On vertex disjoint cycles of different length in 3-regular digraphs

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Abstract

Henning and Yeo [SIAM J. Discrete Math. **26** (2012) 687–694] conjectured that a 3-regular digraph D contains two vertex disjoint directed cycles of different length if either D is of sufficiently large order or D is bipartite. In this paper, we disprove the first conjecture. Further, we give support for the second conjecture by proving that every bipartite 3-regular digraph, which either possesses a cycle factor with at least two directed cycles or has a Hamilton cycle $C = v_0, v_1, \ldots, v_{n-1}, v_0$ and a spanning 1-circular subdigraph $D(n, S)$ where $S = \{s\}$ with $s > 1$, does indeed have two vertex disjoint directed cycles of different length.

Key words: 3-regular digraph, bipartite digraph, vertex disjoint cycles, cycles of different length, cycle factor

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1 Introduction

In this paper, the term digraph always means a *finite simple digraph*, i.e., a digraph that has a finite number of vertices, no loops and no multiple arcs. Unless otherwise indicated, our graph-theoretic terminology will follow [3].

Let D be a digraph. Then the vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$ (or by V and A for short), respectively. A vertex $v \in V$

is called an *outneighbor* of a vertex $u \in V$ if $(u, v) \in A$. We denote the set of all outneighbors of u by $N_D^+(u)$. The *outdegree* of $u \in V$, denoted by $d_D^+(u)$, is $|N_D^+(u)|$. Similarly, a vertex $w \in V$ is called an *inneighbor* of a vertex $u \in V$ if $(w, u) \in A$. We denote the set of all inneighbors of u by $N_D^-(u)$. The *indegree* of $u \in V$, denoted by $d_D^-(u)$, is $|N_D^-(u)|$. If $W \subseteq V$, then the subdigraph of D induced by W is denoted by $D[W]$.

By a *cycle* (resp., *path*) in a digraph $D = (V, A)$ we always mean a directed cycle (resp., directed path). By *disjoint cycles* in D we always mean vertex disjoint cycles. A cycle factor in D is a spanning subdigraph F of D such that every connected component of F is a cycle. Thus, a subdigraph F of D is called a cycle factor in D with ℓ cycles if $F = C_1 \cup C_2 \cup \ldots \cup C_\ell$, where $C_1, C_2, \ldots, C_\ell$ are cycles in D such that every vertex of D lies in exactly one of these cycles.

An oriented graph is a digraph with no cycles of length 2.

A digraph $D = (V, A)$ is called bipartite if the vertex set V has a bipartition $V = U \cup W$ such that for every vertex $v \in U$ (resp., $v \in W$) both $N_D^+(v)$ and $N_D^-(v)$ are subsets of W (resp., U). The subsets U and W are called parts of this bipartition for D. For a natural number k, a digraph $D = (V, A)$ is called k-regular if $d_D^+(v) = d_D^-(v) = k$ for every vertex $v \in V$.

Let $n \geq 2$ be an integer. Then all integers modulo n are $0, 1, 2, \ldots, n-1$. Further, let $S \subseteq \{1, 2, \ldots, n-1\}$. We define $D(n, S)$ to be a digraph with the vertex set $V(D(n, S)) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and the arc set $A(D(n, S)) = \{(v_i, v_j) \mid (j - i) \pmod{n} \in S\}$. A digraph $D = (V, A)$ is called *d-circular* if there exists a subset $S \subseteq \{1, 2, \ldots, n-1\}$ with $|S| = d$, where $n = |V|$, such that D is isomorphic to the digraph $D(n, S)$. For simplicity, we will identify a d-circular digraph with its isomorphic digraph $D(n, S)$. By definition, it is clear that $d^-_{D(n,S)}(v_i) = d^+_{D(n,S)}(v_i) = |S|$ for every vertex $v_i \in V$, i.e., $D(n, S)$

is $|S|$-regular. It is not difficult to show that a d-circular digraph $D(n, S)$ is an oriented graph if and only if $S \cap (-S) = \emptyset$, where $-S = \{-x \pmod n \mid x \in S\}$.

In [4], Henning and Yeo have posed several conjectures about the existence of two disjoint cycles of different length in digraphs. Among them, there are the following conjectures.

Conjecture 1. *A 3-regular digraph of sufficiently large order contains two disjoint cycles of different length.*

Conjecture 2. *A bipartite 3-regular digraph contains two disjoint cycles of different length.*

We would like to mention that Conjecture 2 has a connection with 2-colorings of hypergraphs (see [4]).

In Section 2 of this paper, for any natural number $n \geq 2$ we will construct a 3-regular digraph of order $2n$, in which any two disjoint cycles have the same length. By this, we will disprove Conjecture 1. In Section 3 we will give support for Conjecture 2 by proving that every bipartite 3-regular digraph, possessing a cycle factor with at least 2 cycles, contains two disjoint cycles of different length. We note that by [5] every 3-regular digraph contains a cycle factor. So, by the result obtained in Section 3, we don't know whether Conjecture 2 is true or not only for those bipartite 3-regular digraphs D which are hamiltonian and only Hamilton cycles in which are their cycle factors. Perhaps, this remaining case is the most challenging one for Conjecture 2. In Section 4, we will investigate this case. We will prove there that a hamiltonian bipartite 3-regular digraph $D = (V, A)$ with a Hamilton cycle $C = v_0, v_1, \ldots, v_{n-1}, v_0$, having a spanning 1-circular subdigraph $D(n, S)$ where $S = \{s\}$ with $s > 1$, contains two disjoint cycles of different length. Thus, the result of Section 4 also supports Conjecture 2 for the remaining case.

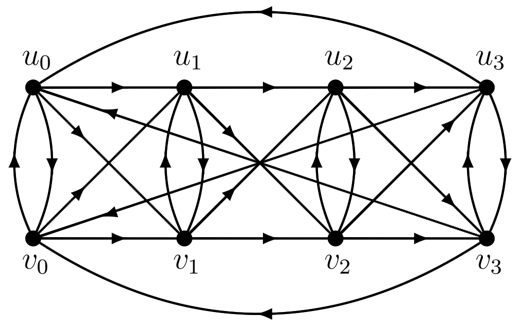

Figure 1: The digraph D_8

Notation. Let $D = (V, A)$ be a digraph. Then for short, we will write uv for an arc $(u, v) \in A$. If $C = v_0, v_1, \ldots, v_{m-1}, v_0$ is a cycle of length m in D and $v_i, v_j \in V(C)$, then v_iCv_j denotes the sequence $v_i, v_{i+1}, v_{i+2}, \ldots, v_j$, where all indices are taken modulo m. We will consider v_iCv_j both as a path and as a vertex set. If $w \in V(C)$, then w_C^- and w_C^+ denote the predecessor and the successor of w on C, respectively.

2 Disproving Conjecture 1

Let $n \geq 2$ be an integer and $D_{2n} = (V_{2n}, A_{2n})$ be a digraph with the vertex set $V_{2n} = \{u_i, v_i \mid i = 0, 1, \ldots, n-1\}$ and the arc set $A_{2n} = \{u_iv_i, v_iu_i, u_iu_{i+1}, u_iv_{i+1}, v_iu_{i+1}, v_iv_{i+1} \mid i = 0, 1, \ldots, n-1\}$, where $i+1$ is always taken modulo n.

The digraph D_4 is the complete digraph on 4 vertices. The digraph D_8 is illustrated on Figure 1.

Now we prove the following result.

Theorem 1. *For any integer $n \geq 2$, the digraph D_{2n} is a 3-regular digraph of order $2n$, in which any two disjoint cycles have the same length.*

Proof. It is clear that D_{2n} is a 3-regular digraph of order $2n$. We prove now that any two disjoint cycles in D_{2n} have the same length.

For $i = 0, 1, \dots, n-1$, let $S_i = \{u_i, v_i\}$, $\bar{u}_i = v_i$ and $\bar{v}_i = u_i$. We have the following remarks.

(i) If a cycle C in D_{2n} contains an arc from S_i to S_{i+1}, where $i \in \{0, 1, \dots, n-1\}$ and $i+1$ is always taken modulo n, then for every $j \in \{0, 1, \dots, n-1\}$ the cycle C contains at least one vertex of S_j.

In fact, the remark is trivial if $n = 2$. So, we assume further that $n > 2$. Let $C = x_0, x_1, x_2, x_3, \dots, x_{m-1}, x_0$ be a cycle with x_0x_1 an arc from S_i to S_{i+1}. Then by the construction of D_{2n}, the vertex x_2 which is the successor of x_1 on C must be either $\bar{x}_1$ or a vertex in S_{i+2}. Moreover, if x_2 is $\bar{x}_1$ then again by the construction of D_{2n}, x_3 must be a vertex in S_{i+2} because $\bar{x}_2 = x_1$ already is a vertex in C. By continuing this process we can see that Remark (i) is true.

(ii) If a cycle C in D_{2n} contains an arc from S_i to S_{i+1} and both vertices of S_k, where $i, k \in \{0, 1, \dots, n-1\}$, then for every cycle C' in D_{2n}, $V(C) \cap V(C') \neq \emptyset$.

In fact, if C' contains an arc from S_i to S_{i+1} for some $i \in \{0, 1, \dots, n-1\}$, then for every $j \in \{0, 1, \dots, n-1\}$, by Remark (i), C' contains at least one vertex of S_j. Therefore, C and C' contain a common vertex in S_k. If C' contains no arcs from S_i to S_{i+1} for any $i \in \{0, 1, \dots, n-1\}$, then $C' = u_r, v_r, u_r$ for some $r \in \{0, 1, \dots, n-1\}$. Therefore, since C contains at least one vertex of S_j for every $j \in \{0, 1, \dots, n-1\}$ by Remark (i), C and C' contain a common vertex in S_r.

We continue to prove Theorem 1. Let C and C' be two disjoint cycles in D_{2n}.

First, assume that C contains an arc from S_i to S_{i+1} for some $i \in \{0, 1, \dots, n-1\}$. Then by Remark (ii), C cannot contain both vertices of

S_k for any $k \in \{0, 1, \dots, n-1\}$. Together with Remark (i), this implies that for every $j \in \{0, 1, \dots, n-1\}$, C contains exactly one vertex of S_j. So, $C = x_0, x_1, x_2, \dots, x_{n-1}, x_0$, where $x_i \in S_i$ for $i = 0, 1, \dots, n-1$. Now, if C' contains no arcs from S_i to S_{i+1} for any $i \in \{0, 1, \dots, n-1\}$, then $C' = u_r, v_r, u_r$ for some $r \in \{0, 1, \dots, n-1\}$. Thus, C and C' have a common vertex in S_r, a contradiction. It follows that C' contains an arc from S_i to S_{i+1} for some $i \in \{0, 1, \dots, n-1\}$. By Remark (i), C' contains at least one vertex of every S_j, $j \in \{0, 1, \dots, n-1\}$. Since C and C' are disjoint, C' must contain exactly one vertex of every S_j, $j \in \{0, 1, \dots, n-1\}$ and therefore $C' = \bar{x}_0, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}, \bar{x}_0$. Thus, C and C' have the same length n.

Next, assume that C contains no arcs from S_i to S_{i+1} for any $i \in \{0, 1, \dots, n-1\}$. Then $C = u_r, v_r, u_r$ for some $r \in \{0, 1, \dots, n-1\}$. Since C and C' are disjoint, by Remark (i), C' also contains no arcs from S_i to S_{i+1} for any $i \in \{0, 1, \dots, n-1\}$. So, $C' = u_s, v_s, u_s$ for some $s \in \{0, 1, \dots, n-1\}$ with $s \neq r$. Thus, C and C' have the same length 2.

The proof of Theorem 1 is complete. □

Theorem 1 shows that Conjecture 1 is false.

3 Bipartite 3-regular digraphs possessing a cycle factor with at least 2 cycles

In the two remaining sections of this paper, we will consider Conjecture 2. The results obtained in these sections will give support for this conjecture. First, we prove the following result.

Theorem 2. *Let $D = (V, A)$ be a bipartite 3-regular digraph which possesses a cycle factor with at least two cycles. Then D contains two disjoint cycles of different length.*

We note that by [5] every 3-regular digraph contains a cycle factor. So, by Theorem 2 we don't know whether Conjecture 2 is true or not only for those bipartite 3-regular digraphs D which are hamiltonian and only Hamilton cycles in which are their cycle factors. This remaining case for Conjecture 2 will be considered in Section 4.

Proof. Suppose, on the contrary, that Theorem 2 is false and let $D = (V, A)$ be a bipartite 3-regular digraph such that D possesses a cycle factor with at least two cycles, but any two disjoint cycles in D have the same length. Then we have the following claim.

Claim 1. *D must be an oriented graph.*

Proof. Suppose, on the contrary, that D is not an oriented graph. Then D contains a cycle C of length 2, say $C = u, v, u$, where $u, v \in V$. Let $D' = D[V \setminus \{u, v\}] = (V', A')$. Since D is bipartite, each vertex of V' is adjacent in D to at most one of u and v. So, each vertex of D' has at least two outneighbors in D' because D is 3-regular. Therefore, it is not difficult to see that D' has a cycle C' of length at least 3. It is clear that $V(C) \cap V(C') = \emptyset$. So, C and C' are two disjoint cycles of different length in D. This contradicts our assumption about D. Thus, D must be an oriented graph. □

By Claim 1, if $uv \in A$ then $vu \notin A$. The reader should remember this because further we will use it without mention.

Let $F = C_0 \cup C_1 \cup \ldots \cup C_{\ell-1}$ with $\ell \geq 2$ be a cycle factor of D with at least two cycles. By our assumption about D, $|V(C_0)| = |V(C_1)| = \cdots = |V(C_{\ell-1})| = k$. Moreover, since D is bipartite, k must be an even number. Further, since $\ell \geq 2$, each of the cycles $C_0, C_1, \ldots, C_{\ell-1}$ must be chordless and therefore each vertex of C_i, $i = 0, 1, \ldots, \ell - 1$, has exactly two outneighbors and exactly two inneighbors not in $V(C_i)$. For $i = 0, 1, \ldots, \ell - 1$, let

$$\begin{aligned} V(C_i) &= \{v_0^i, v_1^i, \ldots, v_{k-1}^i\}, \\ C_i &= v_0^i, v_1^i, \ldots, v_{k-1}^i, v_0^i. \end{aligned}$$

Claim 2. *For $i = 0, 1, 2, \ldots, \ell - 1$, we may assume without loss of generality that in D all arcs out of C_i go to C_{i+1}, where indices are always taken modulo ℓ.*

Proof. The claim is trivial for $\ell = 2$. So, we assume from now on that $\ell \geq 3$. Since D is 3-regular and $C_0, \ldots, C_{\ell-1}$ are chordless, every vertex of C_i, $i \in \{0, 1, \ldots, \ell - 1\}$, has two outneighbors not in $V(C_i)$. By renaming the cycles $C_1, \ldots, C_{\ell-1}$ and their vertices, if necessary, without loss of generality, we may assume that $v_0^0 v_1^1 \in A$. If v_0^1, the predecessor of v_1^1 on C_1, has an outneighbor in $V(C_0)$, say v_j^0, then $C' = v_0^0, v_1^1 C_1 v_0^1, v_j^0 C_0 v_0^0$ has $|V(C')| > |V(C_{\ell-1})|$, and therefore C' and $C_{\ell-1}$ are two disjoint cycles of different lengths in D, a contradiction. Thus, v_0^1 has no outneighbors in $V(C_0)$. It follows that it has an outneighbor not in $V(C_0) \cup V(C_1)$. Again, by renaming cycles $C_2, \ldots, C_{\ell-1}$ and their vertices, if necessary, without loss of generality, we may assume that $v_0^1 v_1^2 \in A$. If $\ell > 3$, then as before we can show that v_0^2, the predecessor of v_1^2 on C_2, has no outneighbors in $V(C_0) \cup V(C_1)$. In fact, if v_0^2 has an outneighbor v_j^0 in $V(C_0)$, then $C' = v_0^0, v_1^1 C_1 v_0^1, v_1^2 C_2 v_0^2, v_j^0 C_0 v_0^0$ and $C_{\ell-1}$ are two disjoint cycles of different length in D; and if v_0^2 has an outneighbor v_j^1 in $V(C_1)$, then $C'' = v_0^1, v_1^2 C_2 v_0^2, v_j^1 C_1 v_0^1$ and $C_{\ell-1}$ are two disjoint cycles of different length in D, a contradiction. Thus, if $\ell > 3$, then v_0^2 has no outneighbors in $V(C_0) \cup V(C_1)$ and therefore it has an outneighbor not in $V(C_0) \cup V(C_1) \cup V(C_2)$. Without loss of generality, we may assume that $v_0^2 v_1^3 \in A$. By continuing this process, we get $v_0^3 v_1^4, \ldots, v_0^{\ell-2} v_1^{\ell-1}$ are arcs in D. Further, if $v_0^{\ell-1}$, the predecessor of $v_1^{\ell-1}$ on $C_{\ell-1}$, has an outneighbor v_j^i in $V(C_i)$ with $1 \leq i < \ell-1$, then $C' = v_0^i, v_1^{i+1}C_{i+1}v_0^{i+1}, v_1^{i+2}C_{i+2}v_0^{i+2}, \ldots, v_1^{\ell-1}C_{\ell-1}v_0^{\ell-1}, v_j^iC_iv_0^i$ and C_0 are two disjoint cycles of different length in D, a contradiction again. So, both two outneighbors of $v_0^{\ell-1}$, that are not in $V(C_{\ell-1})$, are in $V(C_0)$, say $v_{j_1}^0$ and $v_{j_2}^0$.

Now let u be a vertex in $V(C_0)$. If u has an outneighbor v not in $V(C_0) \cup V(C_1)$, say $v \in V(C_i)$ with $i \in \{2, \ldots, \ell-1\}$, then

$$C' = u, vC_iv_0^i, v_1^{i+1}C_{i+1}v_0^{i+1}, \ldots, v_1^{\ell-1}C_{\ell-1}v_0^{\ell-1}, v_{j_1}^0C_0u \quad \text{and}$$
$$C'' = u, vC_iv_0^i, v_1^{i+1}C_{i+1}v_0^{i+1}, \ldots, v_1^{\ell-1}C_{\ell-1}v_0^{\ell-1}, v_{j_2}^0C_0u$$

are two cycles of different length because one of $v_{j_1}^0C_0u$ and $v_{j_2}^0C_0u$ is a proper subpath of the other. Since both C' and C'' are disjoint from C_2, either C' and C_2 or C'' and C_2 are two disjoint cycles of different length in D, a contradiction. Thus, all arcs out of C_0 go to C_1. Now if C_i plays the role of C_{i-1} for $i = 0, \ldots, \ell-1$, where $i-1$ is taken modulo ℓ, then the above argument shows that all arcs out of C_1 go to C_2. By continuing this process, we can see that Claim 2 is true. □

Claim 3. *D has no cycle factors with two cycles.*

Proof. Suppose, on the contrary, that D has a cycle factor F with two cycles C_0 and C_1. Let $D' = (V, A')$ be the subdigraph of D obtained from D by deleting all arcs of both C_0 and C_1. Then D' is a bipartite 2-regular oriented graph. Let $U \cup W$ be a bipartition for D and for $i = 0, 1$ let $U_i = U \cap V(C_i), W_i = W \cap V(C_i)$. Then in D' every vertex of U_0 (resp. W_0) has its outneighbors and inneighbors in W_1 (resp., U_1) and vice versa every vertex of W_1 (resp., U_1) has its outneighbors and inneighbors in U_0 (resp., W_0). Therefore, D' has at least two connected components.

Let H be a connected component of D'. We show now that H has two cycles of different length. Here these cycles are not required to be disjoint.

Since D' is a 2-regular digraph, H is also 2-regular. So, by [5] H has a cycle factor $F_H = X_0 \cup X_1 \cup \ldots \cup X_{m-1}$. If $m = 1$, then $F_H = X_0$. Therefore, X_0 is a Hamilton cycle for H. Since H is 2-regular, X_0 must possess a chord uv. Then $X'_0 = u, vX_0u$ is a cycle in H with $|V(X'_0)| \neq |V(X_0)|$, i.e., X_0 and X'_0 are two cycles of different length in H. So, we assume further that $m \geq 2$. If there are two cycles X_i and X_j of different length in F_H or there is a cycle X_i with a chord in F_H, then it is clear that H has two cycles of different length. So, we may assume further that all cycles $X_i, i = 0, 1, \ldots, m-1$, in F_H have the same length t and are chordless. Let $V(X_i) = \{x^i_0, x^i_1, \ldots, x^i_{t-1}\}$ and $X_i = x^i_0, x^i_1, \ldots, x^i_{t-1}, x^i_0$ for $i = 0, 1, \ldots, m-1$.

We continue to prove our claim by applying the arguments which are already used in the proof of Claim 2. Since H is 2-regular and $X_0, \ldots, X_{m-1}$ are chordless, every vertex x^i_j, $i \in \{0, \ldots, m-1\}, j \in \{0, \ldots, t-1\}$ of H has an outneighbor not in $V(X_i)$. By renaming the cycles $X_1, \ldots, X_{m-1}$ and their vertices, if necessary, we may assume that x^1_1 is an outneighbor of x^0_0. If x^1_0, the predecessor of x^1_1 on X_1, has an outneighbor in $V(X_0)$, say x^0_j, then $X' = x^0_0, x^1_1X_1x^1_0, x^0_jX_0x^0_0$ has $|V(X')| > |V(X_0)|$. Therefore, X_0 and X' are two cycles of different length in H. So, we may assume further that $m \geq 3$ and x^1_0 has an outneighbor not in $V(X_0) \cup V(X_1)$. By renaming the cycles $X_2, \ldots, X_{m-1}$ and their vertices, if necessary, we may assume that $x^1_0x^2_1 \in A(H)$. Now if x^2_0, the predecessor of x^2_1 on X_2, has an outneighbor in $V(X_0)$, say x^0_j, then $X'' = x^0_0, x^1_1X_1x^1_0, x^2_1X_2x^2_0, x^0_jX_0x^0_0$ and X_0 are two cycles of different length in H; and if x^2_0 has an outneighbor in $V(X_1)$, say x^1_j, then $X''' = x^1_0, x^2_1X_2x^2_0, x^1_jX_1x^1_0$ and X_0 are two cycles of different length in H. So, we again may assume further that x^2_0 has an outneighbor not in $V(X_0) \cup V(X_1) \cup V(X_2)$. By continuing similar arguments, we can see that either we already find two cycles of different length for H or by renaming cycles of F_H

and their vertices, if necessary, we can get $x_0^2x_1^3, x_0^3x_1^4, \ldots, x_0^{m-2}x_1^{m-1} \in A(H)$. Now x_0^{m-1}, the predecessor of x_1^{m-1} on X_{m-1}, must have an outneighbor not in $V(X_{m-1})$, say $x_j^i \in V(X_i)$ with $i \in \{0, \ldots, m-2\}$. Then $X^* = x_0^i, x_1^{i+1}X_{i+1}x_0^{i+1}, x_1^{i+2}X_{i+2}x_0^{i+2}, \ldots, x_1^{m-1}X_{m-1}x_0^{m-1}, x_j^iX_ix_0^i$ and X_0 are two cycles of different length in H. Thus, in any situation, we can find two cycles of different length in H, say Y_1 and Y_2.

We have noted before that D' has at least two connected components. So, besides H, D' possesses another connected component $K \neq H$. It is clear that K has at least one cycle, say Z. Then either Y_1 and Z or Y_2 and Z are two disjoint cycles of different length in D'. Since D' is a subdigraph of D, these two cycles are also two disjoint cycles of different length in D. This final contradiction shows that Claim 3 must be true. □

Claim 4. *D has no cycle factors with three cycles.*

Proof. Suppose, on the contrary, that D has a cycle factor F with three cycles C_0, C_1 and C_2. In this proof, we always have $i \in \{0, 1, 2\}$ and indices $i+1$ and $i+2$ are always taken modulo 3. By Claim 2, all arcs out of C_i go to C_{i+1}. We consider cycles in D of the following form:

$$C = x_1, y, z, x_2C_ix_1, \tag{1}$$

where x_1, x_2 are vertices in $V(C_i)$, y is an outneighbor of x_1 in $V(C_{i+1})$, z is an outneighbor of y in $V(C_{i+2})$ and x_2 is an outneighbor of z in $V(C_i)$. We note that since D is bipartite, the length of a cycle C of the form (1) must be even. So, $x_1 \neq x_2$. Further we consider separately the following two cases.

Case 1. There exists a cycle C of the form (1) such that $z^-_{C_{i+2}}$, the predecessor of z on C_{i+2}, has an outneighbor, say x_3, in $V(C_i) \setminus x_2C_ix_1$.

Consider the predecessor $y^-_{C_{i+1}}$ of y on C_{i+1}. Since both $y^-_{C_{i+1}}$ and z are adjacent to y, they are in the same part of the bipartition for D. So, they are not adjacent in D because D is bipartite. It follows that both two outneighbors of $y^-_{C_{i+1}}$ in $V(C_{i+2})$, say z_1 and z_2, are different from z.

Further, since x_3 has two outneighbors in $V(C_{i+1})$, at least one of these outneighbors, say y_1, is different from y. Now we construct two cycles C' and C'' in D as follows:

$$\begin{aligned} C' &= z^-_{C_{i+2}}, x_3, y_1 C_{i+1} y^-_{C_{i+1}}, z_1 C_{i+2} z^-_{C_{i+2}}, \\ C'' &= z^-_{C_{i+2}}, x_3, y_1 C_{i+1} y^-_{C_{i+1}}, z_2 C_{i+2} z^-_{C_{i+2}}. \end{aligned}$$

It is clear that $|V(C')| \neq |V(C'')|$ and both C' and C'' are disjoint from C. So, either C' and C or C'' and C are two vertex disjoint cycles of different lengths in D, a contradiction. Thus, this case cannot occur.

Case 2. For every cycle $C = x_1, y, z, x_2 C_i x_1$ of the form (1), the predecessor $z^-_{C_{i+2}}$ of z on C_{i+2} has no outneighbors in $V(C_i) \setminus x_2 C_i x_1$.

In this case, both two outneighbors of $z^-_{C_{i+2}}$ in $V(C_i)$ are in $x_2 C_i x_1$. Let $C^* = x^*_1, y^*, z^*, x^*_2 C_i x^*_1$ be a cycle of the form (1) with the number of vertices in $x^*_2 C_i x^*_1$ minimum. Further, let x^*_3 be the inneighbor in $V(C_i)$ of y^* which is different from x^*_1. If $x^*_3 \in x^*_2 C_i x^*_1$, then the cycle $C' = x^*_3, y^*, z^*, x^*_2 C_i x^*_3$ has the number of vertices in $x^*_2 C_i x^*_3$ less than the number of vertices in $x^*_2 C_i x^*_1$. This contradicts the choice of C^*. Thus, x^*_3 is in $V(C_i) \setminus x^*_2 C_i x^*_1$.

Let z^*_1 be an inneighbor in $V(C_{i+2})$ of x^*_3. Since both x^*_3 and z^* are adjacent to y^*, they are in the same part of the bipartition for D. It follows that z^*_1 and z^* are in different parts of this bipartition. In particular, $z^*_1 \neq z^*$. Consider the cycle $C'' = z^*_1, x^*_3, y^*, z^* C_{i+2} z^*_1$. Then C'' is a cycle of the form (1). By the assumption of this case, the predecessor $(y^*)^-_{C_{i+1}}$ of y^* on C_{i+1}, has no outneighbors in $V(C_{i+2}) \setminus z^* C_{i+2} z^*_1$. Let z^*_2 and z^*_3 be two outneighbors of $(y^*)^-_{C_{i+1}}$ in $V(C_{i+2})$. Then both z^*_2 and z^*_3 are in $z^* C_{i+2} z^*_1$. Further, since both $(y^*)^-_{C_{i+1}}$ and z^* are adjacent to y^*, they are in the same part of the bipartition for D. So, $(y^*)^-_{C_{i+1}}$ and z^* are not adjacent in D. It follows that both z_2^* and z_3^* are different from z^*.

Let y_1^* be the outneighbor of x_3^* in $V(C_{i+1})$ different from y^*. Consider the following cycles C^{**} and C^{***} in D:

$$\begin{aligned} C^{**} &= z_1^*, x_3^*, y_1^* C_{i+1} (y^*)^-_{C_{i+1}}, z_2^* C_{i+2} z_1^*, \\ C^{***} &= z_1^*, x_3^*, y_1^* C_{i+1} (y^*)^-_{C_{i+1}}, z_3^* C_{i+2} z_1^*. \end{aligned}$$

Then it is clear that $|V(C^{**})| \neq |V(C^{***})|$ and both C^{**} and C^{***} are disjoint from C^*. So, either C^* and C^{**} or C^* and C^{***} are two disjoint cycles of different length in D, a contradiction again. This final contradiction shows that Claim 4 must be true. □

Claim 5. *If D possesses a cycle factor with at least 4 cycles, then for any vertex sets of size two $\{v^0_{t_1}, v^0_{t_2}\} \subseteq V(C_0)$ and $\{v^3_{m_1}, v^3_{m_2}\} \subseteq V(C_3)$, there exist two disjoint paths P_1 and P_2 from $\{v^3_{m_1}, v^3_{m_2}\}$ to $\{v^0_{t_1}, v^0_{t_2}\}$ in $D[V \setminus (V(C_1) \cup V(C_2))]$.*

Proof. The proofs of this claim and Claim III in [4] are just the same. So, we omit the proof of Claim 5 here. The readers who are interested in its details can see the proof of Claim III in [4]. □

Now we complete the proof of Theorem 2. By our assumption about D and by Claims 3 and 4, we may assume further that D possesses a cycle factor with at least 4 cycles. Let $v^1_w \in V(C_1)$ be arbitrary, $v^0_{t_1}$ and $v^0_{t_2}$ be two inneighbors of v^1_w in $V(C_0)$ and $v^2_{y_1}$ and $v^2_{y_2}$ be two outneighbors of v^1_w in $V(C_2)$. Further, let $(v^1_w)^-_{C_1}$ be the predecessor of v^1_w on C_1 and $v^2_{y_3}$ be any outneighbor of $(v^1_w)^-_{C_1}$ in $V(C_2)$. Since D is bipartite and both v^1_w and $v^2_{y_3}$ are adjacent to $(v^1_w)^-_{C_1}$ in D, v^1_w and $v^2_{y_3}$ belong to the same part of the bipartition for D. Therefore, v^1_w and $v^2_{y_3}$ are not adjacent in D. This implies that $v^2_{y_3} \neq v^2_{y_1}$ and $v^2_{y_3} \neq v^2_{y_2}$. Without loss of generality, we may assume that by going along C_2 from $v^2_{y_3}$ in the direction specified by the direction of arcs on C_2 we first encounter $v^2_{y_2}$. Let $v^3_{m_1}$ be an outneighbor of $v^2_{y_1}$ and $v^3_{m_2}$ be an outneighbor of $v^2_{y_3}$ in $V(C_3)$. Then since $v^2_{y_1}$ and $v^2_{y_3}$ belong to different parts of bipartition for D, we have $v^3_{m_1} \neq v^3_{m_2}$. By Claim 5, there exist two disjoint paths P_1 and P_2 from $\{v^3_{m_1}, v^3_{m_2}\}$ to $\{v^0_{t_1}, v^0_{t_2}\}$ in $D[V \setminus (V(C_1) \cup V(C_2))]$.

First assume that P_1 is a path from $v^3_{m_1}$ to $v^0_{t_1}$ and P_2 is a path from $v^3_{m_2}$ to $v^0_{t_2}$. Let v^1_z be the outneighbor of $v^0_{t_2}$ in $V(C_1)$ different from v^1_w. Further, let Q_1, Q_2 and Q_3 be the following paths in D:

$Q_1 = v^0_{t_1}, v^1_w, v^2_{y_1}, v^3_{m_1}$,

$Q_2 = v^0_{t_1}, v^1_w, v^2_{y_2} C_2 v^2_{y_1}, v^3_{m_1}$, and

$Q_3 = v^0_{t_2}, v^1_z C_1 (v^1_w)^-_{C_1}, v^2_{y_3}, v^3_{m_2}$.

We set $C' = Q_1 \cup P_1$, $C'' = Q_2 \cup P_1$ and $C''' = Q_3 \cup P_2$. Then by construction, $|V(C')| \neq |V(C'')|$ and both C' and C'' are disjoint from C'''. So, either C' and C''' or C'' and C''' are two disjoint cycles of different length in D. This contradicts our assumption about D.

Next assume that P_1 is a path from $v^3_{m_1}$ to $v^0_{t_2}$ and P_2 is a path from $v^3_{m_2}$ to $v^0_{t_1}$. Then by analogous arguments we can get two disjoint cycles of different length in D. This again contradicts our assumption about D.

Thus, Theorem 2 must be true. □

4 Hamiltonian bipartite 3-regular digraphs

Following [1, 2] a hamiltonian digraph, in which every cycle factor is a Hamilton cycle, is called *2-factor hamiltonian*. By [5] every 3-regular digraph contains a cycle factor. Therefore, by Theorem 2, Conjecture 2 is true if we can show that every 2-factor hamiltonian bipartite 3-regular digraph contains two disjoint cycles of different length. On the other hand, we don't know whether

2-factor hamiltonian bipartite 3-regular digraphs exist or not. In [1], an infinite family of 2-factor hamiltonian 3-regular digraphs has been constructed. The 3-circular digraph $D(7, S)$ with $S = \{1, 2, 4\}$ is one of digraphs in the family. But all digraphs in this constructed family are not bipartite because all they have odd orders. Until now we don't know any examples of 2-factor hamiltonian bipartite 3-regular digraphs. So, one way to prove Conjecture 2 for the remaining case is to prove that the set of 2-factor hamiltonian bipartite 3-regular digraphs is empty, i.e., every hamiltonian bipartite 3-regular digraph possesses a cycle factor with at least 2 cycles. It seems to us that this is not easier than proving that every hamiltonian bipartite 3-regular digraph contains two disjoint cycles of different length, which is another way to prove Conjecture 2 for the remaining case. In this section, we will follow the last approach to tackle the remaining case for Conjecture 2.

It is clear that a hamiltonian digraph $D = (V, A)$ with a Hamilton cycle $C = v_0, v_1, \ldots, v_{n-1}, v_0$ always can be considered to contain the 1-circular digraph $D' = D(n, S')$ with $S' = \{1\}$ as its spanning subdigraph. In this section, we will show that if besides D' a hamiltonian bipartite 3-regular digraph $D = (V, A)$ with a Hamilton cycle $C = v_0, v_1, \ldots, v_{n-1}, v_0$ contains another 1-circular digraph $D(n, S)$, where $S = \{s\}$ with $s > 1$, as its spanning subdigraph, then D contains two disjoint cycles of different length. This again supports Conjecture 2 for the remaining case considered in this section.

Theorem 3. *Let $D = (V, A)$ be a hamiltonian bipartite 3-regular digraph with a Hamilton cycle $C = v_0, v_1, v_2, \ldots, v_{n-1}, v_0$. Further, let D contain a 1-circular subdigraph $D(n, S)$, where $S = \{s\}$ with $s > 1$. Then D contains two disjoint cycles of different length.*

Proof. Let $D = (V, A)$ be a hamiltonian bipartite 3-regular digraph with a Hamilton cycle $C = v_0, v_1, v_2, \ldots, v_{n-1}, v_0$. Further, let $D(n, S)$ where

$S = \{s\}$ with $s > 1$ be a subdigraph of D. If $a = (v_i, v_j)$ is an arc of D, then the value $(j-i)(\text{mod } n)$ is called the length of the arc a. Thus, every arc of the Hamilton cycle C has length 1 and every arc of the 1-circular subdigraph $D(n, S)$ with $S = \{s\}$ has length s.

If D has a cycle C' of length 2, then as in Claim 1 of Section 3 we can show that D contains a cycle C'' of length at least 3, which is disjoint from C', i.e., D contains two disjoint cycles of different length and Theorem 3 is true for this case.

Thus, from now on we may assume that D is an oriented graph. We continue to consider separately the following two cases.

Case 1. There exists an arc in D with its length greater than s.

Let m be the maximum of lengths of arcs in D. Then $m > s$ in this case. Without loss of generality, we may assume that v_0v_m is an arc of maximum length m. Now we construct a cycle C_1 in D as follows. Let i_0 be the greatest among all non-negative integers i such that $m + is \leq n$. Then we have $n - (m + i_0 s) \leq s - 1$. We again divide this case into two subcases.

Subcase 1.1. $n - (m + i_0 s) \leq s - 2$.

In this subcase, $v_{m+i_0 s-1+s}$ is a vertex in $\{v_1, v_2, \ldots, v_{m-1}\}$. Therefore,

$$\begin{aligned} C_1 &= v_0, v_m, v_{m+s}, v_{m+2s}, \ldots, v_{m+(i_0-1)s}, v_{m+i_0 s} C v_0, \text{ and} \\ C_2 &= v_{m-1}, v_{m+s-1}, v_{m+2s-1}, \ldots, v_{m+i_0 s-1}, v_{m+i_0 s-1+s} C v_{m-1} \end{aligned}$$

are two disjoint cycles in D. Further, we have $|V(C_1)| = (1+i_0) + [n-(m+i_0 s)] = (1+i_0)+(n-i_0 s-m)$ and $|V(C_2)| = (i_0+1)+[(m-1)-(m+i_0 s-1+s)(\text{mod } n) = (1+i_0)+(n-i_0 s-s)$. Since $m > s$, we get $|V(C_1)| < |V(C_2)|$ and therefore C_1 and C_2 are two disjoint cycles of different length in D in this subcase.

Subcase 1.2. $n - (m + i_0 s) = s - 1$.

In this subcase, $v_{m+i_0s+s} = v_1$. Since D is hamiltonian bipartite with a Hamilton cycle $C = v_0, v_1, v_2, \ldots, v_{n-1}, v_0$, n must be even and therefore both m and s must be odd. It follows that i_0 must be at least 1 in Subcase 1.2. Further, together with $s > 1$, we get $s \geq 3$. Therefore,

$C_3 = v_0, v_m, v_{m+1}, v_{m+s+1}, v_{m+2s+1}, \ldots, v_{m+(i_0-1)s+1}, v_{m+i_0s+1}Cv_0$, and

$C_4 = v_{m-1}, v_{m+s-1}, v_{m+s}, v_{m+2s}, \ldots, v_{m+i_0s}, v_1Cv_{m-1}$

are two disjoint cycles in D. We have $|V(C_3)| = (2+i_0)+[n-(m+i_0s+1)] = (2+i_0)+[n-(m+i_0s)]-1 = (2+i_0)+(s-1)-1 = i_0+s$. Here we use the equality $n-(m+i_0s) = s-1$ which holds in this subcase. On the other hand, $|V(C_4)| = i_0+2+[(m-1)-1] = i_0+m$. It follows that $|V(C_3)| < |V(C_4)|$ because $m > s$. So, C_3 and C_4 are two disjoint cycles of different length in D in this subcase.

Case 2. There exist no arcs in D with their lengths greater than s.

Then the length of any arc in $A' = A \setminus [A(C) \cup A(D(n, S))]$ is greater than 1 and less than s. Since D is 3-regular, $A' \neq \emptyset$. It follows that s must be at least 5. Let j_0 be the greatest among all positive integers j such that $js \leq n$. Then $n - j_0s \leq s-1$. We again consider separately several subcases.

Subcase 2.1. $j_0 \geq 2$ and $n - j_0s \leq s-2$.

In this subcase, v_{j_0s-1+s} is a vertex in $\{v_1, v_2, \ldots, v_{s-1}\}$. Further, since D is 3-regular, there is an arc in A' with the tail v_{s-1}. As we have noted before, the length t of this arc satisfies $1 < t < s$. So, v_{s-1+t} is a vertex in $\{v_{s+1}, v_{s+2}, \ldots, v_{2s-1}\}$. Therefore,

$$\begin{aligned} C_5 &= v_0, v_s, v_{2s}, \ldots, v_{(j_0-1)s}, v_{j_0s}Cv_0, \text{ and} \\ C_6 &= v_{j_0s-1}, v_{j_0s-1+s}Cv_{s-1}, v_{s-1+t}Cv_{2s-1}, v_{3s-1}, v_{4s-1}, \ldots, v_{j_0s-1} \end{aligned}$$

are disjoint cycles in D. Further, $|V(C_5)| = j_0 + (n - j_0s)$ and $|V(C_6)| = 1+[(s-1)-(j_0s-1+s)]+1+[(2s-1)-(s-1+t)]+(j_0-2)(\text{mod } n) =$

$j_0+(n-j_0s)+(s-t)$. Since $t < s$, this implies that $|V(C_5)| < |V(C_6)|$. Thus, C_5 and C_6 are two disjoint cycles of different length in D for this subcase.

Subcase 2.2. $j_0 \geq 2$ and $n - j_0s = s - 1$.

In this subcase, $v_{j_0s+s} = v_1$. As in Subcase 2.1, let $v_{s-1}v_{s-1+t}$ with $1 < t < s$ be the arc in A' with the tail v_{s-1}. Then v_{s-1+t} is a vertex in $\{v_{s+1}, v_{s+2}, \ldots, v_{2s-1}\}$. Further, since t must be odd and $t > 1$, we have $t \geq 3$. Therefore,

$$\begin{aligned} C_7 &= v_0, v_s, v_{s+1}, v_{2s+1}, \ldots, v_{(j_0-1)s+1}, v_{j_0s+1}Cv_0, \text{ and} \\ C_8 &= v_1Cv_{s-1}, v_{s-1+t}Cv_{2s}, v_{3s}, v_{4s}, \ldots, v_{j_0s}, v_1 \end{aligned}$$

are disjoint cycles in D. We have $|V(C_7)| = 1 + j_0 + [n - (j_0s + 1)] = j_0 + (n - j_0s) = j_0 + s - 1$. Here we use the equality $n - j_0s = s - 1$, which holds in this subcase. On the other hand, $|V(C_8)| = [(s - 1) - 1] + 1 + [2s - (s - 1 + t)] + (j_0 - 1) = j_0 + s - 1 + (s - t)$. Since $t < s$, we again have $|V(C_7)| < |V(C_8)|$. Thus, C_7 and C_8 are two disjoint cycles of different length in D for this subcase.

Subcase 2.3. $j_0 = 1$.

Since D is hamiltonian bipartite with a Hamilton cycle $C = v_0, v_1, v_2, \ldots, v_{n-1}, v_0$, n must be even and lengths of arcs must be odd. In this subcase, we have $2s > n$. This implies that $2s \pmod{n} \geq 2$, i.e., the vertex v_{2s-1} is a vertex in $\{v_1, v_2, \ldots, v_{s-1}\}$. Let t_0 be the minimum of lengths of arcs in $A' = A \setminus [A(C) \cup A(D(n, S))]$. Since D is 3-regular, for every vertex $u \in V$ there exists exactly one arc in A' with the tail u. By renaming vertices of V, if necessary, without loss of generality we may assume that the arc in A' with the tail v_s has length t_0.

If $s + t_0 \leq n$, then in fact $s + t_0 < n$ because D is an oriented graph.

Consider the cycles

$$C_9 = v_0, v_s, v_{s+t_0} C v_0, \text{ and}$$
$$C_{10} = v_{s-1}, v_{2s-1} C v_{s-1}.$$

Since v_{2s-1} is a vertex in $\{v_1, v_2, \ldots, v_{s-1}\}$, these cycles are disjoint from each other. We have $|V(C_9)| = 2 + [n - (s + t_0)] = (n - s + 1) - (t_0 - 1)$ and $|V(C_{10})| = 1 + [(s-1) - (2s-1)](\text{mod } n) = n - s + 1$. Since $t_0 > 1$, we have $t_0 - 1 > 0$. So, $|V(C_9)| < |V(C_{10})|$ and therefore C_9 and C_{10} are two disjoint cycles of different length in this situation.

If $s + t_0 > n$, then $s + t_0 \geq n + 2$ because n is even and both s and t_0 are odd. If the arc in A' with the tail v_{s-1} has length t, then $t \geq t_0$ because t_0 is the minimum of lengths of arcs in A'. Therefore, $s + t \geq s + t_0 \geq n + 2$. It follows that $(s-1+t)(\text{mod } n) \geq 1$, i.e., v_{s-1+t} is a vertex in $\{v_1, v_2, \ldots, v_{s-1}\}$. Therefore,

$$C_{11} = v_0, v_s C v_0, \text{ and}$$
$$C_{12} = v_{s-1}, v_{s-1+t} C v_{s-1}$$

are disjoint cycles in D. We have $|V(C_{11})| = 1 + [n - s]$ and $|V(C_{12}| = 1 + [(s-1) - (s-1+t)](\text{mod } n) = 1 + [n - t]$. Since $t < s$, we have $|V(C_{11})| < |V(C_{12})|$ and therefore C_{11} and C_{12} are two disjoint cycles of different length in this situation.

The proof of Theorem 3 is complete. □

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