Periods & billiards on the triaxial ellipsoid

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Billiards on the triaxial ellipsoid

To an ellipsoid with 3 different axes

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– elliptic and abelian period spaces - Schneider's 3^{rd} problem revisited
The questions lead to study linear independence of elliptic and abelian integrals of first, second and third kind.

Confocal family of ellipsoids

Considern an ellipsoid $E_{a,b,c}$ in $\mathbb{R}^3$ with three different axes $a > b > c > 0$ given by

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and its confocal deformations

$$E_{a,b,c}(\lambda) : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1$$

for real $\lambda \neq a, b, c$.

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The family is non-trivial only when $-\infty \leq \lambda \leq a$. For $\lambda \in (-\infty, c)$ we get ellipsoids, for $\lambda \in (c, b)$ 1-sheeted hyperboloids and 2-sheeted hyperboloids for $\lambda \in (b, a)$.

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$$(a-\lambda)x^2 + (b-\lambda)y^2 + (c-\lambda)z^2 = 1. \tag{1}$$

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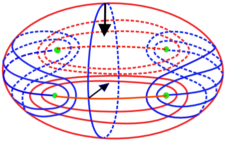

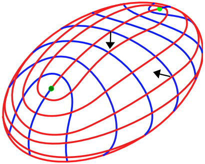

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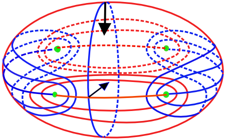

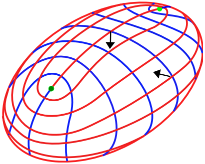

The **principal curvature lines** are the base loci of the family (1).

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The construction via contact spheres leads to two vector fields which, when integrated leads to principal curvature lines.

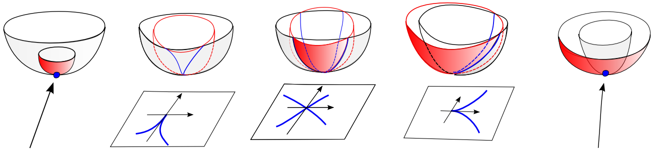

Figure : Tangent spheres and contact type. A center for small and big values of r; saddle for intermediate values of r and cuspidal or more degenerate contact for $r = 1/k_1$ and $r = 1/k_2$.

curvature lines

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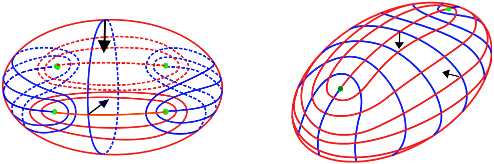

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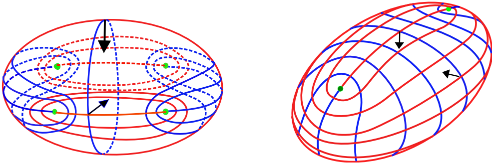

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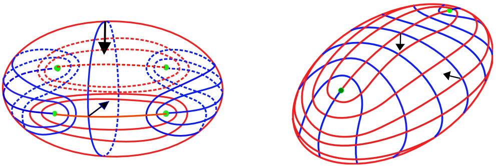

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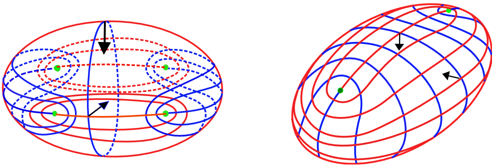

There are 2 further elliptic curves attached to the ellipsoid:
– the 2-fold covering of the ellipsoid ramified in the umbillics
– the characteristic curve The latter is the elliptic curve curve given by

$$\mu^2 = \chi(\lambda)$$

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Theorem (R. Garcia, G.W.)

For $\lambda \neq a, b, c$ the curve $C(\lambda)$ is isomorphic over $\mathbb{Q}(a, b, c; \lambda, \mu)$ to the curve with equation

$$y^2 = x(x-1)(x-\delta)$$

where μ satisfies $\lambda^{12} - (abc(b-a)\chi(\lambda))^3\mu^2 = 0$ and $\delta = \frac{a-c}{a-b}$ is the cross ratio.

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The proof makes among others use of the Edwards form of an elliptic curve

$$X^2 + y^2 = a^2 + a^2x^2y^2$$

by which we transform the space curve into a plane elliptic curve.

α-curvature lines

Let X_1, X_2 be the principal vector fields defined by the tangent vectors of the principal curvature lines. For $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we introduce transversal foliations $\mathcal{F}_{\pm\alpha}$ given by the integral curves of the vector field $X_\alpha := \cos\alpha\, X_1 + \sin\alpha\, X_2$.

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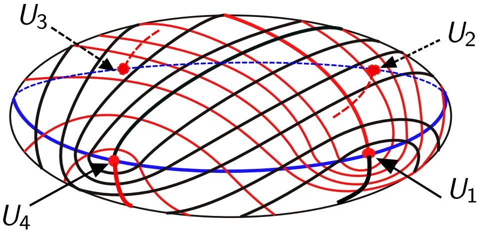

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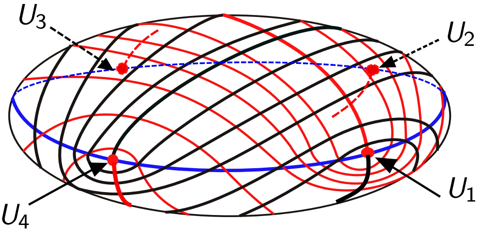
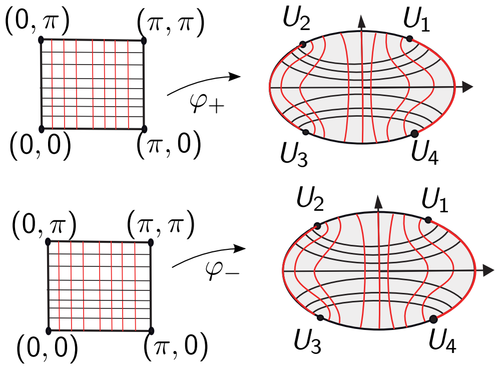


A basic problem in dynamical systems is to determine for which α the leaves of the foliation $\mathcal{F}_{\pm\alpha}$ are dense.

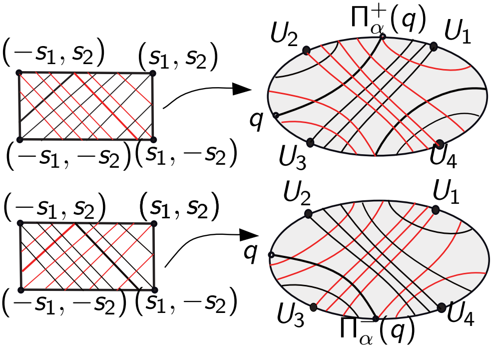
Using elliptic coordinates and integrals of the third kind Jacobi introduced a conformal map from the ellipsoid to the plane



in which the principal curvature lines are the images of lines parallel to the coordinate axes.

Cartography on the ellipsoid

α-**curvature lines** are parametrized by lines with angle α to one of the axes and billiard on the ellipsoid becomes classical billiard



Characteristic curve

The **characteristic polynomial** $\chi(\lambda)$ naturally associated with the pencil defines a family $C = \{C_{a,b,c}\}_{a,b,c \in \mathbb{P}^1}$ of elliptic curves

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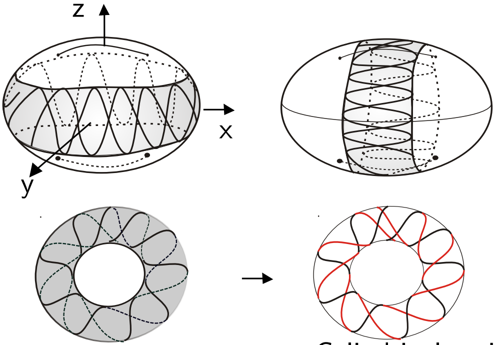
Theorem (R Garcia, G.W.)

For $\alpha \in (0, \pi/2) \cap 2\pi\mathbb{Q}$ the foliations $\mathcal{F}_\alpha$ on E do not contain any compact leaf.

Intersection of quadrics in $\mathbb{P}^5$

Geodesics

A basic problem in dynamical systems: which geodesics are closed.



Foliated torus

Cylindrical region

The envelopes (caustics) of the geodesic on the ellipsoid are principal curvature lines.

Intersection of quadrics in $\mathbb{P}^5$

For dealing with geodesics on a triaxial ellipsoid we look at the two quadrics

$$\begin{aligned} w^2 &= ax^2 + by^2 + cz^2 - \mu_1 w_1^2 - \mu_2 w_2^2 \\ 0 &= x^2 + y^2 + z^2 - w_1^2 - w_2^2 \end{aligned}$$

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The Jacobian $J(C)$ of C is an abelian surface with the property that it contains a curve of genus 2. We fix an embedding $\nu : C \to J$ by choosing once for all a Weierstrass point denoted by O.

Closed geodesics

The tangent in O of the embedded curve plays a special role. In order to determine the tangent we put

$$W = \{\omega \in H^0(C, \Omega^1_C), \omega(O) = 0\}.$$

This is a vector space of dimension 1 and we chose a generator ω_0 which can be expressed as $\nu^*\omega$ for a uniquely determined $\omega \in H^0(J, \Omega^1_J)$. Let γ be a geodesic on the ellipsoid. The following theorem gives a necessary and sufficient condition for the existence of a closed geodesic in $E_{a,b,c} \subset \mathbb{P}^3$.

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Theorem (R Garcia, G.W.)

The geodesic γ is closed if and only if there exists an elliptic curve $\iota : E \hookrightarrow J$ with the property that $\iota^*\omega = 0$

The Theorem above reduces the question on closed geodesics to decide when the Jacobian of C contains an elliptic curve.

Humbert Surface

In a series of papers Kani, partly with Frey, has studied problems of this type. A key role for answering the question is played by the socalled Humbert invariant of a divisor in $Div(J)$. Let $NS(J)$ be the Néron-Severi group of J and $\Theta \in Div(J)$ a divisor. Define for curves C its degree with respect to Θ as

$$\deg(C) = \deg_{\Theta}(C) = (C \cdot \Theta)$$

where the term $(C \cdot \Theta)$ is the intersection product on the surface. We obtain a quadratic form

$$\Delta(D) = (D \cdot \Theta)^2 - (\Theta \cdot \Theta)(D \cdot D) \tag{4}$$

which descends to $NS(J, \Theta) = NS(J) \cdot D)$ and gives a positive definite quadratic form.

A polarized abelian surface is said to satisfy a singular relation with invariant N if there exists a primitive class $[D] \in NS(A, \Theta)$ with Humbert invariant $\Delta(D) = N$.

Let

$$H_N = \{(A, \Theta); (A, \Theta) \text{ satisfies singular relation with invariant N}\}$$

be the Humbert surface introduced by Humbert. Then

Theorem (Biermann-Humbert)

A principally polarized abelian surface (A, Θ) has an elliptic subgroup of degree N if and only if $(A, \Theta) \in H_{N^2}$.

An immediate consequence is the following

Corollary

A necessary condition for a geodesic to be closed is that $J \in H_{N^2}$ for some N.

It would be interesting to see under which conditions the Corollary also gives a sufficient condition. It is conceivable that one has to impose the condition on the differential given in Theorem 3. A discussion of the case $N = 1$ is given below.

Kani's Classification

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To begin with it is relatively easy to see that if the two elliptic curves are non-isogenous there are no closed geodesics. This implies that closed geodesics on the ellipsoid can only exist when the characteristic curve attached to the ellipsoid in $\mathbb{P}^5$ defines a point in the moduli space of abelian surfaces lying on a countable union modular curves on the surface and we have to distinguish between complex multiplication and non-complex multiplication.

No CM

In the case when E_1 and E_2 do not have complex multiplication and such that $Hom(E_1, E_2) = d\mathbb{Z} \neq 0$ then there is no genus 2 curve on $E_1 \times E_2$ if and only if d is in the set Σ_{NCM} given by

$1, 2, 4, 6, 10, 12, 18, 22, 30, 42, 58, 60, 70, 78, 102, 130, 190, 210, 330, 462$

together with at most one more value $d = d^* > 462$. In other words: closed geodesics can only exist in this case when d is not in the set.

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An integer $n \geq 1$ is *idoneal* if it has the following property: if m is an odd integer prime to n properly represented by $q(x, y) = x^2 + ny^2$ and if the equation $q(x, y) = m$ has only one solution, then m is prime.

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Theorem (R. Garcia, G.W.)

There are no closed geodesics on the ellipsoid if and only if $d \in \Sigma_{NCM}$ *or* $\Delta(A) \in \Sigma_{CM}$ *and a possible further isolated value* $d^* > 462$.

Kani's Classification

The proof of the Theorem makes use of the determination of the slope of a geodesics in terms of the slope of the tangent line in a Weierstrass point of the curve C embedded in the Jacobian.
One direction follows directly from the previous Theorem. For the converse one has to go back to the description of the tangent in terms of the differentials.

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A different but incomplete approach to the problem was published by Yu. Fedorov and S. Abenda through KdV techniques. They studied a family of so-called hyperelliptic tangential coverings $C \to E$ which arise in the spectral theory of Lamé potentials or as some spectral curves of elliptic Moser-Calogero systems which describe the motions in an n-body system.

Some remarks on the proofs (geodsics)

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Similar to curvature lines, by the work of M. Chasles, C. G. J. Jacobi, M. Reid, R. Donagi and H. Knörrer geodesics on the ellipsoid can be parametrized by geodesics on $J(C)$ which are linearized by affine lines in the Lie algebra $\mathrm{Lie}(J(C))$ (*Linearization of the geodesic flow*). Again elliptic billiard is reduced to billiard on a billiard table.

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The problem then becomes to determine when the tangent in a Weierstrass point of the characteristic curve C embedded in its Jacobian intersects non-trivially the period lattice of $J(C)$. This is the case if and only if the quotient of two periods of an abelian integral is rational which leads a linear form in abelian logarithms.

Proofs

The underlying differential is

$$\omega(u) = \frac{udu}{\sqrt{-u(u-a)(u-b)(u-c)(u-\lambda)}} = \frac{udu}{\sqrt{p(u)}}$$

on the genus 2 hyperelliptic curve

$$C : \mu^2 = -u(u-a)(u-b)(u-c)(u-\lambda).$$

which is holomorphic.

Some remarks on the proofs (α-curvature lines)

Linear independence of periods

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The α-curvature line problem leads us to consider the elliptic differentials $\omega = dx/y$, $\eta = x\omega$, and ξ on the elliptic curve $y^2 = x(x-1)(x-\lambda)$ with ξ a differential of 3^{rd} kind with period matrix

$$\begin{pmatrix} \langle \omega, \gamma_0 \rangle & \langle \omega, \gamma_1 \rangle & 0 \\ \langle \eta, \gamma_0 \rangle & \langle \eta, \gamma_1 \rangle & 0 \\ \langle \xi, \gamma_0 \rangle & \langle \xi, \gamma_1 \rangle & 2\pi i \; \mathrm{res}_P(\xi) \end{pmatrix}$$

and with $\gamma_0, \gamma_1, \gamma$ cycles on the elliptic curve.

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and with $\gamma_0, \gamma_1, \gamma$ cycles on the elliptic curve.

The differential ξ has two simple poles and determines a point P in $\mathrm{Ext}^1(E, \mathbb{G}_m) = E$ which corresponds to an extension

$$1 \to \mathbb{G}_m \to G_m \to E \to 0$$

of E by $\mathbb{G}_m$.

Linear independence of periods

The differential η determines an extension

$$0 \to \mathbb{G}_a \to \mathrm{G_a} \to \mathrm{E} \to 0$$

of E by $\mathbb{G}_a$ and the entries of the period matrix together with 1 generate a vector space V over $\overline{\mathbb{Q}}$.

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of E by $\mathbb{G}_a$ and the entries of the period matrix together with 1 generate a vector space V over $\overline{\mathbb{Q}}$. Possible relations come from endomorphisms of $\mathrm{G_a}$ and $\mathrm{G_m}$ in terms of the endomorphisms of E which have to be determined. Out of the two extensions a commutative algebraic group G is constructed to apply the Analytic Subgroup Theorem.
We denote by $\mathcal{O}$ the ring of integers in the endomorphism algebra $\mathrm{End(E)}$ which is either $\mathbb{Q}$ or an imaginary quadratic field K. Then

Linear independence of periods

Theorem

$$\mathrm{End(G_a)} = \begin{cases} \mathcal{O} \times \mathbb{G}_a & \textit{if } [G] \textit{ is trivial,} \\ \mathrm{End(E)} & \textit{otherwise} \end{cases}$$

and

$$\mathrm{End(G_m)} = \begin{cases} \mathcal{O} \times \mathbb{Z} & \textit{if } [G] \textit{ is torsion,} \\ \mathbb{Z} & \textit{otherwise} \end{cases}$$

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The first statement is due to Masser using elementary complex analysis. We give a more conceptional proof using Hodge theory which extends easily to abelian varieties of higher dimension. For the second statement our proof uses Serre's criterion for endomorphisms of extensions.

Linear independence of periods

With the previous theorem it is easy to give an upper bound for the dimension and the analytic subgroup theorem then shows that the dimensions are equal to the upper bound.

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$$\dim V \otimes \overline{\mathbb{Q}} = \begin{cases} 8 & \text{if } E \text{ has no CM and } P \text{ is non-torsion} \\ 6 & \text{if } E \text{ has no CM and } P \text{ is torsion} \\ 6 & \text{if } E \text{ has CM and } P \text{ is non-torsion} \\ 4 & \text{if } E \text{ has CM and } P \text{ is torsion} \end{cases}$$

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This is used in the proof of the theorem about α – curvature lines. The link between differentials and commutative algebraic groups which come up is the **generalized Jacobian** introduced by Rosenlicht and Serre which implies that all our differentials are pullback if invariant differentials on the generalized Jacobian.

Ellipsoid of revolution

If two of the half axes of the ellipsoid coincide we are in the case of an ellipsoid of revolution. We consider only the case

$$\frac{x^2}{b} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \tag{5}$$

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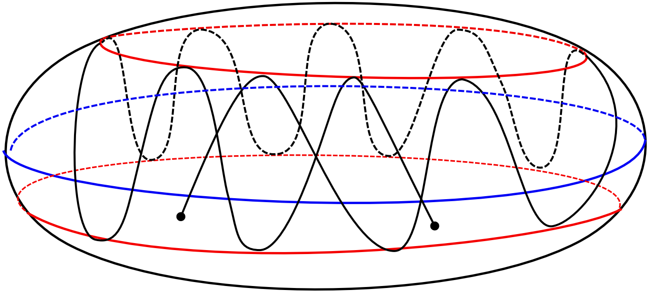

Figure : Geodesic curves in an ellipsoid (oblate) of revolution.

Ellipsoid of revolution

Let ξ be the differential form

$$\xi = \frac{vdv}{(b-v)\sqrt{v(v-c)(\lambda-v)}} \tag{6}$$

of the third kind with $\lambda \in (c, b)$ parametrizing the possible geodesics. In this situation we prove by a deformation argument the following

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Theorem

On an ellipsoid of revolution defined over a number field K there are no non-trivial closed geodesics.

Null geodesic

If the underlying space is replaced by a Minkowski space with Laurentzian metric the geodesics become **Null geodesics** and our problem then turns into the question under which conditions on a, b, c Null geodesics are closed.

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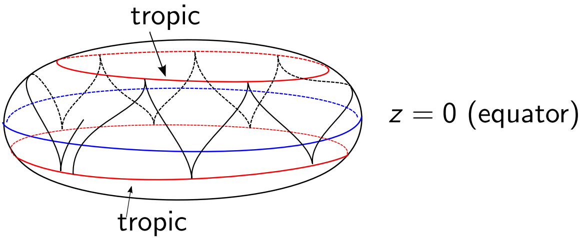

Figure : Null-chain on the ellipsoid $E_{a,b,c}$.

Null geodesics

Here we consider a triaxial ellipsoid $E_{a,b,c}$ in $\mathbb{R}^{3,1}$ with metric $ds^2 = dx^2 + dy^2 - dz^2$. In this case the null geodesic are lying in a cylindric region bounded by two regular curves (tropics). In this strip the null geodesics behaves like a singular billiard and a Poncelet type theorem was proved by Griffiths, Harris, Tabachnikov and alii.

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Theorem

On an ellipsoid in Minkowski space defined over a number field there are closed Null geodesics if and only if $J \in H_{n^2}$ for some N.