

# Some Positivstellensätze for polynomial matrices

Lê Công-Trình

Received: date / Accepted: date

**Abstract** In this paper we give a version of Krivine-Stengle's Positivstellensatz, Schweighofer's Positivstellensatz, Scheiderer's local-global principle, Scheiderer's Hessian criterion and Marshall's boundary Hessian conditions for polynomial matrices, i.e. matrices with entries from the ring of polynomials in the variables  $x_1, \dots, x_d$  with real coefficients. Moreover, we characterize Archimedean quadratic modules of polynomial matrices, and study the relationship between the compactness of a subset in  $\mathbb{R}^d$  with respect to a subset  $\mathcal{G}$  of polynomial matrices and the Archimedean property of the preordering and the quadratic module generated by  $\mathcal{G}$ .

**Keywords** Positive polynomials · Matrix polynomials · Sum of squares · Positivstellensätze · Local-Global principle · Hessian conditions · Boundary Hessian conditions

**Mathematics Subject Classification (2000)** 14P99 · 13J30 · 15B33 · 15B48

## 1 Introduction

Let  $\mathbb{R}[X] := \mathbb{R}[x_1, \dots, x_d]$  be the ring of polynomials in the variables  $x_1, \dots, x_d$  with real coefficients. Denote by  $\sum \mathbb{R}[X]^2$  the set of *sums of squares* in  $\mathbb{R}[X]$ , i.e. the set of finite sums  $\sum f_i^2, f_i \in \mathbb{R}[X]$ . For a subset  $G = \{g_1, \dots, g_m\} \subseteq \mathbb{R}[X]$ , let us consider the basic closed semi-algebraic set associated to  $G$ ,

$$K_G := \{x \in \mathbb{R}^d \mid g_i(x) \geq 0, i = 1, \dots, m\},$$

---

Lê Công-Trình  
Department of Mathematics, Quy Nhon University  
170 An Duong Vuong, Quy Nhon, Binh Dinh, Vietnam  
E-mail: lecongtrinh@qnu.edu.vn

the quadratic module generated by  $G$ ,

$$M_G = \{t_0 + \sum_{i=1}^m t_i g_i \mid t_i \in \sum \mathbb{R}[X]^2, i = 0, 1, \dots, m\}$$

and the preordering generated by  $G$ ,

$$T_G = \left\{ \sum_{\sigma=(\sigma_1, \dots, \sigma_m) \in \{0,1\}^m} t_\sigma g_1^{\sigma_1} \cdots g_m^{\sigma_m} \mid t_\sigma \in \sum \mathbb{R}[X]^2 \right\}.$$

For a polynomial  $f \in \mathbb{R}[X]$ , it is obvious that if  $f \in M_G$  or  $f \in T_G$  then  $f(x) \geq 0$  for all  $x \in K_G$  (in this case we say  $f \geq 0$  on  $K_G$ ). The converse is in general not true. The Positivstellensatz of Krivine-Stengle ([7, 1964], [16, 1974]) characterizes polynomials which are positive (resp. non-negative, vanished) on a basic closed semi-algebraic set, but with a "denominator" (for example,  $f > 0$  on  $K_G$  if and only if  $pf = 1 + q$  for some  $p, q \in T_G$ , that is,  $f \in \frac{1}{p}(1 + T_G)$  with denominator  $p \in T_G$ ).

A "denominator-free" version of this result is due to Schmüdgen (1991) which asserts that any positive polynomial on a compact set  $K_G$  belongs to  $T_G$ . To ensure for  $f > 0$  on  $K_G$  to be in  $M_G$ , Putinar (1993) required the Archimedean property of  $M_G$ . Note that the compactness of  $K_G$  is equivalent to the Archimedean property of  $T_G$  (cf. [8, Theorem 6.1.1]), and if  $M_G$  is Archimedean then so is  $T_G$ , hence  $K_G$  is compact. However the converse is not true in general (see, for example, [8, Putinar's question, chapter 7]).

If  $K_G$  is not assumed to be compact, Schweighofer ([15]) has given a Positivstellensatz which asserts that if  $f \in \mathbb{R}[X]$  is a bounded, positive polynomial on  $K_G$  and if it has only finitely many asymptotic values on  $K_G$  such that all of them are positive then  $f \in T_G$ .

The case where  $K_G$  is compact (resp.  $M_G$  is Archimedean), but  $f$  is assumed to have finitely many zeros in  $K_G$ , Scheiderer ([9], [10]) has given a Hessian criterion at each zero of  $f$  in  $K_G$  for  $f$  to be in  $T_G$  (resp.  $M_G$ ), using his local-global criterion. Marshall (cf. [8]) has also given boundary Hessian conditions at each zero of  $f$  in  $K_G$  to ensure for  $f$  to be in  $M_G$ .

The aim of this paper is to study all of these Positivstellensätze for polynomial matrices, that is for matrices with entries from  $\mathbb{R}[X]$ . A matrix version of Krivine-Stengle's Positivstellensatz was given by Schmüdgen ([14, 2009], for non-negative polynomial matrices) and Cimprič ([2, 2012]). Hol-Scherer ([11, 2006], or [6, 2010]) has given a matrix version of Putinar's Positivstellensatz. Cimprič has also given a version of Schmüdgen's Positivstellensatz for polynomial matrices in [3, 2013].

In section 2 we recall definition of quadratic modules and preorderings in the algebra  $\mathcal{M}_n(\mathbb{R}[X])$  of polynomial matrices, which is proposed by Schmüdgen ([12], [13], [14]) and Cimprič ([1], [2]), and some basic facts used in the paper. In particular, we recall a basic result of Cimprič (see Lemma 2) which tells us that any subset  $\mathbf{K}_G$  of  $\mathbb{R}^d$  associated to  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  can be determined again by a subset  $G$  of polynomials in  $\mathbb{R}[X]$  such that the preordering  $\mathcal{T}_G$

(resp. the quadratic module  $\mathcal{M}_{\mathcal{G}}$ ) contains the preordering  $(T_{\mathcal{G}})^n$  (resp. the quadratic module  $(M_{\mathcal{G}})^n$ ). Moreover we recall also a basic result of Schmüdgen (see Lemma 3) which asserts that any symmetric polynomial matrix, together with a square of a non-zero polynomial in  $\mathbb{R}[X]$ , can be diagonalized. This allows us to prove many results of this paper firstly with diagonal matrices, and then with arbitrary symmetric matrices.

In section 3 we give a matrix version of Krivine-Stengle's Positivstellensatz (Proposition 1 and Theorem 1). This version for polynomial matrices is simpler than the one given in [14] (for positive semidefinite polynomial matrices), however in general more complicated than the one given in [2]. But in our version, the existence of diagonal polynomial matrices in the representation of  $\mathbf{F}$  in  $(T_{\mathcal{G}})^n$  is more convenient.

In section 4 we give a matrix version of Schweighofer's Positivstellensatz (Proposition 2 and Theorem 3). We have a nice representation for diagonal polynomial matrices, however in the representation of an arbitrary symmetric polynomial matrix we need a "denominator", namely, a square of a non-zero polynomial in  $\mathbb{R}[X]$  or a conjugation of a matrix in  $\mathcal{M}_n(\mathbb{R}[X])$ .

In section 5 we recall definition of Archimedean quadratic modules in  $\mathcal{M}_n(\mathbb{R}[X])$  and characterize Archimedean quadratic modules via the ring of bounded elements with respect to these quadratic modules. We show that the Archimedean property of a quadratic module  $M$  in  $\mathbb{R}[X]$  is the same as that of the quadratic module  $M^n$  in  $\mathcal{M}_n(\mathbb{R}[X])$ , and the compactness of the set  $\mathbf{K}_{\mathcal{G}}$  is equivalent to the Archimedean property of the preordering  $\mathcal{T}_{\mathcal{G}}$ . Moreover, we show that if the quadratic module  $\mathcal{M}_{\mathcal{G}}$  of univariate polynomial matrices is Archimedean then the set  $\mathbf{K}_{\mathcal{G}}$  is compact.

The last section deals with a matrix version of Scheiderer's local-global principle (Proposition 7 and Theorem 5), Scheiderer's Hessian criterion (Proposition 8 and Theorem 7) and Marshall's boundary Hessian conditions (Proposition 9 and Theorem 9). Similar to the matrix version of Schweighofer's Positivstellensatz given in section 4, we have a nice representation of diagonal polynomial matrices, but for an arbitrary symmetric polynomial matrices we need a denominator.

## 2 Preliminaries

In this section we shall recall some basic concepts and facts in Real algebraic geometry for matrices over commutative rings which are proposed by Schmüdgen ([12], [13], [14]) and Cimprič ([1], [2]).

For  $n \in \mathbb{N}^*$ , let  $\mathcal{M}_n(R)$  denote the ring of  $n \times n$  matrices with entries from a commutative unital ring  $R$ . Denote by  $\mathcal{S}_n(R)$  the subset of  $\mathcal{M}_n(R)$  consisting of all symmetric matrices. A subset  $\mathcal{M}$  of  $\mathcal{S}_n(R)$  is called a *quadratic module*<sup>1</sup> if

$$\mathbf{I}_n \in \mathcal{M}, \quad \mathcal{M} + \mathcal{M} \subseteq \mathcal{M}, \quad A^T \mathcal{M} A \subseteq \mathcal{M}, \quad \forall A \in \mathcal{M}_n(R).$$

<sup>1</sup> In [12] and [13], the term *m-admissible wedge* was used.

The smallest quadratic module which contains a given subset  $\mathcal{G}$  of  $\mathcal{S}_n(R)$  will be denoted by  $\mathcal{M}_{\mathcal{G}}$ . It is clear that

$$\mathcal{M}_{\mathcal{G}} = \left\{ \sum_{i,j} A_{ij}^T G_i A_{ij} \mid G_i \in \mathcal{G} \cup \{\mathbf{I}_n\}, A_{ij} \in \mathcal{M}_n(R) \right\}.$$

In particular, a subset  $M \subseteq R$  is a quadratic module if  $1 \in M, M + M \subseteq M$ , and  $a^2 M \subseteq M$  for all  $a \in R$ . The smallest quadratic module of  $R$  which contains a given subset  $G \subseteq R$  will be denoted by  $M_G$ , and it consists of all finite sums of the form  $\sum_{i,j} a_{ij}^2 g_i, g_i \in G, a_{ij} \in R$ .

A subset  $\mathcal{T}$  of  $\mathcal{S}_n(R)$  is called a *preordering* if  $\mathcal{T}$  is a quadratic module in  $\mathcal{M}_n(R)$  and the set  $\mathcal{T} \cap (R \cdot \mathbf{I}_n)$  is closed under multiplication. The smallest preordering which contains a given subset  $\mathcal{G}$  of  $\mathcal{S}_n(R)$  will be denoted by  $\mathcal{T}_{\mathcal{G}}$ . We have

**Lemma 1** ([2, Lemma 2]) *For every subset  $\mathcal{G}$  of  $\mathcal{S}_n(R)$ ,*

$$\mathcal{T}_{\mathcal{G}} = \mathcal{M}_{\mathcal{G} \cup (\prod \mathcal{G}' \cdot \mathbf{I}_n)},$$

where  $\prod \mathcal{G}'$  is the set of all finite product of elements from the set  $\mathcal{G}' := \{\mathbf{v}^T \mathbf{G} \mathbf{v} \mid \mathbf{G} \in \mathcal{G}, \mathbf{v} \in R^n\}$ .

In particular, a subset  $T \subseteq R$  is a preordering if  $T + T \subseteq T, T \cdot T \subseteq T, a^2 \in T$  for every  $a \in R$ . The smallest preordering of  $R$  which contains a given subset  $G \subseteq R$  will be denoted by  $T_G$ . It is clear that

$$T_G = \left\{ \sum_{\sigma=(\sigma_1, \dots, \sigma_m) \in \{0,1\}^m} s_{\sigma} g_1^{\sigma_1} \cdots g_m^{\sigma_m} \mid m \in \mathbb{N}, g_i \in G, s_{\sigma} \in \sum R^2 \right\},$$

where  $\sum R^2$  is the set of all sums of squares of finite elements from  $R$ .

In the case  $\mathcal{G} = \emptyset$ ,  $\sum_n R := \mathcal{M}_{\emptyset} = \mathcal{T}_{\emptyset}$  is the set of all finite sums of elements of the form  $A^T A$ , where  $A \in \mathcal{M}_n(R)$ , and which is the smallest quadratic module in  $\mathcal{M}_n(R)$ .

For a quadratic module  $M$  in  $R$ , denote

$$M^n := \left\{ \sum_i m_i A_i^T A_i \mid m_i \in M, A_i \in \mathcal{M}_n(R) \right\}.$$

Then  $M^n$  is the smallest quadratic module in  $\mathcal{M}_n(R)$  whose intersection with  $R \cdot \mathbf{I}_n$  is equal to  $M \cdot \mathbf{I}_n$  ([2, Proposition 3]).

*Remark 1* Let  $M$  be a quadratic module of  $R$ . Denote by  $D(d_1, \dots, d_r), r \leq n$ , the  $n \times n$  diagonal matrix with diagonal entries  $d_1, \dots, d_r, 0, \dots, 0$ , where  $d_i \in M$  for every  $i = 1, \dots, r$ . Then  $D(d_1, \dots, d_r) \in M^n$ .

In fact, for every  $i, j = 1, \dots, n$ , let  $\mathbf{E}_{ij}$  be the coordinate matrices in  $\mathcal{M}_n(R)$ . Note that for each  $i = 1, \dots, n$ , we have  $\mathbf{E}_{ii} = \mathbf{E}_{ii}^T \mathbf{E}_{ii}$ . Hence

$$D(d_1, \dots, d_r) = \sum_{i=1}^r d_i \mathbf{E}_{ii} = \sum_{i=1}^r d_i \mathbf{E}_{ii}^T \mathbf{E}_{ii} \in M^n.$$

For any matrix  $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ , the notation  $\mathbf{A} \succcurlyeq \mathbf{0}$  means  $\mathbf{A}$  is *positive semidefinite*, i.e.  $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$  for every  $\mathbf{x} \in \mathbb{R}^n$ , and  $\mathbf{A} \succ \mathbf{0}$  means  $\mathbf{A}$  is *positive definite*, i.e.  $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$  for every  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ .

In the following we consider  $R$  to be the ring  $\mathbb{R}[X] := \mathbb{R}[x_1, \dots, x_d]$  of polynomials in  $d$  variables  $x_1, \dots, x_d$  with real coefficients. Then each element  $\mathbf{A} \in \mathcal{M}_n(\mathbb{R}[X])$  is a matrix whose entries are polynomials from  $\mathbb{R}[X]$ , called a *polynomial matrix*. Each element  $\mathbf{A} \in \mathcal{M}_n(\mathbb{R}[X])$  is also called a *matrix polynomial*, because it can be viewed as a polynomial in  $x_1, \dots, x_d$  whose entries from  $\mathcal{M}_n(\mathbb{R})$ . Namely, we can write  $\mathbf{A}$  as

$$\mathbf{A} = \sum_{|\alpha|=0}^N \mathbf{A}_\alpha X^\alpha,$$

where  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ ,  $|\alpha| := \alpha_1 + \dots + \alpha_d$ ,  $X^\alpha := x_1^{\alpha_1} \dots x_d^{\alpha_d}$ ,  $\mathbf{A}_\alpha \in \mathcal{M}_n(\mathbb{R})$ ,  $N$  is the maximum over all degree of entries of  $\mathbf{A}$ . To unify notation, throughout the paper each element of  $\mathcal{M}_n(\mathbb{R}[X])$  is called a *polynomial matrix*.

To every  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  we associate the set

$$\mathbf{K}_\mathcal{G} := \{x \in \mathbb{R}^d \mid \mathbf{G}(x) \succcurlyeq \mathbf{0}, \forall \mathbf{G} \in \mathcal{G}\}.$$

In particular, for a subset  $G$  of  $\mathbb{R}[X]$ ,

$$K_G = \{x \in \mathbb{R}^d \mid g(x) \geq 0, \forall g \in G\}.$$

The following result of Cimpric ([2]) shows that the set  $\mathbf{K}_\mathcal{G}$  can be determined by *scalars*, i.e. by polynomials in  $\mathbb{R}[X]$ .

**Lemma 2 ([2, Proposition 5])** *Let  $\mathcal{G} \subset \mathcal{S}_n(\mathbb{R}[X])$ . Then there exists a subset  $G$  of  $\mathbb{R}[X]$  with the following properties:*

- (1)  $\mathbf{K}_\mathcal{G} = K_G$ ;
- (2)  $(M_G)^n \subseteq \mathcal{M}_\mathcal{G}$ ;
- (3)  $(T_G)^n \subseteq \mathcal{T}_\mathcal{G}$ .

Moreover, if  $\mathcal{G}$  is finite then  $G$  can be chosen to be finite.

It is well-known that every symmetric scalar matrix  $A \in \mathcal{S}_n(\mathbb{R})$  can be diagonalized by an orthogonal matrix  $\mathbf{O} \in \mathcal{M}_n(\mathbb{R})$ . For a polynomial matrix  $\mathbf{A}$  in  $\mathcal{S}_n(\mathbb{R}[X])$ , it is in general no longer true, because the matrix  $\mathbf{O}$  may have rational entries (quotients of two polynomials in  $\mathbb{R}[X]$ ). However, Schmüdgen ([14]) has showed that every symmetric polynomial matrix can be diagonalized by an invertible matrix in  $\mathcal{M}_n(\mathbb{R}[X])$  with a quotient by a non-zero polynomial in  $\mathbb{R}[X]$ . Moreover, in some special cases (e.g. that symmetric polynomial is in *standard form*), that invertible matrix can be chosen to be lower triangular.

**Lemma 3 ([14, Corollary 9])** *Let  $\mathbf{A} \in \mathcal{S}_n(\mathbb{R}[X])$ . Then there exist non-zero polynomials  $b, d_j \in \mathbb{R}[X]$ ,  $j = 1, \dots, r$ ,  $r \leq n$ , and matrices  $\mathbf{X}_+, \mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that*

$$\mathbf{X}_+ \mathbf{X}_- = \mathbf{X}_- \mathbf{X}_+ = b \mathbf{I}_n, \quad b^2 \mathbf{A} = \mathbf{X}_+ \mathbf{D} \mathbf{X}_+^T, \quad \mathbf{D} = \mathbf{X}_- \mathbf{A} \mathbf{X}_-^T,$$

where  $\mathbf{D}$  is the  $n \times n$  diagonal matrix  $D(d_1, \dots, d_r)$ .

This lemma deduces a matrix version of the theorem of Artin on Hilbert's seventeenth problem.

**Corollary 1** ([14, Proposition 10], [4]) *Let  $\mathbf{F} \in \mathcal{S}_n(\mathbb{R}[X])$ . Then the following are equivalent:*

- (1)  $\mathbf{F} \succ \mathbf{0}$  (i.e.  $\mathbf{F}(x) \succ \mathbf{0}$  for every  $x \in \mathbb{R}^d$ );
- (2)  $b^2\mathbf{F} \in \sum_n \mathbb{R}[X]$  for some non-zero polynomial  $b \in \mathbb{R}[X]$ .

### 3 Krivine-Stengle's Positivstellensatz for polynomial matrices

In this section we shall give a matrix version of Krivine-Stengle's Positivstellensatz (cf. [7], [16], [8, Positivstellensatz 2.2.1]). Let  $\mathcal{G} = \{\mathbf{G}_1, \dots, \mathbf{G}_m\} \subseteq \mathcal{S}_n(\mathbb{R}[X])$ . Then by Lemma 2, there exists a subset  $G = \{g_1, \dots, g_k\}$  of  $\mathbb{R}[X]$  such that  $\mathbf{K}_{\mathcal{G}} = K_G$  and  $(T_G)^n \subseteq \mathcal{T}_{\mathcal{G}}$ . For diagonal polynomial matrices, we have the following

**Proposition 1** *Let  $\mathbf{D} = D(d_1, \dots, d_r)$ ,  $r \leq n$ , be an  $n \times n$  diagonal matrix in  $\mathcal{S}_n(\mathbb{R}[X])$ . Then*

- (1)  $\mathbf{D} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$  if and only if there exist diagonal matrices  $\mathbf{S}$  and  $\mathbf{T}$  whose entries are in  $T_G$  such that  $\mathbf{SD} = \mathbf{DS} = \mathbf{I}_n + \mathbf{T}$ .
- (2)  $\mathbf{D} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$  if and only if there exist an integer  $m \geq 0$  and diagonal matrices  $\mathbf{S}$  and  $\mathbf{T}$  whose entries are in  $T_G$  such that  $\mathbf{SD} = \mathbf{DS} = \mathbf{D}^{2m} + \mathbf{T}$ .
- (3)  $\mathbf{D} = \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$  if and only if there exist an integer  $m \geq 0$  such that  $-\mathbf{D}^{2m} \in (T_G)^n$ .
- (4)  $\mathbf{K}_{\mathcal{G}} = \emptyset$  if and only if  $-\mathbf{I}_n \in (T_G)^n$ .

*Proof* Note that in each of (1), (2), (3), (4), the "if" part is trivial. Therefore we shall prove the "only if" part in these statements.

(1) Assume  $\mathbf{D} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ . Then  $r = n$  and  $d_i > 0$  on  $\mathbf{K}_{\mathcal{G}} = K_G$  for all  $i = 1, \dots, n$ . It follows from Krivine-Stengle's Positivstellensatz that for each  $i = 1, \dots, n$ , there exist  $s_i$  and  $t_i$  in  $T_G$  such that  $s_i d_i = 1 + t_i$ . Then the matrices  $\mathbf{S} = D(s_1, \dots, s_r)$  and  $\mathbf{T} = D(t_1, \dots, t_r)$  satisfy (1).

(2) Assume  $\mathbf{D} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ . Then  $d_i \geq 0$  on  $\mathbf{K}_{\mathcal{G}} = K_G$  for all  $i = 1, \dots, r$ . It follows from Krivine-Stengle's Positivstellensatz that for each  $i = 1, \dots, r$ , there exist an integer  $m_i \geq 0$  and elements  $s_i$  and  $t_i$  in  $T_G$  such that  $s_i d_i = d_i^{2m_i} + t_i$ . Let  $m = \max\{m_i, i = 1, \dots, r\}$ . Then for every  $i = 1, \dots, r$ , we have

$$(s_i d_i^{2(m-m_i)})d_i = d_i^{2m} + (t_i d_i^{2(m-m_i)}).$$

Denote  $s'_i := s_i d_i^{2(m-m_i)}$ ,  $t'_i := t_i d_i^{2(m-m_i)}$ . Then  $\mathbf{S} = D(s'_1, \dots, s'_r)$  and  $\mathbf{T} = D(t'_1, \dots, t'_r)$  satisfy (2).

(3) Assume  $\mathbf{D} = \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ . Then  $d_i = 0$  on  $\mathbf{K}_{\mathcal{G}} = K_G$  for all  $i = 1, \dots, r$ . It follows from Krivine-Stengle's Positivstellensatz that for each  $i = 1, \dots, r$ , there exists an integer  $m_i \geq 0$  such that  $-d_i^{2m_i} \in T_G$ . Then for  $m = \max\{m_i, i = 1, \dots, r\}$  we have  $-d_i^{2m} \in T_G$  for every  $i = 1, \dots, r$ . Then  $-\mathbf{D}^{2m} \in (T_G)^n$  by Remark 1.

(4) follows from Krivine-Stengle's Positivstellensatz and Remark 1.

For arbitrary symmetric polynomial matrices, we have the following

**Theorem 1** *Let  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[X])$ ,  $G \subseteq \mathbb{R}[X]$ ,  $\mathbf{K}_{\mathcal{G}}$ ,  $K_G$ ,  $\mathcal{T}_{\mathcal{G}}$  and  $T_G$  be determined as above. Then for  $\mathbf{F} \in \mathcal{S}_n(\mathbb{R}[X])$ , we have*

- (1)  $\mathbf{F} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$  if and only if there exist a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  and diagonal matrices  $\mathbf{S}$  and  $\mathbf{T}$  whose entries are in  $T_G$  such that  $\mathbf{S}(\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T) = (\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T) \mathbf{S} = \mathbf{I}_n + \mathbf{T}$ .
- (2)  $\mathbf{D} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$  if and only if there exist an integer  $m \geq 0$ , a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  and diagonal matrices  $\mathbf{S}$  and  $\mathbf{T}$  whose entries are in  $T_G$  such that  $\mathbf{S}(\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T) = (\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T) \mathbf{S} = \mathbf{D}^{2m} + \mathbf{T}$ .
- (3)  $\mathbf{D} = \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$  if and only if there exist an integer  $m \geq 0$  and a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that  $-(\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T)^{2m} \in (T_G)^n$ .

*Proof* By Lemma 3, there exist non-zero polynomials  $b, d_j \in \mathbb{R}[X]$ ,  $j = 1, \dots, r$ ,  $r \leq n$ , and a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T = D(d_1, \dots, d_r) =: \mathbf{D}$ . Note that  $\mathbf{F} \succ \mathbf{0}$  (resp.  $\succcurlyeq \mathbf{0}$ ,  $= \mathbf{0}$ ) if and only if  $\mathbf{D} \succ \mathbf{0}$  (and  $r = n$ ) (resp.  $\succcurlyeq \mathbf{0}$ ,  $= \mathbf{0}$ ). Therefore the theorem follows from Proposition 1, applying for  $\mathbf{D} = \mathbf{X}_- \mathbf{F} \mathbf{X}_-^T$ .

- Remark 2* (1) Theorem 1 (1) gives a simpler representation of the positive definite polynomial matrix  $\mathbf{F}$  on  $\mathbf{K}_{\mathcal{G}}$ , comparing to the non-commutative version of Krivine-Stengle' Positivstellensatz given in [14, sections 4.2 and 4.4].
- (2) In [2] the author has given a matrix version of Krivine-Stengle's Positivstellensatz without the matrix  $\mathbf{X}_-$  in representation of  $\mathbf{F}$ . He requires also  $\mathbf{S}, \mathbf{T} \in (T_G)^n$ , however they are in general not diagonal.

#### 4 Schweighofer's Positivstellensatz for polynomial matrices

In this section we give a matrix version of Schweighofer's Positivstellensatz ([15]) which is recalled as follows. For a polynomial  $f \in \mathbb{R}[X]$  and a subset  $S \subseteq \mathbb{R}^d$ , a real number  $y \in \mathbb{R}$  is called an *asymptotic value* of  $f$  on  $S$  if there exists a sequence  $(x_k)_{k \in \mathbb{N}} \subseteq S$  such that  $\lim_{k \rightarrow \infty} \|x_k\| = \infty$  and  $\lim_{k \rightarrow \infty} f(x_k) = y$ . Denote by  $R_{\infty}(f, S)$  the set of all asymptotic values of  $f$  on  $S$ . Then we have

**Theorem 2** ([15, Theorem 9]) *Let  $G = \{g_1, \dots, g_m\} \subseteq \mathbb{R}[X]$ , and  $f \in \mathbb{R}[X]$ . Assume*

- (1)  $f > 0$  on  $K_G$ ;
- (2)  $f$  is bounded on  $K_G$ ;
- (3)  $R_{\infty}(f, K_G)$  is a finite subset of  $\mathbb{R}_+$ .

*Then  $f \in T_G$ .*

We give firstly a version of this theorem for diagonal polynomial matrices.

**Proposition 2** *Let  $\mathcal{G} = \{\mathbf{G}_1, \dots, \mathbf{G}_m\} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $\mathbf{D} = D(d_1, \dots, d_n)$  be an  $n \times n$  diagonal matrix in  $\mathcal{S}_n(\mathbb{R}[X])$  with  $d_i \neq 0$  for every  $i = 1, \dots, n$ . Assume*

- (1)  $\mathbf{D} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ ;
- (2)  $\mathbf{D}$  is bounded on  $\mathbf{K}_{\mathcal{G}}$  (i.e. there exists a number  $N \in \mathbb{R}_+$  such that  $N \cdot \mathbf{I}_n \pm \mathbf{D} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ );
- (3) For every  $i = 1, \dots, n$ ,  $R_{\infty}(d_i, \mathbf{K}_{\mathcal{G}})$  is a finite subset of  $\mathbb{R}_+$ .

Then there exists a finite subset  $G$  of  $\mathbb{R}[X]$  such that  $D \in (T_G)^n \subseteq \mathcal{T}_{\mathcal{G}}$ .

*Proof* By Lemma 2, there exists a finite subset  $G$  of  $\mathbb{R}[X]$  such that  $K_{\mathcal{G}} = K_G$  and  $(T_G)^n \subseteq \mathcal{T}_{\mathcal{G}}$ . By hypothesis, for every  $i = 1, \dots, n$  we have

- $d_i > 0$  on  $K_G$ ;
- $d_i$  is bounded on  $K_G$ ;
- $R_{\infty}(d_i, K_G)$  is a finite subset of  $\mathbb{R}_+$ .

Then it follows from Theorem 2 that  $d_i \in T_G$  for every  $i = 1, \dots, n$ . This implies that  $\mathbf{D} \in (T_G)^n$  by Remark 1.

For arbitrary symmetric polynomial matrices we have the following

**Theorem 3** Let  $\mathcal{G} = \{\mathbf{G}_1, \dots, \mathbf{G}_m\} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $\mathbf{F} \in \mathcal{S}_n(\mathbb{R}[X])$ . Assume

- (1)  $\mathbf{F} \succ \mathbf{0}$  on  $\mathcal{K}_{\mathcal{G}}$ ;
- (2)  $\mathbf{F}$  is bounded on  $\mathbf{K}_{\mathcal{G}}$  (i.e. there exists a number  $N \in \mathbb{R}_+$  such that  $N \cdot \mathbf{I}_n \pm \mathbf{F} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ );
- (3) for every  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ ,  $R_{\infty}(\mathbf{x}^T \mathbf{F} \mathbf{x}, \mathbf{K}_{\mathcal{G}})$  is a finite subset of  $\mathbb{R}_+$ .

Then there exist a finite subset  $G$  of  $\mathbb{R}[X]$  and

- (i) a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T \in (T_G)^n \subseteq \mathcal{T}_{\mathcal{G}}$ ;
- (ii) a non-zero polynomial  $b \in \mathbb{R}[X]$  such that  $b^2 \mathbf{F} \in (T_G)^n \subseteq \mathcal{T}_{\mathcal{G}}$ .

*Proof* By Lemma 3, there exist non-zero polynomials  $b, d_j \in \mathbb{R}[X]$ ,  $j = 1, \dots, r$ ,  $r \leq n$ , and matrices  $\mathbf{X}_+, \mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that

$$\mathbf{X}_+ \mathbf{X}_- = \mathbf{X}_- \mathbf{X}_+ = b \mathbf{I}_n, \quad b^2 \mathbf{F} = \mathbf{X}_+ \mathbf{D} \mathbf{X}_+^T, \quad \mathbf{D} = \mathbf{X}_- \mathbf{F} \mathbf{X}_-^T,$$

where  $\mathbf{D} = D(d_1, \dots, d_r)$  is the  $n \times n$  diagonal polynomial matrix. Since  $\mathbf{F} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ ,  $r = n$ . Note that for every  $i = 1, \dots, n$ ,

$$d_i = \mathbf{e}_i^T \mathbf{D} \mathbf{e}_i = (\mathbf{X}_-^T \mathbf{e}_i)^T \mathbf{F} (\mathbf{X}_-^T \mathbf{e}_i), \quad (1)$$

where  $\mathbf{e}_i$ ,  $i = 1, \dots, n$ , are the coordinate vectors in  $\mathbb{R}^n$ . Since  $\mathbf{v}_i := \mathbf{X}_-^T \mathbf{e}_i \in \mathbb{R}^n \setminus \{0\}$  and  $\mathbf{F} \succ \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ , it follows that  $d_i > 0$  on  $\mathbf{K}_{\mathcal{G}}$  for every  $i = 1, \dots, n$ .

By (2) and in view of (1), for each  $i = 1, \dots, n$ , we have

$$N(\mathbf{v}_i^T \mathbf{v}_i) \pm d_i \geq 0 \text{ on } \mathbf{K}_{\mathcal{G}}.$$

It follows that each  $d_i$ ,  $i = 1, \dots, n$ , is bounded on  $\mathbf{K}_{\mathcal{G}}$ . Moreover, by (3) and in view of (1),  $R_{\infty}(d_i, \mathbf{K}_{\mathcal{G}})$  is a finite subset of  $\mathbb{R}_+$  for each  $i = 1, \dots, n$ . Then it follows from Proposition 2 that there exists a finite subset  $G$  of  $\mathbb{R}[X]$  such that  $\mathbf{D} \in (T_G)^n \subseteq \mathcal{T}_{\mathcal{G}}$ , hence  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T \in (T_G)^n$ , i.e. we have (i). Moreover, since  $(T_G)^n$  is a quadratic module of  $\mathcal{M}_n(\mathbb{R}[X])$ , by definition we have  $b^2 \mathbf{F} = \mathbf{X}_+ \mathbf{D} \mathbf{X}_+^T \in (T_G)^n$ , i.e. we have (ii). The proof is complete.



## 5 Archimedean quadratic modules

In this section we deal with Archimedean quadratic modules of polynomial matrices. We recall the definition of Archimedean quadratic modules, and show that the Archimedean property of a quadratic module  $\mathcal{M}$  in  $\mathbb{R}[X]$  is the same as that of the quadratic module  $\mathcal{M}^n$  in  $\mathcal{M}_n(\mathbb{R}[X])$ . Moreover, we also show that the compactness of  $\mathbf{K}_{\mathcal{G}}$  is equivalent to the Archimedean property of the preordering  $\mathcal{T}_{\mathcal{G}}$ .

**Definition 1** ([13], [14], [1]) Let  $\mathcal{M}$  be a quadratic module in  $\mathcal{M}_n(\mathbb{R}[X])$ .

- (1)  $\mathcal{M}$  is called *Archimedean* if for each element  $\mathbf{A} \in \mathcal{M}_n(\mathbb{R}[X])$  there exists a number  $n \in \mathbb{N}$  such that  $n \cdot \mathbf{I}_n - \mathbf{A}^T \mathbf{A} \in \mathcal{M}$ .
- (2) Denote

$$H_{\mathcal{M}} := H_{\mathcal{M}}(\mathcal{M}_n(\mathbb{R}[X])) := \{\mathcal{A} \in \mathcal{M}_n(\mathbb{R}[X]) \mid \exists r \in \mathbb{R}_+ : r^2 \cdot \mathbf{I}_n - \mathcal{A}^T \mathcal{A} \in \mathcal{M}\}.$$

It is clear that the quadratic module  $\mathcal{M}$  in  $\mathcal{M}_n(\mathbb{R}[X])$  is Archimedean if and only if  $H_{\mathcal{M}} = \mathcal{M}_n(\mathbb{R}[X])$ .<sup>2</sup> Moreover,  $H_{\mathcal{M}}$  is a subring of  $\mathcal{M}_n(\mathbb{R}[X])$  (cf. [13, Corollary 2.2], [1, Corollary 5]), and it is called the *ring of bounded elements* of  $\mathcal{M}_n(\mathbb{R}[X])$  with respect to the quadratic module  $\mathcal{M}$ .

The following fact is useful and it is easy to check (cf. [13, Lemma 2.1(ii)], [1, Lemma 3]).

**Lemma 4** Let  $\mathcal{M}$  be a quadratic module in  $\mathcal{M}_n(\mathbb{R}[X])$ . Then for any  $\mathbf{A} \in \mathcal{S}_n(\mathbb{R}[X])$  and for any  $r \in \mathbb{R}_+$ , we have  $r^2 \cdot \mathbf{I}_n - \mathbf{A}^2 \in \mathcal{M}$  if and only if  $r \cdot \mathbf{I}_n \pm \mathbf{A} \in \mathcal{M}$ .

Similar to the case of polynomials (cf. [8, Corollary 5.2.4]), we can check the Archimedean property of quadratic modules of polynomial matrices simply as follows.

**Proposition 3** Let  $\mathcal{M} \subseteq \mathcal{M}_n(\mathbb{R}[X])$  be a quadratic module. Then the following are equivalent:

- (1)  $\mathcal{M}$  is Archimedean.
- (2)  $r \cdot \mathbf{I}_n - \sum_{i=1}^d x_i^2 \cdot \mathbf{I}_n \in \mathcal{M}$  for some positive real number  $r$ .
- (3)  $r \cdot \mathbf{I}_n \pm x_i \cdot \mathbf{I}_n \in \mathcal{M}$  for some positive real number  $r$ .

*Proof* (1)  $\implies$  (2) is clear. If (2) holds, for each  $i = 1, \dots, d$  we have

$$r \cdot \mathbf{I}_n - x_i^2 \cdot \mathbf{I}_n = (r \cdot \mathbf{I}_n - \sum_{i=1}^d x_i^2 \cdot \mathbf{I}_n) + \sum_{j \neq i} x_j^2 \cdot \mathbf{I}_n \in \mathcal{M}.$$

It follows from Lemma 4 that  $\sqrt{r} \cdot \mathbf{I}_n \pm x_i \cdot \mathbf{I}_n \in \mathcal{M}$  for every  $i = 1, \dots, d$ , i.e. we have (3).

<sup>2</sup> In this case, the ring  $\mathcal{M}_n(\mathbb{R}[X])$  is called *algebraically bounded* with respect to the quadratic module  $\mathcal{M}$ , cf. [13].

To show (3)  $\implies$  (1), it suffices to prove that  $H_{\mathcal{M}} = \mathcal{M}_n(\mathbb{R}[X])$ . Since  $\mathcal{M}_n(\mathbb{R}[X])$  is generated as an  $\mathbb{R}$ -algebra by  $x_i, i = 1, \dots, d$ , and the coordinate matrices  $\mathbf{E}_{ij}, i, j = 1, \dots, n$ , of  $M_n(\mathbb{R})$ , and since  $H_{\mathcal{M}}$  is closed under addition and multiplication, it is enough to show that  $x_i \cdot \mathbf{I}_n \in H_{\mathcal{M}}$  for every  $i = 1, \dots, d$  and  $\mathbf{E}_{ij} \in H_{\mathcal{M}}$  for every  $i, j = 1, \dots, n$ .

Since  $r \cdot \mathbf{I}_n \pm x_i \cdot \mathbf{I}_n \in \mathcal{M}$ , it follows from Lemma 4 that  $r^2 \cdot \mathbf{I}_n - x_i^2 \cdot \mathbf{I}_n \in \mathcal{M}$ , hence  $x_i \cdot \mathbf{I}_n \in H_{\mathcal{M}}$  for every  $i = 1, \dots, d$ . On the other hand, for every  $i, j = 1, \dots, n$ , we have  $\mathbf{E}_{ij}^T \mathbf{E}_{ij} = \mathbf{E}_{jj}$ . Therefore,

$$\mathbf{I}_n - \mathbf{E}_{ij}^T \mathbf{E}_{ij} = \sum_{k \neq j} \mathbf{E}_{kk}^T \mathbf{E}_{kk} \in \mathcal{M}.$$

It follows that  $\mathbf{E}_{ij} \in H_{\mathcal{M}}$  for every  $i, j = 1, \dots, n$ . The proof is complete.

Using this criterion we can show now the equivalence of the Archimedean property of a quadratic module  $M$  in  $\mathbb{R}[X]$  and the quadratic module  $M^n$  in  $\mathcal{M}_n(\mathbb{R}[X])$ .

**Proposition 4** *Let  $M$  be a quadratic module in  $\mathbb{R}[X]$ . Then  $M$  is Archimedean if and only if  $M^n$  is an Archimedean quadratic module in  $\mathcal{M}_n(\mathbb{R}[X])$ .*

*Proof* The "only if" part follows easily from the usual criterion for Archimedean property of quadratic modules in  $\mathbb{R}[X]$  (cf. [8, Corollary 5.2.4]) and Proposition 3. Now we prove the "if" part.

Assume  $M^n$  is Archimedean. Then it follows from Proposition 3 that for every  $i = 1, \dots, d$ , we have  $r \cdot \mathbf{I}_n \pm x_i \cdot \mathbf{I}_n \in M^n$  for some  $r \in \mathbb{R}_+$ . Then we can write

$$(r \pm x_i) \cdot \mathbf{I}_n = r \cdot \mathbf{I}_n \pm x_i \cdot \mathbf{I}_n = \sum_{j=1}^m m_j \mathbf{A}_j^T \mathbf{A}_j, \text{ where } m_j \in M, \mathbf{A}_j \in \mathcal{M}_n(\mathbb{R}[X]).$$

Note that  $Tr(\mathbf{A}_j^T \mathbf{A}_j) \in \sum \mathbb{R}[X]^2$  for each  $j = 1, \dots, m$ . Then for every  $i = 1, \dots, d$  we have

$$r \pm x_i = \frac{1}{n} Tr((r \pm x_i) \cdot \mathbf{I}_n) = \frac{1}{n} \sum_{j=1}^m m_j Tr(\mathbf{A}_j^T \mathbf{A}_j) \in M.$$

Hence  $M$  is Archimedean (cf. [8, Corollary 5.2.4]).

It is well-known that the compactness of the basic semi-algebraic set  $K_G \subseteq \mathbb{R}^n$ ,  $G \subseteq \mathbb{R}[X]$ , is equivalent to the Archimedean property of the preordering  $T_G$  in  $\mathbb{R}[X]$  (cf. [8, Theorem 6.1.1]). For polynomial matrices we have also the same result.

**Proposition 5** *Let  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[X])$ . Then  $\mathbf{K}_{\mathcal{G}}$  is compact if and only if  $\mathcal{T}_{\mathcal{G}}$  is Archimedean.*

*Proof* Assume  $\mathcal{T}_{\mathcal{G}}$  is Archimedean. It follows from Proposition 3 that there exists a number  $r \in \mathbb{R}_+$  such that  $r \cdot \mathbf{I}_n - \sum_{i=1}^d x_i^2 \cdot \mathbf{I}_n \in \mathcal{T}_{\mathcal{G}}$ . This implies  $r \cdot \mathbf{I}_n - \sum_{i=1}^d x_i^2 \cdot \mathbf{I}_n \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ . Then for any point  $p = (p_1, \dots, p_d) \in \mathbf{K}_{\mathcal{G}}$ , we have  $r - \sum_{i=1}^d p_i^2 \geq 0$ , i.e.,  $\|p\| \leq \sqrt{r}$ . It follows that  $\mathbf{K}_{\mathcal{G}}$  is bounded, whence compact.

Conversely, assume that  $\mathbf{K}_{\mathcal{G}}$  is compact. By Lemma 2, there exists a subset  $G$  of  $\mathbb{R}[X]$  such that  $\mathbf{K}_{\mathcal{G}} = K_G$  and  $(T_G)^n \subseteq \mathcal{T}_{\mathcal{G}}$ . Then  $K_G$  is compact. It follows that  $T_G$  is an Archimedean quadratic module in  $\mathbb{R}[X]$  (cf. [8, Theorem 6.1.1]). Then  $(T_G)^n \subseteq \mathcal{M}_n(\mathbb{R}[X])$  is Archimedean by Proposition 4. This implies that  $\mathcal{T}_{\mathcal{G}} \supseteq (T_G)^n$  is Archimedean.

*Remark 3* For any  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[X])$ , since  $\mathcal{M}_{\mathcal{G}} \subseteq \mathcal{T}_{\mathcal{G}}$ , if  $\mathcal{M}_{\mathcal{G}}$  is Archimedean then  $\mathcal{T}_{\mathcal{G}}$  is Archimedean, hence  $\mathbf{K}_{\mathcal{G}}$  is compact by Proposition 5. The converse is in general not true, even for polynomials (i.e. for  $n = 1$ ). A natural question, like Putinar's question for polynomials (cf. [8, Chapter 7]), is that in which cases the compactness of  $\mathbf{K}_{\mathcal{G}}$  implies the Archimedean property of  $\mathcal{M}_{\mathcal{G}}$ ? For univariate polynomial matrices, we have a confirmation.

**Proposition 6** *Let  $\mathbb{R}[t]$  be the ring of polynomial in one variable  $t$  with real coefficients. Then, for a finite set  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[t])$ , if  $\mathbf{K}_{\mathcal{G}}$  is compact then  $\mathcal{M}_{\mathcal{G}}$  is Archimedean.*

*Proof* By the same argument as given in the proof of the "only if" part of Proposition 5, using [8, Theorem 7.1.2] instead of [8, Theorem 6.1.1], we obtain the result.

For multivariate polynomial matrices (i.e. for  $d \geq 2$ ), the compactness of  $\mathbf{K}_{\mathcal{G}}$  is in general not sufficient to deduce the Archimedean property of  $\mathcal{M}_{\mathcal{G}}$ . It is even not true for the case of multivariate polynomials (i.e. for  $d \geq 2$  and  $n = 1$ ), see, for example, Jacobi-Prestel's counterexample (cf. [5, Example 4.6]).

## 6 Local-global principle and Hessian conditions for polynomial matrices

For a set  $\mathcal{G}$  and a polynomial matrix  $\mathbf{F}$  in  $\mathcal{S}_n(\mathbb{R}[X])$ , it is obvious that if  $\mathbf{F} \in \mathcal{T}_{\mathcal{G}}$  (resp.  $\mathbf{F} \in \mathcal{M}_{\mathcal{G}}$ ) then  $\mathbf{F} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ . The converse is true only in some special cases. For example, if  $\mathbf{K}_{\mathcal{G}}$  is compact (equivalently,  $\mathcal{T}_{\mathcal{G}}$  is Archimedean by Proposition 5) (resp. if  $\mathcal{M}_{\mathcal{G}}$  is Archimedean) and  $\mathbf{F} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$  then  $\mathbf{F} \in \mathcal{T}_{\mathcal{G}}$  (resp.  $\mathbf{F} \in \mathcal{M}_{\mathcal{G}}$ ). This is a matrix version of Schmüdgen's Positivstellensatz, see, for example [3] (resp. Putinar's Positivstellensatz, see, for example [11] or [6]).

In the case where  $\mathbf{K}_{\mathcal{G}}$  is not compact, we have given in section 4 some special conditions for  $\mathbf{F} \succcurlyeq \mathbf{0}$  to ensure that  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T$  or  $b^2 \mathbf{F}$  belongs to  $\mathcal{T}_{\mathcal{G}}$ . If  $\mathbf{F}$  vanishes at some points in  $\mathbf{K}_{\mathcal{G}}$ , we need some conditions at these zeros to ensure for  $\mathbf{F}$  belonging to  $\mathcal{T}_{\mathcal{G}}$  or  $\mathcal{M}_{\mathcal{G}}$ . In the polynomial case (i.e.  $n = 1$ ), one of the well-known criterion for  $f \geq 0$  on  $K_G$  to be in  $T_G$  (resp.  $M_G$ ) is the *Hessian*

*criterion of Scheiderer* (cf. [9], [10] or [8, section 9.5]), and to prove it, he used his *local-global principle* (cf. [9] or [8, section 9.2]). Moreover, Marshall ([8]) has given *boundary Hessian conditions* for  $f$  to ensure that  $f \in M_G$  whenever it is non-negative on  $K_G$ . Therefore, in this section we give a matrix version of the local-global principle of Scheiderer, Scheiderer's Hessian criterion and the boundary Hessian conditions of Marshall.

### 6.1 Local-global principle for polynomial matrices

First we recall Scheiderer's local-global principle.

**Theorem 4** ([9], [8, Theorem 9.2.1]) *Let  $G = \{g_1, \dots, g_m\} \subseteq \mathbb{R}[X]$  and  $f \in \mathbb{R}[X]$ . Assume*

- (1)  $K_G$  is compact;
- (2)  $f \geq 0$  on  $K_G$ , and  $f$  has only finitely many zeros in  $K_G$ ;
- (3) at each zero  $p$  of  $f$  in  $K_G$ ,  $f \in (\widehat{T_G})_p \subseteq \mathbb{R}[[X - p]]$ , the preordering of  $\mathbb{R}[[X - p]]$  generated by  $G$ .

Then  $f \in T_G$ .

For any subset  $\mathcal{G} = \{G_1, \dots, G_m\}$  of  $\mathcal{S}_n(\mathbb{R}[X])$ , by Lemma 2, there exists a finite subset  $G$  of  $\mathbb{R}[X]$  such that  $\mathbf{K}_G = K_G$  and  $(T_G)^n \subseteq \mathcal{T}_G$ . We firstly give a local-global principle for diagonal polynomial matrices.

**Proposition 7** *Let  $\mathcal{G} = \{G_1, \dots, G_m\} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $G \subseteq \mathbb{R}[X]$  as above. Let  $\mathbf{D} = D(d_1, \dots, d_r)$ ,  $r \leq n$ , be an  $n \times n$  diagonal polynomial matrix in  $\mathcal{S}_n(\mathbb{R}[X])$ . Assume*

- (1)  $\mathbf{K}_G$  is compact;
- (2)  $\mathbf{D} \geq 0$  on  $\mathbf{K}_G$ , and each  $d_i$  has only finitely many zeros in  $\mathbf{K}_G$ ;
- (3) at each zero  $p$  of each  $d_i$  in  $\mathbf{K}_G$ ,  $d_i \in (\widehat{T_G})_p \subseteq \mathbb{R}[[X - p]]$ .

Then  $\mathbf{D} \in (T_G)^n \subseteq \mathcal{T}_G$ .

*Proof* Theorem 4, applying for each  $d_i$ , implies that each  $d_i$  belongs to  $T_G$ . Then  $\mathbf{D} \in (T_G)^n \subseteq \mathcal{T}_G$  by Remark 1.

For arbitrary polynomial matrices, we have the following

**Theorem 5** *Let  $\mathcal{G} = \{G_1, \dots, G_m\} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $G \subseteq \mathbb{R}[X]$  as above. Let  $\mathbf{F} \in \mathcal{S}_n(\mathbb{R}[X])$ . Assume*

- (1)  $\mathbf{K}_G$  is compact;
- (2)  $\mathbf{F} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_G$ ;
- (3) for each  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ ,  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  has only finitely many zeros in  $\mathbf{K}_G$ ;
- (4) for each  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$  and for each zero  $p$  of  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  in  $\mathbf{K}_G$ ,  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  belongs to  $(\widehat{T_G})_p \subseteq \mathbb{R}[[X - p]]$ .

Then

- (i) there exists a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T \in (T_G)^n \subseteq \mathcal{T}_G$ ;  
(ii) there exists a non-zero polynomial  $b \in \mathbb{R}[X]$  such that  $b^2 \mathbf{F} \in (T_G)^n \subseteq \mathcal{T}_G$ .

*Proof* By Lemma 3, there exist non-zero polynomials  $b, d_j \in \mathbb{R}[X]$ ,  $j = 1, \dots, r$ ,  $r \leq n$ , and matrices  $\mathbf{X}_+, \mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that

$$\mathbf{X}_+ \mathbf{X}_- = \mathbf{X}_- \mathbf{X}_+ = b \mathbf{I}_n, \quad b^2 \mathbf{F} = \mathbf{X}_+ \mathbf{D} \mathbf{X}_+^T, \quad \mathbf{D} = \mathbf{X}_- \mathbf{F} \mathbf{X}_-^T,$$

where  $\mathbf{D} = D(d_1, \dots, d_r)$  is the  $n \times n$  diagonal polynomial matrix. Note that for every  $i = 1, \dots, r$ ,

$$d_i = \mathbf{e}_i^T \mathbf{D} \mathbf{e}_i = (\mathbf{X}_-^T \mathbf{e}_i)^T \mathbf{F} (\mathbf{X}_-^T \mathbf{e}_i). \quad (2)$$

Since  $\mathbf{v}_i := \mathbf{X}_-^T \mathbf{e}_i \in \mathbb{R}^n \setminus \{0\}$  and  $\mathbf{F} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_G$ , it follows that  $d_i \geq 0$  on  $\mathbf{K}_G$  for every  $i = 1, \dots, r$ .

By (3) and in view of (2), each  $d_i$  has only finitely many zeros in  $K_G$ . By (4) and in view of (2), at each zero  $p$  of  $d_i$  in  $K_G$ ,  $d_i \in (\widehat{T_G})_p$ . It follows from Proposition 7 that  $\mathbf{D} \in (T_G)^n$ , hence  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T \in (T_G)^n$ , i.e. we have (i). Moreover, since  $(T_G)^n$  is a quadratic module of  $\mathcal{M}_n(\mathbb{R}[X])$ , by definition we have  $b^2 \mathbf{F} = \mathbf{X}_+ \mathbf{D} \mathbf{X}_+^T \in (T_G)^n$ , i.e. we have (ii). The proof is complete.

## 6.2 Hessian criterion for polynomial matrices

We recall firstly Scheiderer's Hessian criterion for polynomials in  $\mathbb{R}[X]$ .

**Theorem 6** ([9, Example 3.18],[10, Corollary 3.6]) *Let  $G = \{g_1, \dots, g_m\}$  be a subset of  $\mathbb{R}[X]$  and  $f \in \mathbb{R}[X]$ . Assume*

- (1)  $K_G$  is compact (resp. the quadratic module  $M_G$  is Archimedean);
- (2)  $f \geq 0$  on  $K_G$ ;
- (3)  $f$  has only finitely many zeros in  $K_G$  and all of them are in the interior of  $K_G$ ;
- (4) at each zero  $p$  of  $f$  in  $K_G$ , the Hessian  $D^2 f(p)$  of  $f$  at  $p$  is positive definite.

Then  $f \in T_G$  (resp.  $f \in M_G$ ).

*Remark 4* (1) Condition (3) in Theorem 6 requires each zero  $p$  of  $f$  in  $K_G$  must be in the interior of  $K_G$ , then it follows that  $p$  is a local minimum of  $f$  in  $K_G$ . Therefore, for a Taylor expansion of  $f$  in a neighborhood of  $p$ ,  $f = f_0 + f_1 + f_2 + \dots \in \mathbb{R}[[X - p]]$ , we have  $f_0 = f_1 = 0$ . Moreover, this condition implies that  $(\widehat{T_G})_p = \sum \mathbb{R}[[X - p]]^2$ .

- (2) The Hessian condition of  $f$  at  $p$  in Theorem 6 implies that in the Taylor expansion of  $f$  in a neighborhood of  $p$ , the quadratic form  $f_2$  can be written as  $x_1^2 + \dots + x_d^2$  (after changing coordinates). Then by some special techniques and using Local-global principle (Theorem 4), we have the conclusion for  $T_G$ .

Like in previous sections, we first give a result for diagonal polynomial matrices.

**Proposition 8** Let  $\mathcal{G} = \{G_1, \dots, G_m\} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $G \subseteq \mathbb{R}[X]$  as in Lemma 2. Let  $\mathbf{D} = D(d_1, \dots, d_r)$ ,  $r \leq n$ , be an  $n \times n$  diagonal polynomial matrix in  $\mathcal{S}_n(\mathbb{R}[X])$ . Assume

- (1)  $\mathbf{K}_{\mathcal{G}}$  is compact (resp.  $M_{\mathcal{G}}$  is Archimedean);
- (2)  $\mathbf{D} \geq 0$  on  $\mathbf{K}_{\mathcal{G}}$ ;
- (3) each  $d_i$  has only finitely many zeros in  $\mathbf{K}_{\mathcal{G}}$ , and all of them lie in the interior of  $\mathbf{K}_{\mathcal{G}}$ ;
- (4) at each zero  $p$  of each  $d_i$  in  $\mathbf{K}_{\mathcal{G}}$ , the Hessian  $D^2 d_i(p)$  is positive definite.

Then  $\mathbf{D} \in (T_{\mathcal{G}})^n \subseteq \mathcal{T}_{\mathcal{G}}$  (resp.  $\mathbf{D} \in (M_{\mathcal{G}})^n \subseteq \mathcal{M}_{\mathcal{G}}$ ).

*Proof* Theorem 6, applying for each  $d_i$ , implies that each  $d_i$  belongs to  $T_{\mathcal{G}}$  (resp.  $M_{\mathcal{G}}$ ). Then  $\mathbf{D} \in (T_{\mathcal{G}})^n$  (resp.  $\mathbf{D} \in (M_{\mathcal{G}})^n$ ) by Remark 1.

For arbitrary polynomial matrices we have the following

**Theorem 7** Let  $\mathcal{G} = \{G_1, \dots, G_m\} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $G \subseteq \mathbb{R}[X]$  as in Lemma 2. Let  $\mathbf{F} \in \mathcal{S}_n(\mathbb{R}[X])$ . Assume

- (1)  $\mathbf{K}_{\mathcal{G}}$  is compact (resp.  $M_{\mathcal{G}}$  is Archimedean);
- (2)  $\mathbf{F} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_{\mathcal{G}}$ ;
- (3) for each  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ ,  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  has only finitely many zeros in  $\mathbf{K}_{\mathcal{G}}$  and each zero lies in the interior of  $\mathbf{K}_{\mathcal{G}}$ ;
- (4) for each  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$  and for each zero  $p$  of  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  in  $\mathbf{K}_{\mathcal{G}}$ , the Hessian  $D^2(\mathbf{x}^T \mathbf{F} \mathbf{x})(p)$  is positive definite.

Then

- (i) there exists a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T \in (T_{\mathcal{G}})^n \subseteq \mathcal{T}_{\mathcal{G}}$  (resp.  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T \in (M_{\mathcal{G}})^n \subseteq \mathcal{M}_{\mathcal{G}}$ );
- (ii) there exists a non-zero polynomial  $b \in \mathbb{R}[X]$  such that  $b^2 \mathbf{F} \in (T_{\mathcal{G}})^n \subseteq \mathcal{T}_{\mathcal{G}}$  (resp.  $b^2 \mathbf{F} \in (M_{\mathcal{G}})^n \subseteq \mathcal{M}_{\mathcal{G}}$ ).

*Proof* By a similar argument to the one given in the proof of Theorem 5, using Proposition 8, we have the proof.

### 6.3 Boundary Hessian conditions for polynomial matrices

Let us recall the boundary Hessian conditions of a polynomial at a point, which is defined by Marshall (cf. [8, section 9.5]). Let  $G \subseteq \mathbb{R}[X]$  and  $f \in \mathbb{R}[X]$ . We say that  $f$  satisfies the *boundary Hessian conditions (BHC)* at a point  $p \in K_G$  with respect to  $t_1, \dots, t_k$ ,  $1 \leq k \leq d$ , which are part of a system of uniformizing parameters  $t_1, \dots, t_d$  at  $p$ , if  $p$  is a non-singular point of  $\mathbb{R}^d$ , and in the completion  $\mathbb{R}[[t_1, \dots, t_d]]$  of  $\mathbb{R}[X]$  at  $p$ ,  $f$  decomposes as  $f = f_0 + f_1 + f_2 + \dots$  (where  $f_j$  is homogeneous of degree  $j$  in the variables  $t_1, \dots, t_d$  with coefficients in  $\mathbb{R}$ ),  $f_1 = a_1 t_1 + \dots + a_k t_k$ ,  $a_i > 0$  for  $i = 1, \dots, k$ , and the quadratic form  $f_2(0, \dots, 0, t_{k+1}, \dots, t_d)$  is positive definite. If  $k = 0$  then these are precisely the Hessian conditions mentioned in Theorem 6 (3), (4).

**Theorem 8 ([8, Theorem 9.5.3])** *Let  $G \subseteq \mathbb{R}[X]$  and  $f \in \mathbb{R}[X]$ . Assume*

- (1)  $M_G$  is Archimedean;
- (2)  $f \geq 0$  on  $K_G$ ;
- (3) each zero  $p$  of  $f$  in  $K_G$  is a non-singular point of  $\mathbb{R}^d$ , and there exist  $g_1, \dots, g_k \in M_G$ ,  $1 \leq k \leq d$ , which are part of a system of uniformizing parameters at  $p$  such that  $f$  satisfies BHC with respect to  $g_1, \dots, g_k$  at  $p$ .

Then  $f \in M_G$ .

Note that in this theorem  $G$  is an arbitrary subset of  $\mathbb{R}[X]$ , not necessarily finite. Using this theorem, we have the following boundary Hessian criterion for diagonal polynomial matrices.

**Proposition 9** *Let  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $G \subseteq \mathbb{R}[X]$  as in Lemma 2. Let  $\mathbf{D} = D(d_1, \dots, d_r)$ ,  $r \leq n$ , be an  $n \times n$  diagonal polynomial matrix in  $\mathcal{S}_n(\mathbb{R}[X])$ . Assume*

- (1)  $M_G$  is Archimedean;
- (2)  $\mathbf{D} \geq 0$  on  $\mathbf{K}_G$ ;
- (3) each zero  $p$  of each  $d_i$  in  $\mathbf{K}_G$  is a non-singular point of  $\mathbb{R}^d$ , and there exist  $g_{i_1}, \dots, g_{i_k} \in M_G$ ,  $1 \leq k \leq d$ , which are part of a system of uniformizing parameters at  $p$  such that  $d_i$  satisfies BHC with respect to  $g_{i_1}, \dots, g_{i_k}$  at  $p$ .

Then  $\mathbf{D} \in (M_G)^n \subseteq \mathcal{M}_G$ .

*Proof* The result follows from Theorem 8, applying for each  $d_i \in \mathbb{R}[X]$ , and Remark 1.

By a similar argument to the one given in the proof of Theorem 5, using Proposition 9, we obtain the following

**Theorem 9** *Let  $\mathcal{G} \subseteq \mathcal{S}_n(\mathbb{R}[X])$  and  $G \subseteq \mathbb{R}[X]$  as in Lemma 2. Let  $\mathbf{F} \in \mathcal{S}_n(\mathbb{R}[X])$ . Assume*

- (1)  $M_G$  is Archimedean;
- (2)  $\mathbf{F} \succcurlyeq \mathbf{0}$  on  $\mathbf{K}_G$ ;
- (3) for each  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ , each zero  $p$  of the polynomial  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  in  $\mathbf{K}_G$  is a non-singular point of  $\mathbb{R}^d$ , and there exist  $g_1, \dots, g_k \in M_G$ ,  $1 \leq k \leq d$ , which are part of a system of uniformizing parameters at  $p$  such that  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  satisfies BHC with respect to  $g_1, \dots, g_k$  at  $p$ .

Then

- (i) there exists a matrix  $\mathbf{X}_- \in \mathcal{M}_n(\mathbb{R}[X])$  such that  $\mathbf{X}_- \mathbf{F} \mathbf{X}_-^T \in (M_G)^n \subseteq \mathcal{M}_G$ ;
- (ii) there exists a non-zero polynomial  $b \in \mathbb{R}[X]$  such that  $b^2 \mathbf{F} \in (M_G)^n \subseteq \mathcal{M}_G$ .

**Acknowledgements** The author would like to thank the anonymous referees for their useful comments and suggestions. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the grant number 101.99-2013.24. This work is finished during the author's postdoctoral fellowship at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He thanks VIASM for financial support and hospitality.

## References

1. Cimprič, J.: A representation theorem for Archimedean quadratic modules on  $*$ -rings. *Canad. Math. Bull.* **52** (1), 39-52 (2009)
2. Cimprič, J.: Real algebraic geometry for matrices over commutative rings. *J. Algebra* **359**, 89-103 (2012)
3. Cimprič, J., Zalar, J.: Moment problems for operator polynomials. *J. Math. Anal. App.* **401**(1), 307-316 (2013)
4. Gondard, D., Ribenboim, P.: Le 17e problème de Hilbert pour les matrices. *Bull. Sci. Math. (2)* **98** (1), 49-56 (1974)
5. Jacobi, T., Prestel, A.: Distinguished representations of strictly positive polynomials. *J. reine angew. Math.* **532**, 223-235 (2001)
6. Klep, I., Schweighofer, M.: Pure states, positive matrix polynomials and sums of Hermitian squares. *Indiana Uni. Math. J.* **59** (3), 857-874 (2010)
7. Krivine, J.-L.: Anneaux préordonnés. *J. Anal. Math.* **12**, 307-326 (1964)
8. Marshall, M.: Positive polynomials and sums of squares. *Mathematical Surveys and Monographs* vol. **146**, American Mathematical Society, Providence, RI (2008)
9. Scheiderer, C.: Sums of squares on real algebraic curves. *Math. Z.* **245**, 725-760 (2003)
10. Scheiderer, C.: Distinguished representations of non-negative polynomials. *J. Algebra* **289**, 558-573 (2005)
11. Scherer, C. W., Hol, C. W. J.: Matrix sum-of-squares relaxations for robust semi-definite programs. *Math. Program.* **107**, no. 1-2, Ser. B, 189-211 (2006)
12. Schmüdgen, K.: Unbounded operator algebras and representation theory. *Operator Theory: Advances and Applications*, 37. Birkhäuser Verlag, Basel-Boston-Berlin (1990)
13. Schmüdgen, K.: A strict Positivstellensatz for the Weyl algebra. *Math. Ann.* **331**, 779-794 (2005)
14. Schmüdgen, K.: Noncommutative real algebraic geometry - some basic concepts and first ideas. In: *Emerging Applications of Algebraic Geometry*, IMA Vol. Math. Appl., vol. **149**, pp. 325-350. Springer, New York (2009)
15. Schweighofer, M.: Global Optimization of polynomials using gradient tentacles and sums of squares. *SIAM J. Optimization* **17**(3), 920-942 (2006)
16. Stengle, G.: A Nullstellensatz and a positivstellensatz in semialgebraic geometry. *Math. Ann.* **207**, 87-97 (1974)