Numerical weighted integration of functions having mixed smoothness

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Abstract

We investigate the approximation of weighted integrals over $\mathbb{R}^d$ for integrands from weighted Sobolev spaces of mixed smoothness. We prove upper and lower bounds of the convergence rate of optimal quadratures with respect to n integration nodes for functions from these spaces. In the one-dimensional case ($d = 1$), we obtain the right convergence rate of optimal quadratures. For $d \geq 2$, the upper bound is performed by sparse-grid quadratures with integration nodes on step hyperbolic crosses in the function domain $\mathbb{R}^d$.

Keywords and Phrases: Numerical multivariate weighted integration; Quadrature; Weighted Sobolev space of mixed smoothness; Step hyperbolic crosses of integration nodes; Convergence rate.

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1 Introduction

The aim of the present paper is to investigate approximation of weighted integrals over $\mathbb{R}^d$ for integrands lying in weighted Sobolev spaces $W^r_{1,w}(\mathbb{R}^d)$ of mixed smoothness $r \in \mathbb{N}$. We want to give upper and lower bounds of the approximation error for optimal quadratures with n integration nodes over the unit ball of $W^r_{1,w}(\mathbb{R}^d)$.

We first introduce weighted Sobolev spaces of mixed smoothness. Let

$$w(\boldsymbol{x}) := w_{\lambda,a,b}(\boldsymbol{x}) := \prod_{i=1}^d w(x_i), \tag{1.1}$$

where

$$w(x) := w_{\lambda,a,b}(x) := \exp(-a|x|^\lambda + b), \quad \lambda > 1, \quad a > 0, \quad b \in \mathbb{R}, \tag{1.2}$$

is a univariate Freud-type weight. The most important parameter in the weight $w_{\lambda,a,b}$ is λ. The parameter b which produces only a possitive constant in the weight $w_{\lambda,a,b}$ is introduced for a certain normalization for instance, for the standard Gaussian weight which is one of the most important weights. In what follows, we fix the parameters λ, a, b and for simplicity, drop them from the notation. Let $1 \le p < \infty$ and Ω be a Lebesgue measurable set on $\mathbb{R}^d$. We denote by $L^p_w(\Omega)$ the weighted space of all functions f on Ω such that the norm

$$\|f\|_{L^p_w(\Omega)} := \left(\int_\Omega |f(\boldsymbol{x})w(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \right)^{1/p} \tag{1.3}$$

is finite. For $r \in \mathbb{N}$, we define the weighted Sobolev space $W^r_{p,w}(\Omega)$ of mixed smoothness r as the normed space of all functions $f \in L^p_w(\Omega)$ such that the weak (generalized) partial derivative $D^{\boldsymbol{k}} f$ belongs to $L^p_w(\Omega)$ for every $\boldsymbol{k} \in \mathbb{N}^d_0$ satisfying the inequality $|\boldsymbol{k}|_\infty \le r$. The norm of a function f in this space is defined by

$$\|f\|_{W^r_{p,w}(\Omega)} := \left(\sum_{|\boldsymbol{k}|_\infty \le r} \|D^{\boldsymbol{k}} f\|^p_{L^p_w(\Omega)} \right)^{1/p}. \tag{1.4}$$

It is useful to notice that any function $f \in W^r_{p,w}(\mathbb{R}^d)$ is equivalent in the sense of the Lesbegue measure to a continuous (not necessarily bounded) function on $\mathbb{R}^d$, see Lemma 3.1 below. Hence throughout the present paper, we always assume that the functions $f \in W^r_{p,w}(\mathbb{R}^d)$ are continuous. We need this assumption for well-defined quadratures for functions $f \in W^r_{p,w}(\mathbb{R}^d)$.

Let γ be the standard d-dimensional Gaussian measure γ with the density function

$$g(\boldsymbol{x}) = (2\pi)^{-d/2} \exp(-|\boldsymbol{x}|_2^2/2).$$

The well-known spaces $L^p(\Omega;\gamma)$ and $W^r_p(\Omega;\gamma)$ which are used in many applications, are defined in the same way by replacing the norm (1.3) with the norm

$$\|f\|_{L^p(\Omega;\gamma)} := \left(\int_\Omega |f(\boldsymbol{x})|^p \gamma(\mathrm{d}\boldsymbol{x}) \right)^{1/p} = \left(\int_\Omega |f(\boldsymbol{x})|^p g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{1/p}.$$

The spaces $L^p(\Omega;\gamma)$ and $W^r_p(\Omega;\gamma)$ can be seen as the $L^p_w(\Omega)$ and $W^r_{p,w}(\Omega)$, where

$$w(\boldsymbol{x}) = w_{\lambda,a,b}(\boldsymbol{x}), \quad \text{with} \quad \lambda = 2/p, \; a = 1/2p, \; b = -(d\log 2\pi)/2p$$

for a fixed $1 \le p < \infty$.

In the present paper, we are interested in approximation of weighted integrals

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \tag{1.5}$$

for functions f lying in the space $W^r_{1,w}(\mathbb{R}^d)$. To approximate them we use quadratures of the form

$$Q_k f := \sum_{i=1}^{k} \lambda_i f(\boldsymbol{x}_i), \tag{1.6}$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^d$ are the integration nodes and $\lambda_1, \ldots, \lambda_k$ the integration weights. For convenience, we assume that some of the integration nodes may coincide.

Let $\boldsymbol{F}$ be a set of continuous functions on $\mathbb{R}^d$. Denote by $\mathcal{Q}_n$ the family of all quadratures Q_k of the form (1.6) with $k \le n$. The optimality of quadratures from $\mathcal{Q}_n$ for $f \in \boldsymbol{F}$ is measured by

$$\mathrm{Int}_n(\boldsymbol{F}) := \inf_{Q_n \in \mathcal{Q}_n} \sup_{f \in \boldsymbol{F}} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} - Q_n f \right|. \tag{1.7}$$

We recall that the space $W^r_p(\Omega)$ is defined as the classical Sobolev space of mixed smoothness by replacing $L^p_w(\Omega)$ with $L^p(\Omega)$ in (1.4), where as usually, $L^p(\Omega)$ denotes the Lebesgue space of functions on Ω equipped with the usual p-integral norm.

For approximation of integrals

$$\int_\Omega f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

over the set Ω, we need natural modifications $Q^\Omega_n f$ for functions f on Ω, and $\mathrm{Int}^\Omega_n(\boldsymbol{F})$ for a set $\boldsymbol{F}$ of functions on Ω, of the definitions (1.6) and (1.7). For simplicity we will drop Ω from these notations if there is no misunderstanding.

We first briefly describe the main results of the present paper and then give comments on related works.

For a normed space X of functions on $\mathbb{R}^d$, the boldface $\boldsymbol{X}$ denotes the unit ball in X. Throughout the present paper we make use of the notation

$$r_\lambda := (1 - 1/\lambda) r.$$

For the set $\boldsymbol{W}^r_{1,w}(\mathbb{R}^d))$, we prove the upper and lower bounds

$$n^{-r_\lambda} (\log n)^{r_\lambda (d-1)} \ll \mathrm{Int}_n(\boldsymbol{W}^r_{1,w}(\mathbb{R}^d)) \ll n^{-r_\lambda} (\log n)^{(r_\lambda + 1)(d-1)}, \tag{1.8}$$

in particular, in the case of Gaussian measure

$$n^{-r/2} (\log n)^{r(d-1)/2} \ll \mathrm{Int}_n(\boldsymbol{W}^r_1(\mathbb{R}^d; \gamma)) \ll n^{-r/2} (\log n)^{(r/2+1)(d-1)}. \tag{1.9}$$

In the one-dimensional case, we prove the right convergence rate

$$\mathrm{Int}_n(\boldsymbol{W}^r_{1,w}(\mathbb{R})) \asymp n^{-r_\lambda}. \tag{1.10}$$

The difference between the upper and lower bounds in (1.8) is the logarithmic factor $(\log n)^{d-1}$.

There is a large number of works on high-dimensional unweighted integration over the unit d-cube $\mathbb{I}^d := [0,1]^d$ for functions having a mixed smoothness (see [2, 5, 12] for results and bibliography). However, there are only a few works on high-dimensional weighted integration for functions having a mixed smoothness. The problem of optimal weighted integration (1.5)–(1.7) has been studied in [6, 7, 4] for functions in certain Hermite spaces, in particular, the space $\mathcal{H}_{d,r}$ which coincides with $W^r_2(\mathbb{R}^d;\gamma)$ in terms of norm equivalence. It has been proven in [4] that

$$n^{-r}(\log n)^{(d-1)/2} \ll \operatorname{Int}_n\big(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma)\big) \ll n^{-r}(\log n)^{d(2r+3)/4-1/2}.$$

Recently, in [1, Theorem 2.3] for the space $W^r_p(\mathbb{R}^d,\gamma)$ with $r \in \mathbb{N}$ and $1 < p < \infty$, we have constructed an asymptotically optimal quadrature Q^γ_n of the form (1.6) which gives the asymptotic order

$$\sup_{f\in \boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x}) - Q^\gamma_n f \right| \asymp \operatorname{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)\big) \asymp n^{-r}(\log n)^{(d-1)/2}. \tag{1.11}$$

The results (1.9) and (1.11) show a substantial difference of the convergence rates between the cases $p = 1$ and $1 < p < \infty$. In constructing the asymptotically optimal quadrature Q^γ_n in (1.11), we used a technique collaging a quadrature for the Sobolev spaces on the unit d-cube to the integer-shifted d-cubes. Unfortunately, this technique is not suitable to constructing a quadrature realizing the upper bound in (1.8) for the space $W^r_1(\mathbb{R}^d;\gamma)$ which is the largest among the spaces $W^r_p(\mathbb{R}^d;\gamma)$ with $1 \le p < \infty$. It requires a different technique based on the well-known Smolyak algorithm. Such a quadrature relies on sparse grids of integration nodes which are step hyperbolic crosses in the function domain $\mathbb{R}^d$, and some generalization of the results on univariate numerical integration by truncated Gaussian quadratures from [3]. To prove the lower bound in (1.8) and (1.10) we adopt a traditional technique to construct for arbitrary n integration nodes a fooling function vanishing at these nodes.

It is interesting to compare the results (1.9) and (1.11) on $\operatorname{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)\big)$ with known results on $\operatorname{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{I}^d)\big)$ for the unweighted Sobolev space $W^r_p(\mathbb{I}^d)$ of mixed smoothness r. For $1 < p < \infty$, there holds the asymptotic order

$$\operatorname{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{I}^d)\big) \asymp n^{-r}(\log n)^{(d-1)/2},$$

and for $p = 1$ and $r > 1$, there hold the bounds

$$n^{-r}(\log n)^{(d-1)/2} \ll \operatorname{Int}_n(\boldsymbol{W}^r_1(\mathbb{I}^d)) \ll n^{-r}(\log n)^{d-1}$$

which are so far the best known (see, e.g., [2, Chapter 8], for detail). Hence we can see that $\operatorname{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)\big)$ and $\operatorname{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{I}^d)\big)$ have the same asymptotic order in the case $1 < p < \infty$, and very different lower and upper bounds in both power and logarithmic terms in the case $p = 1$. The right asymptotic orders of the both $\operatorname{Int}_n\big(\boldsymbol{W}^r_1(\mathbb{I}^d)\big)$ and $\operatorname{Int}_n\big(\boldsymbol{W}^r_1(\mathbb{R}^d;\gamma)\big)$ are still open problems (cf. [2, Open Problem 1.9]).

The problem of numerical integration considered in the present paper is related to the research direction of optimal approximation and integration for functions having mixed smoothness on one hand, and the other research direction of univariate weighted polynomial approximation and integration on $\mathbb{R}$, on the other hand. For survey and bibliography, we refer the reader to the books [2, 12] on the first direction, and [11, 9, 8] on the second one.

The paper is organized as follows. In Section 2, we prove the asymptotic order of $\mathrm{Int}_n(\boldsymbol{W}^r_{1,w}(\mathbb{R}))$ and construct asymptotically optimal quadratures. In Section 3, we prove upper and lower bounds of $\mathrm{Int}_n(\boldsymbol{W}^r_{1,w}(\mathbb{R}^d))$ for $d \geq 2$, and construct quadratures which give the upper bound. Section 4 is devoted to some extentions of the results in the previous sections to Markov-Sonin weights.

Notation. Denote $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^d$; for $\boldsymbol{x} \in \mathbb{R}^d$, $\boldsymbol{x} =: (x_1, ..., x_d)$, $|\boldsymbol{x}|_\infty := \max_{1\leq j\leq d} |x_j|$, $|\boldsymbol{x}|_p := \left(\sum_{j=1}^d |x_j|^p\right)^{1/p}$ $(1 \leq p < \infty)$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, the inequality $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_i \leq y_i$ for every $i = 1, ..., d$. For $x \in \mathbb{R}$, denote $\mathrm{sign}(x) := 1$ if $x \geq 0$, and $\mathrm{sign}(x) := -1$ if $x < 0$. We use letters C and K to denote general positive constants which may take different values. For the quantities $A_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$, we write $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ such that $A_n(f, \boldsymbol{k}) \leq CB_n(f, \boldsymbol{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ (the notation $A_n(f, \boldsymbol{k}) \gg B_n(f, \boldsymbol{k})$ has the obvious opposite meaning), and $A_n(f, \boldsymbol{k}) \asymp B_n(f, \boldsymbol{k})$ if $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k}) \ll A_n(f, \boldsymbol{k})$. Denote by $|G|$ the cardinality of the set G. For a Banach space X, denote by the boldface $\boldsymbol{X}$ the unit ball in X.

2 One-dimensional integration

In this section, for one-dimensional numerical integration, we prove the asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}^r_{1,w}(\mathbb{R})\big)$ and present some asymptotically optimal quadratures. We start this section with a well-known inequality in the following lemma which is implied directly from the definition (1.7) and which is quite useful for lower estimation of $\mathrm{Int}_n(\boldsymbol{F})$.

Lemma 2.1 *Let $\boldsymbol{F}$ be a set of continuous functions on $\mathbb{R}^d$. Then we have*

$$\mathrm{Int}_n(\boldsymbol{F}) \geq \inf_{\{\boldsymbol{x}_1,...,\boldsymbol{x}_n\}\subset\mathbb{R}^d} \ \sup_{f\in\boldsymbol{F}:\ f(\boldsymbol{x}_i)=0,\ i=1,...,n} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x})w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right|. \tag{2.1}$$

We now consider the problem of approximation of integral (1.5) for univariate functions from $W^r_{1,w}(\mathbb{R})$. Let $(p_m(w))_{m\in\mathbb{N}}$ be the sequence of orthonormal polynomials with respect to the weight w. In the classical quadrature theory, a possible choice of integration nodes is to take the zeros of the polynomials $p_m(w)$. Denote by $x_{m,k}$, $1 \leq k \leq \lfloor m/2 \rfloor$ the positive zeros of $p_m(w)$, and by $x_{m,-k} = -x_{m,k}$ the negative ones (if m is odd, then $x_{m,0} = 0$ is also a zero of $p_m(w)$). These zeros are located as

$$-a_m + \frac{Ca_m}{m^{2/3}} < x_{m,-\lfloor m/2\rfloor} < \cdots < x_{m,-1} < x_{m,1} < \cdots < x_{m,\lfloor m/2\rfloor} \leq a_m - \frac{Ca_m}{m^{2/3}}, \tag{2.2}$$

with a positive constant C independent of m (see, e. g., [8, (4.1.32)]). Here a_m is the Mhaskar-Rakhmanov-Saff number which is

$$a_m = a_m(w) = (\gamma_\lambda m)^{1/\lambda}, \quad \gamma_\lambda := \frac{2\Gamma((1+\lambda)/2)}{\sqrt{\pi}\Gamma(\lambda/2)}, \tag{2.3}$$

and Γ is the gamma function. Notice that the formula (2.3) is given in [8, (4.1.4)] for the weight $w(x) = \exp\left(-|x|^\lambda\right)$. Inspecting the definition of Mhaskar-Rakhmanov-Saff number (see, e.g., [8, Page 116]), one easily verify that it still holds true for the general weight w for any $a > 0$ and $b \in \mathbb{R}$.

For a continuous function on $\mathbb{R}$, the classical Gaussian quadrature is defined as

$$Q_m^{\mathrm{G}} f := \sum_{|k| \le \lfloor m/2 \rfloor} \lambda_{m,k}(w) f(x_{m,k}), \tag{2.4}$$

where $\lambda_{m,k}(w)$ are the corresponding Cotes numbers. This quadrature is based on Lagrange interpolation (for details, see, e.g., [11, 1.2. Interpolation and quadrature]). Unfortunately, it does not give the optimal convergence rate for functions from $\boldsymbol{W}_{1,w}^r(\mathbb{R})$, see Remark 2.3 below.

In [3], for the weight $w(x) = \exp\left(-|x|^\lambda\right)$, the authors proposed truncated Gaussian quadratures which not only improve the convergence rate but also give the asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}_{1,w}^r(\mathbb{R})\big)$ as shown in Theorem 2.2 below. Let us introduce in the same manner truncated Gaussian quadratures for the weight $w(x)$ with any $a > 0$ and $b \in \mathbb{R}$.

Throughout this paper, we fix a number θ with $0 < \theta < 1$, and denote by $j(m)$ the smallest integer satisfying $x_{m,j(m)} \ge \theta a_m$. It is useful to remark that

$$d_{m,k} \asymp \frac{a_m}{m} \asymp m^{1/\lambda - 1}, \quad |k| \le j(m); \quad x_{m,j(m)} \asymp m^{1/\lambda}, \tag{2.5}$$

where $d_{m,k} := x_{m,k} - x_{m,k-1}$ is the distance between consecutive zeros of the polynomial $p_m(w)$. These relations were proven in [3, (13)] for the weight $w(x) = \exp\left(-|x|^\lambda\right)$. From their proofs there, one can easily see that they are still hold true for the general case of the weight w. By (2.2) and (2.5), for m sufficiently large we have that

$$Cm \le j(m) \le m/2 \tag{2.6}$$

with a positive constant C depending on λ, a, b and θ only.

For a continuous function on $\mathbb{R}$, consider the truncated Gaussian quadrature

$$Q_{2j(m)}^{\mathrm{TG}} f := \sum_{|k| \le j(m)} \lambda_{m,k}(w) f(x_{m,k}). \tag{2.7}$$

Notice that the number $2j(m)$ of samples in the quadrature $Q_{2j(m)}^{\mathrm{TG}} f$ is strictly smalller than m – the number of samples in the quadrature $Q_m^{\mathrm{G}} f$. However, due to (2.6) it has the asymptotic order as $2j(m) \asymp m$ when m going to infinity.

Theorem 2.2 *For any $n \in \mathbb{N}$, let m_n be the largest integer such that $2j(m_n) \le n$. Then the quadratures $Q_{2j(m_n)}^{\mathrm{TG}} \in \mathcal{Q}_n$, $n \in \mathbb{N}$, are asymptotically optimal for $\boldsymbol{W}_{1,w}^r(\mathbb{R})$ and*

$$\sup_{f \in \boldsymbol{W}_{1,w}^r(\mathbb{R})} \left| \int_{\mathbb{R}} f(x) w(x) \mathrm{d}x - Q_{2j(m_n)}^{\mathrm{TG}} f \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}_{1,w}^r(\mathbb{R})\big) \asymp n^{-r_\lambda}. \tag{2.8}$$

Proof. For $f \in W^r_{1,w}(\mathbb{R})$, there holds the inequality

$$\left| \int_{\mathbb{R}} f(x)w(x)\mathrm{d}x - Q^{\mathrm{TG}}_{2j(m)} f \right| \le C \left(m^{-(1-1/\lambda)r} \left\| f^{(r)} \right\|_{L^1_w(\mathbb{R})} + e^{-Km} \|f\|_{L^1_w(\mathbb{R})} \right) \qquad (2.9)$$

with some constants C and K independent of m and f. This inequality was proven in [3, Corollary 4] for the weight $w(x) = \exp\left(-|x|^\lambda\right)$. Inspecting the proof of [3, Corollary 4], one can easily see that this inequality is also true for a weight of the form (1.1) with any $a > 0$ and $b \in \mathbb{R}$. The inequality (2.9) implies the upper bound in (2.8):

$$\mathrm{Int}_n\big(\boldsymbol{W}^r_{1,w}(\mathbb{R})\big) \le \sup_{f \in \boldsymbol{W}^r_{1,w}(\mathbb{R})} \left| \int_{\mathbb{R}} f(x)w(x)\mathrm{d}x - Q^{\mathrm{TG}}_{2j(m_n)} f \right| \ll n^{-r_\lambda}.$$

The lower bound in (2.8) is already contained in Theorem 3.8 below. Since its proof is much simpler for the case $d = 1$, let us proccess it separately. In order to prove the lower bound in (2.8) we will apply Lemma 2.1. Let $\{\xi_1, ..., \xi_n\} \subset \mathbb{R}$ be arbitrary n points. For a given $n \in \mathbb{N}$, we put $\delta = n^{1/\lambda - 1}$ and $t_j = \delta j$, $j \in \mathbb{N}_0$. Then there is $i \in \mathbb{N}$ with $n + 1 \le i \le 2n + 2$ such that the interval (t_{i-1}, t_i) does not contain any point from the set $\{\xi_1, ..., \xi_n\}$. Take a nonnegative function $\varphi \in C^\infty_0([0, 1])$, $\varphi \ne 0$, and put

$$b_0 := \int_0^1 \varphi(y)\mathrm{d}y > 0, \quad b_s := \int_0^1 |\varphi^{(s)}(y)|\mathrm{d}y, \ s = 1, ..., r.$$

Define the functions g and h on $\mathbb{R}$ by

$$g(x) := \begin{cases} \varphi(\delta^{-1}(x - t_{i-1})), & x \in (t_{i-1}, t_i), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$h(x) := (gw^{-1})(x).$$

Let us estimate the norm $\|h\|_{W^r_{1,w}(\mathbb{R})}$. For a given $k \in \mathbb{N}_0$ with $0 \le k \le r$, we have

$$h^{(k)} = (gw^{-1})^{(k)} = \sum_{s=0}^{k} \binom{k}{s} g^{(k-s)} (w^{-1})^{(s)}. \qquad (2.10)$$

By a direct computation we find that for $x \in \mathbb{R}$,

$$(w^{-1})^{(s)}(x) = (w^{-1})(x)(\mathrm{sign}(x))^s \sum_{j=1}^{s} c_{s,j}(\lambda, a)|x|^{\lambda_{s,j}}, \qquad (2.11)$$

where $\mathrm{sign}(x) := 1$ if $x \ge 0$, and $\mathrm{sign}(x) := -1$ if $x < 0$,

$$\lambda_{s,s} = s(\lambda - 1) > \lambda_{s,s-1} > \cdots > \lambda_{s,1} = \lambda - s, \qquad (2.12)$$

and $c_{s,j}(\lambda, a)$ are polynomials in the variables λ and a of degree at most s with respect to each variable. Hence, we obtain

$$h^{(k)}(x)w(x) = \sum_{s=0}^{k} \binom{k}{s} g^{(k-s)}(x)(\mathrm{sign}(x))^s \sum_{j=1}^{s} c_{s,j}(\lambda, a)|x|^{\lambda_{s,j}} \qquad (2.13)$$

which implies that

$$\int_{\mathbb{R}} |h^{(k)} w|(x)\mathrm{d}x \leq C \max_{0\leq s\leq k} \max_{1\leq j\leq s} \int_{t_{i-1}}^{t_i} |x|^{\lambda_{s,j}} |g^{(k-s)}(x)|\mathrm{d}x.$$

From (2.12), the inequality $n^{1/\lambda} \leq x \leq (2n+2)n^{1/\lambda - 1}$ and

$$\int_{t_{i-1}}^{t_i} |g^{(k-s)}(x)|\mathrm{d}x = b_{k-s}\delta^{-k+s+1} = b_{k-s} n^{(k-s-1)(1-1/\lambda)},$$

we derive

$$\begin{aligned}\int_{\mathbb{R}} |h^{(k)} w|(x)\mathrm{d}x &\leq C \max_{0\leq s\leq k} \int_{t_{i-1}}^{t_i} |x|^{\lambda_{s,s}} |g^{(k-s)}(x)|\mathrm{d}x \\ &\leq C \max_{0\leq s\leq k} \left(n^{1/\lambda}\right)^{s(\lambda-1)} \int_{t_{i-1}}^{t_i} |g^{(k-s)}(x)|\mathrm{d}x \\ &\leq C \max_{0\leq s\leq k} n^{s(\lambda-1)/\lambda} n^{(k-s-1)(1-1/\lambda)} \\ &= Cn^{(1-1/\lambda)(k-1)} \leq Cn^{(1-1/\lambda)(r-1)}.\end{aligned}$$

If we define

$$\bar{h} := C^{-1} n^{-(1-1/\lambda)(r-1)} h,$$

then $\bar{h}$ is nonnegative, $\bar{h} \in \boldsymbol{W}^r_{1,w}(\mathbb{R})$, $\sup \bar{h} \subset (t_{i-1}, t_i)$ and

$$\begin{aligned}\int_{\mathbb{R}} (\bar{h}w)(x)\mathrm{d}x &= C^{-1} n^{-(1-1/\lambda)(r-1)} \int_{t_{i-1}}^{t_i} g(x)\mathrm{d}x \\ &= C^{-1} n^{-(1-1/\lambda)(r-1)} b_0 \delta \gg n^{-(1-1/\lambda)r}\end{aligned}$$

Since the interval (t_{i-1}, t_i) does not contain any point from the set $\{\xi_1, ..., \xi_n\}$, we have $\bar{h}(\xi_k) = 0$, $k = 1, ..., n$. Hence, by Lemma 2.1,

$$\mathrm{Int}_n\big(\boldsymbol{W}^r_{1,w}(\mathbb{R})\big) \geq \int_{\mathbb{R}} \bar{h}(x) w(x)\mathrm{d}x \gg n^{-r_\lambda}.$$

The lower bound in (2.8) is proven. ∎

Remark 2.3 In the case when $w(x) = \exp(-x^2/2)$ is the Gaussian density, the truncated Gaussian quadratures $Q^{\mathrm{TG}}_{2j(m)}$ in Theorem 2.2 give

$$\sup_{f\in \boldsymbol{W}^r_{1,w}(\mathbb{R})} \left| \int_{\mathbb{R}} f(x)w(x)\mathrm{d}x - Q^{\mathrm{TG}}_{2j(m_n)} f \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}^r_{1,w}(\mathbb{R})\big) \asymp n^{-r/2}. \tag{2.14}$$

On the other hand, for the full Gaussian quadratures Q^{G}_n, it has been proven in [3, Proposition 1] the convergence rate

$$\sup_{f\in \boldsymbol{W}^1_{1,w}(\mathbb{R})} \left| \int_{\mathbb{R}} f(x)w(x)\mathrm{d}x - Q^{\mathrm{G}}_n f \right| \asymp n^{-1/6}$$

which is much worse than the convergence rate of $\mathrm{Int}_n\big(\boldsymbol{W}^1_{1,w}(\mathbb{R})\big) \asymp n^{-1/2}$ as in (2.14) for $r = 1$.

3 High-dimensional integration

In this section, for high-dimensional numerical integration ($d \geq 2$), we prove upper and lower bounds of $\mathrm{Int}_n\big(\boldsymbol{W}^r_{1,w}(\mathbb{R}^d)\big)$ and construct quadratures based on step-hyperbolic-cross grids of integration nodes which give the upper bounds. To do this we need some auxiliary lemmata.

Lemma 3.1 *Let $1 \leq p < \infty$. Then any function $f \in W^r_{p,w}(\mathbb{R}^d)$ is equivalent in the sense of the Lebesgue measure to a continuous function on $\mathbb{R}^d$.*

Proof. We prove this lemma in the particular case when $p = 1$, $r = 1$ and $d = 2$. The general case can be proven in a similar way.

Fix $\tau > \lambda$ and define for $\boldsymbol{x} = (x_1, x_2)$,

$$v(\boldsymbol{x}) := \exp(-a|x_1|^\tau + b)\exp(-a|x_2|^\tau + b).$$

We preliminarly prove that $W^r_{1,w}(\mathbb{T}^2)$ is continuously embbeded into the space $C_v(\mathbb{T}^2)$ where $\mathbb{T}^2 := [-T, T]^2$, T is any positive number and $C_v(\mathbb{T}^2)$ is the Banach space of continuous functions f on $\mathbb{T}^2$ equipped with the norm

$$\|f\|_{C_v(\mathbb{T}^2)} := \max_{\boldsymbol{x}\in\mathbb{T}^2} |(vf)(\boldsymbol{x})|.$$

Since the subspace $C^\infty_0(\mathbb{T}^2)$ of infinitely differentiable functions with compact support is dense in both the Banach spaces $C(\mathbb{T}^2)$ and $W^1_{1,w}(\mathbb{T}^2)$, to prove this continuous embbeding, it is sufficient to show the inequality

$$\|f\|_{C_v(\mathbb{T}^2)} \ll \|f\|_{W^1_{1,w}(\mathbb{T}^2)}, \quad f \in C^\infty_0(\mathbb{T}^2). \tag{3.1}$$

For $\boldsymbol{k} \in \mathbb{N}^2_0$, denote by $D^{\boldsymbol{k}} g$ the $\boldsymbol{k}$th partial derivative of g. Taking a function $f \in C^\infty_0(\mathbb{T}^2)$, we have that for $\boldsymbol{x} \in \mathbb{T}^2$,

$$(vf)(\boldsymbol{x}) = \int_{-T}^{x_1}\int_{-T}^{x_2} D^{(1,1)}(vf)(\boldsymbol{t})\mathrm{d}\boldsymbol{t},$$

and

$$\begin{aligned} D^{(1,1)}(vf)(\boldsymbol{x}) = v(\boldsymbol{x})\big[&D^{(1,1)}f(\boldsymbol{x}) - a\tau\,\mathrm{sign}(x_1)|x_1|^{\tau-1}D^{(1,0)}f(\boldsymbol{x}) \\ &- a\tau\,\mathrm{sign}(x_2)|x_2|^{\tau-1}D^{(0,1)}f(\boldsymbol{x}) + a^2\tau^2\,\mathrm{sign}(x_1)|x_1|^{\tau-1}\,\mathrm{sign}(x_2)|x_2|^{\tau-1}f(\boldsymbol{x})\big]. \end{aligned}$$

Hence by using the inequality $\tau > \lambda$ we derive (3.1):

$$\begin{aligned} \|f\|_{C_v(\mathbb{T}^2)} &\ll \int_{\mathbb{T}^2}\big|\big(vD^{(1,1)}f\big)(\boldsymbol{x})\big|\mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^2}\big|\big(vD^{(1,0)}f\big)(\boldsymbol{x})\big||x_1|^{\tau-1}\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\mathbb{T}^2}\big|\big(vD^{(0,1)}f\big)(\boldsymbol{x})\big||x_2|^{\tau-1}\mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^2}\big|\big(vf\big)(\boldsymbol{x})\big||x_1x_2|^{\tau-1}\mathrm{d}\boldsymbol{x} \\ &\ll \int_{\mathbb{T}^2}\big|\big(wD^{(1,1)}f\big)(\boldsymbol{x})\big|\mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^2}\big|\big(wD^{(1,0)}f\big)(\boldsymbol{x})\big|\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\mathbb{T}^2}\big|\big(wD^{(0,1)}f\big)(\boldsymbol{x})\big|\mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^2}\big|\big(wf\big)(\boldsymbol{x})\big|\mathrm{d}\boldsymbol{x} = \|f\|_{W^1_{1,w}(\mathbb{T}^2)} \end{aligned}$$

From the continuous embbeding of $W^r_{1,w}(\mathbb{T}^2)$ into $C_v(\mathbb{T}^2)$ it follows that any function $f \in W^r_{1,w}(\mathbb{T}^2)$ is equivalent in sense of the Lebesgue measure to a continuous (not necessarily bounded) function on $\mathbb{T}^2$. Hence we obtain the claim of the lemma for $p = 1$ since T has been taken arbitrarily and the restriction of a function $f \in W^r_{1,w}(\mathbb{R}^2)$ to $\mathbb{T}^2$ belongs to $W^r_{1,w}(\mathbb{T}^2)$. ∎

Importantly, as noticed in Introduction from Lemma 3.1 we can assume that the functions $f \in W^r_{p,w}(\mathbb{R}^d)$ are continuous. This allows to correctly define quadratures for them.

For $\boldsymbol{x} \in \mathbb{R}^d$ and $e \subset \{1, ..., d\}$, let $\boldsymbol{x}^e \in \mathbb{R}^{|e|}$ be defined by $(x^e)_i := x_i$, and $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$ by $(\bar{x}^e)_i := x_i$, $i \in \{1, ..., d\} \setminus e$. With an abuse we write $(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e) = \boldsymbol{x}$.

Lemma 3.2 *Let $1 \le p \le \infty$, $e \subset \{1, ..., d\}$ and $\boldsymbol{r} \in \mathbb{N}_0^d$. Assume that f is a function on $\mathbb{R}^d$ such that for every $\boldsymbol{k} \le \boldsymbol{r}$, $D^{\boldsymbol{k}} f \in L^p_w(\mathbb{R}^d)$. Put for $\boldsymbol{k} \le \boldsymbol{r}$ and $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$,*

$$g(\boldsymbol{x}^e) := D^{\bar{\boldsymbol{k}}^e} f(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e).$$

Then $D^{\boldsymbol{s}} g \in L^p_w(\mathbb{R}^{|e|})$ for every $\boldsymbol{s} \le \boldsymbol{k}^e$ and almost every $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$.

Proof. Taking arbitrary test functions $\varphi^e \in C_0^\infty(\mathbb{R}^{|e|})$ and $\bar{\varphi}^e \in C_0^\infty(\mathbb{R}^{d-|e|})$ and defining $\varphi(\boldsymbol{x}) := \varphi^e(\boldsymbol{x}^e)\bar{\varphi}^e(\bar{\boldsymbol{x}}^e)$, we have that $\varphi \in C_0^\infty(\mathbb{R}^d)$. For $\boldsymbol{k} \le \boldsymbol{r}$ and $\boldsymbol{s} \in \mathbb{N}_0^d$ with $s_i \le k_i$, $i \in e$ and $s_i := 0$ otherwise, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}^{d-|e|}} \bar{\varphi}^e(\bar{\boldsymbol{x}}^e) \int_{\mathbb{R}^{|e|}} g(\boldsymbol{x}^e) D^{\boldsymbol{s}} \varphi^e(\boldsymbol{x}^e) \mathrm{d}\boldsymbol{x}^e \, \mathrm{d}\bar{\boldsymbol{x}}^e \\
&= \int_{\mathbb{R}^{d-|e|}} \bar{\varphi}^e(\bar{\boldsymbol{x}}^e) \int_{\mathbb{R}^{|e|}} D^{\bar{\boldsymbol{k}}^e} f(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e) D^{\boldsymbol{s}} \varphi^e(\boldsymbol{x}^e) \mathrm{d}\boldsymbol{x}^e \, \mathrm{d}\bar{\boldsymbol{x}}^e \\
&= \int_{\mathbb{R}^d} D^{\bar{\boldsymbol{k}}^e} f(\boldsymbol{x}) D^{\boldsymbol{s}} \varphi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = (-1)^{|\boldsymbol{s}|_1} \int_{\mathbb{R}^d} D^{\bar{\boldsymbol{k}}^e + \boldsymbol{s}} f(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^{d-|e|}} \bar{\varphi}^e(\bar{\boldsymbol{x}}^e)(-1)^{|\boldsymbol{s}|_1} \int_{\mathbb{R}^{|e|}} D^{\bar{\boldsymbol{k}}^e + \boldsymbol{s}} f(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e) \varphi^e(\boldsymbol{x}^e) \mathrm{d}\boldsymbol{x}^e \, \mathrm{d}\bar{\boldsymbol{x}}^e.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^{|e|}} g(\boldsymbol{x}^e) D^{\boldsymbol{s}} \varphi^e(\boldsymbol{x}^e) \mathrm{d}\boldsymbol{x}^e = (-1)^{|\boldsymbol{s}|_1} \int_{\mathbb{R}^{|e|}} D^{\bar{\boldsymbol{k}}^e + \boldsymbol{s}} f(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e) \varphi^e(\boldsymbol{x}^e) \mathrm{d}\boldsymbol{x}^e$$

for almost every $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$. This means that the weak derivative $D^{\boldsymbol{s}} g$ exists for almost every $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$ which coincides with $D^{\bar{\boldsymbol{k}}^e + \boldsymbol{s}} f(\cdot, \bar{\boldsymbol{x}}^e)$. Moreover, $D^{\boldsymbol{s}} g \in L^p_w(\mathbb{R}^{|e|})$ for almost every $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$ since by the assumption $D^{\boldsymbol{k}} f \in L^p_w(\mathbb{R}^d)$ for every $\boldsymbol{k} \le \boldsymbol{r}$. ∎

Assume that there exists a sequence of quadratures $(Q_{2^k})_{k \in \mathbb{N}_0}$ with

$$Q_{2^k} f := \sum_{s=1}^{2^k} \lambda_{k,s} f(x_{k,s}), \quad \{x_{k,1}, \dots, x_{k,2^k}\} \subset \mathbb{R}, \tag{3.2}$$

such that

$$\left| \int_{\mathbb{R}} f(x) w(x) \mathrm{d}x - Q_{2^k} f \right| \le C 2^{-ak} \|f\|_{W^r_{1,w}(\mathbb{R})}, \quad k \in \mathbb{N}_0, \quad f \in W^r_{1,w}(\mathbb{R}), \tag{3.3}$$

for some number $a > 0$ and constant $C > 0$.

Based on a sequence $(Q_{2^k})_{k\in\mathbb{N}_0}$ of the form (3.2) satisfying (3.3), we construct quadratures on $\mathbb{R}^d$ by using the well-known Smolyak algorithm. We define for $k \in \mathbb{N}_0$, the one-dimensional operators

$$\Delta_k^Q := Q_{2^k} - Q_{2^{k-1}},\ k > 0, \quad \Delta_0^Q := Q_1,$$

and

$$E_k^Q f := \int_{\mathbb{R}} f(x) w(x) \mathrm{d}x - Q_{2^k} f.$$

For $\boldsymbol{k} \in \mathbb{N}^d$, the d-dimensional operators $Q_{2^{\boldsymbol{k}}}$, $\Delta_{\boldsymbol{k}}^Q$ and $E_{\boldsymbol{k}}^Q$ are defined as the tensor product of one-dimensional operators:

$$Q_{2^{\boldsymbol{k}}} := \bigotimes_{i=1}^d Q_{2^{k_i}}, \quad \Delta_{\boldsymbol{k}}^Q := \bigotimes_{i=1}^d \Delta_{k_i}^Q, \quad E_{\boldsymbol{k}}^Q := \bigotimes_{i=1}^d E_{k_i}^Q, \tag{3.4}$$

where $2^{\boldsymbol{k}} := (2^{k_1}, \cdots, 2^{k_d})$ and the univariate operators $Q_{2^{k_j}}$, $\Delta_{k_j}^Q$ and $E_{k_j}^Q$ are successively applied to the univariate functions $\bigotimes_{i<j} Q_{2^{k_i}}(f)$, $\bigotimes_{i<j} \Delta_{k_i}^Q(f)$ and $\bigotimes_{i<j} E_{k_i}^Q$, respectively, by considering them as functions of variable x_j with the other variables held fixed. The operators $Q_{2^{\boldsymbol{k}}}$, $\Delta_{\boldsymbol{k}}^Q$ and $E_{\boldsymbol{k}}^Q$ are well-defined for continuous functions on $\mathbb{R}^d$, in particular for ones from $W_{1,w}^r(\mathbb{R}^d)$.

Notice that if f is a continuous function on $\mathbb{R}^d$, then $Q_{2^{\boldsymbol{k}}} f$ is a quadrature on $\mathbb{R}^d$ which is given by

$$Q_{2^{\boldsymbol{k}}} f = \sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}}} \lambda_{\boldsymbol{k},\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}}), \quad \{\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}}\}_{\boldsymbol{1}\le\boldsymbol{s}\le 2^{\boldsymbol{k}}} \subset \mathbb{R}^d, \tag{3.5}$$

where

$$\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} := (x_{k_1,s_1}, ..., x_{k_d,s_d}), \quad \lambda_{\boldsymbol{k},\boldsymbol{s}} := \prod_{i=1}^d \lambda_{k_i,s_i},$$

and the summation $\sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}}}$ means that the sum is taken over all $\boldsymbol{s}$ such that $\boldsymbol{1} \le \boldsymbol{s} \le 2^{\boldsymbol{k}}$. Hence we derive that

$$\Delta_{\boldsymbol{k}}^Q f = \sum_{e\subset\{1,\dots,d\}} (-1)^{d-|e|} Q_{2^{\boldsymbol{k}(e)}} f = \sum_{e\subset\{1,\dots,d\}} (-1)^{d-|e|} \sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}(e)}} \lambda_{\boldsymbol{k}(e),\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}), \tag{3.6}$$

where $\boldsymbol{k}(e) \in \mathbb{N}_0^d$ is defined by $k(e)_i = k_i$, $i \in e$, and $k(e)_i = \max(k_i - 1, 0)$, $i \notin e$. We also have

$$E_{\boldsymbol{k}}^Q f = \sum_{e\subset\{1,\dots,d\}} (-1)^{|e|} \int_{\mathbb{R}^{d-|e|}} Q_{2^{\boldsymbol{k}^e}} f(\cdot, \bar{\boldsymbol{x}}^e) w(\bar{\boldsymbol{x}}^e) \mathrm{d}\bar{\boldsymbol{x}}^e, \tag{3.7}$$

where $w(\bar{\boldsymbol{x}}^e) := \prod_{j\notin e} w(x_j)$.

Notice that as mappings from $C(\mathbb{R}^d)$ to $\mathbb{R}$, the operators $Q_{2^{\boldsymbol{k}}}$, $\Delta_{\boldsymbol{k}}^Q$ and $E_{\boldsymbol{k}}^Q$ possess commutative and associative properties with respect to applying the component operators $Q_{2^{k_j}}$, $\Delta_{k_j}^Q$ and $E_{k_j}^Q$ in the following sense. We have for any $e \subset \{1, ..., d\}$,

$$Q_{2^{\boldsymbol{k}}} f = Q_{2^{\boldsymbol{k}^e}} \left(Q_{2^{\bar{\boldsymbol{k}}^e}} f \right), \quad \Delta_{\boldsymbol{k}}^Q f = \Delta_{\boldsymbol{k}^e}^Q \left(\Delta_{\bar{\boldsymbol{k}}^e}^Q f \right), \quad E_{\boldsymbol{k}}^Q f = E_{\boldsymbol{k}^e}^Q \left(E_{\bar{\boldsymbol{k}}^e}^Q f \right),$$

and for any reordered sequence $\{i(1), ..., i(d)\}$ of $\{1, ..., d\}$,

$$Q_{2^{\boldsymbol{k}}} = \bigotimes_{j=1}^d Q_{2^{k_{i(j)}}}, \quad \Delta_{\boldsymbol{k}}^Q = \bigotimes_{j=1}^d \Delta_{k_{i(j)}}^Q, \quad E_{\boldsymbol{k}}^Q = \bigotimes_{j=1}^d E_{k_{i(j)}}^Q. \tag{3.8}$$

These properties directly follow from (3.5)–(3.7).

Lemma 3.3 *Under the assumption* (3.2)–(3.3)*, we have*

$$\left| E_{\boldsymbol{k}}^Q f \right| \le C 2^{-a|\boldsymbol{k}|_1} \|f\|_{W_{1,w}^r(\mathbb{R}^d)}, \quad \boldsymbol{k} \in \mathbb{N}_0^d, \quad f \in W_{1,w}^r(\mathbb{R}^d).$$

Proof. The case $d = 1$ of the lemma is as in (3.3) by the assumption. For simplicity we prove the lemma for the case $d = 2$. The general case can be proven in the same way by induction on d. We make use of the temporary notation:

$$\|f\|_{W_{1,w}^r(\mathbb{R}),2}(x_1) := \|f(x_1, \cdot)\|_{W_{1,w}^r(\mathbb{R})}.$$

From Lemmata 3.1 and 3.2 it follows that $f(\cdot, x_2) \in W_{1,w}^r(\mathbb{R})$ for every $x_2 \in \mathbb{R}$. Notice that $E_{k_2}^Q f$ is a function in the variable x_1 only. Hence, by (3.3) we obtain that

$$\begin{aligned} \left| E_{(k_1,k_2)}^Q f \right| &= \left| E_{k_1}^Q (E_{k_2}^Q f) \right| \le C 2^{-a k_1} \|E_{k_2}^Q f\|_{W_{1,w}^r(\mathbb{R})} \\ &\le C 2^{-a k_1} \|2^{-a k_2} \|f\|_{W_{1,w}^r(\mathbb{R}),2}(\cdot)\|_{W_{1,w}^r(\mathbb{R})} = C 2^{-a|\boldsymbol{k}|_1} \|f\|_{W_{1,w}^r(\mathbb{R}^2)}. \end{aligned}$$

□

We say that $\boldsymbol{k} \to \infty$, $\boldsymbol{k} \in \mathbb{N}_0^d$, if and only if $k_i \to \infty$ for every $i = 1, ..., d$.

Lemma 3.4 *Under the assumption* (3.2)–(3.3)*, we have that for every* $f \in W_{1,w}^r(\mathbb{R}^d)$*,*

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} \Delta_{\boldsymbol{k}}^Q f \tag{3.9}$$

with absolute convergence of the series, and

$$\left| \Delta_{\boldsymbol{k}}^Q f \right| \le C 2^{-a|\boldsymbol{k}|_1} \|f\|_{W_{1,w}^r(\mathbb{R}^d)}, \quad \boldsymbol{k} \in \mathbb{N}_0^d. \tag{3.10}$$

Proof. The operator $\Delta_{\boldsymbol{k}}^Q$ can be represented in the form

$$\Delta_{\boldsymbol{k}}^Q = \sum_{e \subset \{1,...,d\}} (-1)^{|e|} E_{\boldsymbol{k}(e)}^Q.$$

Therefore, by using Lemma 3.3 we derive that for every $f \in W^r_{1,w}(\mathbb{R}^d)$ and $\boldsymbol{k} \in \mathbb{N}^d_0$,

$$\begin{aligned}\left|\Delta^Q_{\boldsymbol{k}} f\right| &\le \sum_{e \subset \{1,\dots,d\}} \left|E^Q_{\boldsymbol{k}(e)} f\right| \\ &\le \sum_{e \subset \{1,\dots,d\}} C 2^{-a|\boldsymbol{k}(e)|_1} \|f\|_{W^r_{1,w}(\mathbb{R}^d)} \le C 2^{-a|\boldsymbol{k}|_1} \|f\|_{W^r_{1,w}(\mathbb{R}^d)}\end{aligned}$$

which proves (3.10) and hence the absolute convergence of the series in (3.9) follows. Notice that

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_{2^{\boldsymbol{k}}} f = \sum_{e \subset \{1,\dots,d\},\ e \ne \varnothing} (-1)^{|e|} E^Q_{\boldsymbol{k}^e} f,$$

where recall $\boldsymbol{k}^e \in \mathbb{N}^d_0$ is defined by $k^e_i = k_i$, $i \in e$, and $k^e_i = 0$, $i \notin e$. By using Lemma 3.3 we derive for $\boldsymbol{k} \in \mathbb{N}^d_0$ and $f \in W^r_{1,w}(\mathbb{R}^d)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_{2^{\boldsymbol{k}}} f\right| &\le \sum_{e \subset \{1,\dots,d\},\ e \ne \varnothing} \left|E^Q_{\boldsymbol{k}^e} f\right| \\ &\le C \max_{e \subset \{1,\dots,d\},\ e \ne \varnothing} \max_{1 \le i \le d} 2^{-a|k^e_i|} \|f\|_{W^r_{1,w}(\mathbb{R}^d)} \\ &\le C \max_{1 \le i \le d} 2^{-a|k_i|} \|f\|_{W^r_{1,w}(\mathbb{R}^d)},\end{aligned}$$

which is going to 0 when $\boldsymbol{k} \to \infty$. This together with the obvious equality

$$Q_{2^{\boldsymbol{k}}} = \sum_{\boldsymbol{s} \le \boldsymbol{k}} \Delta^Q_{\boldsymbol{s}}$$

proves (3.9). $\square$

We now define an algorithm for quadrature on sparse grids adopted from the alogorithm for sampling recovery initiated by Smolyak (for detail see [2, Sections 4.2 and 5.3]). For $\xi > 0$, we define the operator

$$Q_\xi := \sum_{|\boldsymbol{k}|_1 \le \xi} \Delta^Q_{\boldsymbol{k}}.$$

From (3.6) we can see that Q_ξ is a quadrature on $\mathbb{R}^d$ of the form (1.6):

$$Q_\xi f = \sum_{|\boldsymbol{k}|_1 \le \xi} \sum_{e \subset \{1,\dots,d\}} (-1)^{d-|e|} \sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}(e)}} \lambda_{\boldsymbol{k}(e),\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}) = \sum_{(\boldsymbol{k},e,\boldsymbol{s}) \in G(\xi)} \lambda_{\boldsymbol{k},e,\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}), \tag{3.11}$$

where

$$\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}} := \boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}, \quad \lambda_{\boldsymbol{k},e,\boldsymbol{s}} := (-1)^{d-|e|} \lambda_{\boldsymbol{k}(e),\boldsymbol{s}}$$

and

$$G(\xi) := \{(\boldsymbol{k}, e, \boldsymbol{s}) : \ |\boldsymbol{k}|_1 \le \xi,\, e \subset \{1,\dots,d\}\,,\, \boldsymbol{1} \le \boldsymbol{s} \le \boldsymbol{k}(e)\}$$

is a finite set. The set of integration nodes in this quadrature

$$H(\xi) := \{\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}\}_{(\boldsymbol{k},e,\boldsymbol{s}) \in G(\xi)}$$

is a step hyperbolic cross in the function domain $\mathbb{R}^d$. The number of integration nodes in the quadrature Q_ξ is

$$|G(\xi)| = \sum_{|\boldsymbol{k}|_1 \le \xi} \sum_{e \subset \{1,\dots,d\}} 2^{|\boldsymbol{k}(e)|_1}$$

which can be estimated as

$$|G(\xi)| \asymp \sum_{|\boldsymbol{k}|_1 \le \xi} 2^{|\boldsymbol{k}|_1} \asymp 2^\xi \xi^{d-1}, \quad \xi \ge 1. \tag{3.12}$$

As commented in Introduction, this quadrature plays a crucial role in the proof of the upper bound in the main results of the present paper (1.8).

Lemma 3.5 *Under the assumption* (3.2)–(3.3), *we have that*

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_\xi f \right| \le C 2^{-a\xi} \xi^{d-1} \|f\|_{W^r_{1,w}(\mathbb{R}^d)}, \quad \xi \ge 1, \quad f \in W^r_{1,w}(\mathbb{R}^d). \tag{3.13}$$

Proof. From Lemma 3.4 we derive that for $\xi \ge 1$ and $f \in W^r_{1,w}(\mathbb{R}^d)$,

$$\begin{aligned} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_\xi f \right| &\le \sum_{|\boldsymbol{k}|_1 > \xi} \left| \Delta^Q_{\boldsymbol{k}} f \right| \le C \sum_{|\boldsymbol{k}|_1 > \xi} 2^{-a|\boldsymbol{k}|_1} \|f\|_{W^r_{1,w}(\mathbb{R}^d)} \\ &\le C \|f\|_{W^r_{1,w}(\mathbb{R}^d)} \sum_{|\boldsymbol{k}|_1 > \xi} 2^{-a|\boldsymbol{k}|_1} \le C 2^{-a\xi} \xi^{d-1} \|f\|_{W^r_{1,w}(\mathbb{R}^d)}. \end{aligned}$$

□

Remark 3.6 From Theorem 2.2 we can see that the truncated Gaussian quadratures $Q^{\mathrm{TG}}_{2j(m)}$ form a sequence $(Q_{2^k})_{k \in \mathbb{N}_0}$ of the form (3.2) satisfying (3.3) with $a = r_\lambda$.

Remark 3.7 The technique for proving the upper bound (3.13) is analogous to a general technique for establishing upper bounds of the error of unweighted sampling recovery by Smolyak algorithms of functions having mixed smoothness on a bounded domain (see, e.g., [2, Section 5.3] and [13, Section 6.9] for detail).

Theorem 3.8 *We have that*

$$n^{-r_\lambda} (\log n)^{r_\lambda (d-1)} \ll \mathrm{Int}_n(\boldsymbol{W}^r_{1,w}(\mathbb{R}^d)) \ll n^{-r_\lambda} (\log n)^{(r_\lambda + 1)(d-1)}. \tag{3.14}$$

Proof. Let us first prove the upper bound in (4.2). We will construct a quadrature of the form (3.11) which realizes it. In order to do this, we take the truncated Gaussian quadrature $Q^{\mathrm{TG}}_{2j(m)} f$ defined in (2.4). For every $k \in \mathbb{N}_0$, let m_k be the largest number such that $2j(m_k) \le 2^k$. Then we have $2j(m_k) \asymp 2^k$. For the sequence of quadratures $(Q_{2^k})_{k \in \mathbb{N}_0}$ with

$$Q_{2^k} := Q^{\mathrm{TG}}_{2j(m_k)} \in \mathcal{Q}_{2^k},$$

from Theorem 2.2 it follows that

$$\left| \int_{\mathbb{R}} f(x)w(x)\mathrm{d}x - Q_{2^k} f \right| \le C 2^{-r_\lambda k} \|f\|_{W^r_{1,w}(\mathbb{R})}, \quad k \in \mathbb{N}_0, \quad f \in W^r_{1,w}(\mathbb{R}),$$

This means that the assumption (3.2)–(3.3) holds for $a = r_\lambda$. To prove the upper bound in (4.2) we approximate the integral

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

by the quadrature Q_ξ which is formed from the sequence $(Q_{2^k})_{k \in \mathbb{N}_0}$. For every $n \in \mathbb{N}$, let ξ_n be the largest number such that $|G(\xi_n)| \le n$. Then the corresponding operator Q_{ξ_n} defines a quadrature belonging to $\mathcal{Q}_n$. From (3.12) it follows

$$2^{\xi_n} \xi_n^{d-1} \asymp |G(\xi_n)| \asymp n.$$

Hence we deduce the asymptotic equivalences

$$2^{-\xi_n} \asymp n^{-1} (\log n)^{d-1}, \quad \xi_n \asymp \log n,$$

which together with Lemma 3.5 yield that

$$\begin{aligned} \mathrm{Int}_n(\boldsymbol{W}^r_{1,w}(\mathbb{R}^d)) &\le \sup_{f \in \boldsymbol{W}^r_{1,w}(\mathbb{R}^d)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_{\xi_n} f \right| \\ &\le C 2^{-r_\lambda \xi_n} \xi_n^{d-1} \asymp n^{-r_\lambda} (\log n)^{(r_\lambda + 1)(d-1)}. \end{aligned}$$

The upper bound in (4.2) is proven.

We now prove the lower bound in (4.2) by using the inequality (2.1) in Lemma 2.1. For $M \ge 1$, we define the set

$$\Gamma_d(M) := \left\{ \boldsymbol{s} \in \mathbb{N}^d : \prod_{i=1}^d s_i \le 2M, \ s_i \ge M^{1/d}, \ i = 1, ..., d \right\}.$$

Then we have

$$|\Gamma_d(M)| \asymp M (\log M)^{d-1}, \quad M > 1. \tag{3.15}$$

Indeed, we have the inclusion

$$\Gamma_d(M) \subset \Gamma'_d(M) := \left\{ \boldsymbol{s} \in \mathbb{N}^d : \prod_{i=1}^d s_i \le 2M \right\}$$

and

$$|\Gamma'_d(M)| \asymp M (\log M)^{d-1}.$$

Hence, $|\Gamma_d(M)| \ll M(\log M)^{d-1}$. We prove the converse inequality $|\Gamma_d(M)| \gg M(\log M)^{d-1}$ by induction on the dimension d. It is obvious for $d = 1$. Assuming that this inequality is true for $d-1$, we check it for d, $(d \geq 2)$. Fix a positive number τ with $1 < \tau < d$. We have by induction assumption,

$$
\begin{aligned}
|\Gamma_d(M)| &= \sum_{M^{1/d} \leq s_d \leq 2M} |\Gamma_{d-1}(2Ms_d^{-1})| \gg \sum_{M^{1/d} \leq s_d \leq 2M} \left(2Ms_d^{-1}\right) \left(\log 2Ms_d^{-1}\right)^{d-2} \\
&\gg M \sum_{M^{1/d} \leq s_d \leq 2M^{\tau/d}} s_d^{-1} \left(\log 2Ms_d^{-1}\right)^{d-2} \\
&\geq M \sum_{M^{1/d} \leq s_d \leq 2M^{\tau/d}} s_d^{-1} \left(\log 2M^{1-\tau/d}\right)^{d-2} \\
&\gg M \left(\log M\right)^{d-2} \sum_{M^{1/d} \leq s_d \leq 2M^{\tau/d}} s_d^{-1} \gg M \left(\log M\right)^{d-1}.
\end{aligned}
$$

The asymptotic equivalence (3.15) is proven.

For a given $n \in \mathbb{N}$, let $\{\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n\} \subset \mathbb{R}^d$ be arbitrary n points. Denote by M_n the smallest number such that $|\Gamma_d(M_n)| \geq n+1$. We define the d-parallelepiped $K_{\boldsymbol{s}}$ for $\boldsymbol{s} \in \mathbb{N}_0^d$ of size

$$
\delta := M_n^{\frac{1/\lambda - 1}{d}}
$$

by

$$
K_{\boldsymbol{s}} := \prod_{i=1}^d K_{s_i}, \quad K_{s_i} := (\delta s_i, \delta s_{i-1}).
$$

Since $|\Gamma_d(M_n)| > n$, there exists a multi-index $\boldsymbol{s} \in \Gamma_d(M_n)$ such that $K_{\boldsymbol{s}}$ does not contain any point from $\{\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n\}$.

As in the proof of Theorem 2.2, we take a nonnegative function $\varphi \in C_0^\infty([0,1])$, $\varphi \neq 0$, and put

$$
b_0 := \int_0^1 \varphi(y) \mathrm{d}y > 0, \quad b_s := \int_0^1 |\varphi^{(s)}(y)| \mathrm{d}y, \ s = 1, ..., r. \tag{3.16}
$$

For $i = 1, ..., d$, we define the univariate functions g_i in variable x_i by

$$
g_i(x_i) := \begin{cases} \varphi(\delta^{-1}(x_i - \delta s_{i-1})), & x_i \in K_{s_i}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.17}
$$

Then the mulitivariate functions g and h on $\mathbb{R}^d$ are defined by

$$
g(\boldsymbol{x}) := \prod_{i=1}^d g_i(x_i),
$$

and

$$
h(\boldsymbol{x}) := (gw^{-1})(\boldsymbol{x}) = \prod_{i=1}^d g_i(x_i) w^{-1}(x_i) =: \prod_{i=1}^d h_i(x_i). \tag{3.18}
$$

Let us estimate the norm $\|h\|_{W^r_{1,w}(\mathbb{R}^d)}$. For every $\boldsymbol{k} \in \mathbb{N}_0^d$ with $0 \leq |\boldsymbol{k}|_\infty \leq r$, we prove the inequality

$$
\int_{\mathbb{R}^d} \left|(D^{\boldsymbol{k}} h) w\right|(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \leq C M_n^{(1-1/\lambda)(r-1)}. \tag{3.19}
$$

We have

$$D^{\boldsymbol{k}}h = \prod_{i=1}^{d} h_i^{(k_i)}. \tag{3.20}$$

Similarly to (2.10)–(2.13) we derive that for every $i = 1, ..., d$,

$$h_i^{(k_i)}(x_i)w(x_i) = \sum_{\nu_i=0}^{k_i} \binom{k_i}{\nu_i} g_i^{(k_i-\nu_i)}(x_i)(\operatorname{sign}(x_i))^{\nu_i} \sum_{\eta_i=1}^{\nu_i} c_{\nu_i,\eta_i}(\lambda, a)|x_i|^{\lambda_{\nu_i,\eta_i}},$$

where

$$\lambda_{\nu_i,\nu_i} = \nu_i(\lambda - 1) > \lambda_{\nu_i,\nu_i-1} > \cdots > \lambda_{\nu_i,1} = \lambda - \nu_i,$$

and $c_{\nu_i,\eta_i}(\lambda, a)$ are polynomials in the variables λ and a of degree at most ν_i with respect to each variable. This together with (3.16)–(3.17) and the inequalities $s_i \geq M_n^{\frac{1}{d}}$ and $\lambda_{\nu_i,\nu_i} = \nu_i(\lambda - 1) \geq 0$ yields that

$$\begin{aligned}
\int_{\mathbb{R}} \left|h_i^{(k_i)}(x_i)w(x_i)\right| \mathrm{d}x_i &\leq C \max_{0\leq\nu_i\leq k_i} \max_{1\leq\eta_i\leq\nu_i} \int_{K_{s_i}} |x_i|^{\lambda_{\nu_i,\eta_i}} \left|g^{(k_i-\nu_i)}(x_i)\right| \mathrm{d}x_i \\
&\leq C \max_{0\leq\nu_i\leq k_i} (\delta s_i)^{\lambda_{\nu_i,\nu_i}} \int_{K_{s_i}} \left|g^{(k_i-\nu_i)}(x_i)\right| \mathrm{d}x_i \\
&\leq C \max_{0\leq\nu_i\leq k_i} (\delta s_i)^{\nu_i(\lambda-1)} \delta^{-k_i+\nu_i+1} b_{k_i-\nu_i} \\
&= C\delta^{-k_i+1} \max_{0\leq\nu_i\leq k_i} \left(\delta^\lambda s_i^{\lambda-1}\right)^{\nu_i}.
\end{aligned} \tag{3.21}$$

Since $s_i \geq M_n^{\frac{1}{d}}$ and $\delta := M_n^{\frac{1/\lambda-1}{d}}$, we have that $\delta^\lambda s_i^{\lambda-1} \geq 1$, and consequently,

$$\max_{0\leq\nu_i\leq k_i} \left(\delta^\lambda s_i^{\lambda-1}\right)^{\nu_i} = \left(\delta^\lambda s_i^{\lambda-1}\right)^{k_i}.$$

This equality, the estimates (3.21) and the inequalities $0 \leq k_i \leq r$ and $\delta s_i \geq 1$ yield that

$$\begin{aligned}
\int_{\mathbb{R}} \left|h_i^{(k_i)}(x_i)w(x_i)\right| \mathrm{d}x_i &\leq C\delta^{-k_i+1} \left(\delta^\lambda s_i^{\lambda-1}\right)^{k_i} = C\delta \left(\delta s_i\right)^{k_i(\lambda-1)} \\
&\leq C\delta \left(\delta s_i\right)^{r(\lambda-1)} = C\delta^{r(\lambda-1)+1} s_i^{r(\lambda-1)}
\end{aligned}$$

Hence, by (3.20) we deduce

$$\begin{aligned}
\int_{\mathbb{R}^d} |(D^{\boldsymbol{k}}h)w|(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \prod_{i=1}^{d} \int_{\mathbb{R}} \left|h^{(k_i)}(x_i)w(x_i)\right| \mathrm{d}x_i \\
&\leq C \prod_{i=1}^{d} \delta^{r(\lambda-1)+1} s_i^{r(\lambda-1)} \leq C\delta^{d(r(\lambda-1)+1)} \left(\prod_{i=1}^{d} s_i\right)^{r(\lambda-1)}.
\end{aligned}$$

Since $\prod_{i=1}^d s_i \leq 2M_n$, $\delta := M_n^{\frac{1/\lambda-1}{d}}$ and $\lambda > 1$, we can continue the estimation as

$$\int_{\mathbb{R}^d} |(D^{\boldsymbol{k}}h)w|(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \leq CM_n^{(r(\lambda-1)+1)(1/\lambda-1)} M_n^{r(\lambda-1)} = CM_n^{(1-1/\lambda)(r-1)},$$

which completes the proof of the inequality (3.19). This inequality means that $h \in W^r_{1,w}(\mathbb{R}^d)$ and

$$\|h\|_{W^r_{1,w}(\mathbb{R}^d)} \le C M_n^{(1-1/\lambda)(r-1)}.$$

If we define

$$\bar{h} := C^{-1} M_n^{-(1-1/\lambda)(r-1)} h,$$

then $\bar{h}$ is nonnegative, $\bar{h} \in \boldsymbol{W}^r_{1,w}(\mathbb{R})$, $\sup \bar{h} \subset K_{\boldsymbol{s}}$ and by (3.16)–(3.18),

$$\begin{aligned}\int_{\mathbb{R}^d} (\bar{h}w)(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} &= C^{-1} M_n^{-(1-1/\lambda)(r-1)} \int_{\mathbb{R}^d} (hw)(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \prod_{i=1}^d \int_{K_{s_i}} g_i(x_i) \mathrm{d}x_i \\ &= C^{-1} M_n^{-(1-1/\lambda)(r-1)} (b_0\delta)^d = C' M_n^{-r_\lambda}.\end{aligned}$$

From the definition of M_n and (3.15) it follows that

$$M_n (\log M_n)^{d-1} \asymp |\Gamma(M_n)| \asymp n,$$

which implies that $M_n^{-1} \asymp n^{-1} (\log n)^{d-1}$. This allows to receive the estimate

$$\int_{\mathbb{R}^d} (\bar{h}w)(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = C' M_n^{-r_\lambda} \gg n^{-r_\lambda} (\log n)^{r_\lambda (d-1)}. \tag{3.22}$$

Since the interval $K_{\boldsymbol{s}}$ does not contain any point from the set $\{\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n\}$ which has been arbitrarily choosen, we have

$$\bar{h}(\boldsymbol{\xi}_k) = 0, \quad k = 1, ..., n.$$

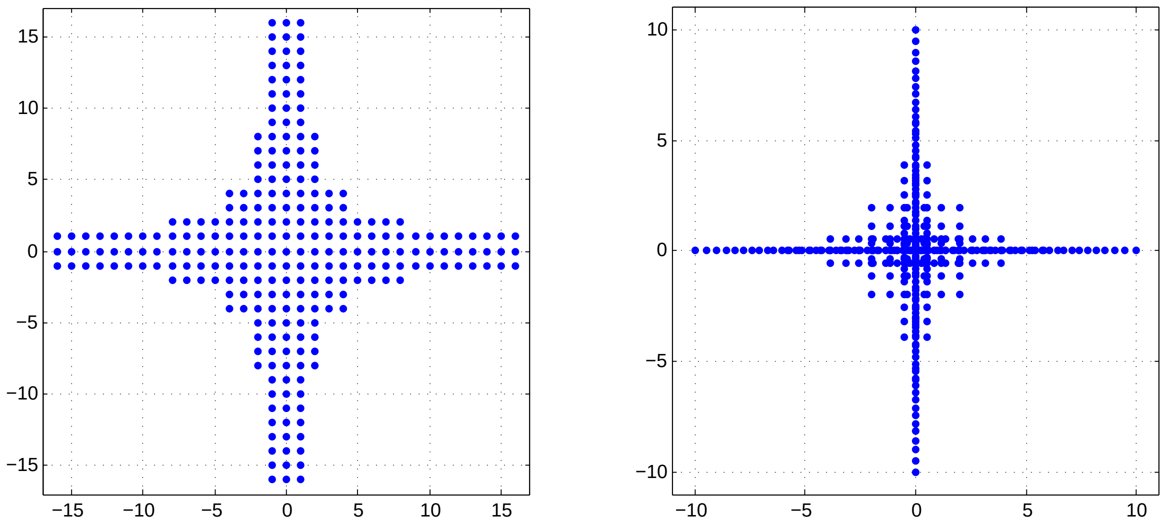
Hence, by Lemma 2.1 and (3.22) we have that

$$\mathrm{Int}_n(\boldsymbol{W}^r_{1,w}(\mathbb{R}^d)) \ge \int_{\mathbb{R}^d} \bar{h}(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \gg n^{-r_\lambda} (\log n)^{r_\lambda (d-1)}.$$

The lower bound in (4.2) is proven. $\square$

Remark 3.9 Let us analyse some properties of the quadratures Q_ξ and their integration nodes $H(\xi)$ which give the upper bound in (4.2).

1. The set of integration nodes $H(\xi)$ in the quadratures Q_ξ which are formed from the non-equidistant zeros of the orthonormal polynomials $p_m(w)$, is a step hyperbolic cross *on the function domain* $\mathbb{R}^d$. This is a contrast to the classical theory of approximation of multivariate periodic functions having mixed smoothness for which the classical step hyperbolic crosses of integer points are *on the frequency domain* $\mathbb{Z}^d$ (see, e.g., [2, Section 2.3] for detail). The terminology 'step hyperbolic cross' of intergration nodes is borrowed from there. In Figure 1, in particular, the step hyperbolic cross in the right picture is designed for the Hermite weight $w(\boldsymbol{x}) = \exp(-x_1^2 - x_2^2)$ $(d = 2)$. The set $H(\xi)$ also completely differs from the classical Smolyak grids of fractional dyadic points *on the function domain* $[-1,1]^d$ (see Figure 2 for $d = 2$) which are used in sparse-grid sampling recovery and numerical integration for functions having a mixed smoothness (see, e.g., [2, Section 5.3]



A classical step hyperbolic cross   A Hermite step hyperbolic cross

Figure 1: Step hyperbolic crosses ($d = 2$)

for detail).

2. The set $H(\xi)$ is very sparsely distributed inside the d-cube

$$K(\xi) := \left\{ \boldsymbol{x} \in \mathbb{R}^d : \, |x_i| \le C 2^{\xi/\lambda}, \; i = 1, ..., d \right\},$$

for some constant $C > 0$. Its diameter which is the length of its symmetry axes is $2C2^{\xi/\lambda}$, i.e., the size of $K(\xi)$. The number of integration nodes in $H(\xi)$ is $|G(\xi)| \asymp 2^\xi \xi^{d-1}$. For the integration nodes $H(\xi) = \{\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}\}_{(\boldsymbol{k},e,\boldsymbol{s}) \in G(\xi)}$, we have that

$$\min_{\substack{(\boldsymbol{k},e,\boldsymbol{s}),(\boldsymbol{k}',e',\boldsymbol{s}') \in G(\xi) \\ (\boldsymbol{k},e,\boldsymbol{s}) \ne (\boldsymbol{k}',e',\boldsymbol{s}')}} \; \min_{1 \le i \le d} \left| \left(x_{\boldsymbol{k},e,\boldsymbol{s}}\right)_i - \left(x_{\boldsymbol{k}',e',\boldsymbol{s}'}\right)_i \right| \; \asymp \; 2^{-(1-1/\lambda)\xi} \; \to 0, \text{ when } \xi \to \infty.$$

On the other hand, the diameter of $H(\xi)$ is going to ∞ when $\xi \to \infty$.

4 Extension to Markov-Sonin weights

In this section, we extend the results of the previous sections to Markov-Sonin weights. A univariate Markov-Sonin weight is a function of the form

$$w_\beta(x) := |x|^\beta \exp(-a|x|^2 + b), \quad \beta > 0, \quad a > 0, \quad b \in \mathbb{R},$$

(here β is indicated in the notation to distinguish Markov-Sonin weights w_β and Freud-type weight w). A d-dimensional Markov-Sonin weight is defined as

$$w_\beta(\boldsymbol{x}) := \prod_{i=1}^d w_\beta(x_i).$$

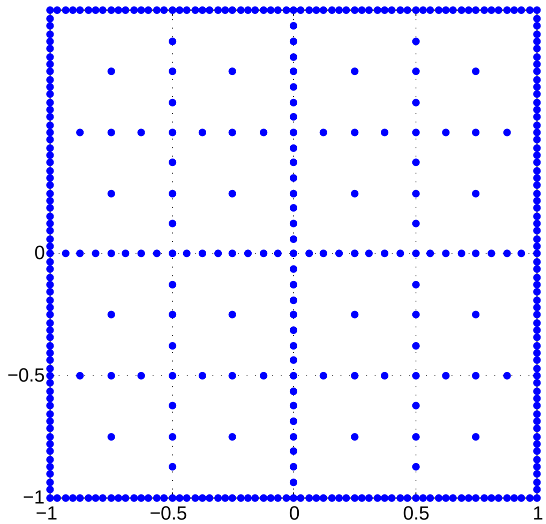


Figure 2: A Smolyak grid ($d = 2$)

Markov-Sonin weights are not of the form (1.2) and have a singularity at 0. We will keep all the notations and definitions in Sections 1–3 with replacing w by w_β, pointing some modifications.

Denote $\mathring{\mathbb{R}}^d := (\mathbb{R} \setminus \{0\})^d$ and $\mathring{\Omega} := \Omega \cap \mathring{\mathbb{R}}^d$. Besides the spaces $L^p_{w_\beta}(\Omega)$ and $W^r_{p,w_\beta}(\Omega)$ we consider also the spaces $L^p_{w_\beta}(\mathring{\Omega})$ and $W^r_{p,w_\beta}(\mathring{\Omega})$ which are defined in a similar manner. For the space $W^r_{p,w_\beta}(\mathring{\Omega})$, we require one of the following restrictions on r and β to be satisfied:

(i) $\beta > r - 1$;

(ii) $0 < \beta < r - 1$ and β is not an integer, for $f \in W^r_{p,w_\beta}(\mathring{\Omega})$, the derivative $D^{\boldsymbol{k}} f$ can be extended to a continuous function on Ω for all $\boldsymbol{k} \in \mathbb{N}_0^d$ such that $|\boldsymbol{k}|_\infty \le r - 1 - \lceil \beta \rceil$.

Let $(p_m(w_\beta))_{m \in \mathbb{N}}$ be the sequence of orthonormal polynomials with respest to the weight w_β. Denote again by $x_{m,k}$, $1 \le k \le \lfloor m/2 \rfloor$ the positive zeros of $p_m(w_\beta)$, and by $x_{m,-k} = -x_{m,k}$ the negative ones (if m is odd, then $x_{m,0} = 0$ is also a zero of $p_m(w_\beta)$). If m is even, we add $x_{m,0} := 0$. These nodes are located as

$$-\sqrt{m}+Cm^{-1/6} < x_{m,-\lfloor m/2 \rfloor} < \cdots < x_{m,-1} < x_{m,0} < x_{m,1} < \cdots < x_{m,\lfloor m/2 \rfloor} \le \sqrt{m}-Cm^{-1/6},$$

with a positive constant C independent of m (the Mhaskar-Rakhmanov-Saff number is $a_m(w_\beta) = \sqrt{m}$).

In the case (i), the truncated Gaussian quadrature is defined by

$$Q^{\mathrm{TG}}_{2j(m)} f := \sum_{1 \le |k| \le j(m)} \lambda_{m,k}(w_\beta) f(x_{m,k}),$$

and in the case (ii) by

$$Q^{\mathrm{TG}}_{2j(m)} f := \sum_{0 \le |k| \le j(m)} \lambda_{m,k}(w_\beta) f(x_{m,k}),$$

where $\lambda_{m,k}(w_\beta)$ are the corresponding Cotes numbers.

In the same ways, by using related results in [10] we can prove the following counterparts of Theorems 2.2 and 3.8 for the unit ball $\boldsymbol{W}^r_{1,w_\beta}(\mathring{\mathbb{R}}^d)$ of the Markov-Sonin weighted Sobolev space $W^r_{1,w_\beta}(\mathring{\mathbb{R}}^d)$ of mixed smoothness $r \in \mathbb{N}$.

Theorem 4.1 *For any $n \in \mathbb{N}$, let m_n be the largest integer such that $2j(m_n) \le n$. Then the quadratures $Q^{\mathrm{TG}}_{2j(m_n)} \in \mathcal{Q}_n$, $n \in \mathbb{N}$, are asymptotically optimal for $\boldsymbol{W}^r_{1,w_\beta}(\mathring{\mathbb{R}})$ and*

$$\sup_{f \in \boldsymbol{W}^r_{1,w_\beta}(\mathring{\mathbb{R}})} \left| \int_{\mathbb{R}} f(x)w(x)\mathrm{d}x - Q^{\mathrm{TG}}_{2j(m_n)} f \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}^r_{1,w_\beta}(\mathring{\mathbb{R}})\big) \asymp n^{-r/2}.$$

Theorem 4.2 *We have that*

$$n^{-r/2}(\log n)^{(d-1)r/2} \ll \mathrm{Int}_n \boldsymbol{W}^r_{1,w_\beta}(\mathring{\mathbb{R}}^d)) \ll n^{-r/2}(\log n)^{(d-1)(r/2+1)}.$$

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