Vietnam Institute for Advanced Study in Mathematics

Mini-Course
Moment Problems, Sums of Squares and Polynomial Optimization

Hanoi, March 4 to 8, 2019

Mini-Course

Moment Problems, Sums of Squares and Polynomial Optimization

Program

Monday, March 4

8:30 – 9:00	Registration/Coffee
9:00 – 9:10	Welcoming
9:10–10:00	Konrad Schmüdgen (University of Leipzig, Germany)
	Integral representation of linear functionals
10:00–10:30	Coffee Break
10:30–11:20	Krzysztof Kurdyka (University Savoie Mont Blanc, France)
	Composed univariate sum of squares approximation of polynomials
11:20–14:00	Lunch
14:00–14:50	Konrad Schmüdgen
	Multidimensional moment problem on compact semi-algebraic sets
14:50–15:20	Coffee Break
15:20–16:10	Tutorial

Tuesday, March 5

9:00–9:50	Phạm Tiến Sơn (Dalat University)
	Optimization of polynomials on basic closed semi-algebraic sets
9:50–10:20	Coffee Break
10:20–11:10	Grzegorz Oleksik (University of Lodz, Poland)
	Conjecture on the Łojasiewicz exponent
11:10–14:00	Lunch
14:00–14:50	Phạm Tiến Sơn
	Optimization of polynomials on basic closed semi-algebraic sets (cont.)
14:50–15:20	Coffee Break
15:20–16:10	Tutorial

Wednesday, March 6

9:00–9:50	Konrad Schmüdgen
	Polynomial optimization and semidefinite programming
9:50–10:20	Coffee Break
10:20–11:10	Krzysztof Kurdyka
	Polynomial and exponential convexifying of positive polynomials
11:10–14:00	Lunch
14:00–14:50	Konrad Schmüdgen
	Truncated multidimensional moment problem: Existence via positivity
14:50–15:20	Coffee Break
15:20–16:10	Tutorial

Thursday, March 7

9:00–9:50	Jean Bernard Lasserre (LAAS-CNRS, Toulouse, France)
	The Moment-SOS hierarchy
9:50–10:20	Coffee Break
10:20–11:10	Grzegorz Oleksik
	The Łojasiewicz exponent at infinity of non-negative and non-degenerate polynomials
11:10–14:00	Lunch
14:00–14:50	Jean Bernard Lasserre
	The Moment-SOS hierarchy (cont.)
14:50–15:20	Coffee Break
15:20–16:10	Tutorial

Friday, March 8

9:00–9:50	Konrad Schmüdgen
	Truncated multidimensional moment problem: Existence via flat extension
9:50–10:20	Coffee Break
10:20–11:10	Tutorial
11:10–14:00	Lunch
14:00–14:50	Konrad Schmüdgen
	Truncated multidimensional moment problem: The moment cone
14:50–15:20	Coffee Break
15:20–16:10	Tutorial

Composed univariate sum of squares approximation of polynomials

Krzysztof Kurdyka

University Savoie Mont Blanc, France

We show that if a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is nonnegative on a basic closed semialgebraic set

$$X = \{x \in \mathbb{R}^n \colon g_1(x) \geq 0, \ldots, g_r(x) \geq 0\},$$

where $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$, then f can be approximated uniformly on compact sets by polynomials of the form

$$\delta_0 + \varphi(g_1)g_1 + \ldots + \varphi(g_r)g_r,$$

where $\delta_0 \in \mathbb{R}[x_1, \ldots, x_n],\ \varphi \in \mathbb{R}[t]$ are sums of squares of polynomials. In particular, if X is compact, and $h(x) := R^2 - (x_1^2 + \ldots + x_n^2)$ is positive on X, then

$$f = \delta_0 + \delta_1 h + \varphi(g_1)g_1 + \ldots + \varphi(g_r)g_r$$

for some sums of squares δ_0, δ_1. This is a joint work with S. Spodzieja.

Polynomial and exponential convexifying of positive polynomials

Krzysztof Kurdyka

University Savoie Mont Blanc, France

Let $X \subset \mathbb{R}^n$ be a convex closed and semialgebraic set and let f be a polynomial positive on X. We prove that there exists an exponent $N \geq 1$, such that for any $\xi \in \mathbb{R}^n$ the function $\varphi_N(x) = e^{N\|x-\xi\|^2} f(x)$ is strongly convex on X. When X is unbounded we have to assume also that the leading form of f is positive in $\mathbb{R}^n \setminus \{0\}$. We obtain strong convexity of $\Phi_N(x) = e^{e^{N\|x\|^2}} f(x)$ on possibly unbounded X, provided N is sufficiently large, assuming only that f is positive on X. We apply these results for searching critical points of polynomials on convex closed semialgebraic sets. This is a joint work with K. Rudnicka and S. Spodzieja.

The Moment-SOS hierarchy

Jean Bernard Lasserre

Laboratoire d'Analyse et d'Architecture des Systèmes (LAAS-CNRS), Toulouse, France

The Moment-SOS hierarchy initially introduced in optimization in 2000, is based on the theory of the K-moment problem and its dual counterpart, polynomials that are positive on K. It turns out that this methodology can be also applied to solve problems with positivity constraints "$f(x) \geq 0$ for all $x \in K$" and/or linear constraints on Borel measures. Such problems can be viewed as specific instances of the "Generalized Problem of Moments" (GPM) whose list of important applications in various domains is endless. In these two lectures, I will describe this methodology and outline some of its applications in various domains.

More precisely, in the first lecture, I introduce the Moment-SOS hierarchy, positivity certificates, semidefinite relaxations, sparsity, etc. In the second lecture, I describe a few applications outside optimization (Optimal control, non-linear hyperbolic PDEs', super-resolution, polynomial interpolation, etc.

Lectures on moment problems

Konrad Schmüdgen

University of Leipzig, Germany

In this course I will give a series of six lectures on moment problems.

Lecture 1. *Integral representation of linear functionals*

An integral representation theorem of positive functionals on Choquet's adapted spaces is obtained. As applications, Haviland's theorem is derived and existence results for moment problems on intervals are developed. Moment problems of $*$-semigroups are briefly discussed.

Lecture 2. *Multidimensional moment problem on compact semi-algebraic sets*

The interplay between the moment problem and Positivstellensätze of real algebraic geometry on compact semi-algebraic sets is developed. Existence criteria for special semi-algebraic sets are given.

Lecture 3. *Polynomial optimization and semidefinite programming*

Semidefinite programming is introduced and the Lasserre relaxations are defined. From the Archimedean Positivstellensatz convergence results are obtained.

Lecture 4. *Truncated multidimensional moment problem: Existence via positivity*

The theorem of Richter-Tchakaloff is proved. Existence criteria by positivity conditions are formulated. Stochel's theorem is mentioned.

Lecture 5. *Truncated multidimensional moment problem: Existence via flat extension*

Hankel matrices and their basic properties are treated. The flat extension theorem of Curto and Fialkow is stated and explained.

Lecture 6. *Truncated multidimensional moment problem: The moment cone*

The core variety is defined and basic results about the core variety are obtained. Results on the structure of the moment cone are discussed.

References

K. Schmüdgen, *The moment problem*, Springer: Berlin/Heidelberg, Germany, 2017.

Conjecture on the Łojasiewicz exponent

Grzegorz Oleksik

University of Lodz, Poland

In this talk, we propose a conjecture that the Łojasiewicz exponent of a nondegenerate (in the Kouchnirenko sense) isolated singularity (complex case) could be read off from its Newton diagram. We also discuss the history and progress of this problem.

The Łojasiewicz Exponent at infinity of non-negative and non-degenerate polynomials

Grzegorz Oleksik

University of Lodz, Poland

Let f be a real polynomial, non-negative at infinity with non-compact zero set. Suppose that f is non-degenerate in the Kouchnirenko sense at infinity. In this talk we give a formula for the Łojasiewicz exponent at infinity of f and a formula for the exponent of growth of f in terms of its Newton polyhedron.

Optimization of polynomials on basic closed semi-algebraic sets

Phạm Tiến Sơn

Dalat University, Dalat, Vietnam

In this lecture, we address the problem of minimizing a polynomial function over a basic closed semi-algebraic set. The cases where the constraint set is bounded and unbounded are considered. In each case we construct a sequence of semidefinite programs whose optimal values converge monotonically, increasing to the optimal value of the original problem. Finally, some generic properties of polynomial optimization problems are also discussed.