4.* Since $\varphi_4([\theta_2]) = [\theta_1]$, we have

$$\gamma_{31}[\theta_1] = 0.$$

Using this equality and by a same argument as given in Step 3, we get $\gamma_{31} = 0$.

For $t > 1$, we have $|B_4^+(n)| = m(t)$ with $m(2) = 95, m(3) = 128$ and $m(t) = 139$ for $t \geqslant 4$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{m(t)} \gamma_i d_i \equiv 0, \tag{5.4.6.6}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n,i}$. A direct computation from the relations $p_{(j;J)}(\mathcal{S}) \equiv 0$, for $(j;J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i$. The proposition is proved. □

### 5.5. The case of degree $2^{s+t} + 2^s - 2$.

For $s \geqslant 1$ and $t \geqslant 2$, the space $(QP_4)_n$ was determined in [32]. Hence, in this subsection we need only to compute $(QP_4)_n$ for $n = 2^{s+1} + 2^s - 2$ with $s > 1$.

Recall that, the homomorphism

$$\widetilde{Sq}^0_* : (QP_4)_{2^{s+1}+2^s-2} \to (QP_4)_{2^s+2^{s-1}-3}$$

is an epimorphism. Hence we have

$$(QP_4)_{2m+4} \cong (QP_4)_m \oplus (QP_4^0)_{2m+4} \oplus (\mathrm{Ker}\widetilde{Sq}^0_* \cap (QP_4^+)_{2m+4}),$$

where $m = 2^s + 2^{s-1} - 3$. So it suffices to compute $\mathrm{Ker}\widetilde{Sq}^0_* \cap (QP_4^+)_n$ for $s > 1$.

For $s > 1$, denote by $C(s)$ the set of all the following monomials:

$$\begin{array}{lll}
x_1x_2x_3^{2^s-2}x_4^{2^{s+1}-2}, & x_1x_2x_3^{2^{s+1}-2}x_4^{2^s-2}, & x_1x_2^{2^s-2}x_3x_4^{2^{s+1}-2},\\
x_1x_2^{2^{s+1}-2}x_3x_4^{2^s-2}, & x_1x_2^2x_3^{2^s-4}x_4^{2^{s+1}-1}, & x_1x_2^2x_3^{2^{s+1}-1}x_4^{2^s-4},\\
x_1x_2^{2^{s+1}-1}x_3^2x_4^{2^s-4}, & x_1^{2^{s+1}-1}x_2x_3^2x_4^{2^s-4}, & x_1x_2^2x_3^{2^{s+1}-3}x_4^{2^s-2},\\
x_1x_2^3x_3^{2^{s+1}-4}x_4^{2^s-2}, & x_1^3x_2x_3^{2^{s+1}-4}x_4^{2^s-2}. &
\end{array}$$

For $s > 2$, denote by $D(s)$ the set of all the following monomials:

$$\begin{array}{lll}
x_1x_2^2x_3^{2^s-3}x_4^{2^{s+1}-2}, & x_1x_2^2x_3^{2^s-1}x_4^{2^{s+1}-4}, & x_1x_2^2x_3^{2^{s+1}-4}x_4^{2^s-1},\\
x_1x_2^{2^s-1}x_3^2x_4^{2^{s+1}-4}, & x_1^{2^s-1}x_2x_3^2x_4^{2^{s+1}-4}, & x_1x_2^3x_3^{2^s-4}x_4^{2^{s+1}-2},\\
x_1x_2^3x_3^{2^{s+1}-2}x_4^{2^s-4}, & x_1^3x_2x_3^{2^s-4}x_4^{2^{s+1}-2}, & x_1^3x_2x_3^{2^{s+1}-2}x_4^{2^s-4},\\
x_1x_2^3x_3^{2^s-2}x_4^{2^{s+1}-4}, & x_1^3x_2x_3^{2^s-2}x_4^{2^{s+1}-4}, & x_1^3x_2^{2^{s+1}-3}x_3^2x_4^{2^s-4},\\
x_1^3x_2^{2^s-3}x_3^2x_4^{2^{s+1}-4}, & x_1^3x_2^5x_3^{2^{s+1}-6}x_4^{2^s-4}. &
\end{array}$$

Set $E(2) = C(2) \cup \{x_1^3x_2^4x_3x_4\}$, $E(3) = C(3) \cup D(3) \cup \{x_1^3x_2^5x_3^6x_4^8\}$ and $E(s) = C(s) \cup D(s) \cup \{x_1^3x_2^5x_3^{2^s-6}x_4^{2^{s+1}-4}\}$, for $s > 3$.

**Proposition 5.5.1.** *For any integer $s > 1$, $E(s) \cup \Phi^0(B_3(n)) \cup \psi(B_4(m))$ is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n = 2m + 4$ with $m = 2^s + 2^{s-1} - 3$.*

**Lemma 5.5.2.** *Let $x$ be an admissible monomial of degree $n = 2^{s+t} + 2^s - 2$ in $P_4$. If $[x] \in \mathrm{Ker}\widetilde{Sq}^0_*$, then either $\omega(x) = (2^{(s)}, 1)$.*

*Proof.* We prove the lemma by induction on $s$. Since $n = 2^{s+1} + 2^s - 2$ is even, we get either $\omega_1(x) = 0$ or $\omega_1(x) = 2$ or $\omega_1(x) = 4$. If $\omega_1(x) = 0$, then $x = Sq^1(y)$ for some monomial $y$. If $\omega_1(x) = 4$, then $x = X_\emptyset y^2$ for some monomial $y$. Since $x$ is admissible, $y$ also is admissible. This implies $\mathrm{Ker}\widetilde{Sq}^0_*([x]) = [y] \neq 0$ and we have a contradiction. So $\omega_1(x) = 2$ and $x = x_ix_jy^2$ with $1 \leqslant i < j \leqslant 4$, and $y$ a monomial of degree $2^s + 2^{s-1} - 2$ in $P_4$. Using Proposition 2.10 we get $\omega_i(x) = 2$ for $1 \leqslant i \leqslant s$. Then $x = x'z^{2^s}$ with $x', z$ monomials in $P_4$ and $\deg z = 2^t - 1$. By a direct computation we see that if $w$ is a monomial such that either $\omega(w) = (2, 1, 3)$ or $\omega(w) = (2, 2, 3)$ or $\omega(w) = (2, 3, 2, 2)$ then $w$ is strictly inadmissible. Now, the lemma follows from this fact, Lemma 5.3.1 and Theorem 2.9. □

The following is proved by a direct computation.

**Lemma 5.5.3.** *The following monomials are strictly inadmissible:*

i) $x_i^2x_jx_m,\ x_i^3x_j^4x_m^3,\ x_i^7x_j^7x_m^8,\ 1 \leqslant i < j < m \leqslant 4.$

ii) $x_1x_2^7x_3^{10}x_4^4$, $x_1^7x_2x_3^{10}x_4^4$, $x_1x_2^6x_3^7x_4^8$, $x_1x_2^7x_3^6x_4^8$, $x_1^7x_2x_3^6x_4^8$, $x_1^3x_2^3x_3^4x_4^{12}$, $x_1^3x_2^3x_3^{12}x_4^4$, $x_1^7x_2^9x_3^2x_4^4$, $x_1^7x_2^8x_3^3x_4^4$, $x_1^3x_2^5x_3^8x_4^6$.

*Proof of Proposition 5.5.1.* Let $x$ be an admissible monomial of degree $n = 2^{s+1} + 2^s - 2$ in $P_4$ and $[x] \in \mathrm{Ker}\widetilde{Sq}^0_*$. By Lemma 5.5.2, $\omega_i(x) = 2$, for $1 \leqslant i \leqslant s$, $\omega_{s+1}(x) = 1$ and $\omega_i(x) = 0$ for $i > s + 1$. By induction on $s$, we see that if $x \notin E(s) \cup \Phi^0(B_3(n))$ then there is a monomial $w$ which is given in one of Lemmas 5.2.3, 5.5.3 such that $x = wy^{2^u}$ for some monomial $y$ and positive integer $u$. By Theorem 2.9, $x$ is inadmissible. Hence $\mathrm{Ker}\widetilde{Sq}^0_*$ is spanned by the set $[E(s) \cup \Phi^0(B_3(n))]$ in degree $n = 2^{s+1} + 2^s - 2$. Now, we prove that set $[E(s) \cup \Phi^0(B_3(n))]$ is linearly independent.

It suffices to prove that the set $[E(s)]$ is linearly independent. For $s = 2, |E(2)| = 12$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{12} \gamma_i d_i \equiv 0, \tag{5.1}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{10,i}$. A direct computation from the relations $p_{(1;j)}(\mathcal{S}) \equiv 0$, for $j = 1, 2, 3$, we obtain $\gamma_i = 0$ for all $i$.

For $s > 2, |E(s)| = 26$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{26} \gamma_i d_i \equiv 0, \tag{5.2}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n,i}$. A direct computation from the relations $p_{(r;j)}(\mathcal{S}) \equiv 0$, for $1 \leqslant r < j \leqslant 4$, we obtain $\gamma_i = 0$ for all $i$. The proposition is proved. $\square$

5.6. **The case of degree** $2^{s+t+u} + 2^{s+t} + 2^s - 3$**.**

First, we determine the $\omega$-vector of an admissible monomial of degree $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$.

**Lemma 5.6.1.** *If $x$ is an admissible monomial of degree $2^{s+t+u} + 2^{s+t} + 2^s - 3$ in $P_4$ then $\omega(x) = (3^{(s)}, 2^{(t)}, 1^{(u)})$.*

*Proof.* Observe that $z = x_1^{2^{s+t+u}-1} x_2^{2^{s+t}-1} x_3^{2^s-1}$ is the minimal spike of degree $2^{s+t+u} + 2^{s+t} + 2^s - 3$ and $\omega(z) = (3^{(s)}, 2^{(t)}, 1^{(u)})$. Since $2^{s+t+u} + 2^{s+t} + 2^s - 3$ is odd and $x$ is admissible, using Proposition 2.10 and Theorem 2.12, we get $\omega_i(x) = 3$ for $1 \leqslant i \leqslant s$. Set $x' = \prod_{1 \leqslant i \leqslant s} X_{I_{i-1}(x)}^{2^{i-1}}$. Then $x = x'y^{2^s}$ for some monomial $y$. We have $\omega_j(y) = \omega_{j+s}(x)$ for all $j \geqslant 1$ and

$$\begin{aligned} 2^{s+t+u} + 2^{s+t} + 2^s - 3 = \deg x &= \sum_{i \geqslant 1} 2^{i-1}\omega_i(x) \\ &= 3(2^s - 1) + 2^s \sum_{j \geqslant 1} 2^{j-1}\omega_{j+s}(x) \\ &= 3.2^s - 3 + 2^s \deg y. \end{aligned}$$

This equality implies $\deg y = 2^{t+u} + 2^u - 2$. Since $x$ is admissible, using Theorem 2.9, we see that $y$ is also admissible. By a direct computation we see that if $w$ is a monomial such that $\omega(w) = (3, 2, 3)$ then $w$ is strictly inadmissible. Combining this fact, Lemma 5.3.1, Proposition 2.10 and Theorem 2.9, we obtain $\omega(y) = (2^{(t)}, 1^{(u)})$. The lemma is proved. $\square$

Applying Theorem 1.3, we get the following.

**Proposition 5.6.2.** *Let $s, t, u$ be positive integers. If $s \geqslant 3$, then $\Phi(B_3(n))$ is a minimal set of generators for $\mathcal{A}$-module $P_4$ in degree $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$.*

So, we need only to consider the cases $s = 1$ and $s = 2$.

5.6.1. **The subcase** $s = t = 1$**.**

For $s = 1, t = 1$, we have $n = 2^{u+2} + 3$. According to Theorem 4.3, we have

$$B_3(n) = \begin{cases} \psi(\Phi(B_2(2^{u+1}))), & \text{if } u \neq 2, \\ \psi(\Phi(B_2(8)) \cup \{x_1^7x_2^9x_3^3\}, & \text{if } u = 2. \end{cases}$$

**Proposition 5.6.3.**

i) $\Phi(B_3(11)) \cup \{x_1^3x_2^4x_3x_4^3, x_1^3x_2^4x_3^3x_4\}$ *is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree* $11$.

ii) $\Phi(B_3(19)) \cup \{x_1^7x_2^9x_3^2x_4,\ x_1^3x_2^{12}x_3x_4^3,\ x_1^3x_2^{12}x_3^3x_4,\ x_1^3x_2^4x_3x_4^{11},\ x_1^3x_2^4x_3^{11}x_4, x_1^3x_2^7x_3^8x_4, x_1^7x_2^3x_3^8x_4, x_1^7x_2^8x_3x_4^3, x_1^7x_2^8x_3^3x_4, x_1^3x_2^4x_3^3x_4^9, x_1^3x_2^4x_3^9x_4^3\}$ *is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree* $19$.

iii) $\Phi(B_3(n)) \cup \{x_1^3x_2^4x_3x_4^{2^{u+2}-5}, x_1^3x_2^4x_3^{2^{u+2}-5}x_4, x_1^3x_2^4x_3^3x_4^{2^{u+2}-7}\}$ *is the set of all the admissible monomials for* $\mathcal{A}$*-module* $P_4$ *in degree* $n = 2^{u+2}+3$*, with any positive integer* $u \geqslant 3$.

By a direct computation, we can easy obtain the following lemma.

**Lemma 5.6.4.** *The following monomials are strictly inadmissible:*
i) $x_1^3x_2^4x_3^4x_4^4x_ix_j^3,\ i,j > 1,\ i \neq j,\ x_1^7x_2^3x_3^4x_4^4x_j,\ x_1^3x_2^5x_3^5x_4^5x_j,\ j = 3,4.$
ii) $X_2x_1^2x_j^2x_2^{28},\ X_jx_1^2x_4^2x_2^4x_3^{24},\ X_2x_1^2x_j^2x_4^4x_3^8x_4^{16},\ X_jx_1^2x_2^4x_3^8x_4^{18},\ X_jx_1^2x_2^4x_3^{10}x_4^{16},$ $X_jx_1^2x_2^2x_i^4x_3^8x_4^{16},\ X_3x_1^2x_2^2x_i^4x_3^{24},\ X_2x_1^2x_4^2x_4^4x_2^4x_3^{24},\ i = 1,2,\ j = 3,4.$

*Proof of Theorem 5.6.3.* Let $x$ be an admissible monomial of degree $n = 2^{u+2}+3$ in $P_4$. By Lemma 5.6.1, $\omega_1(x) = 3$. So $x = X_iy^2$ with $y$ a monomial of degree $2^{u+1}$. Since $x$ is admissible, by Theorem 2.9, $y \in B_4(2^{u+1})$. By a direct computation, we see that if $x = X_iy^2$ with $y \in B_4(2^{u+1})$ and $x$ not belongs to the set $C_4(n)$ as given in the proposition, then there is a monomial $w$ which is given in one of Lemmas 5.3.3, 5.6.4 such that $x = wy^{2^r}$ for some monomial $y$ and integer $r > 1$. By Theorem 2.9, $x$ is inadmissible. Hence $(QP_4)_n$ is spanned by the set $[C_4(n)]$.

Set $|C_4(2^{u+2}+3) \cap P_4^+| = m(u)$, where $m(1) = 32,\ m(2) = 80,\ m(u) = 64$ for all $u > 2$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{m(u)} \gamma_i d_i = 0, \tag{5.6.1}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n,i}$. By a direct computation from the relations $p_{(j;J)}(\mathcal{S}) \equiv 0$ with $(j;J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i$ if $u \neq 2$.

For $u = 2$, $\gamma_j = 0$ for $j$ =1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 21, 23, 26, 27, 28, 29, 30, 31, 32, 35, 36, 38, 40, 43, 45, 51, 54, 55, 60, 61, 62, 68, 71, 79, 80, and $\gamma_2 = \gamma_i, i = 5,24,25,41,42,52,53,\ \gamma_{13} = \gamma_i, i = 13,33,20,56,48,58,$ $\gamma_{15} = \gamma_i, i = 22,34,49,57,59,\ \gamma_{37} = \gamma_i, i = 67,70,75,\ \gamma_{46} = \gamma_i, i = 69,72,76,$ $\gamma_{65} = \gamma_i, i = 66,73,74,77,78,\ \gamma_{46} = \gamma_{39}+\gamma_2,\ \gamma_{44} = \gamma_{37}+\gamma_2,\ \gamma_{65} = \gamma_{47}+\gamma_{13},$ $\gamma_{65} = \gamma_{50}+\gamma_{22},\ \gamma_{63} = \gamma_{37}+\gamma_{13},\ \gamma_{64} = \gamma_{46}+\gamma_{22}.$

Substituting the above equalities into the relation (5.6.1), we have

$$\gamma_{37}[\theta_1] + \gamma_{46}[\theta_2] + \gamma_{13}[\theta_3] + \gamma_{22}[\theta_4] + \gamma_{65}[\theta_5] + \gamma_2[\theta_6] = 0, \tag{5.6.2}$$

where

$$\begin{aligned}
\theta_1 &= d_{37} + d_{44} + d_{63} + d_{67} + d_{70} + d_{75},\\
\theta_2 &= d_{39} + d_{46} + d_{64} + d_{69} + d_{72} + d_{76},\\
\theta_3 &= d_{13} + d_{20} + d_{33} + d_{47} + d_{48} + d_{56} + d_{58} + d_{63},\\
\theta_4 &= d_{15} + d_{22} + d_{34} + d_{49} + d_{50} + d_{57} + d_{59} + d_{64},\\
\theta_5 &= d_{47} + d_{50} + d_{65} + d_{66} + d_{73} + d_{74} + d_{77} + d_{78},\\
\theta_6 &= d_2 + d_5 + d_{24} + d_{25} + d_{39} + d_{41} + d_{42} + d_{44} + d_{52} + d_{53}.
\end{aligned}$$

We need to prove $\gamma_2 = \gamma_{13} = \gamma_{22} = \gamma_{37} = \gamma_{46} = \gamma_{65} = 0$. The proof is divided into 4 steps.

*Step 1.* First we prove $\gamma_{65} = 0$ by showing the polynomial $[\theta] = [\beta_1\theta_1 + \beta_2\theta_2 + \beta_3\theta_3 + \beta_4\theta_4 + \theta_5 + \beta_6\theta_6] \neq 0$ for all $\beta_1,\beta_2,\beta_3,\beta_4,\beta_6 \in \mathbb{F}_2$. Suppose the contrary that this polynomial is hit. Then we have

$$\theta = Sq^1(A) + Sq^2(B) + Sq^4(C) + Sq^8(D),$$

for some polynomials $A, B, C, D$ in $P_4^+$. Let $(Sq^2)^3$ act on the both sides of this equality. Using the relations $(Sq^2)^3Sq^1 = 0, (Sq^2)^3Sq^2 = 0$, we get

$$(Sq^2)^3(\theta) = (Sq^2)^3Sq^4(C) + (Sq^2)^3Sq^8(D).$$

The monomial $x_1^7x_2^{12}x_3^4x_4^2$ is a term of $(Sq^2)^3(\theta)$. If $x_1^7x_2^{12}x_3^4x_4^2$ is a term of the polynomial $(Sq^2)^3Sq^8(y)$ with $y$ a monomial of degree 11 in $P_4$, then $y = x_1^7f_1(z)$ with $z$ a monomial of degree 4 in $P_3$. Then $x_1^7x_2^{12}x_3^4x_4^2$ is a term of $x_1^7(Sq^2)^3Sq^8(f_1(z)) = 0$. This is a contradiction. So $x_1^7x_2^{12}x_3^4x_4^2$ is not a term of $(Sq^2)^3Sq^8(D)$ for all $D$. Hence $x_1^7x_2^{12}x_3^4x_4^2$ is a term of $(Sq^2)^3Sq^4(C)$, then either $x_1^7x_2^5x_3x_4^2$ or $x_1^7x_2^5x_3^2x_4$ or $x_1^7x_2^6x_3x_4$ is a term of $C$.

Suppose $x_1^7x_2^5x_3^2x_4$ is a term of $C$. Then

$$(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4)) = (Sq^2)^3(Sq^4(C') + Sq^8(D)),$$

where $C' = C + x_1^7x_2^5x_3^2x_4$. We see that the monomial $x_1^{16}x_2^6x_3^2x_4$ is a term of $(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4))$. This monomial is not a term of $(Sq^2)^3Sq^8(D)$ for all $D$. So it is a term of $(Sq^2)^3Sq^4(C')$. Then either $x_1^7x_2^5x_3^2x_4$ or $x_1^7x_2^6x_3x_4$ is a term of $C$. Since $x_1^7x_2^5x_3^2x_4$ is a term of $C'$, $x_1^7x_2^6x_3x_4$ is s term of $C'$. Hence we obtain

$$(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4)) = (Sq^2)^3(Sq^4(C'') + Sq^8(D)),$$

where $C'' = C + x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4$. Now $x_1^7x_2^{12}x_3^4x_4^2$ is a term of

$$(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4))$$

So either $x_1^7x_2^5x_3x_4^2$ or $x_1^7x_2^5x_3^2x_4$ or $x_1^7x_2^6x_3x_4$ is a term of $C''$. Since $x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4$ is a summand of $C''$, $x_1^7x_2^5x_3x_4^2$ is s term of $C''$. Then $x_1^{16}x_2^6x_3^2x_4$ is a term of $(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4 + x_1^7x_2^5x_3x_4^2 + x_1^7x_2^6x_3x_4))$. So either $x_1^7x_2^5x_3x_4^2$ or $x_1^7x_2^5x_3^2x_4$ or $x_1^7x_2^6x_3x_4$ is a term of $C'' + x_1^7x_2^5x_3x_4^2$ and we have a contradiction.

By a same argument, if either $x_1^7x_2^5x_3x_4^2$ or $x_1^7x_2^6x_3x_4$ is a term of $C$ then we have also a contradiction. Hence $[\theta] \neq 0$ and $\gamma_{65} = 0$.

*Step 2.* By a direct computation, we see that the homomorphism $\varphi_3$ sends (5.6.2) to

$$\gamma_{37}[\theta_1] + \gamma_2[\theta_3] + \gamma_{22}[\theta_4] + \gamma_{46}[\theta_5] + \gamma_{13}[\theta_6] = 0.$$

By Step 1, we obtain $\gamma_{46} = 0$.

*Step 3.* The homomorphism $\varphi_2$ sends (5.6.2) to

$$\gamma_{13}[\theta_1] + \gamma_{22}[\theta_2] + \gamma_{37}[\theta_3] + \gamma_2[\theta_6] = 0.$$

By Step 2, we obtain $\gamma_{22} = 0$.

*Step 4.* Now the homomorphism $\varphi_3$ sends (5.6.2) to $\gamma_{37}[\theta_2] + \gamma_{13}[\theta_4] + \gamma_2[\theta_6] = 0$. Combining Step 2 and Step 3, we obtain $\gamma_{13} = \gamma_{37} = 0$.

Since $\varphi_2([\theta_3]) = [\theta_6]$, we get $\gamma_2 = 0$. So we obtain $\gamma_j = 0$ for all $j$. The proposition follows. □

5.6.2. **The subcase $s = 1,\ t = 2$.**

For $s = 1, t = 2$, we have $n = 2^{u+3} + 7 = 2m + 3$ with $m = 2^{u+2} + 2$. Combining Theorem 1.3 and Theorem 4.3, we have $B_3(n) = \psi(\Phi(B_2(m)))$. where

$$B_2(m) = \begin{cases} \{x_1^3x_2^7, x_1^7x_2^3\}, & \text{if } u = 1, \\ \{x_1^3x_2^{2^{u+2}-1}, x_1^{2^{u+2}-1}x_2^3, x_1^7x_2^{2^{u+2}-5}\}, & \text{if } u > 1. \end{cases}$$

Denote by $F(u)$ the set of all the following monomials:

$$x_1^3x_2^4x_3x_4^{2^{u+3}-1},\ x_1^3x_2^4x_3^{2^{u+3}-1}x_4,\ x_1^3x_2^{2^{u+3}-1}x_3^4x_4,\ x_1^{2^{u+3}-1}x_2^3x_3^4x_4,$$
$$x_1^3x_2^7x_3^{2^{u+3}-4}x_4,\ x_1^7x_2^3x_3^{2^{u+3}-4}x_4,\ x_1^7x_2^{2^{u+3}-5}x_3^4x_4,\ x_1^7x_2^7x_3^{2^{u+3}-8}x_4,$$
$$x_1^3x_2^4x_3^3x_4^{2^{u+3}-3},\ x_1^3x_2^4x_3^{2^{u+3}-5}x_4^5,\ x_1^3x_2^4x_3^7x_4^{2^{u+3}-7},\ x_1^3x_2^7x_3^4x_4^{2^{u+3}-7},$$
$$x_1^7x_2^3x_3^4x_4^{2^{u+3}-7},\ x_1^3x_2^7x_3^8x_4^{2^{u+3}-11},\ x_1^7x_2^3x_3^8x_4^{2^{u+3}-11}.$$

**Proposition 5.6.5.**
i) $\Phi(B_3(23)) \cup F(1) \cup \{x_1^7x_2^9x_3^2x_4^5,\ x_1^7x_2^9x_3^3x_4^4\}$ *is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree* 23.
ii) $\Phi(B_3(n)) \cup F(u) \cup \{x_1^7x_2^7x_3^8x_4^{2^{u+3}-15}, x_1^7x_2^7x_3^9x_4^{2^{u+3}-16}, x_1^3x_2^4x_3^{11}x_4^{2^{u+3}-11}\}$ *is the set of of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n = 2^{u+3}+7$ with any positive integer $u > 1$.*

By a direct computation, we can easy obtain the following lemma.

**Lemma 5.6.6.** *The following monomials are strictly inadmissible:*

i) $X_2x_1^2x_j^6x_2^{12}, X_jx_1^2x_2^4x_3^8x_4^6, X_2x_1^2x_i^4x_2^8x_3^2x_4^4, X_2x_1^2x_2^4x_3^8x_4^6,\ i = 1, 2, j = 3, 4.$
ii) $X_3x_1^2x_2^2x_i^{12}x_3^{20},\ X_3x_1^2x_2^2x_i^4x_3^{20}x_4^4,\ X_jx_1^2x_2^2x_i^{12}x_3^4x_4^{16},\ X_jx_1^2x_2^4x_i^{14}x_3^{16},$
$X_jx_1^6x_2^{10}x_3^4x_4^{16},\ X_jx_1^6x_2^{10}x_3^{16}x_4^4,\ X_3x_1^6x_2^{10}x_3^{20},\ X_2x_1^2x_2^4x_3^{14}x_4^{16}, i = 1, 2, j = 3, 4.$

*Proof of Proposition 5.6.5.* Let $x$ be an admissible monomial of degree $n = 2^{u+3}+7$ in $P_4$.

By Lemma 5.6.1, $\omega_1(x) = 3$. So $x = X_iy^2$ with $y$ a monomial of degree $2^{u+2}+2$. Since $x$ is admissible, by Theorem 2.9, $y \in B_4(2^{u+2}+2)$.

By a direct computation, we see that if $x = X_iy^2$ with $y \in B_4(2^{u+2}+2)$ and $x$ not belongs to the set $C_4(n)$ as given in the proposition, then there is a monomial $w$ which is given in one of Lemmas 5.6.6, 5.3.3 such that $x = wy^{2^r}$ for some monomial $y$ and integer $r > 1$.

By Theorem 2.9, $x$ is inadmissible. Hence $(QP_4)_n$ is spanned by the set $[C_4(n)]$.

For $u = 1$, we have, $|C_4^+(23) \cap P_4^+| = 99$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{99} \gamma_i d_i = 0, \tag{5.6.1}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{23,i}$. By a direct computation from the relations $p_{(j;J)}(\mathcal{S}) \equiv 0$ with $(j;J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i \in E$, with some $E \subset \mathbb{N}_{99}$ and the relation (5.6.2) becomes

$$\sum_{i=1}^{15} c_i[\theta_i] = 0, \tag{5.6.2}$$

where $c_1 = \gamma_1, c_2 = \gamma_4, c_3 = \gamma_{33}, c_4 = \gamma_{94}, c_5 = \gamma_2, c_6 = \gamma_{22}, c_7 = \gamma_{74}, c_8 = \gamma_{29}, c_9 = \gamma_{81}, c_{10} = \gamma_{68}, c_{11} = \gamma_{10}, c_{12} = \gamma_{43}, c_{13} = \gamma_{54}, c_{14} = \gamma_{70}, c_{15} = \gamma_{11}$ and

$$\begin{aligned}
\theta_1 &= d_1 + d_{17} + d_{37} + d_{49},\\
\theta_2 &= d_4 + d_{21} + d_{44} + d_{53},\\
\theta_3 &= d_{33} + d_{36} + d_{72} + d_{73},\\
\theta_4 &= d_{94} + d_{97} + d_{98} + d_{99},\\
\theta_5 &= d_2 + d_{19} + d_{40} + d_{51},\\
\theta_6 &= d_{22} + d_{25} + d_{62} + d_{63},\\
\theta_7 &= d_{74} + d_{77} + d_{82} + d_{83},\\
\theta_8 &= d_{12} + d_{14} + d_{26} + d_{29} + d_{66} + d_{67},\\
\theta_9 &= d_{40} + d_{42} + d_{78} + d_{81} + d_{86} + d_{87},\\
\theta_{10} &= d_{10} + d_{15} + d_{24} + d_{27} + d_{46} + d_{47} + d_{64} + d_{65},\\
\theta_{11} &= d_{38} + d_{43} + d_{46} + d_{47} + d_{76} + d_{79} + d_{84} + d_{85},\\
\theta_{12} &= d_{62} + d_{67} + d_{68} + d_{71} + d_{88} + d_{89} + d_{92} + d_{93},\\
\theta_{13} &= d_{47} + d_{54} + d_{57} + d_{62} + d_{69} + d_{82} + d_{85} + d_{88} + d_{90},\\
\theta_{14} &= d_{12} + d_{15} + d_{19} + d_{20} + d_{46} + d_{47} + d_{51} + d_{52} + d_{58} + d_{61}\\
&\quad + d_{64} + d_{66} + d_{67} + d_{70} + d_{84} + d_{87} + d_{89} + d_{91},\\
\theta_{15} &= d_{11} + d_{12} + d_{18} + d_{20} + d_{24} + d_{25} + d_{26} + d_{27} + d_{38} + d_{40} + d_{45}\\
&\quad + d_{47} + d_{48} + d_{50} + d_{52} + d_{57} + d_{61} + d_{63} + d_{64} + d_{65} + d_{66}\\
&\quad + d_{67} + d_{69} + d_{77} + d_{78} + d_{83} + d_{85} + d_{86} + d_{87} + d_{89} + d_{90}.
\end{aligned}$$

Now, we show that $c_i = 0$ for $i = 1, 2, \ldots, 15$. The proof is divided into 6 steps.

*Step 1.* Set $\theta = \theta_1 + \sum_{i=2}^{15} \beta_i\theta_i$ for $\beta_i \in \mathbb{F}_2, i = 2, 3, \ldots, 15$. We prove that $[\theta] \neq 0$. Suppose the contrary that $\theta$ is hit. Then we have

$$\theta = Sq^1(A) + Sq^2(B) + Sq^4(C) + Sq^8(D)$$

for some polynomials $A, B, C, D \in P_4^+$. Let $(Sq^2)^3$ act to the both sides of the above equality, we obtain

$$(Sq^2)^3(\theta) = (Sq^2)^3 Sq^4(C) + (Sq^2)^3 Sq^8(D).$$

By a similar computation as in the proof of Proposition 5.4.5, we see that the monomial $x_1^8x_2^4x_3^2x_4^{15}$ is a term of $(Sq^2)^3(\theta)$. This monomial is not a term of $(Sq^2)^3(Sq^4(C) + Sq^8(D))$ for all polynomials $C, D$ and we have a contradiction. So $[\theta] \neq 0$ and we get $c_1 = \gamma_1 = 0$.

By an argument analogous to the previous one, we get $c_2 = c_3 = c_4 = 0$. Now, the relation (5.6.2) becomes

$$\sum_{i=5}^{15} c_i[\theta_i] = 0. \tag{5.6.3}$$

*Step 2.* The homomorphisms

$$\varphi_1, \varphi_1\varphi_3, \varphi_1\varphi_3\varphi_4, \varphi_1\varphi_3\varphi_2, \varphi_1\varphi_3\varphi_2\varphi_4, \varphi_1\varphi_3\varphi_4\varphi_2\varphi_3$$

send (5.6.3) respectively to

$$\begin{aligned}
c_{10}[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \ldots, [\theta_{15}]\rangle,\\
c_{9}[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \ldots, [\theta_{15}]\rangle,\\
c_{7}[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \ldots, [\theta_{15}]\rangle,\\
c_{8}[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \ldots, [\theta_{15}]\rangle,\\
c_{6}[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \ldots, [\theta_{15}]\rangle,\\
c_{5}[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \ldots, [\theta_{15}]\rangle.
\end{aligned}$$

Using the results in Step 1, we get $c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0$. So the relation (5.6.3) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0. \tag{5.6.4}$$

*Step 3.* The homomorphism $\varphi_1$ sends (5.6.4) to

$$\begin{aligned}
&c_{13}[\theta_6] + (c_{14} + c_{15})[\theta_7] + (c_{11} + c_{12})[\theta_{11}]\\
&\quad + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0.
\end{aligned}$$

By Step 2, we get $c_{13} = 0$ and $c_{14} = c_{15}$. So the relation (5.6.4) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{14}[\theta_{14}] + c_{14}[\theta_{15}] = 0. \tag{5.6.5}$$

*Step 4.* The homomorphism $\varphi_3$ sends (5.6.5) to

$$c_{11}[\theta_{11}] + c_{14}[\theta_{12}] + (c_{12} + c_{14})[\theta_{13}] + c_{14}[\theta_{14}] + c_{14}[\theta_{15}] = 0.$$

By Step 3, we get $c_{12} = c_{14}$. Then the relation (5.6.5) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{12}[\theta_{14}] + c_{12}[\theta_{15}] = 0. \tag{5.6.6}$$

*Step 5.* The homomorphism $\varphi_2$ sends (5.6.6) to

$$(c_{11} + c_{12})[\theta_{12}] + c_{12}[\theta_{14}] + c_{12}[\theta_{15}] = 0.$$

From the result in Step 4, we get $c_{11} = 0$. Then the relation (5.6.6) becomes

$$c_{12}([\theta_{12}] + [\theta_{14}] + [\theta_{15}]) = 0. \tag{5.6.7}$$

*Step 6.* The homomorphism $\varphi_1$ sends (5.6.7) to

$$c_{12}[\theta_{11}] + c_{12}([\theta_{12}] + [\theta_{14}] + [\theta_{15}]) = 0.$$

By the result in Step 5, we have $c_{12} = 0$. The case $u = 1$ of the proposition is completely proved.

For $u > 1$, we have $|C_4(n) \cap P_4^+| = 141$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{141} \gamma_i d_i = 0, \tag{5.6.8}$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n,i} \in B_4^+(n)$. By a direct computation from the relations $p_{(j;J)}(\mathcal{S}) \equiv 0$ with $(j;J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i \notin E$, with some $E =\subset \mathbb{N}_{141}$ and the relation (5.6.8) becomes

$$\sum_{i=1}^{15} c_i[\theta_i] = 0, \tag{5.6.9}$$

where $c_1 = \gamma_1, c_2 = \gamma_6, c_3 = \gamma_{51}, c_4 = \gamma_{136}, c_5 = \gamma_2, c_6 = \gamma_{31}, c_7 = \gamma_{107}, c_8 = \gamma_{40}, c_9 = \gamma_{116}, c_{10} = \gamma_{101}, c_{11} = \gamma_{14}, c_{12} = \gamma_{56}, c_{13} = \gamma_{79}, c_{14} = \gamma_{23}, c_{15} = \gamma_{15}$ and

$$\begin{aligned}
\theta_1 &= d_1 + d_{25} + d_{55} + d_{73},\\
\theta_2 &= d_6 + d_{30} + d_{66} + d_{78},\\
\theta_3 &= d_{51} + d_{54} + d_{105} + d_{106},\\
\theta_4 &= d_7 + d_8 + d_{47} + d_{48},\\
\theta_5 &= d_2 + d_{27} + d_{58} + d_{75},\\
\theta_6 &= d_{31} + d_{34} + d_{89} + d_{90},\\
\theta_7 &= +d_{107} + d_{110} + d_{117} + d_{118},\\
\theta_8 &= d_{16} + d_{22} + d_{35} + d_{40} + d_{94} + d_{95},\\
\theta_9 &= d_{58} + d_{64} + d_{111} + d_{116} + d_{122} + d_{123},\\
\theta_{10} &= d_{89} + d_{95} + d_{101} + d_{104} + d_{124} + d_{127} + d_{129} + d_{130},\\
\theta_{11} &= d_{14} + d_{19} + d_{33} + d_{36} + d_{68} + d_{69} + d_{91} + d_{92},\\
\theta_{12} &= d_{56} + d_{61} + d_{68} + d_{69} + d_{109} + d_{112} + d_{119} + d_{120},\\
\theta_{13} &= d_{67} + d_{69} + d_{79} + d_{82} + d_{89} + d_{90} + d_{117} + d_{118} + d_{124} + d_{125},\\
\theta_{14} &= d_{16} + d_{23} + d_{27} + d_{29} + d_{70} + d_{71} + d_{72} + d_{75} + d_{77}\\
&\quad + d_{83} + d_{88} + d_{94} + d_{95} + d_{122} + d_{123} + d_{126} + d_{127},\\
\theta_{15} &= d_{15} + d_{19} + d_{26} + d_{27} + d_{33} + d_{34} + d_{35} + d_{36} + d_{58}\\
&\quad + d_{61} + d_{68} + d_{69} + d_{70} + d_{74} + d_{75} + d_{82} + d_{83} + d_{91}\\
&\quad + d_{92} + d_{109} + d_{110} + d_{111} + d_{112} + d_{119} + d_{120} + d_{125}.
\end{aligned}$$

Now, we prove $c_i = 0$ for $i = 1, 2, \ldots, 15$. The proof is divided into 6 steps.

*Step 1.* First, we prove $c_1 = 0$. Set $\theta = \theta_1 + \sum_{j=2}^{15} c_j\theta_j$. We show that $[\theta] \neq 0$ for all $c_j \in \mathbb{F}_2, j = 2, 3, \ldots, 15$. Suppose the contrary that $\theta$ is hit. Then we have

$$\theta = \sum_{m=0}^{u+2} Sq^{2^m}(A_m),$$

for some polynomials $A_m, m = 0, 1, \ldots, u + 2$. Let $(Sq^2)^3$ act on the both sides of this equality. Since $(Sq^2)^3Sq^1 = 0, (Sq^2)^3Sq^2 = 0$, we get

$$(Sq^2)^3(\theta) = \sum_{m=2}^{u+2} (Sq^2)^3Sq^{2^m}(A_m).$$

It is easy to see that the monomial $x = x_1^8x_2^4x_3^2x_4^{2^{u+3}-1}$ is a term of $(Sq^2)^3(\theta)$, hence it is a term of $(Sq^2)^3Sq^{2^m}(y)$ for some monomial $y$ of degree $2^{u+3} - 2^m + 7$ with $m \geqslant 2$. Then $y = x_2^{2^{u+3}-1}f_2(z)$ with $z$ a monomial of degree $8 - 2^m \leqslant 4$ in $P_3$ and $x$ is a term of $x_2^{2^{u+3}-1}(Sq^2)^3Sq^{2^m}(z)$. If $m > 2$ then $Sq^{2^m}(z) = 0$. If $m = 2$ the $Sq^{2^2}(z) = z^2$, hence $(Sq^2)^3Sq^{2^m}(z) = (Sq^2)^3(z^2) = 0$. So $x$ is not a term of

$$(Sq^2)^3(\theta) = \sum_{m=2}^{u+2} (Sq^2)^3Sq^{2^m}(A_m),$$

for all polynomial $A_m$ with $m > 1$. This is a contradiction. So we get $c_1 = 0$.

By an argument analogous to the previous one, we get $c_2 = c_3 = c_4 = 0$. Then the relation (5.6.9) becomes

$$\sum_{i=5}^{15} c_i[\theta_i] = 0. \tag{5.6.10}$$

*Step 2.* The homomorphisms

$$\varphi_1, \varphi_1\varphi_3, \varphi_1\varphi_3\varphi_4, \varphi_1\varphi_3\varphi_2, \varphi_1\varphi_3\varphi_2\varphi_4, \varphi_1\varphi_3\varphi_4\varphi_2\varphi_3$$

send (5.6.3) respectively to

$$\begin{aligned}
c_{10}[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \dots, [\theta_{15}]\rangle,\\
c_9[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \dots, [\theta_{15}]\rangle,\\
c_7[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \dots, [\theta_{15}]\rangle,\\
c_8[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \dots, [\theta_{15}]\rangle,\\
c_6[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \dots, [\theta_{15}]\rangle,\\
c_5[\theta_3] &= 0 \quad \mathrm{mod}\langle[\theta_5], [\theta_6], \dots, [\theta_{15}]\rangle.
\end{aligned}$$

By Step 1, we get $c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0$. So the relation (5.6.3) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0. \tag{5.6.11}$$

*Step 3.* Applying the homomorphism $\varphi_1$ to (5.6.11), we get

$$c_{13}[\theta_6] + c_{14}[\theta_8] + (c_{11} + c_{12} + c_{15})[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0.$$

By the results in Step 2, we obtain $c_{13} = c_{14} = 0$. Then the relation (5.6.11) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{14}[\theta_{15}] = 0. \tag{5.6.12}$$

*Step 4.* Applying the homomorphism $\varphi_3$ to the relation (5.6.12) we obtain

$$c_{11}[\theta_{11}] + c_{12}[\theta_{13}] + c_{15}[\theta_{15}] = 0.$$

By the results in Step 3, we get $c_{12} = 0$. So the relation (5.6.12) becomes

$$c_{11}[\theta_{11}] + c_{15}[\theta_{15}] = 0. \tag{5.6.13}$$

*Step 5.* Applying the homomorphism $\varphi_2$ to the relation (5.6.12) one gets

$$c_{11}[\theta_{13}] + c_{15}[\theta_{15}] = 0.$$

By Step 4, we get $c_{10} = \gamma_{41} = 0$. So the relation 5.6.13 becomes

$$c_{15}[\theta_{15}] = 0. \tag{5.6.14}$$

*Step 6.* Applying the homomorphism $\varphi_1$ to the relation 5.6.14 we obtain

$$c_{15}[\theta_{11}] + c_{15}[\theta_{15}] = 0.$$

By Step 5, we get $c_{15}$. The proposition is completely proved. □

### 5.6.3. **The subcase $s = 1,\ t > 2$.**

For $s = 1, t > 2$, we have $n = 2^{t+u+1} + 2^{t+1} - 1 = 2m + 3$ with $m = 2^{t+u} + 2^t - 2$. From Theorem 4.3, we have $B_3(n) = \psi(\Phi(B_2(m)))$.

**Proposition 5.6.7.**

i) $\Phi(B_3(n))\cup\{x_1^3x_2^4x_3^{2^{t+1}-5}x_4^{2^{t+2}-3},\ x_1^3x_2^4x_3^{2^{t+2}-5}x_4^{2^{t+1}-3}\}$ *is the set of of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n = 2^{t+2}+2^{t+1}-1$ with any positive integer $t>2$.*

ii) $\Phi(B_3(n))\cup A(t,u)$ *is the set of of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n = 2^{t+u+1}+2^{t+1}-1$ with any positive integers $t>2, u>1$, where $A(t,u)$ is the set consisting of 3 monomials:*

$$x_1^3x_2^4x_3^{2^{t+1}-5}x_4^{2^{t+u+1}-3},\ x_1^3x_2^4x_3^{2^{t+u+1}-5}x_4^{2^{t+1}-3},\ x_1^3x_2^4x_3^{2^{t+2}-5}x_4^{2^{t+u+1}-2^{t+1}-3}.$$

By a direct computation, we can easy obtain the following lemma.

**Lemma 5.6.8.** *The following monomials are strictly inadmissible:*

$$X_3x_1^2x_2^2x_3^8x_4^{28}x_i^4,\ X_3x_1^2x_2^2x_3^8x_4^{12}x_i^4,\ i=1,2,\ X_4x_1^6x_2^{10}x_3^{12}x_4^{16}.$$

*Proof of Proposition 5.6.7.* Let $x\in P_4$ be an admissible monomial of degree $n = 2^{t+u+1}+2^{t+1}-1$.

By Lemma 5.6.1, $\omega_1(x)=3$. So $x=X_iy^2$ with $y$ a monomial of degree $2^{t+u}+2^t-2$. Since $x$ is admissible, by Theorem 2.9, $y\in B_4(2^{t+u}+2^t-2)$.

By a direct computation, we see that if $x=X_iy^2$ with $y\in B_4(2^{t+u}+2^t-2)$ and $x$ not belongs to the set $C_4(n)$ as given in the proposition, then there is a monomial $w$ which is given in one of Lemmas 5.6.8 and 5.3.3 such that $x=wy^{2^r}$ for some monomial $y$ and integer $r>1$.

By Theorem 2.9, $x$ is inadmissible. Hence $(QP_4)_n$ is spanned by the set $[C_4(n)]$.

We set $|C_4(n)\cap P_4^+| = m(t,u)$ with $m(t,1)=84$ for $u=1$ and $m(t,u)=126$ for $u>1$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{m(t,u)}\gamma_i d_i = 0,$$

with $\gamma_i\in\mathbb{F}_2$ and $d_i = d_{n,i}$. By a direct computation from the relations $p_{(j;J)}(\mathcal{S})\equiv 0$ with $(j;J)\in\mathcal{N}_4$, we obtain $\gamma_i=0$ for all $i$. □

5.6.4. **The subcase $s=2,\ t=1$.**

For $s=2, t=1$, we have $n=2^{u+3}+9$. According to Theorem 4.3, we have

$$B_3(n)=\begin{cases}\psi^2(\Phi(B_2(2^{u+1}))), & \text{if } u\neq 2,\\ \psi^2(\Phi(B_2(8)))\cup\{x_1^{15}x_2^{19}x_3^7\}, & \text{if } u=2.\end{cases}$$

Denote by $G(u)$ the set of 7 monomials:

$$x_1^3x_2^7x_3^{2^{u+3}-5}x_4^4,\ x_1^7x_2^3x_3^{2^{u+3}-5}x_4^4,\ x_1^7x_2^{2^{u+3}-5}x_3^3x_4^4,$$
$$x_1^3x_2^7x_3^7x_4^{2^{u+3}-8},\ x_1^7x_2^3x_3^7x_4^{2^{u+3}-8},\ x_1^7x_2^7x_3^3x_4^{2^{u+3}-8},\ x_1^7x_2^7x_3^{2^{u+3}-8}x_4^3,$$

**Proposition 5.6.9.**

i) $\Phi(B_3(25))\cup G(1)\cup\{x_1^7x_2^9x_3^3x_4^6\}$ *is the set of of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree* 25.

ii) $\Phi(B_3(n))\cup G(u)\cup H(u)$ *is the set of of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n=2^{u+3}+9$ with any positive integer $u>1$, where $H(u)$*

*is the set consisting of* 5 *monomials:*

$$x_1^3x_2^7x_3^{11}x_4^{2^{u+3}-12},\ x_1^7x_2^3x_3^{11}x_4^{2^{u+3}-12},\ x_1^7x_2^{11}x_3^3x_4^{2^{u+3}-12},$$
$$x_1^7x_2^7x_3^8x_4^{2^{u+3}-13},\ x_1^7x_2^7x_3^{11}x_4^{2^{u+3}-16}.$$

The following is proved by a direct computation.

**Lemma 5.6.10.** *The following monomials are strictly inadmissible:*

i) $X_3X_2^2x_1^4x_2^8x_4^4,\ X_jX_2^2x_1^4x_2^8x_4^4,\ X_3^3x_i^4x_3^8x_4^4,\ X_2^3x_1^4x_2^8x_j^4,\ i=1,2,\ j=3,4.$

ii) $X_4X_3^2x_1^{12}x_2^{16}x_3^4, X_4X_2^2x_1^4x_2^{24}x_4^4,\ X_4^3x_i^{12}x_3^{16}x_4^4, X_4X_2^2x_1^{12}x_2^{16}x_4^4, X_4X_3x_1^4x_2^4x_i^8x_3^{16},$ $X_jX_2^2x_1^{12}x_2^{16}x_3^4,\ X_jX_2^2x_1^{12}x_2^{16}x_4^4,\ X_4X_2^2x_1^4x_2^8x_4^{20},\ X_j^3x_1^4x_2^4x_i^8x_j^{16},\ X_2^3x_1^{12}x_2^{16}x_j^4,$ $X_4^3x_i^4x_3^{12}x_4^{16},\ X_4^3x_i^{12}x_3^4x_4^{16},\ X_3^3x_i^{12}x_3^{16}x_4^4,\ X_j^3x_1^4x_2^8x_3^{16}x_4^4,\ X_4X_2^2x_1^4x_2^8x_3^{16}x_4^4$ $X_4^3x_1^4x_2^8x_3^4x_4^{16},\ i=1,2,\ j=3,4.$

*Proof of Proposition 5.6.9.* Let $x$ be an admissible monomial of degree $n=2^{u+3}+9$ in $P_4$.

By Lemma 5.6.1, $\omega_1(x)=\omega_2(x)=3$. So $x=X_iX_j^2y^4$ with $y$ a monomial of degree $2^{u+1}$. Since $x$ is admissible, by Theorem 2.9, $y\in B_4(2^{t+u}+2^t-2)$.

By a direct computation, we see that if $x=X_iX_j^2y^4$ with $y\in B_4(2^{t+u}+2^t-2)$ and $x$ not belongs to the set $C_4(n)$ given in the proposition, then there is a monomial $w$ which is given in one of Lemmas 5.6.10, 5.3.3 such that $x=wy^{2^r}$ for some monomial $y$ and integer $r>1$.

By Theorem 2.9, $x$ is inadmissible. Hence $(QP_4)_n$ is spanned by the set $[C_4(n)]$.

We denote $|C_4(n)\cap P_4^+|=m(u)$ with $m(1)=88$, $m(2)=165$ and $m(u)=154$ for $u\geqslant 3$. Suppose that there is a linear relation

$$\mathcal{S}=\sum_{i=1}^{m(u)}\gamma_id_i=0,$$

with $\gamma_i\in\mathbb{F}_2$ and $d_i=d_{n,i}$. By a direct computation from the relations $p_{(j;J)}(\mathcal{S})\equiv 0$ with $(j;J)\in\mathcal{N}_4$, we obtain $\gamma_i=0$ for all $i$. □

### 5.6.5. The subcase $s=2,\ t\geqslant 2$.

For $s=2, t\geqslant 2$, we have $n=2^{t+u+2}+2^{t+2}+1=4m+9$ with $m=2^{t+u}+2^t-2$. From Theorem 1.3, we have

$$B_3(n)=\psi^2(\Phi(B_2(m))).$$

Denote by $B(t,u)$ the set of 8 monomials:

$$x_1^3x_2^7x_3^{2^{t+2}-5}x_4^{2^{t+u+2}-4},\ x_1^7x_2^3x_3^{2^{t+2}-5}x_4^{2^{t+u+2}-4},\ x_1^7x_2^{2^{t+2}-5}x_3^3x_4^{2^{t+u+2}-4},$$
$$x_1^3x_2^7x_3^{2^{t+u+2}-5}x_4^{2^{t+2}-4},\ x_1^7x_2^3x_3^{2^{t+u+2}-5}x_4^{2^{t+2}-4},\ x_1^7x_2^{2^{t+u+2}-5}x_3^3x_4^{2^{t+2}-4},$$
$$x_1^7x_2^7x_3^{2^{t+2}-8}x_4^{2^{t+u+2}-5},\ x_1^7x_2^7x_3^{2^{t+u+2}-8}x_4^{2^{t+2}-5},$$

and by $C(t,u)$ the set of 4 monomials:

$$x_1^3x_2^7x_3^{2^{t+3}-5}x_4^{2^{t+u+2}-2^{t+2}-4},\ x_1^7x_2^3x_3^{2^{t+3}-5}x_4^{2^{t+u+2}-2^{t+2}-4},$$
$$x_1^7x_2^{2^{t+3}-5}x_3^3x_4^{2^{t+u+2}-2^{t+2}-4},\ x_1^7x_2^7x_3^{2^{t+3}-8}x_4^{2^{t+u+2}-2^{t+2}-5}.$$

**Proposition 5.6.11.**

i) $\Phi(B_3(n))\cup B(t,1)$ *is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n=2^{t+3}+2^{t+2}+1$.*

ii) *For any positive integer $t,u>1$, $\Phi(B_3(n))\cup B(t,u)\cup C(t,u)$ is the set of all the admissible monomials for $\mathcal{A}$-module $P_4$ in degree $n=2^{t+u+2}+2^{t+2}+1$.*

By a direct computation, we get the following.

**Lemma 5.6.12.** *The following monomials are strictly inadmissible:*

$$X_jX_3^2x_1^{12}x_2^{12}x_3^{16}, X_4^3x_i^{12}x_3^{12}x_4^{16},\ X_4^3x_1^{12}x_2^{12}x_4^{16}, X_4^3x_1^4x_2^4x_3^8x_4^8x_j^{16}, X_4X_3^2x_3^4x_1^{12}x_4^8x_2^{16},$$
$$X_4X_3^2x_1^4x_2^4x_4^8x_i^8x_3^{16}, X_j^3x_1^4x_2^4x_3^8x_i^8x_4^{16}, X_4^3x_1^4x_3^4x_2^8x_3^8x_4^{16},\ i=1,2,\ j=3,4.$$

*Proof of Proposition 5.6.11.* Let $x\in P_4$ be an admissible monomial of degree $n=2^{t+u+2}+2^{t+2}+1$. By Lemma 5.6.1, $\omega_1(x)=\omega_2(x)=3$. So $x=X_iX_j^2y^4$ with $y$ a monomial of degree $2^{t+u}+2^t-2$.

Since $x$ is admissible, by Theorem 2.9, $y\in B_4(2^{t+u}+2^t-2)$. By a direct computation, we see that if $x=X_iX_j^2y^4$ with $y\in B_4(2^{t+u}+2^t-2)$ and $x$ not belongs to the set $C_4(n)$ as given in the proposition, then there is a monomial $w$ which is given in one of Lemmas 5.6.12, 5.1.3 such that $x=wy^{2^r}$ for some monomial $y$ and integer $r>1$.

By Theorem 2.9, $x$ is inadmissible. Hence $(QP_4)_n$ is spanned by the set $[C_4(n)]$.

We set $|C_4(n)\cap P_4^+|=m(t,u)$ with $m(t,1)=154$ and $m(t,u)=231$ for $t\geqslant 2$. Suppose that there is a linear relation

$$\mathcal{S}=\sum_{i=1}^{m(t,u)}\gamma_id_i=0,$$

with $\gamma_i\in\mathbb{F}_2$ and $d_i=d_{n,i}$. By a direct computation from the relations $p_{(j;J)}(\mathcal{S})\equiv 0$ with $(j;J)\in\mathcal{N}_4$, we obtain $\gamma_i=0$ for all $i$. □

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