First Vietnam Workshop on Graph Theory and Discrete Geometry

Hanoi, September 7-10, 2016

Abstracts

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

General information

- First Vietnam Workshop on Graph Theory and Discrete Geometry, Hanoi, Vietnam will take place at Vietnam Institute for Advanced Study in Mathematics in September 2016.
- The workshop will start on Wednesday morning 7 September and will finish on Friday 9 September. The workshop will be followed by an excursion on September 10.

Venue

Vietnam Institute for Advanced Study in Mathematics
Tang 7, Thu Vien Ta Quang Buu
Dai Hoc Bach Khoa
Ha Noi
More information online at:
http://viasm.edu.vn/en/about-the-institute

Keynote Speakers

Alex Iosevich

János Pach

Andrei Raigorodskii

Günter Rote

Gábor Tardos

Le Anh Vinh

David Wood

Invited speakers

Pham Hoang Ha

Vu Dinh Hoa

Pham Tuan Huy

Ben Lund

Thang Pham

Nguyen Duy Phuong

Tran Dang Phuc

Tran Nam Trung

Steven Senger

Schedule

Tentative Schedule	Wednesday 7	Thursday 8	Friday 9
8:00-8:30	Registration		
8:30-9:30	**Iosevich**	**Raigorodskii**	**Wood**
9:30-10:30	**Tardos**	**Pach**	**Hoa**
10:30-10:45	Coffee	Coffee	Coffee
10:45-11:45	**Pach**	**Rote**	**Trung**
12:00-13:30	Lunch	Lunch	Lunch
13:30-14:30	**Rote**	**Tardos**	**Raigorodskii**
14:30-15:30	**Wood**	**Iosevich**	**Vinh**
15:30-15:45	Coffee	Coffee	Coffee
15:45-16:15	**Lund**	**Senger**	**Ha**
16:15-16:45	**Huy**	**Phuc**	**Phuong**
16:45-17:15			**Thang**
18:00-20:00			Conference Dinner

List of abstracts

(in alphabetical order of the speakers)

Speaker: **Pham Hoang Ha**

Department of Mathematics, Hanoi National University of Education

A new progress on Weak Dirac conjecture

Abstract: In 2014, Payne-Wood proved that every non-collinear set P of n points in the Euclidean plane contains a point in at least $\frac{n}{37}$ lines determined by P. This is a remarkable answer for the weak Dirac conjecture, which was proposed by Erdős, that every non-collinear set P of n points contains a point in at least $\frac{n}{c_1}$ lines determined by P, for some constant c_1. In this talk, we would like to discuss some problems on the weak Dirac conjecture. Firstly, we show that every non-collinear set P of n points contains a point in at least $\frac{n}{26} + 2$ lines determined by P . After that, we discuss some relations on theorem Beck to show that every set P of n points with at most l collinear determines at least $\frac{1}{61}n(n-l)$ lines and at least $\frac{1}{122}n(n-l)$ lines with at most three points. This is joint work with Phi Tien Cuong

Speaker: **Vu Dinh Hoa**

Department of Computer Sciences, Hanoi University of Education, Vietnam.

Decomposition of complete graphs into cubic graphs

Abstract: We say that a graph G decomposes the graph H if the edges of H can be covered by edge-disjoint copies of G. This talk will present some recent results and open problems of the decomposition of complete graphs into cubic graphs.

Speaker: **Pham Tuan Huy**

Stanford University

Tight tower-type bounds for progressions with popular common differences in dense sets

Abstract: Szemerédi's regularity lemma is one of the most powerful tools in combinatorics, giving a rough structural decomposition for all graphs. Green proved an arithmetic analogue of the regularity lemma in order to prove applications in number theory, and in particular used it to answer a question of Bergelson, Host, and Kra. Observe that a random subset A of $\mathbb{F}_3^n$ of density α almost surely satisfies that for every nonzero $d \in \mathbb{F}_3^n$, the density of three-term arithmetic progressions with common difference d that are in A is at least roughy α^3. Simple constructions show that there are sets with density α whose density of three-term arithmetic progressions is substantially smaller than α^3. However, Green used the arithmetic regularity lemma to prove that there is a nonzero d for which the density of three-term arithmetic progressions with common difference d is at least roughly α^3. Precisely, for each $\epsilon > 0$, there is a least positive integer $n_0(\epsilon)$ such that for each $n \geq n_0(\epsilon)$ and subset A of $\mathbb{F}_3^n$ of density α, the density of three-term arithmetic progressions of common difference d that are in A is at least $\alpha^3 - \epsilon$. Due to the application of the regularity lemma, Green's proof gives an upper bound on $n_0(\epsilon)$ which is an exponential tower of twos of height $\epsilon^{-O(1)}$.

We prove new lower and upper bounds which show that $n_0(\epsilon)$ grows as an exponential tower of twos of height $\Theta(\log(1/\epsilon))$. This is the first example of an application of a regularity lemma where the tower-type bound is shown to be necessary.

This is joint work with professor Jacob Fox.

Speaker: **Alex Iosevich**

Department of Mathematics, University of Rochester

Erdős type problems in vector spaces over finite fields

Abstract: We shall discuss the Erdos/Falconer distance problem in $\mathbb{F}_q^d$, which asks how large $E \subset \mathbb{F}_q^d$ needs to be to ensure that $\Delta(E) = \{||x-y|| : x, y \in E\}$, $||x|| = x_1^2 + \cdots + x_d^2$, is all of $\mathbb{F}_q$, or at least a positive proportion. We shall ask the same question when $\Delta(E)$ is replaced by $\prod(E) = \{x \cdot y : x, y \in E\}$. The similarities and difference in these problems harken back to the issues that arise in the study of Fourier Integral Operators in the Euclidean settings. These ideas will be explained in a completely elementary fashion.

Interaction of combinatorial, analytic and number theoretic ideas in geometric combinatorics

Abstract: The study of the distribution of simplexes in $\mathbb{F}_q^d$ generated some valuable elementary ideas that eventually made their way into the Euclidean setting. Those results, in turn, found surprising applications in geometric combinatorics, namely the question of often an equilateral triangle may arise among n points in $\mathbb{R}^2$. We shall describe these connections and outline a number of unsolved problems and possible future directions.

Speaker: **Ben Lund**

Rutgers University

Recent progress on finite field Nikodym sets

Abstract: This talk will cover recent joint work with Shubhangi Saraf and Charles Wolf.

Let F_q be the finite field of order q. A subset of F_q^n is a *Kakeya set* if it contains a line in every direction. A subset $E \subseteq F_q^n$ is a *Nikodym set* if, for each point $x \in F_q^n$, there is a line ℓ through x such that $\ell \setminus \{x\}$ is contained in E.

As an analog to a deep (and still open) question in real Euclidean space, in 1999 Wolff [6] asked for the minimum possible size of a Kakeya set in F_q^n. Ten years later, Dvir [1] proved, with a wonderfully simple application of the polynomial method, that any Kakeya set in F_q^n must have size at least $(1/n!)q^n$, giving the correct exponent of q. Following refinements of this proof, by Saraf and Sudan [5], and by Dvir, Kopparty, Saraf, and Sudan [2], improved the lower bound to $q^n/2^n$. Dvir et. al. also constructed a Kakeya set of size $q^n/2^{n-1} + O(q^{n-1})$, and so the gap between the upper and lower bounds is a factor of at most 2.

The situation for Nikodym sets is less settled. The lower bound arguments for Kakeya sets can easily be adapted to give lower bounds on the size of Nikodym sets, so it is known that any Nikodym set in F_q^n must have size at least $q^n/2^n$. However, no Nikodym set is known to exist smaller than $(1 - o(1))q^n$, and $(1 - o(1))q^n$ is likely to be the correct bound.

In this talk, I will discuss some recent results and conjectures on the structure of Nikodym sets, including a new lower bound for Nikodym sets in F_q^3, constructions of Nikodym sets, and a connection between Nikodym sets in F_q^3 and the question of finding the minimum number of points that must be contained in a collection of lines in F_q^3, not too many of which lie in any plane. Incidence questions of this type have recently been investigated in vector spaces over finite fields by Kollár [4], and by Hablisek and Ellenberg [3].

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Speaker: **János Pach**

EPF Lausanne, and Rényi Institute, Hungarian Academy of Sciences

Nearly perfect graphs

Abstract: Given a set of (geometric) objects, their *intersection graph* is a graph whose vertices correspond to the objects, two vertices being connected by an edge if and only if their intersection is nonempty. Intersection graphs of intervals on a line [H57], more generally, chordal graphs and comparability graphs, turned out to be *perfect graphs*, that is, for them and for all of their induced subgraph H, we have $\chi(H) = \omega(H)$, where $\chi(H)$ and $\omega(H)$ denote the chromatic number and the clique number of H, respectively. It was shown [HS58] that the complements of these graphs are also perfect, and based on these results, Berge [B61] conjectured and Lovász [Lo72] proved that the complement of every perfect graph is perfect. By now, we have a complete characterization of all perfect graphs, which immediately implies the Lovász theorem.

Most geometrically defined intersection graphs are not perfect. However, in many cases they still have nice coloring properties. For example, Asplund and Grünbaum [AG60] proved that every intersection graph G of axis-parallel rectangles in the plane satisfies $\chi(G) = O((\omega(G))^2)$. The best known lower bound for $\chi(G)$ is linear in $\omega(G)$. For intersection graphs of chords of a circle, Gyárfás [G85] established the bound $\chi(G) = O((\omega(G))^2 4^{\omega(G)})$, which was improved to $O(2^{\omega(G)})$ in [KoK97]. Here we have a slightly superlinear lower bound. In some cases, there is no functional dependence between χ and ω. The first such example was found by Burling: there are sets of axis-parallel boxes in $\mathbb{R}^3$, whose intersection graphs are *triangle-free* ($\omega = 2$), but their chromatic numbers are arbitrarily large. Following Gyárfás and Lehel [GL83], we call a family $\mathcal{G}$ of graphs χ-*bounded* if there exists a function f such that all elements $G \in \mathcal{G}$ satisfy the inequality $\chi(G) \leq f(\omega(G))$. The function f is called a *bounding function* for $\mathcal{G}$. Heuristically, if a family of graphs is χ-bounded, then its members can be regarded "nearly perfect". Consult [G87, Ko04] for surveys.

At first glance, one might believe that, in analogy to perfect graphs, a family of intersection graphs is χ-bounded if and only if the family of their complements is. Burling's above mentioned constructions show that this is not the case: the family of complements of intersection graphs of axis-parallel boxes in $\mathbb{R}^d$ is χ-bounded with bounding function $f(x) = O(\log^{d-1} x)$. More recently, Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [PKK14] have proved that Burling's triangle-free graphs can be realized as intersection graphs of segments in the plane. Consequently, the family of these intersection graphs is *not* χ-bounded either. On the other hand, the family of their complements is.

To simplify the exposition, we call the complement of the intersection graph of a set of objects their *disjointness graph*. That is, in the disjointness graph two vertices are connected by an edge if and only if the correspoinding objects are disjoint. Using this terminology, Larman, Matoušek, Pach, and Törőcsik proved the following result.

Theorem 1. [LMPT94] *The family of disjointness graphs of segments in the plane is χ-bounded.*

For the proof of Theorem 1, one has to introduce four partial orders on the family of segments four times. Although this method does not seem to generalize to higher dimensions, the statement does. We establish the following.

Theorem 2. P.-Tardos-Tóth [PTT16] *The family of disjointness graphs of segments in $\mathbb{R}^d, d \geq 2$ is χ-bounded.*

Theorem 3. P.-Tardos-Tóth [PTT16]

(i) For every n, there is a system of lines in $\mathbb{R}^3$ such that their disjointness graphs G_n satisfy $\lim_{n\to\infty} \frac{\chi(G_n)}{\omega(G_n)} = \infty$.

(ii) For infinitely many values of n, there is a system of n lines in $\mathbb{P}^3$ whose disjointness graph G'_n satisfies $\chi(G'_n) \geq 2\omega(G'_n) - 1$.

A continuous arc in the plane is called a *string*. One may wonder whether Theorem 1 can be extended to disjointness graphs of strings in place of segments. The answer is no, in a very strong sense.

Theorem 4. P.-Tardos-Tóth [PTT16] *There exist triangle-free disjointness graphs of n strings in the plane with arbitrarily large chromatic numbers. Moreover, we can assume that these strings are polygonal paths consisting of at most 4 segments.*

The following problems remain open.

Problem 5.

(i) Is the family of disjointness graphs of strings in the plane, any pair of which intersect in at most one point, χ-bounded?

(ii) Is this true for strings that are polygonal paths consisting of at most k segments, where $k > 1$ is fixed?

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Speaker: **Thang Pham**

Department of Mathematics, EPF Lausanne

Right angles in finite spaces

Abstract: We study the distribution of right angles determined by points in a set $\mathcal{E} \subseteq \mathbb{F}_q^d$. More precisely, we prove that if $|\mathcal{E}| \geq q^{\frac{d}{2}}$, then the number of right angles determined by points in $\mathcal{E}$ is larger than the expected value, and when $q^{\frac{d+1}{2}} = o(|\mathcal{E}|)$, the

number of right angles is $(1 - o(1))\frac{|\mathcal{E}|^3}{q}$. This is an improvement of a recent result due to Bennett (2016). This is joint work with Gábor Tardos and Nguyen Minh Sang.

Speaker: **Nguyen Duy Phuong**

Vietnam National University Hanoi

Incidences between points and spheres in $\mathbb{F}_q^d$

Abstract: Let $\mathbb{F}_q$ be a finite field of q elements where q is a large odd prime power and $Q = a_1x_1^{c_1} + \cdots + a_dx_d^{c_d} \in \mathbb{F}_q[x_1, \ldots, x_d]$, where $2 \le c_i \le N$, $\gcd(c_i, q) = 1$, and $a_i \in \mathbb{F}_q$ for all $1 \le i \le d$. A Q-*sphere* is a set of the form $\left\{x \in \mathbb{F}_q^d \mid Q(x - b) = r\right\}$, where $b \in \mathbb{F}_q^d, r \in \mathbb{F}_q$. We prove bounds on the number of incidences between a point set $\mathcal{P}$ and a Q-sphere set $\mathcal{S}$, denoted by $I(\mathcal{P}, \mathcal{S})$, as the following.

$$\left| I(\mathcal{P}, \mathcal{S}) - \frac{|\mathcal{P}||\mathcal{S}|}{q} \right| \le q^{d/2}\sqrt{|\mathcal{P}||\mathcal{S}|}.$$

We prove this estimate by studying the spectra of directed graphs. This is a joint work with Thang Pham and Le Anh Vinh.

Speaker: **Tran Dang Phuc**

Vietnam National University Hanoi

Expending Hall's Theorem in Tripartite Graph

Abstract: In this note we are concerned with a packing and covering problem for triangles in graphs. Let a graph G be given. We say that a family F of triangles in G is independent if the elements of F are pairwise edge-disjoint. Moreover, we say that a set $C \subset G$ of edges of G is a transversal of the set of triangles of G, or simply a transversal for G, if every triangle of G contains at least one element of C. We denote by $\nu(G)$ the maximum cardinality of an independent family of triangles in G, and by $\tau(G)$ the minimum cardinality of a transversal for G. It is clear that for every graph G we have: $\nu(G) \le \tau(G) \le 3 \cdot \nu(G)$. In this paper, we are concerned with equation: $\nu(G) = \tau(G)$. If G is bipartite Graph, Hall's theorem, Konig theorem and Min-Max Theorem completely answer our concern by using "perfect cover" and "complete matching". In our work, we attempt to expand Hall's condition and solve equation in tripartite graph, and then in triangle graph. Our conjecture: Now assume that G is tripartite, with parts X, Y , and Z. Let E be the set of all edges on one side of G, between vertex sets X and Y. Call the

edges in E Hall edges, and label the Hall edges. Every triangle in G has exactly one side that is a Hall edge, and if two triangles are adjacent then their respective Hall edges must be adjacent, either on a vertex from X or from Y. If E is a minimum triangle edge cover of G (the "Hall's condition"), then there is a complete packing of triangles in G on every edge in E.

Speaker: **Andrei Raigorodskii**

Department of Discrete Mathematics, Moscow Institute of Physics and Technology

Borsuk's problem

Abstract: In 1933 Borsuk asked whether any bounded non-singleton set in $\mathbb{R}^n$ can be partitioned into $n+1$ parts of smaller diameter. Positive answers have been quickly received for the dimensions $n \leq 3$. However, in general case, the question has been remaining open until 1993 when Kahn and Kalai found a very nice combinatorial construction showing that $n+1$ is certainly not sufficient in high dimensions. In our lecture, we will exhibit a colorful history of Borsuk's problem and its relatives.

Coloring random graphs

Abstract: In our lecture, we will consider various questions concerning colorings of random graphs in different models.

Speaker: **Günter Rote**

Institut für Informatik, Freie Universität Berlin

The Computational Geometry of Congruence Testing

Abstract

Part I. Testing two geometric objects for congruence, i.e., whether they are the same up to translations and rotations (and possibly reflections) is a fundamental question of geometry.

In the first part, I will survey the various algorithmic techniques that have been used since the 1970s to solve the problem in two and three dimensions in $O(n \log n)$ time for two n-point sets, such as string matching, planar graph isomorphism (Sugihara [6]), and the reduction technique of Atkinson [3].

In d-dimensions, for small constant d, the best previous algorithm takes $O(n^{\lceil d/3 \rceil} \log n)$ time (Brass and Knauer [4]). There is also a randomized Monte Carlo algorithm of Akutsu [1] and Matoušek, which takes $O(n^{\lfloor d/2 \rfloor /2} \log n)$ time but which may miss to find

a congruence, with small probability. I will review the involved techniques: the basic dimension reduction technique of Alt, Mehlhorn, Wagener, and Welzl [2], the canonical forms of Akutsu [1], the closest-pair graph of Matoušek.

Part II. In the second part, I will introduce our recent algorithm for solving the 4-dimensional problem in $O(n \log n)$ time (joint work with Heuna Kim [5]). This algorithm will require the study of four-dimensional geometry, in particular the structure of four-dimensional rotations, Hopf fibrations, and the regular polytopes.

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Speaker: **Steven Senger**

Department of Mathematics, Missouri State University.

Polychromatic point configurations

Abstract: Suppose that a vector space, V, has been partitioned into k color classes of roughly equal size. We explore conditions on V and k to guarantee the existence of point configurations with each point from a different color class. One example is looking for triples of points which form the vertices of a unit equilateral triangle (with an appropriately defined notion of distance) for colorings of a vector space over a finite field, with no pair of points from the triple belonging to the same color class. This should be held in sharp contrast to traditional Ramsey theoretical problems, where configurations are sought within a single color class.

Speaker: **Gábor Tardos**

Rényi Institute, Hungarian Academy of Sciences

Pattern avoidance in ordered graphs and matrices

Abstract: Pattern avoidance is a rich topic and it shows up in many different forms. In this abstract we focus on graphs and ordered graphs. Other settings in which pattern avoidance shows up include 0-1 matrices and permutations some of which will come up during the talk. Here we focus on the extremal theory, but enumerative and Ramsey-type questions are also widely studied. All these theories are closely related to each other. Some of the quoted results are not formulated originally for ordered graph but in some other setting. In these cases the quoted result follows from known equivalences between these related extremal theories.

No prior knowledge is required beyond the familiarity with graphs. The talk will contain full proofs of several of the results mentioned here and many other results will also be mentioned.

Let us start with a brief introduction to classical (also called Turán type) extremal graph theory. Given a simple graph H we ask what is the maximal number $\mathrm{ex}(n, H)$ of edges a simple graph on n vertices can have if it has no subgraph isomorphic to H. The earliest result in this area is Mantel's Theorem from 1907:

Mantel's Theorem. $\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$

Turán's Theorem extends this to larger forbidden cliques:

Turán's Theorem. $\mathrm{ex}(n, K_r) = \left(1 - \frac{1}{r-1}\right) \frac{n^2}{2} - O(r)$

(Turán's theorem gives the exact value of $\mathrm{ex}(n, K_r)$ and also the unique extremal graph but here we are satisfied with this weaker form.)

A far reaching further generalization is the Erdős-Stone-Simonovits Theorem that relates the extremal function $\mathrm{ex}(n, H)$ to the chromatic number $\chi(H)$ for any graph H:

Erdős-Stone-Simonovits Theorem. $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right) \frac{n^2}{2} + o_H(n^2)$

This result establishes the exact asymptotics of $\mathrm{ex}(n, H)$ for any fixed non-bipartite graph H. For bipartite graphs H, however, the above result only states $\mathrm{ex}(n, H) = o_H(n^2)$. The Kővári-Sós-Turán theorem establishes a much stronger upper bound for complete bipartite graphs — and by monotonicity to all bipartite graphs, but despite a large body of research over several decades there are still many bipartite graphs H for which not even the order of magnitude of $\mathrm{ex}(n, H)$ is known. Among them $K_{4,4}$ is one of the most notorious.

In this talk I will focus on an extension of this classical theory to *ordered graphs*. We give additional structure to simple graphs by specifying a linear order on the set of vertices to obtain ordered graphs. Subgraphs naturally inherit this order. Using the notions of ordered graphs and ordered subgraphs we ask the same pattern avoidance questions: Given an ordered graph H let $\mathrm{ex}_<(n, H)$ stand for the maximum number of edges an ordered graph on n vertices can have if it has no ordered subgraph isomorphic to H. If $\mathcal{H}$ is a family of ordered graphs we can similarly define $\mathrm{ex}_<(n, \mathcal{H})$ as the maximum number of edges an n vertex ordered graph can have if it has no ordered subgraph isomorphic to any member of $\mathcal{H}$. To justify that this is an extension of the Turán type extremal graph theory note that if H is a simple graph and $\mathcal{H}$ is the family of ordered graphs obtained from H by adding all possible linear orders on its vertices, then one clearly has

$$\mathrm{ex}(n, H) = \mathrm{ex}_<(n, \mathcal{H}).$$

Ordered graphs allow us to ask many more extremal questions and some of these are better suited for applications in discrete geometry and other fields where an order of the vertices may arise naturally. Distinct ordered graphs having the same underlying simple graph typically have wildly different extremal functions. As an example consider the path P_4 on four vertices. With a suitable linear order on its vertices the extremal function of the resulting ordered graph falls into any one of the following categories: $n^2/3 + O(1)$, $n^2/4 + O(1)$, $\Theta(n \log n)$, $\Theta(n)$. Note that no extremal function of the type $\Theta(n \log n)$ show up in the classical theory. Indeed, it is easy to see that for forests H we have $\mathrm{ex}(n, H) = O(n)$, while for graphs H containing a cycle C_m we have $\mathrm{ex}(n, H) \geq \mathrm{ex}(n, C_m) = \Omega(n^{1+1/(m-1)})$.

To find the extension of the Erdős-Stone-Simonovits theorem to this setting we introduce the *ordered chromatic number* $\chi_<(H)$ (also called *interval chromatic number*) of an ordered graph H: this is the smallest number of colors in a proper coloring of the simple graph underlying H in which the color classes form consecutive intervals in the ordering. With this notation we have the following for any ordered graph H:

Erdős-Stone-Simonovits Theorem for ordered graphs. [4]

$$\mathrm{ex}_<(n, H) = \left(1 - \frac{1}{\chi_<(H) - 1}\right)\frac{n^2}{2} + o_H(n^2)$$

As in the classical case, this result gives the exact asymptotics for the extremal function of the ordered graph H unless H is *ordered bipartite*, that is, $\chi_<(H) \leq 2$.

For ordered bipartite graphs much less is known and there are lot of interesting open problems. Among these is the characterization of ordered graphs H with a linear extremal function (that is, satisfying $\mathrm{ex}_<(n, H) = O(n)$). Marcus and Tardos [3] proved that matchings are among them:

[3] **For an ordered bipartite matching H one has $\mathrm{ex}_<(n, H) = O(n)$.**

Based on Balázs Keszeg's work [2], Jesse Geneson [1] proved that there are an infinite number of *minimal non-linear* ordered graphs (i.e., ordered graphs whose extremal function is not linear, but whose all proper ordered subgraphs have linear extremal functions). This, however, should not be considered as an unbreachable obstacle in the characterization. The corresponding problem in the classical theory is very easy, but one still has infinitely many minimal graphs with non-linear extremal functions: the cycles C_m.

My personal favorite open problem is about ordered bipartite trees, that is about ordered bipartite graphs H with a tree as their underlying simple graph. Even P_4 has an ordered bipartite ordering with extremal function $\Theta(n \log n)$, but finding examples with higher extremal function is difficult. Seth Pettie [5] found an ordered bipartite tree H with a slightly larger extremal function: $\Omega(n \log n \log \log n)$.

Conjecture 1. *For any ordered bipartite tree H one has $c = c(H)$ such that $\mathrm{ex}_<(n, H) = O(n \log^c n)$.*

The above conjecture may very well be true for $c = 2$ (independent of H), but even proving a weaker estimate $\mathrm{ex}_<(n, H) = O(n^{1+\epsilon})$ for all $\epsilon > 0$ would be a breakthrough.

For partial results see Pach and Tardos [4], where the conjecture is established for all very small trees. The smallest graphs H for which this conjecture is open are two distinct orderings of the path P_7 (see them also in [4]). Meanwhile, an unpublished paper of Tardos and Weidert [6] established an $n2^{O(\sqrt{\log n \log\log n})}$ bound for the extremal function of one of these two ordered graphs and a similar bound was established for the other one too, but for several bipartite orderings of P_8 no meaningful upper bound is known.

It would be interesting to find out how far the extremal function of ordered bipartite graphs can be from the extremal function of the underlying simple graph. The latter is always a lower bound corresponding to avoiding *all possible orderings* of the underlying graph. Pach and Tardos [4] show examples of bipartite orderings of C_{2m} for any m that are all avoided in some ordered graphs with n vertices and $\Omega(n^{4/3})$ edges. This establishes a strong separation as by the Even Circuit Theorem of Bondy, Simonovits and Erdős the corresponding classical extremal function satisfies $\text{ex}(n, C_{2m}) = O(n^{1+1/m})$. Finding examples where the classical and the ordered extremal function differ by a factor of $n^{1/3}$ is still open, but I believe that much larger separation is also possible:

Conjecture 2. *For every $\epsilon > 0$ there exists a bipartite graph H and a bipartite ordering $H_<$ of H such that $\text{ex}_<(n, H_<) = \Omega(n^{2-\epsilon})$, but $\text{ex}(n, H) = O(n^{1+\epsilon})$.*

There is just one bipartite ordering of C_4, so its extremal function is well understood, but not even the order of magnitude is known of any bipartite ordering of any other even cycle.

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Speaker: **Tran Nam Trung**

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An algebraic characterization of Cameron-Walker graphs

Abstract: Let $R = K[x_1, ..., x_n]$ is the polynomial ring over a field K and M a finitely generated graded R-module. By Hilbert syzygy theorem, M has a minimal graded free resolution of the form:

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_p \longleftarrow 0.$$

Let $t_i(M)$ be the maximal degree of generators of F_i. Then, the Castelnuovo-Mumford regularity (regularity for short) of M is defined by

$$reg(M) = \max\{t_i(M) - i \mid i = 0, \ldots, p\}.$$

Let $G = (V, E)$ be a simple graph on the vertex set $V = [n]$. We associated to G an ideal

$$I(G) = (x_i x_j \mid \{i, j\} \in E)$$

in R, which is called the edge ideal of G.

Finding bounds for the regularity of $I(G)$ in terms of combinatorial data of G is an active research program in combinatorial commutative algebra in recent years. Let $\nu(G)$ denote the maximum size of matchings of G and $\nu_0(G)$ that of induced matchings of G. It is known that

$$\nu_0(G) + 1 \leqslant reg(I(G)) \leqslant \nu(G) + 1.$$

Cameron and Walker succeeded in classifying the finite connected simple graphs G with $\nu(G) = \nu_0(G)$. In this case, $reg(I(G)) = \nu(G) + 1$. We say that a finite connected simple graph G is a Cameron-Walker graph if $\nu(G) = \nu_0(G)$. In this talk, we give a graph-theoretic characterization of graphs G such that $reg(I(G)) = \nu(G) + 1$.

Speaker: **Le Anh Vinh**

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Some geometric combinatorial problems in finite vector spaces - A spectral graph theory viewpoint

Abstract: Using spectral graph theory, we will study a unified proof of several geometric combinatorial problems such as the Erdős type problem, point-line incidences, simplexes distributions and their variants in finite vector spaces. We also outline a number of unsolved problems, possible future directions and the connection between this approach and different methods.

N-e.c. graphs

Abstract: For a positive integer n, a graph is *n-existentially closed* or n-e.c. if we can extend all n-subsets of vertices in all possible ways. It is known that almost all finite graphs are n-e.c. Despite this result, until recently, only few explicit examples of n-e.c. graphs are known for $n > 2$. In this talk, we give a survey on the current known constructions and construct explicitly a family of 3-e.c and 4-e.c. graphs via permutation polynomials and multiplicative groups over finite fields. This is a joint work with N. M. Hai and T. D. Phuc.

Speaker: **David Wood**

School of Mathematical Sciences, Monash University, Melbourne, Australia

Nonrepetitive Graph Colouring

Abstract: Graph colouring is a central topic in combinatorial mathematics. Most famously, the Four Colour Theorem states that the regions of a planar map can be 4-coloured so that regions that share a common border receive distinct colours. This is equivalent to the statement that the vertices of every planar graph can be 4-coloured so that adjacent vertices receive distinct colours. There are numerous extensions of the notion of graph colouring in the literature.

One such extension is called nonrepetitive colouring. This notion is best introduced via the theorem of Thue [9] who constructed an arbitrarily long string of three characters containing no substring of even length where the first half of the substring is the same as the second half. We can think of Thue's theorem as a result about 3-colouring paths, which naturally suggests a generalisation for arbitrary graphs. A *nonrepetitive colouring* of a graph G is a function that assigns each vertex of G a colour, such that for every path

P of even length in G, the sequence of colours on the first half of P is distinct from the sequence of colours on the second half of P. The *nonrepetitive chromatic number* of G is the minimum number of colours in a nonrepetitive colouring of G. It follows from Thue's result that the nonrepetitive chromatic number of a path (of length at least 4) equals 3.

What about nonrepetitive colouring of other graph classes? Alon et al. [1] proved that bounded degree graphs have bounded nonrepetitive chromatic number; in particular, graphs with maximum degree Δ are nonrepetitively $O(\Delta^2)$-colourable. The proof is an elegant example of the Lovász Local Lemma. Using a recent technique called 'entropy compression', Dujmović et al. [4] reduced this bound to $(1 + o(1))\Delta^2$. Non-repetitive colourings have been studied for several well-structured graph families. For example, Brešar et al. [2] proved that every tree is nonrepetitively 4-colourable. Kündgen and Pelsmajer [7] generalised this result for graphs of bounded treewidth. Nešetřil et al. [8] studied nonrepetitive colourings of subdivisions, and concluded that graph classes with bounded nonrepetitive chromatic number have bounded expansion.

A challenging open problem is whether planar graphs have bounded nonrepetitive chromatic number [6]. The best known upper bound for n-vertex planar graphs is $O(\log n)$ due to Dujmović et al. [3] This bound was generalised by Dujmović et al. [5] for graphs excluding a fixed topological minor. The proof employs powerful tools such as the Robertson–Seymour graph minor structure theorem.

This talk will survey these results, focusing on the tools used, which should be of wide interest. Only elementary knowledge of graph theory will be assumed.

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Visibility graphs: geometric problems in need of some additive combinatorics

Abstract: I will discuss a number of open problems in combinatorial geometry that seem to need additive combinatorics in their solution. The recurring theme is the notion of the visibility graph of a point set P, which has vertex set P, where distinct points v and w in P are adjacent whenever the line segment vw contains no other point in P. Many interesting results and open problems are obtained by studying graph-theoretic properties of visibility graphs. For example, Székely's simple proof of the Szemerédi–Trotter Theorem can be thought of as applying the crossing lemma to a particular subgraph of the visibility graph. And Beck's Theorem can be thought of as saying that visibility graphs have many edges. Open problems related to elliptic curves and Freiman's Theorem also arise. The following references are relevant [1–9].

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