ON THE GENERATORS OF THE POLYNOMIAL ALGEBRA AS A MODULE OVER THE STEENROD ALGEBRA

SUR LES GÉNÉRATEURS DE L'ALGÈBRE POLYNOMIALE COMME MODULE SUR L'ALGÈBRE DE STENNROD

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Abstract. Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1.

We are interested in the *Peterson hit problem* of finding a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we study the hit problem in degree $(k-1)(2^d-1)$ with d a positive integer. Our result implies the one of Mothebe [4, 5].

Résumé. Soient $\mathcal{A}$ l'algèbre de Steenrod mod-2 et $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ l'algèbre polynomiale graduée à k générateurs sur le corps à deux éléments $\mathbb{F}_2$, chaque générateur étant de degré 1.

Nous étudions le problème suivant soulevé par F. Peterson: déterminer un système minimal de générateurs comme module sur l'algèbre de Steenrod pour P_k, problème appelé *hit problem* en anglais. Dans ce but, nous étudions le *hit problem* en degré $(k-1)(2^d-1)$ avec $d > 0$. Cette solution implique un résultat de Mothebe [4, 5].

1. Introduction

Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$, with the degree of each x_i being 1. This algebra arises as the cohomology with coefficients in $\mathbb{F}_2$ of an elementary abelian 2-group of rank k. Then, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula (see Steenrod and Epstein [12]).

An element g in P_k is called *hit* if it belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ is the augmentation ideal of $\mathcal{A}$. That means g can be written as a finite sum $g = \sum_{u\geqslant 0} Sq^{2^u}(g_u)$ for suitable polynomials $g_u \in P_k$.

We are interested in the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to find a basis of the $\mathbb{F}_2$-vector space $QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

The hit problem was first studied by Peterson [7], Wood [16], Singer [10], and Priddy [8], who showed its relation to several classical problems respectively in cobordism theory, modular representation theory, the Adams spectral sequence for

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the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups.

The vector space QP_k was explicitly calculated by Peterson [7] for $k = 1, 2$, by Kameko [3] for $k = 3$, and recently by the second author [13, 14] for $k = 4$. From the results of Wood [16] and Kameko [3], the hit problem is reduced to the case of degree n of the form

$$n = s(2^d - 1) + 2^d m, \tag{1.1}$$

where s, d, m are non-negative integers and $1 \leqslant s < k$, (see [14].) For $s = k-1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [2], Nam [6], Repka and Selick [9] and the second author [13, 14].

In the present paper, we study the hit problem in degree n of the form (1.1) with $s = k-1$, $m = 0$ and d an arbitrary positive integer.

Denote by $(QP_k)_n$ the subspace of QP_k consisting of the classes represented by the homogeneous polynomials of degree n in P_k. From the result of Carlisle and Wood [1] on the boundedness conjecture, one can see that for d big enough, the dimension of $(QP_k)_n$ does not depend on d; it depends only on k. In this paper, we prove the following.

Main Theorem. *Let $n = (k-1)(2^d - 1)$ with d a positive integer and let $p = \min\{k, d\}$, $q = \min\{k, d-1\}$. If $k \geqslant 3$, then*

$$\dim(QP_k)_n \geqslant c(k,d) := \sum_{t=1}^{p} \binom{k}{t} + (k-3)\binom{k}{2} \sum_{u=1}^{q} \binom{k}{u},$$

with equality if and only if either $k = 3$ or $k = 4$, $d \geqslant 5$ or $k = 5$, $d \geqslant 6$.

Note that $c(k,1) = \binom{k}{1} = k$. If $d > k$, then $c(k,d) = \big((k-3)\binom{k}{2} + 1\big)(2^k - 1)$. At the end of Section 3, we show that our result implies Mothebe's result in [4, 5].

In Section 2, we recall the definition of an admissible monomial in P_k and Singer's criterion on the hit monomials. Our results will be presented in Section 3.

2. Preliminaries

In this section, we recall some needed information from Kameko [3] and Singer [11], which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \ldots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j \in \mathbb{N}_k \setminus \mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k,$$

In particular, $X_{\mathbb{N}_k} = 1$, $X_\emptyset = x_1 x_2 \ldots x_k$, $X_j = x_1 \ldots \hat{x}_j \ldots x_k$, $1 \leqslant j \leqslant k$, and $X := X_k \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 with $i \geqslant 0$. Set $\alpha(a) = \sum_{i \geqslant 0} \alpha_i(a)$.

Let $x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant k$. Set $\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\}$, for $t \geqslant 0$. Then, we have $x = \prod_{t \geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. For a monomial x in P_k, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)},\ i \geqslant 1$. The sequence $\omega(x)$ is called the weight vector of x.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$.

The sets of the weight vectors and the sigma vectors are given the left lexicographical order.

For a weight vector ω, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$. If there are $i_0 = 0, i_1, i_2, \ldots, i_r > 0$ such that $i_1 + i_2 + \ldots + i_r = m$, $\omega_{i_1+\ldots+i_{s-1}+t} = b_s, 1 \leqslant t \leqslant i_s, 1 \leqslant s \leqslant r$, and $\omega_i = 0$ for all $i > m$, then we write $\omega = (b_1^{(i_1)}, b_2^{(i_2)}, \ldots, b_r^{(i_r)})$. Denote $b_u^{(1)} = b_u$. For example, $\omega = (3, 3, 2, 1, 1, 1, 0, \ldots) = (3^{(2)}, 2, 1^{(3)})$.

Denote by $P_k(\omega)$ the subspace of P_k spanned by monomials y such that $\deg y = \deg \omega$, $\omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$ then f is called hit.

ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have $QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega))$ and $(QP_k)_n \cong \bigoplus_{\deg \omega = n} QP_k(\omega)$ (see Walker and Wood [15]).

We note that the weight vector of a monomial is invariant under the permutation of the generators x_i, hence $QP_k(\omega)$ has an action of the symmetric group Σ_k.

For a polynomial $f \in P_k(\omega)$, we denote by $[f]_\omega$ the class in $QP_k(\omega)$ represented by f. Denote by $|S|$ the cardinal of a set S.

Definition 2.4. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds:

i) $\omega(x) < \omega(y)$;

ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.5. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_m$ such that $y_t < x$ for $t = 1, 2, \ldots, m$ and $x - \sum_{t=1}^m y_t \in \mathcal{A}^+P_k$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n. Now, we recall a result of Singer [11] on the hit monomials in P_k.

Definition 2.6. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{d_j} - 1$ for d_j a non-negative integer and $j = 1, 2, \ldots, k$. If z is a spike with $d_1 > d_2 > \ldots > d_{r-1} \geqslant d_r > 0$ and $d_j = 0$ for $j > r$, then it is called the minimal spike.

In [11], Singer showed that if $\alpha(n+k) \leqslant k$, then there exists uniquely a minimal spike of degree n in P_k.

Lemma 2.7.

i) *All the spikes in P_k are admissible and their weight vectors are weakly decreasing.*

ii) *If a weight vector ω is weakly decreasing and $\omega_1 \leqslant k$, then there is a spike z in P_k such that $\omega(z) = \omega$.*

The proof of the this lemma is elementary. The following is a criterion for the hit monomials in P_k.

Theorem 2.8 (See Singer [11])**.** *Suppose $x \in P_k$ is a monomial of degree n, where $\alpha(n+k) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

The following theorem will be used in the next section.

Theorem 2.9 (See [13, 14])**.** *Let $n = \sum_{i=1}^{k-1}(2^{d_i}-1)$ with d_i positive integers such that $d_1 > d_2 > \ldots > d_{k-2} \geqslant d_{k-1}$, and let $m = \sum_{i=1}^{k-2}(2^{d_i-d_{k-1}}-1)$. If $d_{k-1} \geqslant k-1 \geqslant 3$, then*

$$\dim(QP_k)_n = (2^k-1)\dim(QP_{k-1})_m.$$

Note that we correct Theorem 3 in [13] by replacing the condition $d_{k-1} \geqslant k-1 \geqslant 1$ with $d_{k-1} \geqslant k-1 \geqslant 3$.

3. Proof of Main Theorem

Denote $\mathcal{N}_k = \big\{(i;I); I = (i_1, i_2, \ldots, i_r), 1 \leqslant i < i_1 < \ldots < i_r \leqslant k,\ 0 \leqslant r < k\big\}$.

Definition 3.1. Let $(i;I) \in \mathcal{N}_k$, let $r = \ell(I)$ be the length of I, and let u be an integer with $1 \leqslant u \leqslant r$. A monomial $x \in P_{k-1}$ is said to be u-compatible with $(i;I)$ if all of the following hold:

i) $\nu_{i_1-1}(x) = \nu_{i_2-1}(x) = \ldots = \nu_{i_{(u-1)}-1}(x) = 2^r-1$,
ii) $\nu_{i_u-1}(x) > 2^r-1$,
iii) $\alpha_{r-t}(\nu_{i_u-1}(x)) = 1,\ \forall t,\ 1 \leqslant t \leqslant u$,
iv) $\alpha_{r-t}(\nu_{i_t-1}(x)) = 1,\ \forall t,\ u < t \leqslant r$.

Clearly, a monomial x can be u-compatible with a given $(i;I) \in \mathcal{N}_k$ for at most one value of u. By convention, x is 1-compatible with $(i;\emptyset)$.

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i,\\ x_{j+1}, & \text{if } i \leqslant j < k.\end{cases}$$

Definition 3.2. Let $(i;I) \in \mathcal{N}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1}+\ldots+2^{r-u}}\prod_{u<t\leqslant r} x_{i_t}^{2^{r-t}}$ for $r = \ell(I) > 0$, $x_{(\emptyset,1)} = 1$. For a monomial x in P_{k-1}, we define the monomial $\phi_{(i;I)}(x)$ in P_k by setting

$$\phi_{(i;I)}(x) = \begin{cases} (x_i^{2^r-1}f_i(x))/x_{(I,u)}, & \text{if there exists } u \text{ such that}\\ & x \text{ is } u\text{-compatible with } (i,I),\\ 0, & \text{otherwise.}\end{cases}$$

Then we have an $\mathbb{F}_2$-linear map $\phi_{(i;I)} : P_{k-1} \to P_k$. In particular, $\phi_{(i;\emptyset)} = f_i$.

For a positive integer b, denote $\omega_{(k,b)} = ((k-1)^{(b)})$ and $\bar\omega_{(k,b)} = ((k-1)^{(b-1)}, k-3, 1)$.

Lemma 3.3 (See [14])**.** *Let b be a positive integer and let $j_0, j_1, \ldots, j_{b-1} \in \mathbb{N}_k$. We set $i = \min\{j_0, \ldots, j_{b-1}\}$, $I = (i_1, \ldots, i_r)$ with $\{i_1, \ldots, i_r\} = \{j_0, \ldots, j_{b-1}\} \setminus \{i\}$. Then, we have*

$$\prod_{0\leqslant t<b} X_{j_t}^{2^t} \equiv_{\omega_{(k,b)}} \phi_{(i;I)}(X^{2^b-1}).$$

Definition 3.4. For any $(i;I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ \sum_{s\in I} x_{s-1}, & \text{if } j = i, \\ x_{j-1}, & \text{if } i < j \leqslant k. \end{cases}$$

Then, $p_{(i;I)}$ is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = \emptyset$, $p_{(i;\emptyset)}(x_i) = 0$ and $p_{(i;I)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

Lemma 3.5. *If x is a monomial in P_k, then $p_{(i;I)}(x) \in P_{k-1}(\omega(x))$.*

Proof. Set $y = p_{(i;I)}\left(x/x_i^{\nu_i(x)}\right)$. Then, y is a monomial in P_{k-1}. If $\nu_i(x) = 0$, then $y = p_{(i;I)}(x)$ and $\omega(y) = \omega(x)$. Suppose $\nu_i(x) > 0$ and $\nu_i(x) = 2^{t_1} + \ldots + 2^{t_c}$, where $0 \leqslant t_1 < \ldots < t_c,\ c \geqslant 1$.

If $I = \emptyset$, then $p_{(i;I)}(x) = 0$. If $I \neq \emptyset$, then $p_{(i;I)}(x)$ is a sum of monomials of the form $\bar y := \big(\prod_{u=1}^{c} x_{s_u-1}^{2^{t_u}}\big)y$, where $s_u \in I$, $1 \leqslant u \leqslant c$. If $\alpha_{t_u}(\nu_{s_u-1}(y)) = 0$ for all u, then $\omega(\bar y) = \omega(x)$. Suppose there is an index u such that $\alpha_{t_u}(\nu_{s_u-1}(y)) = 1$. Let u_0 be the smallest index such that $\alpha_{t_{u_0}}(\nu_{s_{u_0}-1}(y)) = 1$. Then, we have

$$\omega_i(\bar y) = \begin{cases} \omega_i(x), & \text{if } i \leqslant t_{u_0}, \\ \omega_i(x) - 2, & \text{if } i = t_{u_0} + 1. \end{cases}$$

Hence, $\omega(\bar y) < \omega(x)$ and $\bar y \in P_{k-1}(\omega(x))$. The lemma is proved. $\square$

Lemma 3.5 implies that if ω is a weight vector and $x \in P_k(\omega)$, then $p_{(i;I)}(x) \in P_{k-1}(\omega)$. Moreover, $p_{(i;I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$. In particular, we have

Lemma 3.6 (See [14]). *Let b be a positive integer and let $(j;J),(i;I) \in \mathcal{N}_k$ with $\ell(I) < b$.*

i) *If $(i;I) \subset (j;J)$, then $p_{(j;J)}\phi_{(i;I)}(X^{2^b-1}) = X^{2^b-1} \mod(P_{k-1}^-(\omega_{(k,b)}))$.*
ii) *If $(i;I) \not\subset (j;J)$, then $p_{(j;J)}\phi_{(i;I)}(X^{2^b-1}) \in P_{k-1}^-(\omega_{(k,b)})$.*

For $0 < h \leqslant k$, set $\mathcal{N}_{k,h} = \{(i;I) \in \mathcal{N}_k : \ell(I) < h\}$. Then, $|\mathcal{N}_{k,h}| = \sum_{t=1}^{h}\binom{k}{t}$.

Proposition 3.7. *Let d be a positive integer and let $p = \min\{k,d\}$. Then, the set*

$$B(d) := \left\{\left[\phi_{(i;I)}(X^{2^d-1})\right]_{\omega_{(k,d)}} : (i;I) \in \mathcal{N}_{k,p}\right\}$$

is a basis of the $\mathbb{F}_2$-vector space $QP_k(\omega_{(k,d)})$. Consequently $\dim QP_k(\omega_{(k,d)}) = \sum_{t=1}^{p}\binom{k}{t}$.

Proof. Let x be a monomial in $P_k(\omega_{(k,d)})$ and $[x]_{\omega_{(k,d)}} \neq 0$. Then, we have $\omega(x) = \omega_{(k,d)}$. So, there exist $j_0, j_1, \ldots, j_{d-1} \in \mathbb{N}_k$ such that $x = \prod_{0\leqslant t<d} X_{j_t}^{2^t}$. According to Lemma 3.3, there is $(i;I) \in \mathcal{N}_k$ such that $x = \prod_{0\leqslant t<d} X_{j_t}^{2^t} \equiv_{\omega_{(k,d)}} \phi_{(i;I)}(X^{2^d-1})$, where $r = \ell(I) < p = \min\{k,d\}$. Hence, $QP_k(\omega_{(k,d)})$ is spanned by the set $B(d)$.

Now, we prove that the set $B(d)$ is linearly independent in $QP_k(\omega_{(k,d)})$. Suppose that there is a linear relation

$$\sum_{(i;I)\in\mathcal{N}_{k,p}} \gamma_{(i;I)}\phi_{(i;I)}(X^{2^d-1}) \equiv_{\omega_{(k,d)}} 0,$$

where $\gamma_{(i;I)} \in \mathbb{F}_2$. By induction on $\ell(I)$, using Lemma 3.5 and Lemma 3.6 with $b = d$, we can easily show that $\gamma_{(i;I)} = 0$ for all $(i;I) \in \mathcal{N}_{k,p}$. The proposition is proved. $\square$

Set $C_k = \{x_{j_1}x_{j_2}\ldots x_{j_{k-3}}x_j^2 : 1 \leqslant j_1 < j_2 < \ldots < j_{k-3} < k,\ j_1 \leqslant j < k\} \subset P_{k-1}$. It is easy to see that $|C_k| = (k-3)\binom{k}{2}$.

Lemma 3.8. *C_k is the set of the admissible monomials in P_{k-1} such that their weight vectors are $\bar\omega_{(k,1)} = (k-3,1)$. Consequently,* $\dim QP_{k-1}(\bar\omega_{(k,1)}) = (k-3)\binom{k}{2}$.

Proof. Let z be a monomial in P_{k-1} such that $\omega(z) = (k-3,1)$. Then, $z = x_{j_1}x_{j_2}\ldots x_{j_{k-3}}x_j^2$ with $1 \leqslant j_1 < j_2 < \ldots < j_{k-3} < k$ and $1 \leqslant j < k$. If $z \notin C_k$, then $j < j_1$. Then, we have

$$z = \sum_{s=1}^{k-3} x_{j_s}^2 x_{j_1}x_{j_2}\ldots \hat{x}_{j_s}\ldots x_{j_{k-3}}x_j + Sq^1(x_{j_1}x_{j_2}\ldots x_{j_{k-3}}x_j).$$

Since $x_{j_s}^2 x_{j_1}x_{j_2}\ldots \hat{x}_{j_s}\ldots x_{j_{k-3}}x_j < z$ for $1 \leqslant s \leqslant k-3$, z is inadmissible.

Suppose that $z \in C_k$. If there is an index s such that $j = j_s$, then z is a spike. Hence, by Lemma 2.7, it is admissible. Assume that $j \neq j_s$ for all s. If z is inadmissible, then there exist monomials $y_1,\ldots,y_m$ in P_{k-1} such that $y_t < z$ for all t and $z = \sum_{t=1}^m y_t + \sum_{u\geqslant 0} Sq^{2^u}(g_u)$, where g_u are suitable polynomials in P_{k-1}. Since $y_t < z$ for all t, z is a term of $\sum_{u\geqslant 0} Sq^{2^u}(g_u)$, (recall that a monomial x in P_k is called *a term* of a polynomial f if it appears in the expression of f in terms of the monomial basis of P_k.) Based on the Cartan formula, we see that z is not a term of $Sq^{2^u}(g_u)$ for all $u > 0$. If z is a term of $Sq^1(y)$ with y a monomial in P_{k-1}, then $y = x_{j_1}x_{j_2}\ldots x_{j_{k-3}}x_j := \tilde y$. So, $\tilde y$ is a term of g_0. Then, we have

$$\begin{aligned}\bar y := x_{j_1}^2 x_{j_2}\ldots x_{j_{k-3}}x_j = &\sum_{s=2}^{k-3} x_{j_s}^2 x_{j_1}x_{j_2}\ldots \hat{x}_{j_s}\ldots x_{j_{k-3}}x_j\\ &+ \sum_{t=1}^m y_t + Sq^1(g_0 + \tilde y) + \sum_{u\geqslant 1} Sq^{2^u}(g_u).\end{aligned}$$

Since $j_1 < j$, we have $y_t < z < \bar y$ for all t. Hence, $\bar y$ is a term of $Sq^1(g_0+\tilde y) + \sum_{u\geqslant 1} Sq^{2^u}(g_u)$. By an argument analogous to the previous one, we see that $\tilde y$ is a term of $g_0 + \tilde y$. This contradicts the fact that $\tilde y$ is a term of g_0. The lemma is proved. $\square$

Proposition 3.9. *Let d be a positive integer and let $q = \min\{k, d-1\}$. Then, the set*

$$\bar B(d) := \bigcup_{z\in C_k} \big\{\big[\phi_{(i;I)}(X^{2^{d-1}-1}z^{2^{d-1}})\big]_{\bar\omega_{(k,d)}} :\ (i;I) \in \mathcal{N}_{k,q}\big\}$$

is linearly independent in $QP_k(\bar\omega_{(k,d)})$. If $d > k$, then $\bar B(d)$ is a basis of $QP_k(\bar\omega_{(k,d)})$. Consequently $\dim QP_k(\bar\omega_{(k,d)}) \geqslant (k-3)\binom{k}{2}\sum_{u=1}^q \binom{k}{u}$ *with equality if $d > k$.*

Proof. We prove the first part of the proposition. Suppose there is a linear relation

$$\mathcal{S} := \sum_{((i;I),z)\in \mathcal{N}_{k,q}\times C_k} \gamma_{(i;I),z}\phi_{(i;I)}(X^{2^{d-1}-1}z^{2^{d-1}}) \equiv_{\bar\omega_{(k,d)}} 0,$$

where $\gamma_{(i;I),z} \in \mathbb{F}_2$. We prove $\gamma_{(j;J),z} = 0$ for all $(j;J) \in \mathcal{N}_{k,q}$ and $z \in C_k$. The proof proceeds by induction on $m = \ell(J)$. Let $(i;I) \in \mathcal{N}_{k,q}$. Since $r = \ell(I) < q =$

$\min\{k, d-1\}$, $X^{2^{d-1}-1}z^{2^{d-1}}$ is 1-compatible with $(i;I)$ and $x_i^{2^r-1}f_i(X^{2^{d-1}-1})$ is divisible by $x_{(I,1)}$. Hence, using Definition 3.2, we easily obtain

$$\phi_{(i;I)}(X^{2^{d-1}-1}z^{2^{d-1}}) = \phi_{(i;I)}(X^{2^{d-1}-1})f_i(z^{2^{d-1}}).$$

A simple computation show that if $g \in P_{k-1}^-(\omega_{(k,d-1)})$, then $gz^{2^{d-1}} \in P_{k-1}^-(\bar\omega_{(k,d)})$; if $(i;I) \subset (j;\emptyset)$, then $(i;I) = (j;\emptyset)$; by Lemma 3.5, $p_{(j;\emptyset)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} 0$. Hence, applying Lemma 3.6 with $b = d-1$, we get

$$p_{(j,\emptyset)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} \sum_{z\in C_k} \gamma_{(j;\emptyset),z}X^{2^{d-1}-1}z^{2^{d-1}} \equiv_{\bar\omega_{(k,d)}} 0.$$

Since z is admissible in P_{k-1}, $X^{2^{d-1}-1}z^{2^{d-1}}$ is also admissible in P_{k-1}. Hence, the last relation implies $\gamma_{(j;\emptyset),z} = 0$ for all $z \in C_k$. Suppose $0 < m < q$ and $\gamma_{(i;I),z} = 0$ for all $z \in C_k$ and $(i;I) \in \mathcal{N}_{k,q}$ with $\ell(I) < m$. Let $(j;J) \in \mathcal{N}_{k,q}$ with $\ell(J) = m$. Note that by Lemma 3.5, $p_{(j;J)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} 0$; if $(i;I) \in \mathcal{N}_{k,q}$, $\ell(I) \geqslant m$ and $(i;I) \subset (j;J)$, then $(i;I) = (j;J)$. So, using Lemma 3.6 with $b = d-1$ and the inductive hypothesis, we obtain

$$p_{(j,J)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} \sum_{z\in C_k} \gamma_{(j;J),z}X^{2^{d-1}-1}z^{2^{d-1}} \equiv_{\bar\omega_{(k,d)}} 0.$$

From this equality, one gets $\gamma_{(j;J),z} = 0$ for all $z \in C_k$. The first part of the proposition follows.

The proof of the second part is similar to the one of Proposition 3.3 in [14]. However, the relation $\equiv_{\bar\omega_{(k,d)}}$ is used in the proof instead of $\equiv$. □

For $k = 5$, we have the following result.

Theorem 3.10. *Let $n = 4(2^d-1)$ with d a positive integer. The dimension of the $\mathbb{F}_2$-vector space $(QP_5)_n$ is determined by the following table:*

$n = 4(2^d-1)$	$d=1$	$d=2$	$d=3$	$d=4$	$d\geqslant 5$
$\dim(QP_5)_n$	45	190	480	650	651

Since $n = 4(2^d-1) = 2^{d+1}+2^d+2^{d-1}+2^{d-1}-4$, for $d \geqslant 5$, the theorem follows from Theorem 2.9 and a result in [14]. For $1 \leqslant d \leqslant 4$, the proof of this theorem is based on Theorem 2.8 and some results of Kameko [3]. It is long and very technical. The detailed proof of it will be published elsewhere.

Proof of Main Theorem. For $k = 3$, the theorem follows from the results of Kameko [3]. For $k = 4$, it follows from the results in [13, 14]. Theorem 3.10 implies immediately this theorem for $k = 5$.

Suppose $k \geqslant 6$. Lemma 3.8 implies that $QP_k(\bar\omega_{(k,1)}) \neq 0$. Hence,

$$\begin{aligned}\dim(QP_k)_{k-1} &\geqslant \dim QP_k(\omega_{(k,1)}) + \dim QP_k(\bar\omega_{(k,1)})\\ &> \dim QP_k(\omega_{(k,1)}) = k = c(k,1).\end{aligned}$$

So, the theorem holds for $d = 1$.

Now, let $d > 1$ and $\widetilde\omega_{(k,d)} = ((k-1)^{(d-2)}, k-3, k-4, 2)$. Since $\widetilde\omega_{(k,d)}$ is weakly decreasing, by Lemma 2.7, $QP_k(\widetilde\omega_{(k,d)}) \neq 0$. We have $\deg(\omega_{(k,d)}) = \deg(\bar\omega_{(k,d)}) = \deg(\widetilde\omega_{(k,d)}) = (k-1)(2^d-1) = n$ and $(QP_k)_n \cong \bigoplus_{\deg\omega=n} QP_k(\omega)$. Hence, using

Propositions 3.7 and 3.9, we get

$$\begin{aligned}\dim(QP_k)_n &= \sum_{\deg\omega = n}\dim QP_k(\omega)\\ &\geqslant \dim QP_k(\omega_{(k,d)}) + \dim QP_k(\bar\omega_{(k,d)}) + \dim QP_k(\widetilde\omega_{(k,d)})\\ &> \dim QP_k(\omega_{(k,d)}) + \dim QP_k(\bar\omega_{(k,d)}) \geqslant c(k,d).\end{aligned}$$

The theorem is proved. □

Denote by $N(k,n)$ the number of spikes of degree n in P_k. Note that if $(i;I) \in \mathcal{N}_k$ and $I \neq \emptyset$, then $\phi_{(i;I)}(x)$ is not a spike for any monomial x. Hence, using Propositions 3.7 and 3.9, we easily obtain the following.

Corollary 3.11. *Under the hypotheses of Main Theorem,*

$$\dim(QP_k)_n \geqslant N(k,n) + \sum_{t=2}^{p}\binom{k}{t} + (k-3)\binom{k}{2}\sum_{u=2}^{q}\binom{k}{u}.$$

This corollary implies Mothebe's result.

Corollary 3.12 (See Mothebe [4, 5])**.** *Under the above hypotheses,*

$$\dim(QP_k)_n \geqslant N(k,n) + \sum_{t=2}^{p}\binom{k}{t}.$$

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