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Parallel hybrid iterative methods for variational inequalities, equilibrium problems and common fixed point problems

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Dedicated to Professor Nguyen Khoa Son's 65th Birthday

Abstract In this paper we propose two strongly convergent parallel hybrid iterative methods for finding a common element of the set of fixed points of a family of asymptotically quasi ϕ-nonexpansive mappings, the set of solutions of variational inequalities and the set of solutions of equilibrium problems in uniformly smooth and 2-uniformly convex Banach spaces. A numerical experiment is given to verify the efficiency of the proposed parallel algorithms.

Keywords Asymptotically quasi ϕ-nonexpansive mapping · Variational inequality · Equilibrium problem · Hybrid method · Parallel computation

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1 Introduction

Let C be a nonempty closed convex subset of a Banach space E. The variational inequality for a possibly nonlinear mapping $A : C \to E^*$, consists of finding $p^* \in C$ such as

$$\langle Ap^*, p - p^* \rangle \geq 0, \quad \forall p \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(A, C)$.
Takahashi and Toyoda [19] proposed a weakly convergent method for finding a

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common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an α-inverse strongly monotone mapping in a Hilbert space.

Theorem 1.1 *[19] Let K be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$. Let A be an α-inverse strongly-monotone mapping of K into H, and let S be a nonexpansive mapping of K into itself such that $F(S) \bigcap VI(K,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_K(x_n - \lambda_n A x_n), \end{cases}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda_n \in [a,b]$ for some $a, b \in (0, 2\alpha)$ and $\alpha_n \in [c,d]$ for some $c, d \in (0,1)$. Then, $\{x_n\}$ converges weakly to $z \in F(S) \bigcap VI(K,A)$, where $z = \lim_{n\to\infty} P_{F(S)\bigcap VI(K,A)} x_n$.

In 2008, Iiduka and Takahashi [8] considered problem (1.1) in a 2-uniformly convex, uniformly smooth Banach space under the following assumptions:

(V1) A is α-inverse-strongly-monotone.
(V2) $VI(A,C) \neq \emptyset$.
(V3) $||Ay|| \leq ||Ay - Au||$ for all $y \in C$ and $u \in VI(A,C)$.

Theorem 1.2 *[8] Let E be a 2-uniformly convex, uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous, and let C be a nonempty, closed convex subset of E . Assume that A is a mapping of C into E^* satisfing conditions $(V1) - (V3)$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If λ_n is chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, then the sequence $\{x_n\}$ converges weakly to some element z in $VI(C,A)$. Here $1/c$ is the 2-uniform convexity constant of E, and $z = \lim_{n\to\infty} \Pi_{VI(A,C)} x_n$.

In 2009, Zegeye and Shahzad [22] studied the following hybrid iterative algorithm in a 2-uniformly convex and uniformly smooth Banach space for finding a common element of the set of fixed points of a weakly relatively nonexpansive mapping T and the set of solutions of a variational inequality involving an α-inverse strongly monotone mapping A:

$$\begin{cases} y_n = \Pi_C \left(J^{-1}(Jx_n - \lambda_n Ax_n) \right), \\ z_n = Ty_n, \\ H_0 = \{v \in C : \phi(v, z_0) \le \phi(v, y_0) \le \phi(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \bigcap W_{n-1} : \phi(v, z_n) \le \phi(v, y_n) \le \phi(v, x_n)\}, \\ W_0 = C, \\ W_n = \{v \in H_{n-1} \bigcap W_{n-1} : \langle x_n - v, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = P_{H_n \bigcap W_n} x_0, n \ge 1, \end{cases}$$

where J is the normalized duality mapping on E. The strong convergence of $\{x_n\}$ to $\Pi_{F(T) \bigcap VI(A,C)} x_0$ has been established.
Kang, Su, and Zhang [9] extended this algorithm to a weakly relatively nonexpansive mapping, a variational inequality and an equilibrium problem. Recently, Saewan and Kumam [14] have constructed a sequential hybrid block iterative algorithm for an infinite family of closed and uniformly asymptotically quasi ϕ-nonexpansive mappings, a variational inequality for an α -inverse-strongly monotone mapping, and a system of equilibrium problems.

Qin, Kang, and Cho [12] considered the following sequential hybrid method for a pair of inverse strongly monotone and a quasi ϕ-nonexpansive mappings in a 2-uniformly convex and uniformly smooth Banach space:

$$\begin{cases} x_0 = E, C_1 = C, x_1 = \Pi_{C_1} x_0, \\ u_n = \Pi_C \left(J^{-1}(Jx_n - \eta_n Bx_n) \right), \\ z_n = \Pi_C \left(J^{-1}(Ju_n - \lambda_n Au_n) \right), \\ y_n = Tz_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, z_n) \le \phi(v, u_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \ge 0. \end{cases}$$

They proved the strong convergence of the sequence $\{x_n\}$ to $\Pi_F x_0$, where $F = F(T) \bigcap VI(A,C) \bigcap VI(B,C)$.

Let f be a bifunction from $C \times C$ to a set of real numbers $\mathbb{R}$. The equilibrium problem for f consists of finding an element $\widehat{x} \in C$, such that

$$f(\widehat{x}, y) \ge 0, \ \forall y \in C. \tag{1.2}$$

The set of solutions of the equilibrium problem (1.2) is denoted by $EP(f)$. Equilibrium problems include several problems such as: variational inequalities, optimization problems, fixed point problems, ect. In recent years, equilibrium problems have been studied widely and several solution methods have been proposed (see [3, 9, 14, 15, 18]). On the other hand, for finding a common element in $F(T)\bigcap EP(f)$, Takahashi and Zembayashi [20] introduced the following algorithm in a uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTy_n), \\ u_n \in C, \text{ s.t., } f(u_n, y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \ge 0 \quad \forall y \in C, \\ H_n = \{v \in C : \phi(v,u_n) \le \phi(v,x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \bigcap W_n} x_0, n \ge 1. \end{cases}$$

The strong convergence of the sequences $\{x_n\}$ and $\{u_n\}$ to $\Pi_{F(T)\bigcap EP(f)}x_0$ has been established.
Recently, the above mentioned algorithms have been generalized and modified for finding a common point of the set of solutions of variational inequalities, the set of fixed points of (asymptotically) quasi ϕ-nonexpansive mappings, and the set of solutions of equilibrium problems by several authors, such as Takahashi and Zembayashi [20], Wang et al. [21] and others.
Very recently, Anh and Chung [4] have considered the following parallel hybrid method for a finite family of relatively nonexpansive mappings $\{T_i\}_{i=1}^N$:

$$\begin{cases} x_0 \in C, \\ y_n^i = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i x_n), \quad i = 1,\ldots,N, \\ i_n = \arg\max_{1\le i\le N}\left\{\left\|y_n^i - x_n\right\|\right\}, \quad \bar{y}_n := y_n^{i_n}, \\ C_n = \{v \in C : \phi(v,\bar{y}_n) \le \phi(v,x_n)\}, \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0, n \ge 0. \end{cases}$$

This algorithm was extended, modified and generelized by Anh and Hieu [5] for a finite family of asymptotically quasi ϕ-nonexpansive mappings in Banach spaces. Note that the proposed parallel hybrid methods in [4,5] can be used for solving simultaneous systems of maximal monotone mappings. Other parallel methods for solving accretive operator equations can be found in [3].
In this paper, motivated and inspired by the above mentioned results, we propose two novel parallel iterative methods for finding a common element of the set of fixed points of a family of asymptotically quasi ϕ-nonexpansive mappings $\{F(S_j)\}_{j=1}^N$, the set of solutions of variational inequalities $\{VI(A_i, C)\}_{i=1}^M$, and the set of solutions of equilibrium problems $\{EP(f_k)\}_{k=1}^K$ in uniformly smooth and 2-uniformly convex Banach spaces, namely:
Method A

$$\begin{cases} x_0 \in C \quad \text{is chosen arbitrarily,} \\ y_n^i = \Pi_C\left(J^{-1}(Jx_n - \lambda_n A_i x_n)\right), i = 1, 2, \dots M, \\ i_n = \arg\max\left\{||y_n^i - x_n|| : i = 1, \dots, M\right\}, \bar{y}_n = y_n^{i_n}, \\ z_n^j = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J S_j^n \bar{y}_n\right), j = 1, \dots, N, \\ j_n = \arg\max\left\{||z_n^j - x_n|| : j = 1, \dots, N\right\}, \bar{z}_n = z_n^{j_n}, \\ u_n^k = T_{r_n}^k \bar{z}_n, k = 1, \dots, K, \\ k_n = \arg\max\left\{||u_n^k - x_n|| : k = 1, 2, \dots K\right\}, \bar{u}_n = u_n^{k_n}, \\ C_{n+1} = \left\{z \in C_n : \phi(z, \bar{u}_n) \le \phi(z, \bar{z}_n) \le \phi(z, x_n) + \epsilon_n\right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \ge 0, \end{cases} \tag{1.3}$$

where, $T_r x := z$ is a unique solution to a regularized equlibrium problem $f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$
Further, the control parameter sequences $\{\lambda_n\}, \{\alpha_n\}, \{r_n\}$ satisfy the conditions

$$0 \le \alpha_n \le 1, \lim\sup_{n\to\infty} \alpha_n < 1, \quad \lambda_n \in [a, b], \quad r_n \ge d, \tag{1.4}$$

for some $a, b \in (0, \alpha c^2/2), d > 0$, where $1/c$ is the 2-uniform convexity constant of E.
Concerning the sequence $\{\epsilon_n\}$, we consider two cases. If the mappings $\{S_i\}$ are asymptotically quasi ϕ-nonexpansive, we assume that the solution set F is bounded, i.e., there exists a positive number ω, such that $F \subset \Omega := \{u \in C : ||u|| \le \omega\}$ and put $\epsilon_n := (k_n - 1)(\omega + ||x_n||)^2$. If the mappings $\{S_i\}$ are quasi ϕ-nonexapansive, then $k_n = 1$, and we put $\epsilon_n = 0$.

Method B

$$\begin{cases} x_0 \in C \quad \text{is chosen arbitrarily,} \\ y_n^i = \Pi_C\left(J^{-1}(Jx_n - \lambda_n A_i x_n)\right), i = 1, \dots, M, \\ i_n = \arg\max\left\{||y_n^i - x_n|| : i = 1, \dots, M\right\}, \bar{y}_n = y_n^{i_n}, \\ z_n = J^{-1}\left(\alpha_{n,0} J x_n + \sum_{j=1}^N \alpha_{n,j} J S_j^n \bar{y}_n\right), \\ u_n^k = T_{r_n}^k z_n, k = 1, \dots, K, \\ k_n = \arg\max\left\{||u_n^k - x_n|| : k = 1, \dots, K\right\}, \bar{u}_n = u_n^{i_n}, \\ C_{n+1} = \left\{z \in C_n : \phi(z, \bar{u}_n) \le \phi(z, x_n) + \epsilon_n\right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \ge 0, \end{cases} \tag{1.5}$$

where, the control parameter sequences $\{\lambda_n\}, \{\alpha_{n,j}\}, \{r_n\}$ satisfy the conditions

$$0 \le \alpha_{n,j} \le 1, \sum_{j=0}^N \alpha_{n,j} = 1, \quad \lim_{n\to\infty} \inf \alpha_{n,0}\alpha_{n,j} > 0, \quad \lambda_n \in [a, b], \, r_n \ge d. \tag{1.6}$$

In Method A (1.3), knowing x_n we find the intermediate approximations $y_n^i, i = 1, \dots, M$ in parallel. Using the farthest element among y_n^i from x_n, we compute $z_n^j, j = 1, \ldots, N$ in parallel. Further, among z_n^j, we choose the farthest element from x_n and determine solutions of regularized equilibrium problems $u_n^k, k = 1, \ldots, K$ in parallel. Then the farthest from x_n element among u_n^k, denoted by $\bar{u}_n$ is chosen. Based on $\bar{u}_n$, a closed convex subset C_{n+1} is constructed. Finally, the next approximation x_{n+1} is defined as the generalized projection of x_0 onto C_{n+1}.

A similar idea of parallelism is employed in Method B (1.5). However, the subset C_{n+1} in Method B is simpler than that in Method A.

The results obtained in this paper extend and modify the corresponding results of Zegeye and Shahzad [22], Takahashi and Zembayashi [20], Anh and Chung [4], Anh and Hieu [5] and others.

The paper is organized as follows: In Section 2, we collect some definitions and results needed for further investigtion. Section 3 deals with the convergence analysis of the methods (1.3) and (1.5). In Section 4, a novel parallel hybrid iterative method for variational inequalities and closed, quasi ϕ- nonexpansive mappings is studied. Finally, a numerical experiment is considered in Section 5 to verify the efficiency of the proposed parallel hybrid methods.

2 Preliminaries

In this section we recall some definitions and results which will be used later. The reader is refered to [2] for more details.

Definition 1 A Banach space E is called

1) strictly convex if the unit sphere $S_1(0) = \{x \in X : ||x|| = 1\}$ is strictly convex, i.e., the inequality $||x + y|| < 2$ holds for all $x, y \in S_1(0), x \neq y$;
2) uniformly convex if for any given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1, \|x - y\| = \epsilon$ the inequality $\|x + y\| \leq 2(1 - \delta)$ holds;
3) smooth if the limit
$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$
exists for all $x, y \in S_1(0)$;
4) uniformly smooth if the limit (2.1) exists uniformly for all $x, y \in S_1(0)$.

The modulus of convexity of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by
$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \epsilon \right\}$$

for all $\epsilon \in [0,2]$. Note that E is uniformly convex if only if $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$ and $\delta_E(0) = 0$. Let $p > 1$, E is said to be p-uniformly convex if there exists some constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$. It is well-known that spaces L^p, l^p and W_m^p are p-uniformly convex if $p > 2$ and 2 -uniformly convex if $1 < p \leq 2$ and a Hilbert space H is uniformly smooth and 2-uniformly convex.

Let E be a real Banach space with its dual E^*. The dual product of $f \in E^*$ and $x \in E$ is denoted by $\langle x, f\rangle$ or $\langle f, x\rangle$. For the sake of simpicity, the norms of E and E^* are denoted by the same symbol $||.||$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle f, x\rangle = \|x\|^2 = \|f\|^2 \right\}.$$

The following properties can be found in [7]:

i) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one, and onto;

ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm to weak * continuous;

iii) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E;

iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex;

v) Each uniformly convex Banach space E has the Kadec-Klee property, i.e., for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Lemma 2.1 *[22] If E is a 2-uniformly convex Banach space, then*

$$||x - y|| \leq \frac{2}{c^2}||Jx - Jy||, \quad \forall x, y \in E,$$

where J is the normalized duality mapping on E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ is called the 2-*uniform convexity constant* of E.
Next we assume that E is a smooth, strictly convex, and reflexive Banach space. In the sequel we always use $\phi : E \times E \to [0, \infty)$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in E.$$

From the definition of ϕ, we have

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{2.2}$$

Moreover, the Lyapunov functional satisfies the identity

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle z - x, Jy - Jz\rangle \tag{2.3}$$

for all $x, y, z \in E$.
The generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(x,y).$$

In what follows, we need the following properties of the functional ϕ and the generalized projection Π_C.

Lemma 2.2 *[1] Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Then the following conclusions hold:*

i) $\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \le \phi(x,y), \forall x \in C, y \in E$;
ii) *if* $x \in E, z \in C$, *then* $z = \Pi_C(x)$ *iff* $\langle z - y; Jx - Jz\rangle \ge 0, \forall y \in C$;
iii) $\phi(x,y) = 0$ *iff* $x = y$.

Lemma 2.3 *[10] Let C be a nonempty closed convex subset of a smooth Banach E, $x, y, z \in E$ and $\lambda \in [0,1]$. For a given real number a, the set*

$$D := \{v \in C : \phi(v,z) \le \lambda\phi(v,x) + (1-\lambda)\phi(v,y) + a\}$$

is closed and convex.

Lemma 2.4 *[1] Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a uniformly convex and uniformly smooth real Banach space E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.5 *[6] Let E be a uniformly convex Banach space, r be a positive number and $B_r(0) \subset E$ be a closed ball with center at origin and radius r. Then, for any given subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{i=1}^N \lambda_i = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0,\infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$,*

$$\left\| \sum_{k=1}^N \lambda_k x_k \right\|^2 \le \sum_{k=1}^N \lambda_k \|x_k\|^2 - \lambda_i\lambda_j g(||x_i - x_j||).$$

Definition 2 A mapping $A : E \to E^*$ is called

1) monotone, if

$$\langle A(x) - A(y), x - y\rangle \geq 0 \quad \forall x, y \in E;$$

2) uniformly monotone, if there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty), \psi(0) = 0$, such that

$$\langle A(x) - A(y), x - y\rangle \geq \psi(||x - y||) \quad \forall x, y \in E; \tag{2.4}$$

3) η-strongly monotone, if there exists a positive constant η, such that in (2.4), $\psi(t) = \eta t^2$;
4) α-inverse strongly monotone, if there exists a positive constant α, such that

$$\langle A(x) - A(y), x - y\rangle \geq \alpha||A(x) - A(y)||^2 \quad \forall x, y \in E.$$

5) L-Lipschitz continuous if there exists a positive constant L, such that

$$||A(x) - A(y)|| \leq L||x - y|| \quad \forall x, y \in E.$$

If A is α-inverse strongly monotone then it is $\frac{1}{\alpha}$-Lipschitz continuous. If A is η-strongly monotone and L-Lipschitz continuous then it is $\frac{\eta}{L^2}$-inverse strongly monotone.

Lemma 2.6 *[17] Let C be a nonempty, closed convex subset of a Banach space E and A be a monotone, hemicontinuous mapping of C into E^*. Then*

$$VI(C, A) = \{u \in C : \langle v - u, A(v)\rangle \geq 0, \quad \forall v \in C\}.$$

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, $T : C \to C$ be a mapping. The set

$$F(T) = \{x \in C : Tx = x\}$$

is called the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$ as $n \to +\infty$. The set of all asymptotic fixed points of T will be denoted by $\tilde{F}(T)$.

Definition 3 A mapping $T : C \to C$ is called

i) relatively nonexpansive mapping if $F(T) \neq \emptyset, \tilde{F}(T) = F(T)$, and

$$\phi(p, Tx) \leq \phi(p, x), \forall p \in F(T), \forall x \in C;$$

ii) closed if for any sequence $\{x_n\} \subset C, x_n \to x$ and $Tx_n \to y$, then $Tx = y$;

iii) quasi ϕ - nonexpansive mapping (or hemi-relatively nonexpansive mapping) if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \forall p \in F(T), \forall x \in C;$$

iv) asymptotically quasi ϕ-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to +\infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \forall n \geq 1, \forall p \in F(T), \forall x \in C;$$

v) uniformly L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \forall n \geq 1, \forall x, y \in C.$$

The reader is refered to [6,16] for examples of closed and asymptotically quasi ϕ-nonexpansive mappings. It has been shown that the class of asymptotically quasi ϕ-nonexpansive mappings contains properly the class of quasi ϕ-nonexpansive mappings, and the class of quasi ϕ-nonexpansive mappings contains the class of relatively nonexpansive mappings as a proper subset.

Lemma 2.7 *[6] Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi ϕ-nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty), k_n \to 1$. Then $F(T)$ is a closed convex subset of C.*

Next, for solving the equilibrium problem (1.2), we assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) For all $x, y, z \in C$,

$$\lim_{t \to 0^+} \sup f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) For all $x \in C$, $f(x, .)$ is convex and lower semicontinuous.

The following results show that in a smooth (uniformly smooth), strictly convex and reflexive Banach space, the regularized equilibrium problem has a solution (unique solution), respectively.

Lemma 2.8 *[20] Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach space E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(A1)$-$(A4)$ and let $r > 0$, $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.9 *[20] Let C be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach spaces E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(A1)$-$(A4)$. For all $r > 0$ and $x \in E$, define the mapping*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

Then the following hold:

(B1) T_r *is single-valued;*

(B2) T_r *is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(B3) $F(T_r) = \tilde{F}(T) = EP(f)$;

(B4) $EP(f)$ *is closed and convex and T_r is a relatively nonexpansive mapping.*

Lemma 2.10 *[20] Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Let E be a real Banach space. Alber [1] studied the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2.$$

Clearly, $V(x, x^*) = \phi(x, J^{-1}x^*)$.

Lemma 2.11 *[1] Let E be a refexive, strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\left\langle J^{-1}x - x^*, y^*\right\rangle \leq V(x, x^* + y^*), \quad \forall x \in E \quad \text{and} \quad \forall x^*, y^* \in E^*.$$

Consider the normal cone N_C to a set C at the point $x \in C$ defined by

$$N_C(x) = \{x^* \in E^* : \langle x - y, x^* \rangle \geq 0, \quad \forall y \in C\}.$$

We have the following result.

Lemma 2.12 *[13] Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemi-continuous mapping of C into E^* with $D(A) = C$. Let Q be a mapping defined by:*

$$Q(x) = \begin{cases} Ax + N_C(x) & \text{if} \quad x \in C, \\ \emptyset & \text{if} \quad x \notin C. \end{cases}$$

Then Q is a maximal monotone and $Q^{-1}0 = VI(A, C)$.

3 Convergence analysis

Throughout this section, we assume that C is a nonempty closed convex subset of a real uniformly smooth and 2-uniformly convex Banach space E. Denote

$$F = \left(\bigcap_{i=1}^{M} VI(A_i, C) \right) \bigcap \left(\bigcap_{j=1}^{N} F(S_j) \right) \bigcap \left(\bigcap_{k=1}^{K} EP(f_k) \right)$$

and assume that the set F is nonempty.
We prove convergence theorems for methods (1.3) and (1.5) with the control parameter sequences satisfying conditions (1.4) and (1.6), respectively. We also propose similar parallel hybrid methods for quasi ϕ-nonexpansive mappings, variational inequalities and equilibrium problems.

Theorem 3.1 *Let $\{A_i\}_{i=1}^M$ be a finite family of mappings from C to E^* satisfying conditions* (V1)-(V3)*. Let $\{f_k\}_{k=1}^K : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying conditions* (A1)-(A4)*. Let $\{S_j\}_{j=1}^N : C \to C$ be a finite family of uniform L-Lipschitz continuous and asymptotically quasi-ϕ-nonexpansive mappings with the same sequence $\{k_n\} \subset [1, +\infty), k_n \to 1$. Assume that there exists a positive number ω such that $F \subset \Omega := \{u \in C : ||u|| \leq \omega\}$. If the control parameter sequences $\{\alpha_n\}, \{\lambda_n\}, \{r_n\}$ satisfy condition* (1.4)*, then the sequence $\{x_n\}$ generated by* (1.3) *converges strongly to $\Pi_F x_0$.*

Proof We divide the proof of Theorem 3.1 into seven steps.
Step 1. Claim that F, C_n are closed convex subsets of C.
Indeed, since each mapping S_i is uniformly L-Lipschitz continuous, it is closed.

By Lemmas 2.6, 2.7 and 2.9, $F(S_i), VI(A_j,C)$ and $EP(f_k)$ are closed convex sets, therefore, $\bigcap_{j=1}^N (F(S_j)), \bigcap_{i=1}^M VI(A_i,C)$ and $\bigcap_{k=1}^K EP(f_k)$ are also closed and convex. Hence F is a closed and convex subset of C. It is obvious that C_n is closed for all $n \geq 0$. We prove the convexity of C_n by induction. Clearly, $C_0 := C$ is closed convex. Assume that C_n is closed convex for some $n \geq 0$. From the construction of C_{n+1}, we find
$C_{n+1} = C_n \bigcap \{z \in E : \phi(z, \bar{u}_n) \leq \phi(z, \bar{z}_n) \leq \phi(z, x_n) + \epsilon_n\}$.
Lemma 2.3 ensures that C_{n+1} is convex. Thus, C_n is closed convex for all $n \geq 0$. Hence, $\Pi_F x_0$ and $x_{n+1} := \Pi_{C_{n+1}} x_0$ are well-defined.

Step 2. Claim that $F \subset C_n$ for all $n \geq 0$.

By Lemma 2.10 and the relative nonexpansiveness of T_{r_n}, we obtain $\phi(u, \bar{u}_n) = \phi(u, T_{r_n} \bar{z}_n) \leq \phi(u, \bar{z}_n)$, for all $u \in F$. From the convexity of $||.||^2$ and the asymptotical quasi ϕ-nonexpansiveness of S_j, we find

$$\begin{aligned}
\phi(u, \bar{z}_n) &= \phi\left(u, J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J S_{j_n}^n \bar{y}_n\right)\right) \\
&= ||u||^2 - 2\alpha_n \langle u, x_n\rangle - 2(1-\alpha_n)\left\langle u, J S_{j_n}^n \bar{y}_n\right\rangle \\
&\quad + ||\alpha_n J x_n + (1-\alpha_n) J S_{j_n}^n \bar{y}_n||^2 \\
&\leq ||u||^2 - 2\alpha_n \langle u, x_n\rangle - 2(1-\alpha_n)\left\langle u, J S_{j_n}^n \bar{y}_n\right\rangle \\
&\quad + \alpha_n ||x_n||^2 + (1-\alpha_n)||S_{j_n}^n \bar{y}_n||^2 \\
&= \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, S_{j_n}^n \bar{y}_n) \\
&\leq \alpha_n \phi(u, x_n) + (1-\alpha_n) k_n \phi(u, \bar{y}_n) \qquad (3.1)
\end{aligned}$$

for all $u \in F$. By the hypotheses of Theorem 3.1, Lemmas 2.1, 2.2, 2.11 and $u \in F$, we have

$$\begin{aligned}
\phi(u, \bar{y}_n) &= \phi\left(u, \Pi_C\left(J^{-1}(J x_n - \lambda_n A_{i_n} x_n)\right)\right) \\
&\leq \phi(u, J^{-1}(J x_n - \lambda_n A_{i_n} x_n)) \\
&= V(u, J x_n - \lambda_n A_{i_n} x_n) \\
&\leq V(u, J x_n - \lambda_n A_{i_n} x_n + \lambda_n A_{i_n} x_n) \\
&\quad -2\left\langle J^{-1}\left(J x_n - \lambda_n A_{i_n} x_n\right) - u, \lambda_n A_{i_n} x_n\right\rangle \\
&= \phi(u, x_n) - 2\lambda_n \left\langle J^{-1}\left(J x_n - \lambda_n A_{i_n} x_n\right) - J^{-1}\left(J x_n\right), A_{i_n} x_n\right\rangle \\
&\quad -2\lambda_n \langle x_n - u, A_{i_n} x_n - A_{i_n}(u)\rangle - 2\lambda_n \langle x_n - u, A_{i_n} u\rangle \\
&\leq \phi(u, x_n) + \frac{4\lambda_n}{c^2}||J x_n - \lambda_n A_{i_n} x_n - J x_n||||A_{i_n} x_n|| \\
&\quad -2\lambda_n \alpha ||A_{i_n} x_n - A_{i_n} u||^2 \\
&\leq \phi(u, x_n) + \frac{4\lambda_n^2}{c^2}||A_{i_n} x_n||^2 - 2\lambda_n \alpha ||A_{i_n} x_n - A_{i_n} u||^2
\end{aligned}$$

$$\leq \phi(u,x_n) - 2a\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2$$
$$\leq \phi(u,x_n). \tag{3.2}$$

From (3.1), (3.2) and the estimate (2.2), we obtain

$$\begin{aligned}\phi(u,\bar{z}_n) &\leq \alpha_n\phi(u,x_n) + (1-\alpha_n)k_n\phi(u,x_n)\\ &\quad -2a(1-\alpha_n)\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2\\ &\leq \phi(u,x_n) + (k_n-1)\phi(u,x_n)\\ &\quad -2a(1-\alpha_n)\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2\\ &\leq \phi(u,x_n) + (k_n-1)\left(\omega + ||x_n||\right)^2\\ &\quad -2a(1-\alpha_n)\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2\\ &\leq \phi(u,x_n) + \epsilon_n. \end{aligned} \tag{3.3}$$

Therefore, $F \subset C_n$ for all $n \geq 0$.
Step 3. Claim that the sequence $\{x_n\}$, $\{y_n^i\}$, $\{z_n^j\}$ and $\{u_n^k\}$ converge strongly to $p \in C$.
By Lemma 2.2 and $x_n = \Pi_{C_n}x_0$, we have

$$\phi(x_n,x_0) \leq \phi(u,x_0) - \phi(u,x_n) \leq \phi(u,x_0).$$

for all $u \in F$. Hence $\{\phi(x_n,x_0)\}$ is bounded. By (2.2), $\{x_n\}$ is bounded, and so are the sequences $\{\bar{y}_n\}$, $\{\bar{u}_n\}$, and $\{\bar{z}_n\}$. By the construction of C_n, $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$. From Lemma 2.2 and $x_n = \Pi_{C_n}x_0$, we get

$$\phi(x_n,x_0) \leq \phi(x_{n+1},x_0) - \phi(x_{n+1},x_n) \leq \phi(x_{n+1},x_0).$$

Therefore, the sequence $\{\phi(x_n,x_0)\}$ is nondecreasing, hence it has a finite limit. Note that, for all $m \geq n$, $x_m \in C_m \subset C_n$, and by Lemma 2.2 we obtain

$$\phi(x_m,x_n) \leq \phi(x_m,x_0) - \phi(x_n,x_0) \to 0 \tag{3.4}$$

as $m,n \to \infty$. From (3.4) and Lemma 2.4 we have $||x_n - x_m|| \to 0$. This shows that $\{x_n\} \subset C$ is a Cauchy sequence. Since E is complete and C is closed convex subset of E, $\{x_n\}$ converges strongly to $p \in C$. From (3.4), $\phi(x_{n+1},x_n) \to 0$ as $n \to \infty$. Taking into account that $x_{n+1} \in C_{n+1}$, we find

$$\phi(x_{n+1},\bar{u}_n) \leq \phi(x_{n+1},\bar{z}_n) \leq \phi(x_{n+1},x_n) + \epsilon_n \tag{3.5}$$

Since $\{x_n\}$ is bounded, we can put $M=\sup\{||x_n||:n=0,1,2,\ldots\}$, hence

$$\epsilon_n:=(k_n-1)(\omega+||x_n||)^2\le(k_n-1)(\omega+M)^2\to 0. \tag{3.6}$$

By $(3.5),(3.6)$ and $\phi(x_{n+1},x_n)\to 0$, we find that

$$\lim_{n\to\infty}\phi(x_{n+1},\bar{u}_n)=\lim_{n\to\infty}\phi(x_{n+1},\bar{z}_n)=\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \tag{3.7}$$

Therefore, from Lemma 2.4,

$$\lim_{n\to\infty}||x_{n+1}-\bar{u}_n||=\lim_{n\to\infty}||x_{n+1}-\bar{z}_n||=\lim_{n\to\infty}||x_{n+1}-x_n||=0.$$

This together with $||x_{n+1}-x_n||\to 0$ implies that

$$\lim_{n\to\infty}||x_n-\bar{u}_n||=\lim_{n\to\infty}||x_n-\bar{z}_n||=0.$$

By the definitions of j_n and k_n, we obtain

$$\lim_{n\to\infty}||x_n-u_n^k||=\lim_{n\to\infty}||x_n-z_n^j||=0. \tag{3.8}$$

for all $1\le k\le K$ and $1\le j\le N$. Hence

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}u_n^k=\lim_{n\to\infty}z_n^j=p \tag{3.9}$$

for all $1\le k\le K$ and $1\le j\le N$. By the hypotheses of Theorem 3.1, Lemmas $2.1, 2.2$ and 2.11, we also have

$$\begin{aligned}
\phi(x_n,\bar{y}_n)&=\phi\left(x_n,\Pi_C\left(J^{-1}(Jx_n-\lambda_nA_{i_n}x_n)\right)\right)\\
&\le\phi(x_n,J^{-1}(Jx_n-\lambda_nA_{i_n}x_n))\\
&=V(x_n,Jx_n-\lambda_nA_{i_n}x_n)\\
&\le V(x_n,Jx_n-\lambda_nA_{i_n}x_n+\lambda_nA_{i_n}x_n)\\
&\quad-2\left\langle J^{-1}\left(Jx_n-\lambda_nA_{i_n}x_n\right)-x_n,\lambda_nA_{i_n}x_n\right\rangle\\
&=-2\lambda_n\left\langle J^{-1}\left(Jx_n-\lambda_nA_{i_n}x_n\right)-J^{-1}Jx_n,A_{i_n}x_n\right\rangle\\
&\le\frac{4\lambda_n}{c^2}||Jx_n-\lambda_nA_{i_n}x_n-Jx_n||||A_{i_n}x_n||\\
&\le\frac{4\lambda_n^2}{c^2}||A_{i_n}x_n||^2\\
&\le\frac{4b^2}{c^2}||A_{i_n}x_n-A_{i_n}u||^2
\end{aligned} \tag{3.10}$$

for all $u\in\bigcap_{i=1}^M VI(A_i,C)$. From (3.3), we obtain

$$2(1-\alpha_n)a\left(\alpha-\frac{2b}{c^2}\right)||A_{i_n}x_n-A_{i_n}u||^2\le(\phi(u,x_n)-\phi(u,\bar{z}_n))+\epsilon_n$$

$$= 2\langle u, J\bar{z}_n - Jx_n\rangle + (||x_n||^2 - ||\bar{z}_n||^2) + \epsilon_n$$
$$\leq 2||u||||J\bar{z}_n - Jx_n|| + ||x_n - \bar{z}_n||(||x_n|| + ||\bar{z}_n||) + \epsilon_n. \quad (3.11)$$

Using the fact that $||x_n - \bar{z}_n|| \to 0$ and J is uniformly continuous on each bounded set, we can conclude that $||J\bar{z}_n - Jx_n|| \to 0$ as $n \to \infty$. This together with (3.11), and the relations $\limsup_{n\to\infty} \alpha_n < 1$ and $\epsilon_n \to 0$ imply that

$$\lim_{n\to\infty} ||A_{i_n}x_n - A_{i_n}u|| = 0. \quad (3.12)$$

From (3.10) and (3.12), we obtain

$$\lim_{n\to\infty} \phi(x_n, \bar{y}_n) = 0.$$

Therefore $\lim_{n\to\infty} ||x_n - \bar{y}_n|| = 0$. By the definition of i_n, we get
$\lim_{n\to\infty} ||x_n - y_n^i|| = 0$. Hence,

$$\lim_{n\to\infty} y_n^i = p \quad (3.13)$$

for all $1 \leq i \leq M$.
Step 4. Claim that $p \in \bigcap_{j=1}^n F(S_j)$.
The relation $z_n^j = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n)JS_j^n\bar{y}_n\right)$ implies that $Jz_n^j = \alpha_n Jx_n + (1-\alpha_n)JS_j^n\bar{y}_n$. Therefore,

$$||Jx_n - Jz_n^j|| = (1-\alpha_n)||Jx_n - JS_j^n\bar{y}_n||. \quad (3.14)$$

Since $||x_n - z_n^j|| \to 0$ and J is uniformly continuous on each bounded subset of E, $||Jx_n - Jz_n^j|| \to 0$ as $n \to \infty$. This together with (3.14) and $\limsup_{n\to\infty} \alpha_n < 1$ implies that

$$\lim_{n\to\infty} ||Jx_n - JS_j^n\bar{y}_n|| = 0.$$

Therefore,

$$\lim_{n\to\infty} ||x_n - S_j^n\bar{y}_n|| = 0. \quad (3.15)$$

Since $\lim_{n\to\infty} ||x_n - \bar{y}_n|| = 0$, $\lim_{n\to\infty} ||\bar{y}_n - S_j^n\bar{y}_n|| = 0$, hence

$$\lim_{n\to\infty} S_j^n\bar{y}_n = p. \quad (3.16)$$

Further,

$$\begin{aligned}\left\|S_j^{n+1}\bar{y}_n - S_j^n\bar{y}_n\right\| &\leq \left\|S_j^{n+1}\bar{y}_n - S_j^{n+1}\bar{y}_{n+1}\right\| + \left\|S_j^{n+1}\bar{y}_{n+1} - \bar{y}_{n+1}\right\| \\ &\quad + \|\bar{y}_{n+1} - \bar{y}_n\| + \left\|\bar{y}_n - S_j^n\bar{y}_n\right\| \\ &\leq (L+1)\|\bar{y}_{n+1} - \bar{y}_n\| + \left\|S_j^{n+1}\bar{y}_{n+1} - \bar{y}_{n+1}\right\|\end{aligned}$$

$$+ \left\|\bar{y}_n - S_j^n \bar{y}_n\right\| \to 0,$$

therefore

$$\lim_{n\to\infty} S_j^{n+1}\bar{y}_n = \lim_{n\to\infty} S_j\left(S_j^n\bar{y}_n\right) = p. \tag{3.17}$$

From (3.16),(3.17) and the closedness of S_j, we obtain $p \in F(S_j)$ for all $1 \le j \le N$. Hence $p \in \bigcap_{j=1}^N F(S_j)$.

Step 5. Claim that $p \in \bigcap_{i=1}^M VI(A_i, C)$.

Lemma 2.12 ensures that the mapping

$$Q_i(x) = \begin{cases} A_i x + N_C(x) & \text{if} \quad x \in C, \\ \emptyset & \text{if} \quad x \notin C, \end{cases}$$

is maximal monotone, where $N_C(x)$ is the normal cone to C at $x \in C$. For all (x, y) in the graph of Q_i, i.e., $(x, y) \in G(Q_i)$, we have $y - A_i(x) \in N_C(x)$. By the definition of $N_C(x)$, we find that

$$\langle x - z, y - A_i(x)\rangle \ge 0$$

for all $z \in C$. Since $y_n^i \in C$,

$$\left\langle x - y_n^i, y - A_i(x)\right\rangle \ge 0.$$

Therefore,

$$\left\langle x - y_n^i, y\right\rangle \ge \left\langle x - y_n^i, A_i(x)\right\rangle. \tag{3.18}$$

Taking into account $y_n^i = \mathit{\Pi}_C\left(J^{-1}(Jx_n - \lambda_n A_i x_n)\right)$ and Lemma 2.2, we get

$$\left\langle x - y_n^i, Jy_n^i - Jx_n + \lambda_n A_i x_n\right\rangle \ge 0. \tag{3.19}$$

Therefore, from (3.18), (3.19) and the monotonicity of A_i, we find that

$$\begin{aligned} \left\langle x - y_n^i, y\right\rangle &\ge \left\langle x - y_n^i, A_i(x)\right\rangle \\ &= \left\langle x - y_n^i, A_i(x) - A_i(y_n^i)\right\rangle + \left\langle x - y_n^i, A_i(y_n^i) - A_i(x_n)\right\rangle \\ &\quad + \left\langle x - y_n^i, A_i(x_n)\right\rangle \\ &\ge \left\langle x - y_n^i, A_i(y_n^i) - A_i(x_n)\right\rangle + \left\langle x - y_n^i, \frac{Jx_n - Jy_n^i}{\lambda_n}\right\rangle. \end{aligned} \tag{3.20}$$

Since $||x_n - y_n^i|| \to 0$ and J is uniform continuous on each bounded set, $||Jx_n - Jy_n^i|| \to 0$. By $\lambda_n \ge a > 0$, we obtain

$$\lim_{n\to\infty} \frac{Jx_n - Jy_n^i}{\lambda_n} = 0. \tag{3.21}$$

Since A_i is α-inverse strongly monotone, A_i is $\frac{1}{\alpha}$-Lipschitz continuous. This together with $||x_n - y_n^i|| \to 0$ implies that

$$\lim_{n\to\infty} ||A_i(y_n^i) - A_i(x_n)|| = 0. \tag{3.22}$$

From (3.20), (3.21),(3.22), and $y_n^i \to p$, we obtain $\langle x - p, y\rangle \geq 0$ for all $(x, y) \in G(Q_i)$. Therefore $p \in Q_i^{-1}0 = VI(A_i, C)$ for all $1 \leq i \leq M$. Hence, $p \in \bigcap_{i=1}^M VI(A_i, C)$.

Step 6. Claim that $p \in \bigcap_{k=1}^K EP(f_k)$.

Since $\lim_{n\to\infty} \left\|u_n^k - \bar{z}_n\right\| = 0$ and J is uniformly continuous on every bounded subset of E, we have

$$\lim_{n\to\infty} \left\|Ju_n^k - J\bar{z}_n\right\| = 0.$$

This together with $r_n \geq d > 0$ implies that

$$\lim_{n\to\infty} \frac{\left\|Ju_n^k - J\bar{z}_n\right\|}{r_n} = 0. \tag{3.23}$$

We have $u_n^k = T_{r_n}^k \bar{z}_n$, and

$$f_k(u_n^k, y) + \frac{1}{r_n}\left\langle y - u_n^k, Ju_n^k - J\bar{z}_n\right\rangle \geq 0 \quad \forall y \in C. \tag{3.24}$$

From (3.24) and condition $(A2)$, we get

$$\frac{1}{r_n}\left\langle y - u_n^k, Ju_n^k - J\bar{z}_n\right\rangle \geq -f_k(u_n^k, y) \geq f_k(y, u_n^k) \quad \forall y \in C. \tag{3.25}$$

Letting $n \to \infty$, by $(3.23), (3.25)$ and $(A4)$, we obtain

$$f_k(y, p) \leq 0, \forall y \in C. \tag{3.26}$$

Putting $y_t = ty + (1 - t)p$, where $0 < t \leq 1$ and $y \in C$, we get $y_t \in C$. Hence, for sufficiently small t, from $(A3)$ and (3.26), we have

$$f_k(y_t, p) = f_k(ty + (1 - t)p, p) \leq 0.$$

By the properties $(A1), (A4)$, we find

$$\begin{aligned} 0 &= f_k(y_t, y_t) \\ &= f_k(y_t, ty + (1 - t)p) \\ &\leq tf_k(y_t, y) + (1 - t)f_k(y_t, p) \\ &\leq tf_k(y_t, y) \end{aligned}$$

Dividing both sides of the last inequality by $t > 0$, we obtain $f_k(y_t, y) \geq 0$ for all $y \in C$, i.e.,

$$f_k(ty + (1-t)p, y) \geq 0, \forall y \in C.$$

Passing $t \to 0^+$, from $(A3)$, we get $f_k(p, y) \geq 0$, $\forall y \in C$ and $1 \leq k \leq K$, i.e., $p \in \bigcap_{k=1}^{K} EP(f_k)$.

Step 7. Claim that the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Indeed, since $x^\dagger := \Pi_F(x_0) \in F \subset C_n$, $x_n = \Pi_{C_n}(x_0)$ from Lemma 2.2, we have

$$\phi(x_n, x_0) \leq \phi(x^\dagger, x_0) - \phi(x^\dagger, x_n) \leq \phi(x^\dagger, x_0). \tag{3.27}$$

Therefore,

$$\begin{aligned}\phi(x^\dagger, x_0) &\geq \lim_{n\to\infty} \phi(x_n, x_0) = \lim_{n\to\infty} \left\{ \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 \right\} \\ &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \\ &= \phi(p, x_0).\end{aligned}$$

From the definition of $x^\dagger$, it follows that $p = x^\dagger$. The proof of Theorem 3.1 is complete.

Remark 3.1 Assume that $\{A_i\}_{i=1}^M$ is a finite family of η-strongly monotone and L-Lipschitz continuous mappings. Then each A_i is $\frac{\eta}{L}$-inverse strongly monotone and $VI(A_i, C) = A_i^{-1}0$. Hence, $||A_i x|| \leq ||A_i x - A_i u||$ for all $x \in C$ and $u \in VI(A_i, C)$. Thus, all the conditions (V1)-(V3) for the variational inequalities $VI(A_i, C)$ hold.

Theorem 3.2 *Let $\{A_i\}_{i=1}^M$ be a finite family of mappings from C to E^* satisfying conditions* (V1)-(V3)*. Let $\{f_k\}_{k=1}^K : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying conditions* (A1)-(A4)*. Let $\{S_j\}_{j=1}^N : C \to C$ be a finite family of uniform L-Lipschitz continuous and asymptotically quasi-ϕ-nonexpansive mappings with the same sequence $\{k_n\} \subset [1, +\infty), k_n \to 1$. Assume that F is a subset of Ω, and suppose that the control parameter sequences $\{\alpha_n\}, \{\lambda_n\}, \{r_n\}$ satisfy condition* (1.6)*. Then the sequence $\{x_n\}$ generated by method* (1.5) *converges strongly to $\Pi_F x_0$.*

Proof Arguing similarly as in Step 1 of the proof of Theorem 3.1, we conclude that F, C_n are closed convex for all $n \geq 0$. Now we show that $F \subset C_n$ for all $n \geq 0$. For all $u \in F$, by Lemma 2.5 and the convexity of $||.||^2$ we obtain

$$\phi(u, z_n) = \phi\left(u, J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{l=1}^{N} \alpha_{n,l} JS_l^n \bar{y}_n\right)\right)$$

$$
\begin{aligned}
&= ||u||^2 - 2\alpha_{n,0}\langle u, x_n\rangle - 2\sum_{l=1}^{N}\alpha_{n,l}\langle u, S_l^n\bar{y}_n\rangle \\
&\quad + ||\alpha_{n,0}Jx_n + \sum_{l=1}^{N}\alpha_{n,l}JS_l^n\bar{y}_n||^2 \\
&\le ||u||^2 - 2\alpha_{n,0}\langle u, x_n\rangle - 2\sum_{l=1}^{N}\alpha_{n,l}\langle u, S_l^n\bar{y}_n\rangle + \alpha_{n,0}||x_n||^2 \\
&\quad + \sum_{l=1}^{N}\alpha_{n,l}||S_l^n\bar{y}_n||^2 - \alpha_{n,0}\alpha_{n,j}g\left(||Jx_n - JS_j^n\bar{y}_n||\right) \\
&\le \alpha_{n,0}\phi(u, x_n) + \sum_{l=1}^{N}\alpha_{n,l}\phi(u, S_l^n\bar{y}_n) - \alpha_{n,0}\alpha_{n,j}g\left(||Jx_n - JS_j^n\bar{y}_n||\right) \\
&\le \alpha_{n,0}\phi(u, x_n) + \sum_{l=1}^{N}\alpha_{n,l}k_n\phi(u, \bar{y}_n) - \alpha_{n,0}\alpha_{n,j}g\left(||Jx_n - JS_j^n\bar{y}_n||\right).
\end{aligned}
\tag{3.28}
$$

From (3.2), we get

$$
\phi(u, \bar{y}_n) \le \phi(u, x_n) - 2a\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2. \tag{3.29}
$$

Using (3.28), (3.29) and the estimate (2.2), we find

$$
\begin{aligned}
\phi(u, \bar{u}_n) &= \phi(u, T_{r_n}^{k_n}z_n) \\
&\le \phi(u, z_n) \\
&\le \phi(u, x_n) + \sum_{l=1}^{N}\alpha_{n,l}(k_n - 1)\phi(u, x_n) - \alpha_{n,0}\alpha_{n,j}g\left(||Jx_n - JS_j^n\bar{y}_n||\right) \\
&\quad -2\sum_{l=1}^{N}\alpha_{n,l}a\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2 \\
&\le \phi(u, x_n) + (k_n - 1)\left(\omega + ||x_n||\right)^2 - \alpha_{n,0}\alpha_{n,j}g\left(||Jx_n - JS_j^n\bar{y}_n||\right) \\
&\quad -2\sum_{l=1}^{N}\alpha_{n,l}a\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2 \\
&\le \phi(u, x_n) + \epsilon_n.
\end{aligned}
\tag{3.30}
$$

Therefore

$$
\phi(u, \bar{u}_n) \le \phi(u, x_n) + \epsilon_n
$$

for all $u \in F$. This implies that $F \subset C_n$ for all $n \ge 0$. Using (3.30) and arguing similarly as in Steps 3, 5, 6 of Theorem 3.1, we obtain

$$
\lim_{n\to\infty}\bar{u}_n = \lim_{n\to\infty}u_n^k = \lim_{n\to\infty}\bar{y}_n = \lim_{n\to\infty}y_n^i = \lim_{n\to\infty}x_n = p \in C,
$$

and $p \in \left(\bigcap_{k=1}^{K} EP(f_k)\right) \bigcap \left(\bigcap_{i=1}^{M} VI(A_i, C)\right)$.
Next, we show that $p \in \bigcap_{j=1}^{N} F(S_j)$. Indeed, from (3.30), we have

$$\alpha_{n,0}\alpha_{n,j} g\left(||Jx_n - JS_j^n \bar{y}_n||\right) \leq (\phi(u, x_n) - \phi(u, \bar{u}_n)) + \epsilon_n. \tag{3.31}$$

Since $||x_n - \bar{u}_n|| \to 0$, $|\phi(u, x_n) - \phi(u, \bar{u}_n)| \to 0$ as $n \to \infty$. This together with (3.31) and the facts that $\epsilon_n \to 0$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$ imply that

$$\lim_{n\to\infty} g\left(||Jx_n - JS_j^n \bar{y}_n||\right) = 0.$$

By Lemma 2.5, we get

$$\lim_{n\to\infty} ||Jx_n - JS_j^n \bar{y}_n|| = 0.$$

Since J is uniformly continuous on each bounded subset of E, we conclude that

$$\lim_{n\to\infty} ||x_n - S_j^n \bar{y}_n|| = 0.$$

Using the last equality and a similar argument for proving relations (3.15), (3.16), (3.17), and acting as in Step 7 of the proof of Theorem 3.1, we obtain $p \in \bigcap_{j=1}^{N} F(S_j)$ and $p = x^\dagger = \Pi_F x_0$. The proof of Theorem 3.2 is complete.

Next, we consider two parallel hybrid methods for solving variational inequalities, equilibrium problems and quasi ϕ-nonexpansive mappings, when the boundedness of the solution set F and the uniform Lipschitz continuity of S_i are not required.

Theorem 3.3 *Assume that* $\{A_i\}_{i=1}^{M}, \{f_k\}_{k=1}^{K}, \{\alpha_n\}, \{r_n\}$ *and* $\{\lambda_n\}$ *satisfy all conditions of Theorem* 3.1 *and* $\{S_j\}_{j=1}^{N}$ *is a finite family of closed and quasi ϕ-nonexpansive mappings. In addition, suppose that the solution set F is nonempty. For an intitial point* $x_0 \in C$, *define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n^i = \Pi_C\left(J^{-1}(Jx_n - \lambda_n A_i x_n)\right), i = 1, 2, \dots M, \\ i_n = \arg\max\left\{||y_n^i - x_n|| : i = 1, 2, \dots M.\right\}, \bar{y}_n = y_n^{i_n}, \\ z_n^j = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JS_j \bar{y}_n\right), j = 1, 2, \dots N, \\ j_n = \arg\max\left\{||z_n^j - x_n|| : j = 1, 2, \dots N\right\}, \bar{z}_n = z_n^{j_n}, \\ u_n^k = T_{r_n}^k \bar{z}_n, k = 1, 2, \dots K, \\ k_n = \arg\max\left\{||u_n^k - x_n|| : k = 1, 2, \dots K\right\}, \bar{u}_n = u_n^{k_n}, \\ C_{n+1} = \{z \in C_n : \phi(z, \bar{u}_n) \leq \phi(z, \bar{z}_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{3.32}$$

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Proof Since S_i is a closed and quasi ϕ-nonexpansive mapping, it is closed and asymptotically quasi ϕ-nonexpansive mapping with $k_n = 1$ for all $n \geq 0$. Hence, $\epsilon_n = 0$ by definition. Arguing similarly as in the proof of Theorem 3.1, we come to the desired conclusion.

Theorem 3.4 *Assume that $\{A_i\}_{i=1}^M, \{f_k\}_{k=1}^K, \{r_n\}, \{\alpha_{n,j}\}$ and $\{\lambda_n\}$ satisfy all conditions of Theorem 3.2 and $\{S_j\}_{j=1}^N$ is a finite family of closed and quasi ϕ-nonexpansive mappings. In addition, suppose that the solution set F is nonempty. For an initial approximation $x_0 \in C$, let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n^i = \Pi_C \left(J^{-1}(Jx_n - \lambda_n A_i x_n)\right), i = 1, 2, \ldots M, \\ i_n = \arg\max \left\{||y_n^i - x_n|| : i = 1, 2, \ldots M.\right\}, \bar{y}_n = y_n^{i_n}, \\ z_n = J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{j=1}^N \alpha_{n,j} JS_j \bar{y}_n\right), \\ u_n^k = T_{r_n}^k z_n, k = 1, 2, \ldots K, \\ k_n = \arg\max \left\{||u_n^k - x_n|| : k = 1, 2, \ldots K\right\}, \bar{u}_n = u_n^{k_n}, \\ C_{n+1} = \{z \in C_n : \phi(z, \bar{u}_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{3.33}$$

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof The proof is similar to that of Theorem 3.2 for S_i being closed and quasi ϕ- asymptotically nonexpansive mapping with $k_n = 1$ for all $n \geq 0$.

4 A parallel iterative method for quasi ϕ-nonexpansive mappings and variational inequalities

In 2004, using Mann's iteration, Matsushita and Takahashi [11] proposed the following scheme for finding a fixed point of a relatively nonexpansive mapping T:

$$x_{n+1} = \Pi_C J^{-1}\left(\alpha_n Jx_n + (1 - \alpha_n) JTx_n\right), \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

where $x_0 \in C$ is given. They proved that if the interior of $F(T)$ is nonempty then the sequence $\{x_n\}$ generated by (4.1) converges strongly to some point in $F(T)$. Recently, using Halpern's and Ishikawa's iterative processes, Zhang, Li, and Liu [23] have proposed modified iterative algorithms of (4.1) for a relatively nonexpansive mapping.

In this section, employing the ideas of Matsushita and Takahashi [11] and Anh and Chung [4], we propose a parallel hybrid iterative algorithm for finite families of closed and quasi ϕ- nonexpansive mappings $\{S_j\}_{j=1}^N$ and variational inequalities $\{VI(A_i,C)\}_{i=1}^M$:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n^i = \Pi_C \left(J^{-1}(Jx_n - \lambda_n A_i x_n)\right), i = 1, 2, \dots M, \\ i_n = \arg\max \left\{||y_n^i - x_n|| : i = 1, 2, \dots M.\right\}, \bar{y}_n = y_n^{i_n}, \\ z_n^j = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JS_j \bar{y}_n\right), j = 1, 2, \dots N, \\ j_n = \arg\max \left\{||z_n^j - x_n|| : j = 1, 2, \dots N\right\}, \bar{z}_n = z_n^{j_n}, \\ x_{n+1} = \Pi_C \bar{z}_n, n \ge 0, \end{cases} \tag{4.2}$$

where, $\{\alpha_n\} \subset [0,1]$, such that $\lim_{n\to\infty} \alpha_n = 0$.

Remark 4.1 One can employ method (4.2) for a finite family of relatively nonexpansive mappings without assuming their closedeness.

Remark 4.2 Method (4.2) modifies the corresponding method (4.1) in the following aspects:

- A relatively nonexpansive mapping T is replaced with a finite family of quasi ϕ-nonexpansive mappings, where the restriction $F(S_j) = \tilde{F}(S_j)$ is not required.
- A parallel hybrid method for finite families of closed and quasi ϕ- nonexpansive mappings and variational inequalities is considered instead of an iterative method for a relatively nonexpansive mapping.

Theorem 4.1 *Let E be a real uniformly smooth and 2-uniformly convex Banach space with dual space E^* and C be a nonempty closed convex subset of E. Assume that $\{A_i\}_{i=1}^M$ is a finite family of mappings satisfying conditions* (V1)-(V3)*, $\{S_j\}_{j=1}^N$ is a finite family of closed and quasi ϕ-nonexpansive mappings, and $\{\alpha_n\} \subset [0,1]$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\lambda_n \in [a,b]$ for some $a, b \in (0, \alpha c^2/2)$. In addition, suppose that the interior of $F = \left(\bigcap_{i=1}^M VI(A_i,C)\right) \bigcap \left(\bigcap_{j=1}^N F(S_j)\right)$ is nonempty. Then the sequence $\{x_n\}$ generated by* (4.2) *converges strongly to some point $u \in F$. Moreover, $u = \lim_{n\to\infty} \Pi_F x_n$.*

Proof By Lemma 2.7, the subset F is closed and convex, hence the generalized projections Π_F, Π_C are well-defined. We now show that the sequence $\{x_n\}$ is bounded. Indeed, for every $u \in F$, from Lemma 2.2 and the convexity of $\|.\|^2$, we have

$$\begin{aligned} \phi(u, x_{n+1}) &= \phi(u, \Pi_C \bar{z}_n) \\ &\le \phi(u, \bar{z}_n) \\ &= \|u\|^2 - 2\langle u, J\bar{z}_n\rangle + \|\bar{z}_n\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|u\|^2 - 2\alpha_n \langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JS_{j_n}\bar{y}_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JS_{j_n}\bar{y}_n\|^2 \\
&\leq \|u\|^2 - 2\alpha_n \langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JS_{j_n}\bar{y}_n\rangle \\
&\quad + \alpha_n \|\bar{y}_n\|^2 + (1-\alpha_n)\|S_{j_n}\bar{y}_n\|^2 \\
&= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, S_{j_n}\bar{y}_n) \\
&\leq \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, \bar{y}_n).
\end{aligned}$$

Arguing similarly to (3.2) and (3.3), we obtain

$$\phi(u, x_{n+1}) \leq \phi(u, x_n) - 2a(1-\alpha_n)\left(\alpha - \frac{2b}{c^2}\right)||A_{i_n}x_n - A_{i_n}u||^2 \leq \phi(u, x_n). \tag{4.3}$$

Therefore, the sequence $\{\phi(u, x_n)\}$ is decreasing. Hence there exists a finite limit of $\{\phi(u, x_n)\}$. This together with (2.2) and (4.3) imply that the sequences $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} ||A_{i_n}x_n - A_{i_n}u|| = 0. \tag{4.4}$$

Next, we show that $\{x_n\}$ converges strongly to some element u in C. Since the interior of F is nonempty, there exist $p \in F$ and $r > 0$ such that

$$p + rh \in F,$$

for all $h \in E$ and $\|h\| \leq 1$. Since the sequence $\{\phi(u, x_n)\}$ is decreasing for all $u \in F$, we have

$$\phi(p + rh, x_{n+1}) \leq \phi(p + rh, x_n). \tag{4.5}$$

From (2.3), we find that

$$\phi(u, x_n) = \phi(u, x_{n+1}) + \phi(x_{n+1}, x_n) + 2\langle x_{n+1} - u, Jx_n - Jx_{n+1}\rangle,$$

for all $u \in F$. Therefore,

$$\begin{aligned}
\phi(p + rh, x_n) = \phi(p + rh, x_{n+1}) + \phi(x_{n+1}, x_n) \\
+2\langle x_{n+1} - (p + rh), Jx_n - Jx_{n+1}\rangle.
\end{aligned} \tag{4.6}$$

From (4.5), (4.6), we obtain

$$\phi(x_{n+1}, x_n) + 2\langle x_{n+1} - (p + rh), Jx_n - Jx_{n+1}\rangle \geq 0.$$

This inequality is equivalent to

$$\langle h, Jx_n - Jx_{n+1}\rangle \leq \frac{1}{2r}\{\phi(x_{n+1}, x_n) + 2\langle x_{n+1} - p, Jx_n - Jx_{n+1}\rangle\}. \tag{4.7}$$

From (2.3), we also have

$$\phi(p,x_n)=\phi(p,x_{n+1})+\phi(x_{n+1},x_n)+2\left\langle x_{n+1}-p,Jx_n-Jx_{n+1}\right\rangle. \qquad (4.8)$$

From (4.7), (4.8), we obtain

$$\langle h,Jx_n-Jx_{n+1}\rangle \le \frac{1}{2r}\left\{\phi(p,x_n)-\phi(p,x_{n+1})\right\},$$

for all $\|h\|\le 1$. Hence

$$\sup_{\|h\|\le 1}\langle h,Jx_n-Jx_{n+1}\rangle \le \frac{1}{2r}\left\{\phi(p,x_n)-\phi(p,x_{n+1})\right\}.$$

The last relation is equivalent to

$$\|Jx_n-Jx_{n+1}\| \le \frac{1}{2r}\left\{\phi(p,x_n)-\phi(p,x_{n+1})\right\}.$$

Therefore, for all $n,m\in N$ and $n>m$, we have

$$\begin{aligned}\|Jx_n-Jx_m\| &= \|Jx_n-Jx_{n-1}+Jx_{n-1}-Jx_{n-2}+\ldots+Jx_{m+1}-Jx_m\|\\ &\le \sum_{i=m}^{n-1}\|Jx_{i+1}-Jx_i\|\\ &\le \frac{1}{2r}\sum_{i=m}^{n-1}\left\{\phi(p,x_i)-\phi(p,x_{i+1})\right\}\\ &= \frac{1}{2r}\left(\phi(p,x_m)-\phi(p,x_n)\right).\end{aligned}$$

Letting $m,n\to\infty$, we obtain

$$\lim_{m,n\to\infty}\|Jx_n-Jx_m\|=0.$$

Since E is uniformly convex and uniformly smooth Banach space, J^{-1} is uniformly continuous on every bounded subset of E. From the last relation we have

$$\lim_{m,n\to\infty}\|x_n-x_m\|=0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since E is complete and C is closed and convex, $\{x_n\}$ converges strongly to some element u in C. By arguing similarly to (3.10), we obtain

$$\phi(x_n,\bar{y}_n)\le \frac{4b^2}{c^2}||A_{i_n}x_n-A_{i_n}u||^2.$$

This relation together with (4.4) imples that $\phi(x_n, \bar{y}_n) \to 0$. Therefore, $||x_n - \bar{y}_n|| \to 0$. By the definition of i_n, we conclude that $||x_n - y_n^i|| \to 0$ for all $1 \le i \le M$. Hence,

$$\lim_{n\to\infty} y_n^i = u \in C, \tag{4.9}$$

for all $1 \le i \le M$. From $x_{n+1} = \Pi_C \bar{z}_n$ and Lemma 2.2, we have

$$\phi(S_{j_n}\bar{y}_n, x_{n+1}) + \phi(x_{n+1}, \bar{z}_n) = \phi(S_{j_n}\bar{y}_n, \Pi_C \bar{z}_n) + \phi(\Pi_C \bar{z}_n, \bar{z}_n) \le \phi(S_{j_n}\bar{y}_n, \bar{z}_n). \tag{4.10}$$

Using the convexity of $\|.\|^2$ we have

$$\begin{aligned}
\phi(S_{j_n}\bar{y}_n, \bar{z}_n) &= \|S_{j_n}\bar{y}_n\|^2 - 2\langle S_{j_n}\bar{y}_n, J\bar{z}_n\rangle + \|\bar{z}_n\|^2 \\
&= \|S_{j_n}\bar{y}_n\|^2 - 2\alpha_n \langle S_{j_n}\bar{y}_n, Jx_n\rangle - 2(1-\alpha_n)\langle S_{j_n}\bar{y}_n, JS_{j_n}\bar{y}_n\rangle + \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JS_{j_n}\bar{y}_n\|^2 \\
&\le \|S_{j_n}\bar{y}_n\|^2 - 2\alpha_n \langle S_{j_n}\bar{y}_n, Jx_n\rangle - 2(1-\alpha_n)\langle S_{j_n}\bar{y}_n, JS_{j_n}\bar{y}_n\rangle + \\
&\quad + \alpha_n \|x_n\|^2 + (1-\alpha_n)\|S_{j_n}\bar{y}_n\|^2 \\
&= \alpha_n \phi(S_{j_n}\bar{y}_n, x_n) + (1-\alpha_n)\phi(S_{j_n}\bar{y}_n, S_{j_n}\bar{y}_n) \\
&= \alpha_n \phi(S_{j_n}\bar{y}_n, x_n).
\end{aligned}$$

The last inequality together with (4.10) implies that

$$\phi(x_{n+1}, \bar{z}_n) \le \alpha_n \phi(S_{j_n}\bar{y}_n, x_n).$$

Therefore, from the boundedness of $\{\phi(S_{j_n}\bar{y}_n, x_n)\}$ and $\lim_{n\to\infty} \alpha_n = 0$, we get

$$\lim_{n\to\infty} \phi(x_{n+1}, \bar{z}_n) = 0.$$

Hence, $||x_{n+1} - \bar{z}_n|| \to 0$. Since $||x_n - x_{n+1}|| \to 0$, we find $||x_n - \bar{z}_n|| \to 0$, and by the definition of j_n, we obtain $||x_n - z_n^j|| \to 0$ for all $1 \le j \le N$. Thus,

$$\lim_{n\to\infty} z_n^j = p. \tag{4.11}$$

Arguing similarly to Steps 4, 5 in the proof of Theorem 3.1, we obtain

$$p \in F = \left(\bigcap_{i=1}^{M} VI(A_i, C)\right) \bigcap \left(\bigcap_{j=1}^{N} F(S_j)\right).$$

The proof of Theorem 4.1 is complete.

5 A numerical example

Let $E = \mathbb{R}^1$ be a Hilbert space with the standart inner product $\langle x, y\rangle := xy$ and the norm $||x|| := |x|$ for all $x, y \in E$. Let $C := [0, 1] \subset E$. The normalized dual mapping $J = I$ and the Lyapunov functional $\phi(x, y) = |x - y|^2$. It is well known that, the modulus of convexity of Hilbert space E is $\delta_E(\epsilon) = 1 - \sqrt{1 - \epsilon^2/4} \geq \frac{1}{4}\epsilon^2$. Therefore, E is 2-uniformly convex. Moreover, the best constant $\frac{1}{c}$ satisfying relations $|x-y| \leq \frac{2}{c^2}|Jx-Jy| = \frac{2}{c^2}|x-y|$ with $0 < c \leq 1$ is 1. This implies that $c = 1$. Define the mappings $A_i(x) := x - \frac{x^{i+1}}{i+1}, x \in C, i = 1, \ldots, M$, and consider the variational inequalities

$$\langle A_i(p^*), p - p^*\rangle \geq 0, \quad \forall p \in C,$$

for $i = 1, \ldots, M$. Clearly, $VI(A_i, C) = \{0\}, i = 1, \ldots, M$. Since each mapping $U_i(x) := \frac{x^{i+1}}{i+1}$ is nonexpansive, the mapping $A_i = I - U_i, i = 1, \ldots, M$, is $\frac{1}{2}$-inverse strongly monotone. Besides, $|A_i(y)| = |A_i(y) - A_i(0)|$, hence all the assumptions (V1)-(V3) for the variational inequalities are satisfied.
Further, let $\{t_i\}_{i=1}^N$ and $\{s_i\}_{i=1}^N$ be two sequences of positive numbers, such that $0 < t_1 < \ldots < t_N < 1$ and $s_i \in (1, \frac{1}{1-t_i}]; i = 1, \ldots, N$. Define the mappings $S_i : C \to C, i = 1, \ldots, N$, by putting $S_i(x) = 0$, for $x \in [0, t_i]$, and $S_i(x) = s_i(x - t_i)$, if $x \in [t_i, 1]$.
It is easy to verify that $F(S_i) = \{0\}, \phi(S_i(x), 0) = |S_i(x)|^2 \leq |x|^2 = \phi(x, 0)$ for every $x \in C$, and $|S_i(1) - S_i(t_i)| = s_i(1 - t_i) > |1 - t_i|$. Hence, the mappings S_i are quasi ϕ-nonexpansive but not nonexpansive.
Finally, let $0 < \xi_1 < \ldots < \xi_K < 1$ and $\eta_k \in (0, \xi_k), k = 1, \ldots, K$, be two given sequences. Consider K bifunctions $f_k(x, y) := B_k(x)(y - x), k = 1, \ldots, K$, where $B_k(x) = \frac{\eta_k}{\xi_k}x$ if $0 \leq x \leq \xi_k$, and $B_k(x) = \eta_k$ if $\xi_k \leq x \leq 1$.
It is easy to verify that all the assumptions (A1)-(A4) for the bifunctions $f_k(x, y)$ are fulfilled. Besides, $EP(f_k) = \{0\}$. Thus, the solution set

$$F := \left(\bigcap_{i=1}^{M} VI(A_i, C)\right) \bigcap \left(\bigcap_{j=1}^{N} F(S_j)\right) \bigcap \left(\bigcap_{k=1}^{K} EP(f_k)\right) = \{0\}.$$

According to Theorem 3.4, the iteration sequence $\{x_n\}$ generated by

$$y_n^i = \Pi_C\left(x_n - \lambda_n(x_n - \frac{(x_n)^{i+1}}{i+1})\right), i = 1, 2, \ldots M,$$
$$i_n = \arg\max\left\{|y_n^i - x_n| : i = 1, 2, \ldots M\right\}, \bar{y}_n = y_n^{i_n},$$
$$z_n^j = \alpha_n x_n + (1 - \alpha_n)S_j\bar{y}_n, j = 1, 2, \ldots N,$$

$$j_n = \arg\max\left\{|z_n^j - x_n| : j = 1,2,\ldots N\right\}, \bar{z}_n = z_n^{j_n},$$
$$u_n^k = T_{r_n}^k \bar{z}_n, k = 1,2,\ldots K,$$
$$k_n = \arg\max\left\{|u_n^k - x_n| : k = 1,2,\ldots K\right\}, \bar{u}_n = u_n^{k_n},$$
$$C_{n+1} = \{z \in C_n : \phi(z,\bar{u}_n) \le \phi(z,\bar{z}_n) \le \phi(z,x_n)\},$$
$$x_{n+1} = \Pi_{C_{n+1}} x_0, n \ge 0.$$

strongly converges to $x^\dagger := 0$.
A straightforward calculation yields $y_n^i = (1-\lambda_n)x_n - \lambda_n \frac{(x_n)^{i+1}}{i+1}, i = 1,2,\ldots M$. Further, the element $u := u_n^k = T_{r_n}^k \bar{z}_n$ is a solution of the following inequality

$$(y-u)[B_k(u) + u - \bar{z}_n] \ge 0 \quad \forall y \in [0;1]. \tag{5.1}$$

From (5.1), we find that $u = 0$ if and only if $\bar{z}_n = 0$. Therefore, if $\bar{z}_n = 0$ then the algorithm stops and $x^\dagger = 0$. If $\bar{z}_n \ne 0$ then inequality (5.1) is equivalent to a system of inequalities

$$\begin{cases} -u(B_k(u) + u - \bar{z}_n) \ge 0, \\ (1-u)(B_k(u) + u - \bar{z}_n) \ge 0. \end{cases}$$

The last system yields $B_k(u) + u = \bar{z}_n$. Hence, if $0 < \bar{z}_n \le \eta_k + \xi_k$ then $u_n^k := u = \frac{\xi_k}{\xi_k+\eta_k}\bar{z}_n$. Otherwise, if $\xi_k + \eta_k < \bar{z}_n \le 1$, then $u_n^k := u = \bar{z}_n - \eta_k$.
Using the fact that $F = \{0\} \subset C_{n+1}$ we can conclude that $0 \le \bar{u}_n \le \bar{z}_n \le x_n \le 1$. From the definition of C_{n+1} we find

$$C_{n+1} = C_n \bigcap [0, \frac{\bar{z}_n + \bar{u}_n}{2}].$$

According to Step 1 of the proof of Theorem 3.1, C_n is a closed convex subset, hence $[0,x_n] \subset C_n$ because $0, x_n \in C_n$. Further, since $\frac{\bar{u}_n+\bar{z}_n}{2} \le x_n$, it implies that $[0, \frac{\bar{u}_n+\bar{z}_n}{2}] \subset [0,x_n] \subset C_n$. Thus, $C_{n+1} = [0, \frac{\bar{u}_n+\bar{z}_n}{2}]$.
For the sake of comparison between the computing times in the parallel and sequential modes, we choose sufficiently large numbers N, K, M and a slowly convergent to zero sequence $\{\alpha_n\}$.
The numerical experiment is performed on a LINUX cluster 1350 with 8 computing nodes. Each node contains two Intel Xeon dual core 3.2 GHz, 2GBRam. All the programs are written in C. For given tolerances we compare the execution time of algorithm (5.1) in both parallel and sequential modes. We denote by TOL- the tolerance $\|x_k - x^*\|$; T_p - the execution time in parallel mode using 2 CPUs (in seconds), and T_s- the execution time in sequential mode (in seconds). The computing times in both modes are given in Tables 1, 2.
According to Tables 1, 2, in the most favourable cases, the speed-up and the

Table 1 Experiment with $\alpha_n = \frac{1}{\log(\log(n+10))}$

Table 2 Experiment with $\alpha_n = \frac{\log n}{n}$

efficiency of the parallel algorithm are $S_p = T_s/T_p \approx 2; E_p = S_p/2 \approx 1$, respectively.

6 Conclusions

In this paper we proposed two strongly convergent parallel hybrid iterative methods for finding a common element of the set of fixed points of quasi ϕ-asymptotically nonexpansive mappings, the set of solutions of variational inequalities, and the set of solutions of equilibrium problems in uniformly smooth and 2-uniformly convex Banach spaces. A numerical example was given to demonstrate the efficiency of the proposed parallel algorithms.

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References

1. Alber,Ya. I.: Metric and generalized projection operators in Banach spaces: properties and applications, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. A. G. Kartosator, Ed., vol. **178** of Lecture Notes in Pure and Applied Mathematics, pp. 15-50, Dekker, New York, USA, (1996)
2. Alber, Ya. I., Ryazantseva, I.: Nonlinear ill-posed problems of monotone type. Spinger, Dordrecht, (2006)
3. Anh, P.K., Buong, N., and Hieu, D.V.: Parallel methods for regularizing systems of equations involving accretive operators. Appl. Anal. **93** (10), 2136-2157 (2014)
4. Anh, P. K., Chung, C. V.: Parallel hybrid methods for a finite family of relatively nonexpansive mappings. Numer. Funct. Anal. Optim. **35** (6), 649-664 (2014)
5. Anh, P. K., Hieu, D.V.: Parallel and sequential hybrid methods for a finite family of asymptotically quasi ϕ -nonexpansive mappings. J. Appl. Math. Comput. (2014), DOI: 10.1007/s12190-014-0801-6.
6. Chang, S. S., Kim, J. K., Wang, X. R.: Modified Block Iterative Algorithm for Solving Convex Feasibility Problems in B anach Spaces. J. Inequal. Appl. **2010**, 2010:869684. DOI:10.1155/2010/869684.
7. Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, vol. **62** of Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands, (1990)

8. Iiduka, H., Takahashi, W.: Weak convergence of a projection algorithm for variational inequalities in a Banach space. J. M ath. Anal. Appl. **339**, 668-679 (2008)
9. Kang, J., Su, Y., Zhang, X.: Hybrid algorithm for fixed points of weak relatively nonexpansive mappings and applications. Nonlinear Analysis:Hybrid Systems. **4**, 755-765 (2010)
10. Kim, T. H., Lee, H. J.: Strong convergence of modified iteration processes for relatively nonexpansive mappings in Banach Spaces. Kyungpook Math. J. **48**, 685-703 (2008)
11. Matsushita, S., Takahashi, W.: Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **1**, 37-47 (2004)
12. Qin, X., Kang, S. M., Cho, Y. J. Convergence theorems for inverse strongly monotone mappings and quasi ϕ -nonexpansive mappings. Bull. Korean Math. Soc. **46** (5), 885-894 (2009). DOI 10.4134/BKMS.2009.46.5.885
13. Rockafellar, R. T.: On the maximality of sums of nonlinear monotone operators. Trans Amer Math Soc. **149**, 75-88 (1970)
14. Saewan, S., Kumam, P.: The hybrid block iterative algorithm for solving the system of equilibrium problems and variational inequality problems. Saewan and Kumam Springer Plus. **2012** (2012), http://www.springerplus.com/content/1/1/8
15. Su, Y., Li, M., Zhang, H.: New monotone hybrid algorithm for hemi-relatively nonexpansive mappings and maximal monotone operators. Appl. Math. Comput. **217**(12), 5458-5465 (2011)
16. Su, Y. F., Wang, Z. M., Xu, H. K.: Strong convergence theorems for a common fixed point of two hemi-relatively nonexpansive mappings. Nonlinear Anal. **71**, 5616 - 5628 (2009)
17. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers. Yokohama, (2000)
18. Takahashi, S., Takahashi, W.: Viscosity approximation methods for equilibrium problems and fixed point in Hilbert space. J.Math.Anal.Appl. **331** (1), 506-515 (2007)
19. Takahashi, W., Toyoda, M.: Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **118**, 417-428 (2003)
20. Takahashi, W., Zembayashi, K.: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Analysis. **70**(1), 45-57 (2009)
21. Wang, Z., Kang, M. K., Cho, Y. J.: Convergence theorems based on the shrinking projection method for hemi-relatively nonexpansive mappings, variational inequalities and equilibrium problems. Banach J. Math. Anal. **6** (1), 11-34 (2012)
22. Zegeye, H., Shahzad, N.: Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings. Nonlinear Anal. **70** (7), 2707-2716 (2009)
23. Zhang, C., Li, J., Liu, B.: Strong convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Comput. Math. Appl. **61**, 262-276 (2011)