Introduction to the Deligne-Lusztig theory

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Outline of Section 1

1. Reviewing some basics
   - Character tables
   - Why CT?
   - Operations on group characters/representations

G a finite group
$\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ the set of complex irreducible characters of G (trace functions of complex irreducible representations)

The character table (CT) of G is a square $r \times r$-table:
the entry at the intersection of the row of $\chi_i \in \mathrm{Irr}(G)$ and the column labeled by the conjugacy class g_j^G of G with a representative $g_j \in G$ is $\chi_i(g_j)$.

Example 1.1 (Finite abelian groups)

(i) *Fundamental theorem on finite abelian groups:* Each finite abelian group is a direct product of cyclic subgroups.

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The other extreme class of finite groups: non-abelian *simple* groups.

Example 1.2 (A_5)

If you ask **GAP** for $CT(A_5)$, then this is what you get:

$X.1$	1	1	1	1	1
$X.2$	3	-1	0	A	$*A$
$X.3$	3	-1	0	$*A$	A
$X.4$	4	0	1	-1	-1
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$$A = -E(5) - E(5)^4 = -b5 = (1-\sqrt{5})/2.$$

Of course you know how to construct it. But we will do it again...

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What does $CT(G)$ tell you about the structure of G?

$CT(G)$ determines:

- $|G| = \sum_{i=1}^{r} \chi_i(1)^2$, hence the order of Sylow p-subgroups of G for all p.
- $|C_G(g)| = \sum_{i=1}^{r} |\chi_i(g)|^2$ and $|g^G| = [G : C_G(g)]$.
- The complex group algebra $\mathbb{C}G = \oplus_{i=1}^{r} M_{\chi_i(1)}(\mathbb{C})$.
- $Z(G) = \cap_{\chi \in \mathrm{Irr}(G)} Z(\chi)$, where $Z(\chi) := \{x \in G \mid |\chi(x)| = \chi(1)\}$.
- The derived subgroup $[G, G] = \cap_{i:\chi_i(1)=1} \mathrm{Ker}(\chi_i)$.
- All *normal* subgroups of G: each being the intersection of some $\mathrm{Ker}(\chi_i)$.
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• Hence the *simplicity* of G . . .

- ... The *solvability* of G.

Recall (**Galois**): The polynomial equation $f(x) = 0$ for $f \in \mathbb{Q}[t]$ can be solved by radicals if and only if the Galois group $Gal_{\mathbb{Q}}(f)$ is solvable.

- $CT(G/N)$ if $N \lhd G$.
- The *nilpotency* of G.

• . . . Whether $P \in Syl_p(G)$ is abelian.
This was Problem 12 in **Richard Brauer**'s List of Problems (1963), solved by **Camina-Herzog** (for $p = 2$, 1980), **Navarro-T.** (for $p \neq 3, 5$) and **Navarro-Solomon-T.** (for $p = 3, 5$) in 2015.

• Whether G admits abelian or nilpotent Hall subgroups.
$H \leq G$ is a Hall subgroup if $(|H|, [G : H]) = 1$.
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• *Sums of characters α, β of G*: $(\alpha + \beta)(g) = \alpha(g) + \beta(g)$.
Corresponds to direct sum of G-modules: $(U, V) \mapsto U \oplus V$.

• *Product of characters α, β of G:* $(\alpha \cdot \beta)(g) = \alpha(g)\beta(g)$.
Corresponds to tensor product of G-modules:
$(U, V) \mapsto U \otimes_{\mathbb{C}} V$.

• *Restriction from G to a subgroup $H \leq G$:* $\chi \mapsto \chi_H$.
Corresponds to letting H act on G-module V.

• *Inflation from quotient G/N to G:* $\gamma \mapsto \mathrm{Inf}_{G/N}^{G}(\gamma)$, where

$$\mathrm{Inf}_{G/N}^{G}(\gamma)(g) = \gamma(Ng), \quad \forall g \in G.$$

Corresponds to letting g act on W as Ng act on G/N-module W (so that N acts trivially on W).

• *Induction from a subgroup $H \leq G$ to G*: $\gamma \mapsto \mathrm{Ind}_H^G(\gamma) := \gamma^G$.
Corresponds to letting G act on $\mathbb{C}G \otimes_{\mathbb{C}H} W$ for H-module W.

• *Harish-Chandra restriction from G to L, where $U \rtimes L \leq G$*:
$\chi \mapsto {}^*R_L^G(\chi)$, with

$$ {}^*R_L^G(\chi)(l) = \frac{1}{|U|} \sum_{u \in U} \chi(ul), \ \forall l \in L. $$

Letting L act on the subspace of U-fixed points on G-module V.

• *Harish-Chandra induction from L to G, where $U \rtimes L \leq G$:*

$$ \delta \mapsto R_L^G(\delta) := \mathrm{Ind}_{UL}^G(\mathrm{Inf}_L^{UL}(\delta)). $$

Similarly for modules.

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Properties:

- $\oplus$ and $\otimes$ turn $\mathcal{C}(G)$ into a commutative ring, with identity 1_G.
- (H-C) restriction and (H-C) induction respect $\oplus$.
- *Frobenius reciprocity:*

$$\alpha \otimes \mathrm{Ind}_H^G(\gamma) = \mathrm{Ind}_H^G(\alpha|_H \otimes \gamma), \quad [\alpha, \mathrm{Ind}_H^G(\gamma)]_G = [\alpha|_H, \gamma]_H.$$

In the module language:

$$V \otimes \mathrm{Ind}_H^G(U) \cong \mathrm{Ind}_H^G(V|_H \otimes U), \quad \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(U)) \cong \mathrm{Hom}_H(V|_H, U).$$

- *Mackey formula:* If $H, K \leq G$ then

$$(\mathrm{Ind}_H^G(\gamma))|_K = \sum_{HgK \in H\backslash G/K} \mathrm{Ind}_{gHg^{-1}\cap K}^K((\alpha^g)|_{gHg^{-1}\cap K}).$$

In the module language:

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So how do we construct $CT(\mathrm{A}_5)$?

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- A_5 has five conjugacy classes: those of 1, (12)(34), (123), (12345), and $(12345)^2 = (13524)$.
- The first row is 1_G.
- The equation $x^2 + y^2 + z^2 + t^2 = 59$, subject to $x, y, z, t \in \mathbb{N}$, $x \leq y \leq z \leq t$, and $x, y, z, t | 60$, has only one solution $(3, 3, 4, 5)$. So we know the first column.
- The natural action of $G = A_5$ on $\Omega = \{1, 2, 3, 4, 5\}$ is *doubly transitive*, with point stabilizer $H = A_4$. Consider the corresponding permutation character

$$\rho(g) = |\{a \in \Omega \mid g(a) = a\}|.$$

The 2-transitivity implies that $[\rho, 1_G] = 1$ and $[\rho, \rho] = 2$, whence $\rho - 1_G \in \mathrm{Irr}(G)$. This yields the row with $\chi(1) = 4$.

• $H = A_4 = V \rtimes L$, where

$$V = \{1, (12)(34), (13)(24), (14)(23)\} \cong V_4.$$

$L \cong C_3$ has a faithful linear character λ.
$R_L^G(\lambda)$ yields the row with $\chi(1) = 5$.

• It remains to find two characters α, β of degree 3.
They have 3-defect 0, so vanish at (123). Orthogonality relations yield the remaining values.

If we knew how to decompose tensor powers of chars ...
then knowing just one character would be enough to find $CT(G)$:

Proposition 1.3

Let ρ be a faithful character of G taking n distinct values $a_1 = \rho(1), a_2, \ldots, a_n$. Then each $\chi \in \mathrm{Irr}(G)$ is an irreducible constituent of some ρ^k, $0 \leq k \leq n-1$.

Proof.

ρ is faithful means that if $g \neq 1$ then $\rho(g) \neq \rho(1) = a_1$. Hence $[\chi, \prod_{i=2}^{n}(\rho - a_i \cdot 1_G)] = \chi(1)\prod_{i=2}^{n}(a_1 - a_i)/|G| \neq 0$. □

Heide-Saxl-T.-Zalesski: *If G is simple, not an alternating group, and not $PSU_n(q)$ with $n \geq 3$ odd, then there is an irreducible $\varphi \in \mathrm{Irr}(G)$ such that each $\chi \in \mathrm{Irr}(G)$ is an irreducible constituent of φ^2.*

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By the **Jordan-Hölder theorem**, any finite group can be built up from simple groups.
Hence the knowledge of irreducible representations of all simple groups is of fundamental importance in representation theory and structure theory of finite groups.

So, given such a simple group G, how can one construct all irreducible complex representations of G, or at least, $CT(G)$?

A popular saying: *If something is true for S_n and $GL_n(q)$ (and for all solvable/sporadic groups), it is true for all finite groups . . .*

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Outline of Section 2

We aim to construct the character table of

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}_q,\ ad - bc \neq 0 \right\}.$$

This will illustrate the aforementioned techniques as well as some basic ideas of the **Deligne-Lusztig theory**.

Need a Borel subgroup B with unipotent radical U and a (split) maximal torus T_1 of order $(q-1)^2$:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \right\} = U \rtimes T_1,$$

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \right\},\ T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \right\}.$$

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We also need another (non-split) maximal torus T_2 of order $q^2 - 1$.
To this end, view $G = GL(V)$ where $V = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$. Consider the action of $T_2 = \mathbb{F}_{q^2}^\times$ (via multiplications) on $\mathbb{F}_{q^2}$ and then identify $\mathbb{F}_{q^2}$ with V.
This gives an embedding $T_2 \hookrightarrow G$, since the multiplications are $\mathbb{F}_q$-linear.

For $z \in \mathbb{F}_{q^2}^\times$, let d_z denote the corresponding element in G.
Check that d_z is conjugate to $\mathrm{diag}(z, z^q)$ over $\overline{\mathbb{F}}_q$, and $C_G(d_z) = T_2$ if $z \notin \mathbb{F}_q$.

Some more elements in G:

$$a_x = \mathrm{diag}(x, x),\ b_x = x \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$c_{x,y} = \mathrm{diag}(x, y),\ x, y \in \mathbb{F}_q^\times,\ x \neq y.$$

Check that

$$|\mathrm{C}_G(b_x)| = q(q-1),\ \mathrm{C}_G(c_{x,y}) = T_1.$$

$c_{x,y}$ and $c_{y,x}$ are G-conjugate. The same for d_z and d_{z^q}.

The elements listed in the table are pairwise non-conjugate (by JCF). But since

$$1\cdot(q-1)+(q^2-1)\cdot(q-1)+q(q+1)\cdot\frac{(q-1)(q-2)}{2}+q(q-1)\cdot\frac{q^2-q}{2} = |G|,$$

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Table 1. Character table of $GL_2(q)$

Classes / Length / Number	a_x / 1 / $q-1$	b_x / q^2-1 / $q-1$	$c_{x,y}$ / $q(q+1)$ / $\frac{(q-1)(q-2)}{2}$	d_z / $q(q-1)$ / $\frac{q(q-1)}{2}$
U_α $q-1$ chars	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(z^{q+1})$
$\rho = V_{1_{T_1}}$ V_α $q-1$ chars	q $q\alpha(x^2)$	0 0	1 $\alpha(xy)$	-1 $-\alpha(z^{q+1})$
$W_{\alpha,\beta}$ $\frac{(q-1)(q-2)}{2}$ chars	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)$ $+\alpha(y)\beta(x)$	0
γ^G T_γ $\frac{q(q-1)}{2}$ chars	$q(q-1)\gamma(x)$ $(q-1)\gamma(x)$	0 $-\gamma(x)$	0 0	$\gamma(z)+\gamma(z^q)$ $-\gamma(z)-\gamma(z^q)$

• For each $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$, we get

$$U_\alpha := \alpha \circ \det : G \to \mathbb{C}^\times,$$

yielding $q-1$ linear characters.

• G acts 2-transitively on $q+1$ 1-spaces of V, with point stabilizer B.

The corresponding character is $1_G + \rho$ with $\rho \in \mathrm{Irr}(G)$ of degree q. ρ is known as Steinberg character.

This yields $q-1$ irreducible characters $V_\alpha := \rho \cdot U_\alpha$ of degree q.

• Each pair (α, β) with $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ defines a character $L_{\alpha,\beta}$ of $T_1 \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$:

$$L_{\alpha,\beta}(c_{x,y}) = \alpha(x)\beta(y).$$

As $B = U \rtimes T_1$, we can consider

$$W_{\alpha,\beta} := R_{T_1}^G(L_{\alpha,\beta}).$$

Note that $W_{\alpha,\alpha} = U_\alpha + V_\alpha$ and $W_{\alpha,\beta} = W_{\beta,\alpha}$. Check that

$$[W_{\alpha,\beta}, W_{\alpha',\beta'}] = \begin{cases} 1, & \{\alpha,\beta\} = \{\alpha',\beta'\} \\ 0, & \{\alpha,\beta\} \neq \{\alpha',\beta'\} \end{cases}$$

whenever $\alpha \neq \beta$, $\alpha' \neq \beta'$.

Get $(q-1)(q-2)/2$ irreducible characters of degree $q+1$.

The characters of degree 1, q, and $q+1$ all have the property that $[\chi_U, 1_U] > 0$; they belong to the principal series of G.

• Need to find $(q^2-q)/2$ more irreducible characters.
For each $\gamma \in \mathrm{Irr}(T_2)$, compute γ^G:

It vanishes at b_x and $c_{x,y}$: no conjugates of them belong to T_2.

If $g = a_x$ then $g^G \cap T_2 = \{g\}$, hence

$$\gamma^G(a_x) = q(q-1)\gamma(x).$$

Next, if $g = d_z$ with $z \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, then, by looking at eigenvalues, we see that $g^G \cap T_2 = \{d_z, d_{z^q}\}$.
Since $\mathrm{C}_G(d_z) = T_2 = \mathrm{C}_{T_2}(d_z)$, it follows that

$$\gamma^G(d_z) = \gamma(z) + \gamma(z^q).$$

In particular, $(\gamma^q)^G = \gamma^G$.

The characters of degree 1, q, and $q+1$ all have the property that $[\chi_U, 1_U] > 0$; they belong to the principal series of G.

• Need to find $(q^2 - q)/2$ more irreducible characters.
For each $\gamma \in \mathrm{Irr}(T_2)$, compute γ^G:

It vanishes at b_x and $c_{x,y}$: no conjugates of them belong to T_2.

If $g = a_x$ then $g^G \cap T_2 = \{g\}$, hence

$$\gamma^G(a_x) = q(q-1)\gamma(x).$$

Next, if $g = d_z$ with $z \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, then, by looking at eigenvalues, we see that $g^G \cap T_2 = \{d_z, d_{z^q}\}$.
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In particular, $(\gamma^q)^G = \gamma^G$.

Assuming in addition that $\gamma \neq \gamma^q$, check that $[\gamma^G, \gamma^G] = q - 1$, so γ^G is reducible.

For $\alpha = \gamma_{\mathbb{F}_q^\times}$, consider the virtual character

$$T_\gamma := \rho \otimes W_{\alpha,1} - W_{\alpha,1} - \gamma^G.$$

Check that $T_\gamma(1) = q - 1$ and

$$[T_\gamma, T_{\gamma'}] = \begin{cases} 1, & \gamma' \in \{\gamma, \gamma^q\} \\ 0, & \gamma' \notin \{\gamma, \gamma^q\} \end{cases}$$

Thus we have obtained the missing $(q^2 - q)/2$ characters. They all belong to the discrete series, i.e. satisfy $[\chi_U, 1_U]_U = 0$.

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Character table of $GL_2(q)$, again!

Classes / Length / Number	a_x / 1 / $q-1$	b_x / q^2-1 / $q-1$	$c_{x,y}$ / $q(q+1)$ / $\frac{(q-1)(q-2)}{2}$	d_z / $q(q-1)$ / $\frac{q(q-1)}{2}$
U_α, $q-1$ chars	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(z^{q+1})$
$\rho = V_{1_{T_1}}$	q	0	1	-1
V_α, $q-1$ chars	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(z^{q+1})$
$W_{\alpha,\beta}$, $\frac{(q-1)(q-2)}{2}$ chars	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
γ^G	$q(q-1)\gamma(x)$	0	0	$\gamma(z)+\gamma(z^q)$
T_γ, $\frac{q(q-1)}{2}$ chars	$(q-1)\gamma(x)$	$-\gamma(x)$	0	$-\gamma(z)-\gamma(z^q)$

Notice a certain symmetry between conjugacy classes and complex characters of G.
Reason: G is a "Langlands dual" of itself.

Can one get a version of Harish-Chandra induction that would give the characters from the discrete series?

How to generalize all this story to all finite groups of Lie type: **Deligne-Lusztig theory**.

Also note that $CT(GL_2(q))$ depends on q but is uniform for all q. To generalize to all types: the main task of **Chevie**.

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Outline of Section 3

$GL_n(q)$ usually behaves in an “ideal” way in many respects.

To better understand what happens with other finite groups of Lie type, consider the group

$$S = SL_2(q) = \{g \in G \mid \det(g) = 1\} = [G, G],$$

with $G = GL_2(q)$.

If $2|q$: $G = S \times \mathrm{Z}(G)$.
Hence all irreducible characters of S can be obtained by restricting those of G.

Will now assume $q = p^f$ with p an odd prime.
Keep notations a_x, b_x, $c_{x,y}$, and d_z as in §2.

$$a_x = \mathrm{diag}(x,x),\ b_x = x\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$
$$c_{x,y} = \mathrm{diag}(x,y),\ x,y \in \mathbb{F}_q^\times,\ x \neq y.$$

Since $\det(g) = 1$ for $g \in S$, can consider only

- a_x, b_x for $x = \pm 1$,
- $c_x := c_{x,x^{-1}}$ for $x \in \mathbb{F}_q^\times \setminus \{\pm 1\}$, and
- d_z for $z \in \mathbb{F}_{q^2}^\times \setminus \{\pm 1\}$, $z^{q+1} = 1$.

- $a_x = xI_2$ are still central.
- $|C_S(b_x)| = 2q$, so $|b^S| = (q^2-1)/2 = |b^G|/2$.

Hence $b^G = b^S \sqcup (b')^S$, with $b' = x\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix}$, where $\varepsilon \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^{\times 2}$.

- Check that $|g^G| = |g^S|$ if $g = c_x$ or d_z.

$$2+2\cdot\frac{q^2-1}{2}+\frac{q-3}{2}\cdot q(q+1)+\frac{q-1}{2}\cdot q(q-1) = q(q^2-1) = |S|.$$

So we have found all conjugacy classes in S.

Table 2. Character table of $SL_2(q)$

Classes Length Number	a_x 1 2	b_x $(q^2-1)/2$ 2	b'_x $(q^2-1)/2$ 2	c_x $q(q+1)$ $(q-3)/2$	d_z $q(q-1)$ $(q-1)/2$
U	1	1	1	1	1
V	q	0	0	1	-1
W_α, $\alpha^2 \neq 1_F$ $(q-3)/2$ chars	$(q+1)\alpha(x)$	$\alpha(x)$	$\alpha(x)$	$\alpha(x)+\alpha(1/x)$	0
T_γ, $\gamma^2 \neq 1_C$ $(q-1)/2$ chars	$(q-1)\gamma(x)$	$-\gamma(x)$	$-\gamma(x)$	0	$-\gamma(z)-\bar{\gamma}(z)$
W_+	$\alpha_0(x)(q+1)/2$	$A\alpha_0(x)$	$(1-A)\alpha_0(x)$	$\alpha_0(x)$	0
W_-	$\alpha_0(x)(q+1)/2$	$(1-A)\alpha_0(x)$	$A\alpha_0(x)$	$\alpha_0(x)$	0
T_+	$\gamma_0(x)(q-1)/2$	$-A\gamma_0(x)$	$(A-1)\gamma_0(x)$	0	$-\gamma_0(x)$
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$$A := (1+\sqrt{\alpha_0(-1)q})/2.$$

Restrict irreducible characters of G to S.

Need two tori $F := T_1 \cap S \cong C_{q-1}$, $C := T_2 \cap S \cong C_{q+1}$.

- All U_α restrict to $U = 1_S$.
- All V_α restrict to the same irreducible character V of S.
- Let $W_\alpha := W_{\alpha,1}|_S$. Check that $W_\alpha = W_{\bar{\alpha}}$, and

$$[W_\alpha, W_\beta]_S = \begin{cases} 0, & \beta \neq \alpha, \bar{\alpha}, \\ 1, & \beta = \alpha \neq \bar{\alpha}, \\ 2, & \beta = \alpha = \bar{\alpha}. \end{cases}$$

In particular, the W_α with $\alpha^2 \neq 1_F$ yield $(q-3)/2$ characters of degree $q+1$.

- Write γ for $\gamma|_C$ and T_γ for $(T_\gamma)|_S$. Check that $T_\gamma = T_{\bar{\gamma}}$, and

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Thus the T_γ with $\gamma^2 \neq 1_C$ yield $(q-1)/2$ characters of degree $q-1$.

Still need to find 4 more characters of S.

Let $\alpha_0^2 = 1_F \neq \alpha_0$, $\gamma_0^2 = 1_C \neq \gamma_0$.

Shown above: $W_{\alpha_0} = W_+ + W_-$, $T_{\gamma_0} = T_+ + T_-$.

Clifford theory applied to $S \lhd G$ and orthogonality relations yield the remaining character values.

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The representations affording U, V, W_α, and $W_\pm$ (principal series) can be constructed by Harish-Chandra inducing 1-dimensional representations of T_1 (and decomposing them, and some more work for $W_\pm$).

It is harder to construct the ones for T_γ and $T_\pm$ – the discrete series.

$T_\pm$ and $W_\pm$ are also known as Weil representations.

Alperin-James showed that they can be constructed using certain analogues of Bessel functions.

Drinfeld (1974, when he was 20 !): The ones of degree $q-1$ for $SL_2(q)$) can be realized using l-adic cohomology $H^1_c(X_q, \mathbb{Q}_l)$ of the curve

$$X_q : x^q y - x y^q = 1$$

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Outline of Section 4

The "correct" framework to study groups like $GL_n(q)$, $SL_n(q)$: view them as certain subgroups of linear algebraic groups.

Fix a field $k = \overline{k}$ of characteristic $p \geq 0$.

Zariski topology on k^n: closed sets are the zero sets for (finite) collections of polynomials in $k[t_1, \ldots, t_n]$.

An affine (algebraic) variety: a closet subset $X \subseteq k^n$, considered with the induced Zariski topology.
$J = \{f \in k[t_1, \ldots, t_n] \mid f = 0 \text{ on } X\}$
$k[X] = k[t_1, \ldots, t_n]/J$ – the algebra of regular functions on X.

Morphisms between affine varieties $f : X \to Y$ – maps that can be defined by polynomial functions in the coordinates of X and Y.

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A linear algebraic group (LAG): an affine variety **G** endowed with a group structure such that the multiplication

$$\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G},\ (x, y) \mapsto xy$$

and the inversion

$$j : \mathbf{G} \to \mathbf{G},\ j(x) = x^{-1}$$

are morphisms.

Example 4.1

(i) The additive group $\mathbb{G}_a = (k, +)$; $k[\mathbb{G}_a] = k[t]$.
Here $\mu(x, y) = x + y$, $j(x) = -x$.

(ii) The multiplicative group $\mathbb{G}_m = (k^\times, \cdot)$.
Then $\mu(x, y) = xy$. But $j(x) = x^{-1}$???

Consider $\mathbb{G}_m = \{(x, y) \in k^2 \mid xy = 1\}$.
Then $\mu((x, y), (x', y')) = (xx', yy')$ and $j((x, y)) = (y, x)$.
Also, $k[\mathbb{G}_m] = k[x, y]/(xy - 1)$.

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Example 4.2

(i) The general linear group

$$GL_n = \{(A, y) \in k^{n\times n} \times k \mid \det(A)y = 1\}.$$

The rule for matrix multiplication shows that μ is a morphism. The **Cramer's rule** shows

$$(A, y)^{-1} = \left(((-1)^{i+j}\det(A_{ji}))y, \det(A)\right),$$

so j is a morphism.
Also, $k[GL_n] = k[T_{ij}, y \mid 1 \leq i, j \leq n]/(\det(T_{ij})y - 1)$.

(ii) Any closed subgroup of GL_n is a LAG.

(iii) Any finite group!

$$G \hookrightarrow S_N \hookrightarrow GL_N$$

with $N := |G|$.

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Morphisms between LAGs: maps $f : \mathbf{G} \to \mathbf{H}$ that are both group homomorphisms and variety morphisms.

Fact 4.3

Any LAG is isomorphic to a closed subgroup of some GL_n.

So we can view a given LAG $\mathbf{G}$ inside GL_n.

$g \in \mathbf{G}$ is

- semisimple if it is diagonalizable in GL_n,
- unipotent if it is conjugate to an upper unitriangular matrix in GL_n.

Fact 4.4 (Jordan decomposition)

For any LAG $\mathbf{G}$ and $g \in \mathbf{G}$, there is a unique pair $(s, u) \in \mathbf{G} \times \mathbf{G}$ such that s is semisimple, u is unipotent, and $g = su = us$.

Lie algebra version: $x = s + n$ with s semisimple, n nilpotent, and $[s, n] = 0$.

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Irreducibility and connectedness

An affine variety X is
• connected, if it can't be decomposed as disjoint union $X_1 \sqcup X_2$ of proper closed subsets ($\leftrightarrow k[X]$ contains no non-identity idempotents)
• irreducible, if it can't be decomposed as union $X_1 \cup X_2$ of proper closed subsets ($\leftrightarrow k[X]$ contains no zero divisors)

Note: $\{(x, y) \in k^2 \mid xy = 0\}$ is connected but reducible.

For LAGs, connected = irreducible.

Any LAG $\mathbf{G}$ has a unique irreducible component $\mathbf{G}^0$ that contains 1, which is a closed normal subgroup of finite index.

If $\mathbf{H} \lhd \mathbf{G}$ is a closed normal subgroup, then $\mathbf{G}/\mathbf{H}$ is a LAG.

Irreducibility and connectedness

An affine variety X is
• connected, if it can't be decomposed as disjoint union $X_1 \sqcup X_2$ of proper closed subsets ($\leftrightarrow k[X]$ contains no non-identity idempotents)
• irreducible, if it can't be decomposed as union $X_1 \cup X_2$ of proper closed subsets ($\leftrightarrow k[X]$ contains no zero divisors)

Note: $\{(x, y) \in k^2 \mid xy = 0\}$ is connected but reducible.

For LAGs, connected = irreducible.

Any LAG $\mathbf{G}$ has a unique irreducible component $\mathbf{G}^0$ that contains 1, which is a closed normal subgroup of finite index.

If $\mathbf{H} \lhd \mathbf{G}$ is a closed normal subgroup, then $\mathbf{G}/\mathbf{H}$ is a LAG.

Example 4.5

(i) $\mathbb{G}_a$, $\mathbb{G}_m$, and GL_n are connected.

(ii) The special linear group $SL_n := \{X \in GL_n \mid \det(X) = 1\}$ is connected. $SL_n \lhd GL_n$, and $GL_n/SL_n \cong \mathbb{G}_m$.

(iii) Suppose $\mathrm{char}(k) \neq 2$. Then the general orthogonal group

$$GO_n := \{X \in GL_n \mid {}^tXX = I_n\}$$

is disconnected. The connected component is the special orthogonal group

$$SO_n := GO_n \cap SL_n.$$

(iv) Let $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then the symplectic group

$$Sp_{2n} := \{X \in GL_{2n} \mid {}^tXJ_nX = J_n\}$$

is connected.

A LAG $\mathbf{G}$ is called unipotent, if all $g \in \mathbf{G}$ are unipotent.

A LAG $\mathbf{G}$ has a unique
• maximal closed connected solvable normal subgroup $R(\mathbf{G})$ (the solvable radical);
• maximal closed connected unipotent normal subgroup $R_u(\mathbf{G})$ (the unipotent radical).

A connected $\mathbf{G}$ is called
• reductive if $R_u(\mathbf{G}) = 1$,
• semisimple if $R(\mathbf{G}) = 1$, and
• simple if it has no nontrivial proper closed connected normal subgroup.

A semisimple LAG $\mathbf{G}$ is a central product of simple components.

If $\mathbf{G}$ is reductive, then $\mathbf{G} = [\mathbf{G}, \mathbf{G}]\mathrm{Z}(\mathbf{G})^0$, where $[\mathbf{G}, \mathbf{G}]$ is semisimple and $\mathrm{Z}(\mathbf{G})^0$ is a torus (i.e. a finite direct product of copies of $\mathbb{G}_m$).

Structure of any LAG

Example 4.6

(i) $\mathbb{G}_a$ is unipotent. To see: embed $\mathbb{G}_a$ in GL_2 !

(ii) $\mathbb{G}_m$ is a torus, reductive, but not semisimple.

(iii) GL_n is reductive, but not semisimple:
$R_u(GL_n) = Z(GL_n) \cong \mathbb{G}_m$.
$[GL_n, GL_n] = SL_n, \quad GL_n = Z(GL_n)SL_n$.

(iv) SL_n, SO_n, and Sp_{2n} are simple.

Any LAG **G** has a chain of closed normal subgroups

$$1 \leq R_u(\mathbf{G}) \leq R(\mathbf{G}) \leq \mathbf{G}^0 \leq \mathbf{G},$$

with $R_u(\mathbf{G})$ unipotent, $R(\mathbf{G})/R_u(\mathbf{G})$ a torus, $\mathbf{G}^0/R_u(\mathbf{G})$ reductive, $\mathbf{G}^0/R(\mathbf{G})$ semisimple, and $\mathbf{G}/\mathbf{G}^0$ finite.

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Tori and Borel subgroups

Fact 4.7

Let **G** *be a LAG.*
(i) **G** *has a maximal torus, and all maximal tori of* **G** *are conjugate.*
(ii) **G** *has a maximal closed connected solvable subgroup, called a Borel subgroup* **B***. All Borel subgroups of* **G** *are conjugate. If* **G** *is connected, then* $\mathrm{N}_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$*, and* **G** *is the union of its Borels.*
(iii) *Suppose* **G** *is connected reductive. If* $\mathbf{T} \leq \mathbf{G}$ *is a maximal torus, then* $\mathrm{C}_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ *and* $W = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ *is a finite group, called the Weyl group of* **G***. Any semisimple* $s \in \mathbf{G}$ *is contained in a maximal torus of* **G***.*

Fact 4.8

Let **G** *be connected.*
(i) **G** *is solvable iff it has a chain of closed connected normal subgroups with all successive quotients isomorphic to* $\mathbb{G}_a$ *or* $\mathbb{G}_m$*.*
(ii) *If* **G** *is solvable, then* $\mathbf{G} = R_u(\mathbf{G}) \rtimes \mathbf{T}$ *for some maximal torus* **T** *of* **G***.*

Example 4.9

$\mathbf{G} = GL_n > \mathbf{B} = \mathbf{U} \rtimes \mathbf{T} > \mathbf{T}$, *where*
- **B** *upper triangular,*
- **U** *upper unitriangular,*
- **T** *diagonal,* $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ *monomial, and the Weyl group* $W = S_n$*.*

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Parabolic and Levi subgroups

Let **G** be a connected LAG.

Parabolic subgroups **P** of **G**: any closed subgroup that contains a Borel.
Any such **P** has a Levi decomposition: $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$, with $\mathbf{U} := R_u(\mathbf{P})$ and **L** a closed complement to **U**.
L is called a Levi subgroup of **P** and equals $C_{\mathbf{G}}(Z(\mathbf{L})^0)$.
Any two Levi subgroups of **P** are **U**-conjugate.

In general, a Levi subgroup of **G** is any subgroup of the form $C_{\mathbf{G}}(\mathbf{T})$ for some torus **T**.

Example 4.10

Identify $\mathbf{G} = GL_n$ *with* $GL(V)$ *for* $V = k^n$. *Then any parabolic of* $\mathbf{G}$ *is the stabilizer of a flag*

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_m = V,$$

thus conjugate to

$$\mathbf{P} = \left\{ \begin{pmatrix} GL_{a_1} & * & * & \cdots & * \\ 0 & GL_{a_2} & * & \cdots & * \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & GL_{a_m} \end{pmatrix} \right\}$$

with $n = \sum_i a_i$. *Its Levi is conjugate to* $GL_{a_1} \times GL_{a_2} \times \ldots \times GL_{a_m}$.
If the flag is maximal, i.e. $m = n$*:* $\mathbf{P}$ *is a Borel and* $\mathbf{L}$ *is a maximal torus.*

Characters and cocharacters

Let $\mathbf{T} \cong \mathbb{G}_m^r$ be a torus.

Group of characters $X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m) \cong \mathbb{Z}^r$

Group of cocharacters $Y(\mathbf{T}) := \mathrm{Hom}(\mathbb{G}_m, \mathbf{T}) \cong \mathbb{Z}^r$

If $\chi \in X(T)$ and $\gamma \in Y(T)$, then $\chi \circ \gamma \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$ and so is of form $t \mapsto t^n$ for some $n \in \mathbb{Z}$. The map $(\chi, \gamma) \mapsto \langle \chi, \gamma \rangle := n$ gives a perfect pairing between $X(\mathbf{T})$ and $Y(\mathbf{T})$.

Let $\mathbf{G}$ be connected reductive with a Borus $\mathbf{B} \supset \mathbf{T}$.

There is a unique Borel $\mathbf{B}^-$ (the opposite Borel) such that $\mathbf{B} \cap \mathbf{B}^- = \mathbf{T}$.

Example 4.11 ($\mathbf{G} = GL_n$)

$\mathbf{T}$ diagonal torus, $\mathbf{B}$ upper triangular

$X(\mathbf{T}) = \{\chi_{a_1,\ldots,a_n} : \mathrm{diag}(t_1,\ldots,t_n) \mapsto \prod_{i=1}^n t_i^{a_i} \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^n$

$Y(\mathbf{T}) = \{\gamma_{a_1,\ldots,a_n} : t \mapsto \mathrm{diag}(t^{a_1},\ldots,t^{a_n}) \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^n$

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Root systems and root data

Let $\mathcal{U}$ be the set of minimal nontrivial closed subgroups $\mathbf{X}$, where $\mathbf{X} \leq R_u(\mathbf{B})$ or $\mathbf{X} \leq R_u(\mathbf{B}^-)$, and $\mathbf{X}$ is normalized by $\mathbf{T}$.

Each $\mathbf{X} \in \mathcal{U}$ is isomorphic to $\mathbb{G}_a$ and so can be written as $\mathbf{U}_\alpha := \{x_\alpha(c) \mid c \in k\}$ with $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ for all $t \in \mathbf{T}$ and for some $\alpha \in X(\mathbf{T})$.

$\Phi := \{\alpha \mid \mathbf{U}_\alpha \in \mathcal{U}\}$ – root system, and $\mathbf{G} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi \rangle$.

For each $\alpha \in \Phi$ (a root), there is a unique $\alpha^\vee \in Y(\mathbf{T})$ (a coroot) such that $\mathrm{Im}(\alpha^\vee) = \mathbf{T} \cap \langle U_\alpha, U_{-\alpha} \rangle$ and $\langle \alpha, \alpha^\vee \rangle = 2$.
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$(X(\mathbf{T}), \Phi, Y(\mathbf{T}), \Phi^\vee)$ – root datum of $\mathbf{G}$.

Theorem 4.12 (Chevalley)

Connected reductive algebraic groups are uniquely determined by their root data.

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Connected reductive algebraic groups are uniquely determined by their root data.

Example 4.13 ($\mathbf{G} = SL_2$)

B upper triangular, **T** diagonal.

$$X = \mathbb{Z}\gamma,\ \gamma : \mathrm{diag}(x, x^{-1}) \mapsto x, \quad Y = \mathbb{Z}\delta,\ \delta : x \mapsto \mathrm{diag}(x, x^{-1}).$$

Then $\Phi = \{\alpha, -\alpha\}$, with $U_\alpha = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\},\ U_{-\alpha} = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$,

$\alpha = 2\gamma : \mathrm{diag}(x, x^{-1}) \mapsto x^2$,

$\alpha^\vee = \delta : t \mapsto \mathrm{diag}(t, t^{-1}),\ \langle \alpha, \alpha^\vee \rangle = 2$.

$\mathbb{Z}\Phi^\vee = Y$, so **G** is called simply connected.

Example 4.14 ($\mathbf{G} = PGL_2$)

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Frobenius maps

Let $\mathrm{char}(k) = p > 0$ and $\mathbf{G} \leq GL_n$ a closed subgroup.

Standard Frobenius map:

$$F_q : GL_n \to GL_n,\ F_q(a_{ij}) = (a_{ij}^q) \text{ with } q = p^f.$$

Frobenius map: any morphism $F : \mathbf{G} \to \mathbf{G}$ such that some power of F is some F_q.

Example 4.15

Let $\mathbf{G} = GL_n$.
(i) $F = F_q$ is Frobenius, and the fixed point subgroup

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$$

is $GL_n(q)$.
(ii) Let $\tau(X) = {}^tX^{-1}$. Then $F = \tau F_q$ is a Frobenius map, as $F^2 = F_{q^2}$.
$\mathbf{G}^F = GU_n(q)$, the finite general unitary group.

Fact 4.16

If $\mathbf{G}$ *is connected semisimple, then a morphism* $F : \mathbf{G} \to \mathbf{G}$ *is Frobenius iff it is onto and* $\mathbf{G}^F$ *is finite.*

But: false for connected reductive groups.

Theorem 4.17 (Lang-Steinberg)

Let $\mathbf{G}$ *be connected LAG and* $F : \mathbf{G} \to \mathbf{G}$ *be surjective with finite* $\mathbf{G}^F$*. Then the Lang map* $\mathcal{L} : \mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}F(g)$*, is onto.*

But: false for disconnected groups !!

Definition 4.18

A finite group of Lie type in characteristic $p > 0$ is $\mathbf{G}^F$ for some connected reductive $\mathbf{G}$ and a Frobenius map $F : \mathbf{G} \to \mathbf{G}$.

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Theorem 4.17 is of fundamental importance in the study of finite groups of Lie type.

Proposition 4.19

Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius and $\mathbf{H}$ be an F-stable normal closed connected. Then $(\mathbf{G}/\mathbf{H})^F \cong \mathbf{G}^F/\mathbf{H}^F$.

Proof.

Suppose the coset $x\mathbf{H}$ is F-stable. Then $x^{-1}F(x) \in \mathbf{H}$. By 4.17 applied to $F : \mathbf{H} \to \mathbf{H}$, $x^{-1}F(x) = h^{-1}F(h)$ for some $h \in \mathbf{H}$. Now $x\mathbf{H} = xh^{-1}\mathbf{H}$ has an F-stable representative xh^{-1}. This implies that the map

$$\pi : \mathbf{G}^F/\mathbf{H}^F \to (\mathbf{G}/\mathbf{H})^F, \; g\mathbf{H}^F \mapsto g\mathbf{H}$$

is onto. Check that π is an isomorphism. □

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Proof.

Suppose the coset $x\mathbf{H}$ is F-stable. Then $x^{-1}F(x) \in \mathbf{H}$. By 4.17 applied to $F : \mathbf{H} \to \mathbf{H}$, $x^{-1}F(x) = h^{-1}F(h)$ for some $h \in \mathbf{H}$. Now $x\mathbf{H} = xh^{-1}\mathbf{H}$ has an F-stable representative xh^{-1}. This implies that the map

$$\pi : \mathbf{G}^F/\mathbf{H}^F \to (\mathbf{G}/\mathbf{H})^F, \; g\mathbf{H}^F \mapsto g\mathbf{H}$$

is onto. Check that π is an isomorphism. □

Definition 4.20

Let $F : \mathbf{G} \to \mathbf{G}$ be Frobenius. Then $g, h \in \mathbf{G}$ are F-conjugate, if $h = xgF(x)^{-1}$ for some $x \in \mathbf{G}$.

A similar application of Theorem 4.17 yields

Fact 4.21

Let $F : \mathbf{G} \to \mathbf{G}$ be Frobenius.
(i) *$\mathbf{G}$ admits F-stable Borel subgroups, and any two such are $\mathbf{G}^F$-conjugate.*
(ii) *If $\mathbf{B}_0$ is an F-stable Borel of $\mathbf{G}$, then it contains an F-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}$. All such F-stable Bori $\mathbf{B}_0 \supset \mathbf{T}_0$ are $\mathbf{G}^F$-conjugate. Moreover, there is a bijection*

$$\left\{ \begin{array}{c} \mathbf{G}^F\text{-conjugacy classes} \\ \text{of } F\text{-stable maximal tori} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} F\text{-conjugacy classes} \\ \text{in } W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0, \end{array} \right\}$$

induced by $g\mathbf{T}_0 g^{-1} \mapsto g^{-1}F(g)\mathbf{T}_0 \in W$.

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Rational maximal tori in $GL_n(q)$

Let $\mathbf{G} = GL_n$ and $F = F_q$ so that $G = \mathbf{G}^F = GL_n(q)$, and $\mathbf{T}_0$ the diagonal torus.

(i) Recall $W = \mathsf{S}_n$. By 4.17, can find $g \in \mathbf{G}$ with $g^{-1}F(g)$ inducing an n-cycle in S_n. Check that
$T_{(n)} := (g\mathbf{T}_0 g^{-1})^F \cong C_{q^n-1}$.
$g\mathbf{T}_0 g^{-1}$ is Coxeter torus.
One obtains $T_{(n)}$ by viewing $\mathbb{F}_{q^n}$ as n-dimensional vector space V over $\mathbb{F}_q$, G as $GL(V)$, and taking
$T_{(n)} = \{d_z : v \mapsto vz \mid z \in \mathbb{F}_{q^n}^\times\}$ (recalling §2 !)

(ii) F-conjugacy classes in W are conjugate classes, hence labeled by partitions $\lambda = (n_1 \geq n_2 \geq \ldots \geq n_m \geq 1)$ of n.
For a corresponding F-stable maximal torus $\mathbf{T}'$, one has

$$(\mathbf{T}')^F \cong T_{(n_1)} \times T_{(n_2)} \times \ldots \times T_{(n_m)}.$$

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Root data of finite groups of Lie type

Let $\mathbf{G}$ be connected reductive with Frobenius F.
Fix an F-stable Borus $\mathbf{B} \supset \mathbf{T}$ of $\mathbf{G}$, $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.
$(X = X(\mathbf{T}), \Phi, Y = Y(\mathbf{T}), \Phi^\vee)$ the root datum of $\mathbf{G}$.
F acts on X via $(F \cdot \chi)(t) = \chi(F(t))$.

Fact 4.22

There is some $\delta \in \mathbb{N}$ and some fractional power q of $p = \mathrm{char}(k)$ such that $F^\delta = q^\delta 1_X$, and $F = q\phi$ on X for some $\phi \in \mathrm{Aut}(X \otimes_{\mathbb{Z}} \mathbb{R})$ of order δ.

Theorem 4.23

The complete root datum $(X, \Phi, Y, \Phi^\vee, q, W\phi)$ completely determine $\mathbf{G}^F$ up to isomorphism.

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Theorem 4.23

The complete root datum $(X, \Phi, Y, \Phi^\vee, q, W\phi)$ completely determine $\mathbf{G}^F$ up to isomorphism.

Recall from 4.21 that **G** contains an F-stable "Borus" $\mathbf{B} \supset \mathbf{T}$ of F-stable Borel and F-stable maximal torus. Such a **T** is called maximally split (for $\mathbf{G}^F$). Then the $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori in **G** are labeled by F-conjugacy classes of the Weyl group $W = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

Example 4.24 ($GL_n(q)$)

Let $\mathbf{G} = GL_n$. Then **T** is conjugate to $\{\mathrm{diag}(*, \dots, *)\} \cong \mathbb{G}_m^n$ and $W \cong S_n$.
$\mathbf{T} \subset \mathbf{B}$, the upper triangular Borel.
Let $F = F_q$, so that $\mathbf{G}^F = GL_n(q)$. Then **B** is F-stable, **T** is maximally split, and $\mathbf{T}^F \cong C_{q-1}^n$.
Check: $F = q$ on X, so $\phi = 1_X$ in the root datum of $GL_n(q)$.

Example 4.25 ($GU_n(q)$)

Let $F' = \tau F_q$, with $\tau(X) = {}^tX^{-1}$.
But then **B** is not F'-stable...

Define $w := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$, $\sigma(X) = w\tau(X)w^{-1}$ and $F = \sigma F_q$.
Check that $\mathbf{G}^F \cong GU_n(q)$.

Now $\mathbf{B} \supset \mathbf{T}$ is F-stable, so $\mathbf{T}$ is maximally split.
Writing $n = 2m + \kappa$ with $\kappa \in \{0, 1\}$, one has $\mathbf{T}^F \cong C_{q^2-1}^m \times C_{q+1}^\kappa$.

Check that $F = -wq$ on X, so that $W\phi = -Ww = (-1_X)W$ in the root datum of $GU_n(q)$.

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Outline of Section 5

Let G and H be finite groups.
Let M be a (G, H)-bimodule, i.e. M is a left $\mathbb{C}G$-module and right $\mathbb{C}H$-module. Then M gives rise to the functor

$$R_H^G : \text{ left } H\text{-modules} \to \text{ left } G\text{-modules}, \quad W \mapsto M \otimes_{\mathbb{C}H} W,$$

with G acting on $M \otimes_{\mathbb{C}H} W$ on the left. Similarly, $M^* = \mathrm{Hom}(M, \mathbb{C})$ is a (H, G)-bimodule, and so it gives rise to the functor

$$^*R_H^G : \text{ left } G\text{-modules} \to \text{ left } H\text{-modules}, \quad V \mapsto M^* \otimes_{\mathbb{C}G} V.$$

The two functors are adjoint to each other:

$$\mathrm{Hom}_G(R_H^G(W), V) \cong \mathrm{Hom}_H(W, {}^*R_H^G(V)).$$

At the level of characters:

$$[\chi, R_H^G(\alpha)]_G = [{}^*R_H^G(\chi), \alpha]_H.$$

Proposition 5.1

Let M be a (G,H)-bimodule and W a left H-module. Then for $g \in G$

$$\mathrm{Tr}(g \mid R_H^G(W)) = |H|^{-1} \sum_{h \in H} \mathrm{Tr}((g, h^{-1}) \mid M)\mathrm{Tr}(h \mid W).$$

Proof. Let H^{op} denote the group opposite to H, with multiplication $h * k = kh$. Then M is a left H^{op}-module, with $h \cdot m = mh$. Consider

$$\pi := |H|^{-1} \sum_{h \in H} h^{-1} \otimes h \in \mathbb{C}[H^{\mathrm{op}} \times H] \cong \mathbb{C}H^{\mathrm{op}} \otimes \mathbb{C}H.$$

It is an idempotent: $\pi^2 = \pi$, because

$$(\sum_h h^{-1} \otimes h) \cdot (\sum_k k^{-1} \otimes k) = \sum_{h,k} k^{-1} h^{-1} \otimes hk = \sum_{h,k} (hk)^{-1} \otimes hk.$$

Hence π defines a projection on the left $H^{\mathrm{op}} \times H$-module $M \otimes_{\mathbb{C}} W$:

$$\pi : M \otimes W \to M \otimes W = \mathrm{Ker}(\pi) \oplus \mathrm{Im}(\pi).$$

For any $k \in H$, $m \in M$, and $x \in W$:

$$\sum_h h^{-1} \otimes h(mk \otimes x - m \otimes kx) = \sum_h mkh^{-1} \otimes hx - \sum_h mk(hk)^{-1} \otimes (hk)x = 0.$$

Conversely, if $\sum_i m_i \otimes x_i \in \mathrm{Ker}(\pi)$, then $\sum_{i,h} m_i h^{-1} \otimes hx_i = 0$, and so

$$\sum_i m_i \otimes x_i = |H|^{-1} \sum_h \sum_i \left((m_i h^{-1})h \otimes x_i - m_i h^{-1} \otimes hx_i \right).$$

Thus $\mathrm{Ker}(\pi)$ is spanned by the elements of the form $mh \otimes x - m \otimes hx$. But the quotient of $L := M \otimes W$ by the span of the latter elements is exactly $M \otimes_{\mathbb{C}H} W$. Hence $M \otimes_{\mathbb{C}H} W \cong \mathrm{Im}(\pi)$, and

$$\mathrm{Tr}(g \mid R_H^G(W)) = \mathrm{Tr}(g \mid \pi L) = \mathrm{Tr}(g\pi \mid \pi L) = \mathrm{Tr}(g\pi \mid L) =$$

$$= \mathrm{Tr}\left(|H|^{-1} \sum_h (gh^{-1} \otimes h) \mid L \right) = |H|^{-1} \sum_h \mathrm{Tr}(gh^{-1} \mid M)\mathrm{Tr}(h \mid W). \square$$

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Example 5.2

(i) Suppose $H \leq G$ and $M = \mathbb{C}G$, with G acting by left translations and H acting via right translations. Then R_H^G is the induction, and ${}^*R_H^G$ is the restriction.

(ii) Suppose $G \geq H = U \rtimes L$, and take $M = (1_U)^G$ – the permutation module of G on left U-cosets $gU \in G/U$. Then G acts on M via left translations, and L acts on M via right translations, since L normalizes U. Note that

$$\begin{aligned}\mathrm{Tr}((g, l^{-1}) \mid M) &= |\{xU \in G/U \mid gxUl^{-1} = xU\}| \\ &= |\{xU \mid x^{-1}gx \in Ul\}| = |U|^{-1}|\{x \in G \mid x^{-1}gx \in Ul\}|.\end{aligned}$$

So by 5.1,

$$\begin{aligned}\mathrm{Tr}(g \mid R_L^G(W)) &= |L|^{-1} \textstyle\sum_{l \in L} \mathrm{Tr}((g, l^{-1}) \mid M)\mathrm{Tr}(l \mid W) \\ &= |UL|^{-1} \textstyle\sum_{x \in G,\ x^{-1}gx = ul \in UL} \mathrm{Tr}(l \mid W) \\ &= \mathrm{Tr}\left(g \mid \mathrm{Ind}_{UL}^G(\mathrm{Inf}_L^{UL}(W)\right).\end{aligned}$$

Thus R_L^G is the Harish-Chandra induction, and similarly ${}^*R_L^G$ is the Harish-Chandra restriction.

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Example 5.3

Let $\mathbf{G} = GL_n$ and $F = F_q$. Then $G = \mathbf{G}^F = GL_n(q)$.
For $n = a + b$, one has an F-stable parabolic subgroup $\mathbf{P} = \left\{ \begin{pmatrix} GL_a & * \\ 0 & GL_b \end{pmatrix} \right\}$ with an F-stable Levi and radical

$$\mathbf{L} = \left\{ \begin{pmatrix} GL_a & 0 \\ 0 & GL_b \end{pmatrix} \right\}, \mathbf{U} = \left\{ \begin{pmatrix} I_a & * \\ 0 & I_b \end{pmatrix} \right\}.$$

Now $\mathbf{P}^F = \mathbf{U}^F \rtimes \mathbf{L}^F$, and one can define R_L^G and ${}^*R_L^G$ for $L := \mathbf{L}^F \cong GL_a(q) \times GL_b(q)$, which do not depend on the choice of the parabolic $\mathbf{P}$ containing $\mathbf{L}$ (as we will see).

Example 5.4

Now consider $\mathbf{G} = GL_n$ and $F = \tau F_q$, where $\tau(X) = {}^tX^{-1}$, so that $G = \mathbf{G}^F = GU_n(q)$. Again for $n = a + b$ one has F-stable Levi $\mathbf{L} = \left\{ \begin{pmatrix} GL_a & 0 \\ 0 & GL_b \end{pmatrix} \right\}$ with $\mathbf{L}^F \cong GU_a(q) \times GU_b(q)$.
However, there is no F-stable parabolic $\mathbf{P}$ containing $\mathbf{L}$ as its Levi.

So how can one define R_L^G and ${}^*R_L^G$?

The fundamental idea of **Deligne-Lusztig**:
Associate to $\mathbf{P}$ a variety $\mathbf{X}$ and then define R_L^G as the functor corresponding to the virtual $(\mathbf{G}^F, \mathbf{L}^F)$-bimodule $H_c^*(\mathbf{X})$.

Here, H_c^* is the ℓ-adic cohomology with compact support.

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Let $\ell \neq p$. ℓ-adic cohomology was introduced by **M. Artin** and **A. Grothendieck**, with the goal to approach **Weil's conjectures** on the number of points of an algebraic variety over a finite field of characteristic p.

Let $\mathbf{X}$ be an algebraic (affine, projective, quasi-projective, etc.) variety over $k = \overline{\mathbb{F}}_p$. Then one associates to $\mathbf{X}$ the ℓ-adic cohomology groups with compact support $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$, which are finite dimensional vector spaces over $\overline{\mathbb{Q}}_\ell$.
If $\mathbf{X}$ is projective, one can drop "with compact support".

We collect some necessary results on ℓ-adic cohomology.

Fact 5.5

(i) $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) = 0$ *unless* $0 \leq i \leq 2 \dim \mathbf{X}$.
(ii) *Each automorphism of* $\mathbf{X}$ *induces an invertible linear map on* $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$, *so that* $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ *is an* $\mathrm{Aut}(\mathbf{X})$*-module.*

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Fact 5.5

(i) $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) = 0$ *unless* $0 \leq i \leq 2 \dim \mathbf{X}$.
(ii) *Each automorphism of* $\mathbf{X}$ *induces an invertible linear map on* $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$, *so that* $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ *is an* $\mathrm{Aut}(\mathbf{X})$*-module.*

Define $H_c^*(\mathbf{X}) := \sum_{i=0}^{\infty}(-1)^i H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$, a virtual $\mathrm{Aut}(\mathbf{X})$-module.

Fact 5.6

(i) *If* $g \in \mathrm{Aut}(\mathbf{X})$ *is of finite order, then the Lefschetz number*

$$\mathfrak{L}(g, \mathbf{X}) = \mathrm{Tr}(g \mid H_c^*(\mathbf{X})) = \sum_i (-1)^i \mathrm{Tr}(g \mid H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_\ell))$$

is a rational integer, independent from $\ell \neq p$*. Moreover, if* $g = us = su$ *with* $p \nmid |s|$ *and* $|u|$ *a p-power, then* $\mathfrak{L}(g, \mathbf{X}) = \mathfrak{L}(u, \mathbf{X}^s)$*, where* $\mathbf{X}^s := \{x \in \mathbf{X} \mid s(x) = x\}$.

(ii) *If* $\mathbf{X}$ *is finite, then* $H_c^*(\mathbf{X})$ *is isomorphic to the permutation module of* $\mathrm{Aut}(\mathbf{X})$ *acting on* $\mathbf{X}$*. In particular,* $\mathfrak{L}(g, \mathbf{X}) = |\mathbf{X}^g|$ *for* $g \in \mathrm{Aut}(\mathbf{X})$.

(iii) $\mathfrak{L}(F, \mathbf{X}) = |\mathbf{X}^F|$ *for a Frobenius map* $F : \mathbf{X} \to \mathbf{X}$.

(iv) $\mathfrak{L}(g, \mathbf{X})$ *behaves "well" w.r.t. decompositions* $X = \sqcup_{i=1}^n X_i$ *and* $X = X_1 \times X_2$.

From now on, let $\mathbf{G}$ be a connected reductive algebraic group over k, with a Frobenius map $F : \mathbf{G} \to \mathbf{G}$ and the Lang map $\mathcal{L}(g) = g^{-1}F(g)$. Let $G := \mathbf{G}^F$.

Definition 5.7 (Deligne-Lusztig, Lusztig)

Let $\mathbf{P}$ *be a possibly non-F-stable parabolic of* $\mathbf{G}$*, with radical* $\mathbf{U}$ *and an F-stable Levi* $\mathbf{L}$*. Let* $L := \mathbf{L}^F$. Then $G \times L^{\mathrm{op}}$ acts on the affine variety

$$\mathcal{L}^{-1}(\mathbf{U}) = \{x \in \mathbf{G} \mid \mathcal{L}(x) \in \mathbf{U}\}$$

via $(g, l) \cdot x = gxl$. Hence $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))$ is a virtual left $G \times L^{\mathrm{op}}$-module, whence a virtual (G, L)-bimodule.

Then the Lusztig induction $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ *is the functor associated to the* (G, L)*-bimodule* $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))$.
The adjoint functor is Lusztig restriction ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$.

Theorem 5.8

Let $\mathbf{P}$ *be a (possibly non-F-stable) parabolic of* $\mathbf{G}$*, with radical* $\mathbf{U}$ *and F-stable Levi* $\mathbf{L}$*. For any* $\mathbf{G}^F$*-character* χ*, any* $\mathbf{L}^F$*-character* ψ*,* $g \in G$*,* $l \in L$*, one has*

(i) $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\psi(g) = |\mathbf{L}^F|^{-1} \sum_{l\in\mathbf{L}^F} \mathrm{Tr}\left((g,l) \mid H^*_c(\mathcal{L}^{-1}(\mathbf{U}))\right) \psi(l^{-1})$.

(ii) ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\chi(l) = |\mathbf{G}^F|^{-1} \sum_{g\in\mathbf{G}^F} \mathrm{Tr}\left((g,l) \mid H^*_c(\mathcal{L}^{-1}(\mathbf{U}))\right) \chi(g^{-1})$.

Proof.

(i) follows from Definition 5.7 and Proposition 5.1.

For (ii), note by 5.6 that Lefschetz numbers are integers. Thus the character of the virtual bimodule $H^*_c(\mathcal{L}^{-1}(\mathbf{U}))$ is real. One then shows that the character of $H^*_c(\mathcal{L}^{-1}(\mathbf{U}))$ is $\alpha - \beta$, with α, β being real true characters (or 0). Thus the virtual bimodule is self-dual. Hence the claim follows from 5.1. □

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Fact 5.9

Let $\pi : \mathbf{X} \to \mathbf{Y}$ be an epimorphism of varieties whose fibres are all isomorphic to k^n for same n. Let $g \in \mathrm{Aut}(\mathbf{X})$ and $h \in \mathrm{Aut}(\mathbf{Y})$ be of finite order such that $\pi g = h\pi$. Then $\mathfrak{L}(g, \mathbf{X}) = \mathfrak{L}(h, \mathbf{Y})$.

Corollary 5.10

Suppose $\mathbf{P}$ is F-stable parabolic with F-stable Levi $\mathbf{L}$.

(i) *Lusztig induction $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is the same as the Harish-Chandra induction R^G_L.*

(ii) *Lusztig restriction ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is just the Harish-Chandra restriction ${}^*R^G_L$.*

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Proof of Corollary 5.10

Since $\mathbf{P}$ is F-stable, $\mathbf{U} = R_u(\mathbf{P})$ is F-stable.
Suppose $g \in \mathcal{L}^{-1}(\mathbf{U})$. Then $g^{-1}F(g) \in \mathbf{U}$, and so by 4.17 applied to $\mathbf{U}$, $g^{-1}F(g) = u^{-1}F(u)$ for some $u \in \mathbf{U}$, whence $gu^{-1} \in \mathbf{G}^F = G$ and $g \in G\mathbf{U}$. It follows that $g\mathbf{U} \subset \mathcal{L}^{-1}(\mathbf{U})$.
We thus get a map $\pi : \mathcal{L}^{-1}(\mathbf{U}) \to \mathbf{G}/\mathbf{U}$ with $\pi(x) = x\mathbf{U}$.

The fibres of π are $\mathbf{U}$-cosets, hence isomorphic to $k^{\dim \mathbf{U}}$.
The above computation shows
$\pi : \mathcal{L}^{-1}(\mathbf{U}) \twoheadrightarrow \mathbf{Y} := (\mathbf{G}/\mathbf{U})^F \cong \mathbf{G}^F/\mathbf{U}^F$, cf. 4.19.
As $\mathbf{Y}$ is finite, $H_c^*(\mathbf{Y}) \cong$ the permutation module on the $\mathbf{U}^F$-cosets in G by 5.6. Now Fact 5.9 and Example 5.2 imply that

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = R^G_L.$$

The same for the adjoint functor. □

Let $\mathbf{L}$ be an F-stable Levi of some parabolic $\mathbf{P} = \mathbf{UL}$.
Then the actions of $\mathbf{G}^F$ and $(\mathbf{L}^F)^{\mathrm{op}}$ on $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))$ commute.
Hence Theorem 5.8 shows for $\psi \in \mathrm{Irr}(\mathbf{L}^F)$ and $\chi \in \mathrm{Irr}(\mathbf{G}^F)$:

- $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\psi)$ is the $\mathbf{G}^F$-character afforded by the ψ-component of the $(\mathbf{L}^F)^{\mathrm{op}}$-module $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))$;
- ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\chi)$ is the $\mathbf{L}^F$-character afforded by the χ-component of the $\mathbf{G}^F$-module $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))$.

In particular, if $\mathbf{L} = \mathbf{T}$ is an F-stable maximal torus and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, then $R_{\mathbf{T}\subset\mathbf{B}}^{\mathbf{G}}(\theta)$ is the $\mathbf{G}^F$-character afforded by the θ-component $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))_\theta$ of the $\mathbf{T}^F$-module $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))$.
Note: $\mathbf{T}^F$ is commutative, so left and right $\mathbf{T}^F$-modules are the same!

Corollary 5.11

If $\mathbf{L}$ *is an* F*-stable Levi of* $\mathbf{G}$, $G = \mathbf{G}^F$, $L = \mathbf{L}^F$, *then* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(1_G) = 1_L$.

Proof.

Trivial for Harish-Chandra restriction!

General case: as noted above, ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(1_G)$ is the L-character afforded by the virtual module $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))^G$.
The latter can be shown to be isomorphic to $H_c^*(\mathcal{L}^{-1}(\mathbf{U})/G)$.
The map sending $Gx \in \mathcal{L}^{-1}(\mathbf{U})/G$ to $x^{-1}F(x)$ induces a variety isomorphism $\mathcal{L}^{-1}(\mathbf{U})/G \cong \mathbf{U}$, whence $\mathcal{L}^{-1}(\mathbf{U})/G \cong k^{\dim \mathbf{U}}$.
Hence $\mathfrak{L}(g, \mathcal{L}^{-1}(\mathbf{U})/G) = 1$ for any finite automorphism g. □

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General case: as noted above, ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(1_G)$ is the L-character afforded by the virtual module $H^*_c(\mathcal{L}^{-1}(\mathbf{U}))^G$.
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Hence $\mathfrak{L}(g, \mathcal{L}^{-1}(\mathbf{U})/G) = 1$ for any finite automorphism g. □

Theorem 5.12 (Mackey formula)

Let $\mathbf{L}$ *be an* F*-stable Levi of a parabolic* $\mathbf{P}$ *and let* $\mathbf{M}$ *be an* F*-stable Levi of a parabolic* $\mathbf{Q}$ *of* $\mathbf{G}$*. Then*

$$^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} = \sum_x R^{\mathbf{L}}_{\mathbf{L}\cap x\mathbf{M}x^{-1}\subset\mathbf{L}\cap x\mathbf{Q}x^{-1}}\circ{}^*R^{x\mathbf{M}x^{-1}}_{\mathbf{L}\cap x\mathbf{M}x^{-1}\subset\mathbf{P}\cap x\mathbf{M}x^{-1}}\circ\mathrm{ad}(x),$$

if at least one of the following holds:
(i) (**Harish-Chandra**) *both* $\mathbf{P}$ *and* $\mathbf{Q}$ *are* F*-stable;*
(ii)(**Deligne–Lusztig**) *at least one of* $\mathbf{L}$, $\mathbf{M}$ *is a maximal torus;*
(iii) (**Bonnafé–Michel**) $\mathbf{G}^F$ *is defined over* $\mathbb{F}_q$ *with* $q > 2$*;*
(iv) (**Bonnafé–Michel**) *No* F*-stable component of* $\mathbf{G}$ *is* $E_{6,7,8}$*.*
Here, x *runs over a set of representatives of* $\mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F$*, with* $\mathcal{S}(\mathbf{L},\mathbf{M}) := \{y\in\mathbf{G}\mid \mathbf{L}\cap y\mathbf{M}y^{-1}$ *contains a maximal torus of* $\mathbf{G}\}$*.*
Furthermore, if W *is an* $\mathbf{M}^F$*-module, then* $\mathrm{ad}(x)W$ *is the* $x\mathbf{M}^Fx^{-1}$*-module with underlying space* W *and* $xmx^{-1}\cdot v = mv$ *for all* $m\in\mathbf{M}^F$ *and* $v\in W$*.*

Mackey formula 5.12 has important consequences.

Corollary 5.13

For a given F-stable Levi $\mathbf{L}$ of an F-stable parabolic $\mathbf{P}$, Harish-Chandra induction $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and Harish-Chandra restriction ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ do not depend on the choice of the F-stable parabolic $\mathbf{P}$ having $\mathbf{L}$ as its Levi.

Definition 5.14

If $\mathbf{T}$ is an F-stable maximal torus and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, then $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ is called a Deligne-Lusztig character.

Note that $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ is only a virtual character!

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Next we explain the reason we can write $R_{\mathbf{T}}^{\mathbf{G}}$ instead of $R_{\mathbf{T}\subset\mathbf{B}}^{\mathbf{G}}$.

Corollary 5.15

Let $\mathbf{T} \subset \mathbf{B}$ and $\mathbf{T}' \subset \mathbf{B}'$ be two F-stable maximal tori of $\mathbf{G}$, with Borel $\mathbf{B}$ and $\mathbf{B}'$.

(i) *For $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ and $\theta' \in \mathrm{Irr}(\mathbf{T}'^F)$,*

$$[R_{\mathbf{T}\subset\mathbf{B}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'\subset\mathbf{B}'}^{\mathbf{G}}(\theta')]_{\mathbf{G}^F} = \frac{\#\{n \in \mathbf{G}^F \mid n\mathbf{T}n^{-1} = \mathbf{T}', \mathrm{ad}(n)\theta = \theta'\}}{|\mathbf{T}^F|}.$$

(ii) *$R_{\mathbf{T}\subset\mathbf{B}}^{\mathbf{G}}$ does not depend on the choice of the Borel $\mathbf{B}$ containing $\mathbf{T}$.*

Proof of Corollary 5.15

For (i), note that $S := \mathcal{S}(\mathbf{T}, \mathbf{T}')^F = \{x \in \mathbf{G}^F \mid \mathbf{T} = x\mathbf{T}'x^{-1}\}$. Next, if $x \in S$, $t \in \mathbf{T}^F$, $s \in \mathbf{T}'^F$, then $xsx^{-1} \in (x\mathbf{T}'x^{-1})^F = \mathbf{T}^F$, and so $xsx^{-1} = t_1 \in \mathbf{T}^F$, whence $txs = tt_1 x$ with $tt_1 \in \mathbf{T}^F$. Thus the double coset $\mathbf{T}^F x \mathbf{T}'^F$ equals to $\mathbf{T}^F x$ and so has size $|\mathbf{T}^F|$.

By Mackey formula 5.12 and adjointness, we then have

$$[R^{\mathbf{G}}_{\mathbf{T}'\subset\mathbf{B}'}(\theta'), R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\theta)]_{\mathbf{G}^F} = [{}^*R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}} \circ R^{\mathbf{G}}_{\mathbf{T}'\subset\mathbf{B}'}(\theta'), \theta]_{\mathbf{T}^F} = \frac{\sum_{x\in S}[\mathrm{ad}(x)\theta', \theta]_{\mathbf{T}^F}}{|\mathbf{T}^F|}.$$

The claim now follows by labeling $n = x^{-1}$.

For (ii), consider another Borel $\mathbf{B}_1 \supset \mathbf{T}$. By (i) we have

$$[R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\theta), R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\theta)]_{\mathbf{G}^F} = [R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\theta), R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}_1}(\theta)]_{\mathbf{G}^F} = [R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}_1}(\theta), R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}_1}(\theta)]_{\mathbf{G}^F}.$$

Hence $[R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\theta) - R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}_1}(\theta), R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\theta) - R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}_1}(\theta)]_{\mathbf{G}^F} = 0$, and the claim follows by positive definiteness of $[\cdot,\cdot]_{\mathbf{G}^F}$. □

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For (i), note that $S := \mathcal{S}(\mathbf{T}, \mathbf{T}')^F = \{x \in \mathbf{G}^F \mid \mathbf{T} = x\mathbf{T}'x^{-1}\}$. Next, if $x \in S$, $t \in \mathbf{T}^F$, $s \in \mathbf{T}'^F$, then $xsx^{-1} \in (x\mathbf{T}'x^{-1})^F = \mathbf{T}^F$, and so $xsx^{-1} = t_1 \in \mathbf{T}^F$, whence $txs = tt_1x$ with $tt_1 \in \mathbf{T}^F$. Thus the double coset $\mathbf{T}^F x \mathbf{T}'^F$ equals to $\mathbf{T}^F x$ and so has size $|\mathbf{T}^F|$.

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Fix a maximally split torus $\mathbf{T}$, i.e. $\mathbf{B} \supset \mathbf{T}$ is an F-stable Borus. For $w \in W$, let $\mathbf{T}_w$ denote an F-stable maximal torus corresponding to the F-conjugacy class of $w \in W = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

Corollary 5.16

If $w, w' \in W$, then

$$[R^{\mathbf{G}}_{\mathbf{T}_w}(1_{\mathbf{T}_w^F}), R^{\mathbf{G}}_{\mathbf{T}_{w'}}(1_{\mathbf{T}_{w'}^F})]_{\mathbf{G}^F} = \begin{cases} |W^{wF}|, & w, w' \text{ are } F\text{-conjugate}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

By 5.15, the scalar product can be $\neq 0$ only when $\mathbf{T}_w$ and $\mathbf{T}_{w'}$ are $\mathbf{G}^F$-conjugate, i.e. when w and w' are F-conjugate. In the latter case, may assume $w' = w$. By 4.19 and 5.15 the scalar product is $|\mathrm{N}_{\mathbf{G}^F}(\mathbf{T}_w)|/|\mathbf{T}_w^F| = |W(\mathbf{T}_w)^F|$. Since F acts on $W(\mathbf{T}_w)$ as wF acts on W, we are done. □

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Corollary 5.17

Let $\mathbf{B} \supset \mathbf{T}$ be an F-stable Borus and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. Let $\tilde{\theta}$ be the inflation of θ to $\mathbf{B}^F = \mathbf{U}^F \rtimes \mathbf{T}^F$, where $\mathbf{U} = R_u(\mathbf{B})$. Then $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(\tilde{\theta})$.

Proof.

Since $\mathbf{B}$ is F-stable, by 5.10 Lusztig induction $R_{\mathbf{T}}^{\mathbf{G}}$ and Harish-Chandra induction $R_{\mathbf{T}^F}^{\mathbf{G}^F}$ are the same. □

Example 5.18

Consider $\mathbf{G}^F = G = GL_2(q)$. Then the diagonal torus $\mathbf{T}$ is maximally split, and $\mathbf{T}^F = T_1$ in §2. Any $\theta \in \mathrm{Irr}(T_1)$ is of the form $L_{\alpha,\beta}$, and so $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_{T_1}^{G}(L_{\alpha,\beta}) = W_{\alpha,\beta}$.

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Proof.

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Consider $\mathbf{G}^F = G = GL_2(q)$. Then the diagonal torus $\mathbf{T}$ is maximally split, and $\mathbf{T}^F = T_1$ in §2. Any $\theta \in \mathrm{Irr}(T_1)$ is of the form $L_{\alpha,\beta}$, and so $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_{T_1}^{G}(L_{\alpha,\beta}) = W_{\alpha,\beta}$.

Corollary 5.17

Let $\mathbf{B} \supset \mathbf{T}$ be an F-stable Borus and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. Let $\tilde{\theta}$ be the inflation of θ to $\mathbf{B}^F = \mathbf{U}^F \rtimes \mathbf{T}^F$, where $\mathbf{U} = R_u(\mathbf{B})$. Then $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(\tilde{\theta})$.

Proof.

Since $\mathbf{B}$ is F-stable, by 5.10 Lusztig induction $R_{\mathbf{T}}^{\mathbf{G}}$ and Harish-Chandra induction $R_{\mathbf{T}^F}^{\mathbf{G}^F}$ are the same. □

Example 5.18

Consider $\mathbf{G}^F = G = GL_2(q)$. Then the diagonal torus $\mathbf{T}$ is maximally split, and $\mathbf{T}^F = T_1$ in §2. Any $\theta \in \mathrm{Irr}(T_1)$ is of the form $L_{\alpha,\beta}$, and so $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_{T_1}^{G}(L_{\alpha,\beta}) = W_{\alpha,\beta}$.

Definition 5.19

For an F-stable maximal torus $\mathbf{T}$, $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ is in general position, if no non-identity element of $W^F \cong \mathrm{N}_{\mathbf{G}^F}(\mathbf{T})/\mathbf{T}^F$ fixes θ.

Corollary 5.20

If $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ is in general position, then $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is an irreducible character of $\mathbf{G}^F$.

Proof.

By 5.15,

$$[R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)]_{\mathbf{G}^F} = \#\{w \in W^F \mid w \text{ fixes } \theta\} = 1.$$

Since $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is a virtual character, the claim follows. □

Definition 5.19

For an F-stable maximal torus $\mathbf{T}$, $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ is in general position*, if no non-identity element of $W^F \cong \mathrm{N}_{\mathbf{G}^F}(\mathbf{T})/\mathbf{T}^F$ fixes θ.*

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If $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ is in general position, then $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is an irreducible character of $\mathbf{G}^F$.

Proof.

By 5.15,

$$[R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)]_{\mathbf{G}^F} = \#\{w \in W^F \mid w \text{ fixes } \theta\} = 1.$$

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Since $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is a virtual character, the claim follows. □

Example 5.21 (Example 5.18 continued)

The Weyl group is $W = \{1, s\}$. The reflection s is induced by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and so interchanges the two direct factors of

$$T_1 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}.$$

Thus $s(L_{\alpha,\beta}) = L_{\beta,\alpha}$.

• If $\alpha \neq \beta$, then $L_{\alpha,\beta}$ is in general position, and $W_{\alpha,\beta}$ is irreducible (no sign is needed!).

• Suppose $\theta = L_{\alpha,\alpha}$. Then 5.15 yields $[R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)] = 2$, and in fact $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = U_\alpha + V_\alpha$ in §2.

Deligne-Lusztig varieties

The characters $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})$ are important in the study of unipotent characters of $\mathbf{G}^F$. Here is an alternate construction of them.

$\mathcal{B}$ the set of all Borels of $\mathbf{G}$, and fix an F-stable Borus $\mathbf{B}_0 \supset \mathbf{T}_0$. Since $\mathrm{N}_{\mathbf{G}}(\mathbf{B}_0) = \mathbf{B}_0$, the map $g\mathbf{B}_0g^{-1} \mapsto g\mathbf{B}_0$ allows us to identify $\mathcal{B}$ with $\mathbf{G}/\mathbf{B}_0$ as a projective variety.
$\mathbf{G}$ acts on $\mathcal{B} \times \mathcal{B}$ via conjugation, with orbits labeled by $w \in W = \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$. For such w, the corresponding orbit contains a unique pair $(\mathbf{B}_0, n_w\mathbf{B}_0n_w^{-1})$ with $n_w\mathbf{T}_0 = w$ – the orbit of Borels in relative position w.

Deligne-Lusztig varieties:
$\mathcal{B}_w := \{\mathbf{B} \in \mathcal{B} \mid (\mathbf{B}, F(\mathbf{B}))$ in relative position $w \in W\}$.

Theorem 5.22

If $\mathbf{T}$ corresponds to the F-conjugacy class of $w \in W$, then $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})(g) = \mathfrak{L}(g, \mathcal{B}_w)$ for all $g \in \mathbf{G}^F$.

Drinfeld's example

The map $g \mapsto g\mathbf{B}_0 g^{-1}$ yields a surjective morphism

$$\pi : \mathcal{L}^{-1}(n_w\mathbf{B}_0) \twoheadrightarrow \mathcal{B}_w,$$

whose fibres are $\mathbf{B}_0 \cap n_w\mathbf{B}_0 n_w^{-1}$-orbits on $\mathcal{L}^{-1}(n_w\mathbf{B}_0)$.

Now let $\mathbf{G} = SL_2$, $F = F_q$, $\mathbf{G}^F = SL_2(q)$, $w \neq 1$, $n_w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Then $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\mathcal{L}^{-1}(n_w\mathbf{B}_0)$ iff

$ad - bc = 1, \quad a^q d - bc^q = 0, \quad (ac^q - a^q c)(b^q d - bd^q) = -1.$

Solving this system, we get

$$b = -a^q, \quad d = -c^q, \quad a^q c - ac^q = 1.$$

Thus $\mathcal{L}^{-1}(n_w\mathbf{B}_0)$ is the affine curve

$$X_q = \{(x, y) \in k^2 \mid x^q y - xy^q = 1\}.$$

This explains why $H^1_c(X_q, \mathbb{Q}_\ell)$ gives rise to the discrete series of irreducible characters of $SL_2(q)$, cf. §3.

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$\mathcal{L}^{-1}(\mathbf{U})$ vs. $\mathcal{B}_w$: Proof of Theorem 5.22

Let $\mathbf{B}_0 = \mathbf{U}_0\mathbf{T}_0 \supset \mathbf{T}_0$ be F-stable, and fix $w \in W = \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$.
Let $\mathbf{T} = x\mathbf{T}_0x^{-1}$ with $x^{-1}F(x)\mathbf{T}_0 = w$, $\mathbf{U} = x\mathbf{U}_0x^{-1}$.
$F(\mathbf{U})\mathbf{T}$ is a Borel of $\mathbf{G}$ with radical $F(\mathbf{U})$ and containing $\mathbf{T}$. Then

$$\pi : \mathcal{L}^{-1}(F(\mathbf{U})) \to \mathcal{B}_w, \;\; g \to gx\mathbf{B}_0(gx)^{-1}$$

is a surjective morphism. Its fibres are orbits of $\mathcal{L}^{-1}(F(\mathbf{U}))$ under $(\mathbf{U} \cap F(\mathbf{U}))\mathbf{T}^F$ acting by right multiplication.
In particular, elements in a given $\mathbf{T}^F$-orbit lie in the same fibre.
So π factors through

$$\mathcal{L}^{-1}(F(\mathbf{U})) \to \mathcal{L}^{-1}(F(\mathbf{U}))/\mathbf{T}^F \overset{\gamma}{\twoheadrightarrow} \mathcal{B}_w.$$

Fibers of γ are isomorphic to $\mathbf{U} \cap F(\mathbf{U})$, an affine space. Hence 5.9 implies $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})(g)$ is equal to

$$\mathfrak{L}(g, (\mathcal{L}^{-1}(F(\mathbf{U})))^{\mathbf{T}^F}) = \mathfrak{L}(g, \mathcal{L}^{-1}(F(\mathbf{U}))/\mathbf{T}^F) = \mathfrak{L}(g, \mathcal{B}_w).$$

Outline of Section 6

For a LAG $\mathbf{H}$, $\mathbf{H}_u$ denotes the set of unipotent elements of $\mathbf{H}$.

Definition 6.1

Let $\mathbf{T}$ be an F-stable maximal torus. Then the Green function $Q_{\mathbf{T}}^{\mathbf{G}} : \mathbf{G}_u^F \to \mathbb{Z}$ is defined by $Q_{\mathbf{T}}^{\mathbf{G}}(u) = R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})(u)$.

Note by 5.6 that Lefschetz numbers are integers, so by 5.8

$$R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})(u) = |\mathbf{T}^F|^{-1} \sum_{t \in \mathbf{T}^F} \mathfrak{L}((u,t), \mathcal{L}^{-1}(\mathbf{U})) \in \mathbb{Q}.$$

But $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})$ is a virtual character, so $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})(u)$ is an algebraic integer. Thus $Q_{\mathbf{T}}^{\mathbf{G}}(u) \in \mathbb{Z}$.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, $\mathbf{B} \supset \mathbf{T}$ a (possible non-F-stable) Borel with radical $\mathbf{U}$. Set $\mathbf{X} := \mathcal{L}^{-1}(\mathbf{U})$.

Theorem 6.2

Let $g \in \mathbf{G}^F$ have the Jordan decomposition $g = su = us$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. Then

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|(C^0_{\mathbf{G}}(s))^F|} \sum_{x \in \mathbf{G}^F,\ x^{-1}sx \in \mathbf{T}^F} Q^{C^0_{\mathbf{G}}(s)}_{x\mathbf{T}x^{-1}}(u)\theta(x^{-1}sx).$$

Outline of Proof.

Step 1. $C^0_{\mathbf{G}}(s)$ is connected reductive and contains u.
If $x^{-1}sx \in \mathbf{T}^F$, then $s \in x\mathbf{T}x^{-1}$ and so $x\mathbf{T}x^{-1} \leq C^0_{\mathbf{G}}(s)$.
So $Q^{C^0_{\mathbf{G}}(s)}_{x\mathbf{T}x^{-1}}(u)$ makes sense.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, $\mathbf{B} \supset \mathbf{T}$ a (possible non-F-stable) Borel with radical $\mathbf{U}$. Set $\mathbf{X} := \mathcal{L}^{-1}(\mathbf{U})$.

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So $Q_{x\mathbf{T}x^{-1}}^{C^0_{\mathbf{G}}(s)}(u)$ makes sense.

Step 2. By 5.8 and 5.6,

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|\mathbf{T}^F|}\sum_{t\in\mathbf{T}^F}\mathfrak{L}((su,t),\mathbf{X})\bar{\theta}(t) = \frac{1}{|\mathbf{T}^F|}\sum_{t\in\mathbf{T}^F}\mathfrak{L}(u,\mathbf{X}^{(s,t)})\bar{\theta}(t),$$

where $\mathbf{X}^{(s,t)} := \{x \in \mathbf{X} \mid sxt = x\}$.

Step 3. Let $t \in \mathbf{T}^F$ and $x \in \mathbf{X}$ be such that $sxt = x$. Consider affine varieties

$$(\mathbf{G}^F)^{(s,t)} := \{k \in \mathbf{G}^F \mid skt = k\}, \;\; \mathbf{Y}_t := \mathbf{X} \cap C^0_{\mathbf{G}}(t).$$

An application of Lang-Steinberg 4.17 to $C^0_{\mathbf{G}}(t)$ shows:
$\mu : (k,z) \mapsto kz$ is a surjective morphism $(\mathbf{G}^F)^{(s,t)} \times \mathbf{Y}_t \twoheadrightarrow \mathbf{X}^{(s,t)}$.
Next, $C^0_{\mathbf{G}}(t)^F$ acts on $(\mathbf{G}^F)^{(s,t)} \times \mathbf{Y}_t$ via $m \cdot (k,z) = (km^{-1}, mz)$.
The orbits in this action are exactly the fibres of the map μ.

Step 4. Now fix $k \in (\mathbf{G}^F)^{(s,t)}$ and note that $(\mathbf{G}^F)^{(s,t)} = k\mathrm{C}_{\mathbf{G}}(t)^F$.
Write $\mathrm{C}_{\mathbf{G}}(t)^F = \sqcup_{i=1}^m z_i \mathrm{C}^0_{\mathbf{G}}(t)^F$ and $k_i := kz_i$.
Then $(\mathbf{G}^F)^{(s,t)} = \sqcup_{i=1}^m k_i \mathrm{C}^0_{\mathbf{G}}(t)^F$.

By Step 3, $\mathbf{X}^{(s,t)}$ is the disjoint union of the subsets $k_i\mathbf{Y}_t$.
Each of them is a closed subset of $\mathbf{G}$, and so of $\mathbf{X}$.

Thus: $\mathbf{X}^{(s,t)}$ is the disjoint union of m closed subsets $k_i\mathbf{Y}_t$.

Step 5. Direct computation shows that each $k_i\mathbf{Y}_t$ is invariant under the left multiplication by u. Since Lefschetz numbers are additive, $\mathfrak{L}(u, \mathbf{X}^{(s,t)}) = \sum_i \mathfrak{L}(u, k_i\mathbf{Y}_t)$.

Also, u acts on $k_i\mathbf{Y}_t$ as $k_i^{-1}uk_i$ acts on $\mathbf{Y}_t$.

Step 6. Now we have

$$\begin{aligned}
R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) &= |\mathbf{T}^F|^{-1} \sum_{t \in \mathbf{T}^F} \mathfrak{L}(u, \mathbf{X}^{(s,t)}) \bar{\theta}(t) \\
&= |\mathbf{T}^F|^{-1} \sum_{t \in \mathbf{T}^F} \bar{\theta}(t) \sum_{i=1}^{m} \mathfrak{L}(k_i^{-1} u k_i, \mathbf{Y}_t) \\
&= |\mathbf{T}^F|^{-1} \sum_{t \in \mathbf{T}^F} \bar{\theta}(t) |\mathrm{C}^0_{\mathbf{G}}(t)^F|^{-1} \sum_{k \in (\mathbf{G}^F)^{(s,t)}} \mathfrak{L}(k^{-1} u k, \mathbf{Y}_t) \\
&= |\mathbf{T}^F|^{-1} \sum_{t \in \mathbf{T}^F} \bar{\theta}(t) |\mathrm{C}^0_{\mathbf{G}}(t)^F|^{-1} \sum_{k \in \mathbf{G}^F, k^{-1} s k = t^{-1}} \mathfrak{L}(k^{-1} u k, \mathbf{Y}_t) \\
&= |\mathbf{T}^F|^{-1} |\mathrm{C}^0_{\mathbf{G}}(s)^F|^{-1} \sum_{\substack{k \in \mathbf{G}^F, \\ k^{-1} s k \in \mathbf{T}^F}} \theta(k^{-1} s k) \mathfrak{L}(k^{-1} u k, \mathbf{Y}_{k^{-1} s k}).
\end{aligned}$$

Putting $s = 1$, we obtain

$$Q_{\mathbf{T}}^{\mathbf{G}}(u) = R_{\mathbf{T}}^{\mathbf{G}}(\theta)(u) = |\mathbf{T}^F|^{-1} \mathfrak{L}(u, \mathbf{X}).$$

Final Step. Recall $\mathbf{U}$ is a maximal unipotent subgroup of $\mathbf{G}$. As $s' := k^{-1}sk$ is F-stable, we have

$$\mathbf{Y}_{s'} = \mathbf{X} \cap \mathrm{C}^0_{\mathbf{G}}(s') = \mathcal{L}^{-1}(\mathbf{U}) \cap \mathrm{C}^0_{\mathbf{G}}(s') = \mathcal{L}^{-1}\left(\mathbf{U} \cap \mathrm{C}^0_{\mathbf{G}}(s')\right) \cap \mathrm{C}^0_{\mathbf{G}}(s').$$

Also, as $s' \in \mathbf{T}$ and $\mathbf{B} = \mathbf{UT}$, one can show that $\mathbf{U} \cap \mathrm{C}^0_{\mathbf{G}}(s')$ is a maximal unipotent subgroup of $\mathrm{C}^0_{\mathbf{G}}(s')$.

Thus $(\mathbf{T}, \mathbf{Y}_{s'})$ plays the role of $(\mathbf{T}, \mathbf{X})$ for $\mathrm{C}^0_{\mathbf{G}}(s')$.

Hence by Step 6, $Q_{\mathbf{T}}^{\mathrm{C}^0_{\mathbf{G}}(k^{-1}sk)}(k^{-1}uk) = |\mathbf{T}^F|^{-1}\mathfrak{L}(k^{-1}uk, \mathbf{Y}_{k^{-1}sk})$.

Conjugating by k and using Step 6 again, we obtain

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|(\mathrm{C}^0_{\mathbf{G}}(s))^F|} \sum_{k \in \mathbf{G}^F,\ k^{-1}sk \in \mathbf{T}^F} Q_{k\mathbf{T}k^{-1}}^{\mathrm{C}^0_{\mathbf{G}}(s)}(u)\theta(k^{-1}sk). \square$$

Theorem 6.2 reduces the computation of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ to that of Green functions (of $\mathbf{G}$ and its reductive subgroups).

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Conjugating by k and using Step 6 again, we obtain

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|(\mathrm{C}^0_{\mathbf{G}}(s))^F|} \sum_{k \in \mathbf{G}^F,\ k^{-1}sk \in \mathbf{T}^F} Q_{k\mathbf{T}k^{-1}}^{\mathrm{C}^0_{\mathbf{G}}(s)}(u)\theta(k^{-1}sk). \square$$

Theorem 6.2 reduces the computation of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ to that of Green functions (of $\mathbf{G}$ and its reductive subgroups).

Semisimple $\mathbb{F}_q$-rank

Let $\mathbf{G}$ be a connected reductive with Frobenius $F : \mathbf{G} \to \mathbf{G}$.
Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$.
There are $\phi \in \mathrm{Aut}(X(\mathbf{T}))$ (the "twist of F") and q such that F acts on $X(\mathbf{T})$ as $q\phi$, i.e.

$$\alpha(F(t)) = (\phi(\alpha)(t))^q, \ \ \forall t \in \mathbf{T}, \ \forall \alpha \in X(\mathbf{T}).$$

Definition 6.3

(i) For an F-stable torus $\mathbf{T}$, the $\mathbb{F}_q$-rank of $\mathbf{T}$ is
$r_q(\mathbf{T}) := \mathrm{rank}(X(\mathbf{T})^{\phi})$.
(ii) The $\mathbb{F}_q$-rank of $\mathbf{G}$, $r_q(\mathbf{G})$, is the $\mathbb{F}_q$-rank of its maximally split torus.
Set $\varepsilon_{\mathbf{G}} = (-1)^{r_q(\mathbf{G})}$.
(iii) The semisimple $\mathbb{F}_q$-rank of $\mathbf{G}$ is $r(\mathbf{G}) := r_q(\mathbf{G}/R(\mathbf{G}))$.

Example 6.4

(i) Let $\mathbf{B} = \mathbf{UT}$ be F-stable Borel with F-stable maximal torus $\mathbf{T}$. Then $r_q(\mathbf{B}) = r_q(\mathbf{T})$ and $r(\mathbf{B}) = 0$.

(ii) Let $\mathbf{G} = GL_n$ and $F = F_q$, so that $\mathbf{G}^F = GL_n(q)$. Then the diagonal maximal torus $\mathbf{T}$ is maximally split, and $r_q(\mathbf{T}) = n$.

As $R(\mathbf{G}) = \mathrm{Z}(\mathbf{G}) \cong \mathbb{G}_m$, $r(\mathbf{G}) = n - 1$.

(iii) Let $\mathbf{G} = GL_n$ and $F = \tau F_q$, so that $\mathbf{G}^F = GU_n(q)$.
Let $m := \lfloor n/2 \rfloor$.
For a maximally split torus $\mathbf{T}$, $\mathbf{T}^F \cong C_{q^2-1}^m \times C_{q+1}^{n-2m}$, and $r_q(\mathbf{T}) = m$.

As $R(\mathbf{G}) = \mathrm{Z}(\mathbf{G})$ has $r_q = 0$, $r(\mathbf{G}) = m$.

Definition 6.5 (Alvis, Curtis, Kawanaka, Lusztig)

Let $\mathbf{G}$ be connected reductive with F-stable Borel $\mathbf{B}$.
Let $\mathcal{P}$ be the set of F-stable parabolics of $\mathbf{G}$ that contain $\mathbf{B}$.
For each $\mathbf{P} \in \mathcal{P}$, fix an F-stable Levi $\mathbf{L}$ of $\mathbf{P}$.
The Alvis-Curtis duality $D_{\mathbf{G}}$ is the operator

$$D_{\mathbf{G}} := \sum_{\mathbf{P} \in \mathcal{P}} (-1)^{r(\mathbf{P})} R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}$$

on the space of class functions on $\mathbf{G}^F$.

Note: if φ is a virtual character of $\mathbf{G}^F$, then so is $D_{\mathbf{G}}(\varphi)$.

Fact 6.6 (Properties of $D_{\mathbf{G}}$)

(i) $D_{\mathbf{G}}$ *is self-adjoint:* $[D_{\mathbf{G}}\alpha, \beta]_{\mathbf{G}^F} = [\alpha, D_{\mathbf{G}}\beta]_{\mathbf{G}^F}$
(since ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ is adjoint to $R_{\mathbf{L}}^{\mathbf{G}}$).

(ii) **(Curtis)** *If* $\mathbf{L}$ *is an F-stable Levi of some F-stable parabolic of* $\mathbf{G}$*, then* $D_{\mathbf{G}} \circ R_{\mathbf{L}}^{\mathbf{G}} = R_{\mathbf{L}}^{\mathbf{G}} \circ D_{\mathbf{L}}$.

(iii) *If* $\mathbf{T}$ *is an F-stable maximal torus of* $\mathbf{G}$*, then*
$D_{\mathbf{G}} \circ R_{\mathbf{T}}^{\mathbf{G}} = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}$.

((ii) and (iii) follow from Mackey formula).

(iv) **(Alvis)** $D_{\mathbf{G}}^2$ *is the identity operator.*
If $\chi \in \mathrm{Irr}(\mathbf{G}^F)$*, then* $D_{\mathbf{G}}(\chi)$ *is also an irreducible character of* $\mathbf{G}^F$ *up to sign, and*

$$D_{\mathbf{G}}(\chi)(1)_{p'} = \chi(1)_{p'}.$$

Definition 6.7 (Steinberg)

Let $\mathbf{G}$ be connected reductive with Frobenius F, and $G = \mathbf{G}^F$. Then the Steinberg character of G is $\mathrm{St}_G := D_{\mathbf{G}}(1_G)$.

By 6.6, St_G is irreducible up to sign. But $\mathrm{St}_G(1) > 0$ (see next slide), so St_G is an irreducible character of G.

Example 6.8

Consider the case $G = GL_2(q)$: $\mathbf{B} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$. Then $\mathcal{P} = \{\mathbf{G}, \mathbf{B}\}$, and $r(\mathbf{G}) = 1$, $r(\mathbf{B}) = 0$.
Note $\mathbf{T}^F = T_1$, the split torus and $\mathbf{B}^F = UT_1$. Hence

$$D_{\mathbf{G}}(1_G) = R_{T_1}^G \circ {}^*R_{T_1}^G(1_G) - 1_G = R_{T_1}^G(1_{T_1}) - 1_G = \rho,$$

the Steinberg character of G.

Definition 6.7 (Steinberg)

Let $\mathbf{G}$ be connected reductive with Frobenius F, and $G = \mathbf{G}^F$. Then the *Steinberg character* *of G is $\mathrm{St}_G := D_{\mathbf{G}}(1_G)$.*

By 6.6, St_G is irreducible up to sign. But $\mathrm{St}_G(1) > 0$ (see next slide), so St_G is an irreducible character of G.

Example 6.8

Consider the case $G = GL_2(q)$: $\mathbf{B} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$. Then $\mathcal{P} = \{\mathbf{G}, \mathbf{B}\}$, and $r(\mathbf{G}) = 1$, $r(\mathbf{B}) = 0$.
Note $\mathbf{T}^F = T_1$, the split torus and $\mathbf{B}^F = UT_1$. Hence

$$D_{\mathbf{G}}(1_G) = R^G_{T_1} \circ {}^*R^G_{T_1}(1_G) - 1_G = R^G_{T_1}(1_{T_1}) - 1_G = \rho,$$

the Steinberg character of G.

Theorem 6.9 (Values of the Steinberg character)

Let $\mathbf{G}$ be connected reductive with Frobenius F, $G = \mathbf{G}^F$ and $g \in G$. Then

$$\mathrm{St}_G(g) = \varepsilon_{\mathbf{G}} \varepsilon_{\mathrm{C}_{\mathbf{G}}(g)^0} |\mathrm{C}^0_{\mathbf{G}}(g)^F|_p$$

if g is semisimple, and 0 otherwise. In particular, $\mathrm{St}_G(1) = |G|_p$.

Sketch of Proof. Let $\mathbf{C} := \mathrm{C}^0_{\mathbf{G}}(g)$, $C := \mathbf{C}^F$, and let s be the semisimple part of g. First show

$$\mathrm{St}_G(g) = D_{\mathbf{G}}(1_G)(g) = \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{C}} D_{\mathbf{C}}(1_C)(g) = \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{C}} \mathrm{St}_C(g)$$

to reduce to the case $s \in Z(G)$. In the latter case, put g in an F-stable Borus $\mathbf{B} \supset \mathbf{T}$, and show

$$(\mathrm{St}_G)|_{\mathbf{B}^F} = \mathrm{Ind}_{\mathbf{T}^F}^{\mathbf{B}^F}(1_{\mathbf{T}^F}). \square$$

Theorem 6.9 yields the somewhat surprising

Corollary 6.10

The number of p-elements in a finite group of Lie type $G = \mathbf{G}^F$, and the number of F-stable maximal tori in **G***, are both equal to $|G|_p^2$.*

If **G** is simply connected, then $C_{\mathbf{G}}(s)$ is connected for any semisimple element $s \in \mathbf{G}$. So 6.9 yields that $\mathrm{St}_G(g) = \pm|C_G(g)|$ if g is semisimple in $G := \mathbf{G}^F$ and 0 otherwise.

Feit's conjecture: this property uniquely determines the Steinberg character.

Proved by **T.** in 1996.

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Feit's conjecture: this property uniquely determines the Steinberg character.

Proved by **T.** in 1996.

Proposition 6.11 (Degree of Deligne-Lusztig characters)

For any F-stable maximal torus $\mathbf{T}$ *and* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$,

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}|\mathbf{G}^F|_{p'}/|\mathbf{T}^F|.$$

Note. If $\mathbf{T}$ is the w-twist of a maximally split torus $\mathbf{T}_0$ for $w \in W = \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$, then $\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} = (-1)^{\ell(w)} = \det(w)$.

Proof of 6.11. Let $G = \mathbf{G}^F$, $T = \mathbf{T}^F$. Now apply 6.6:

$$\begin{aligned}[R_{\mathbf{T}}^{\mathbf{G}}(\theta), \mathrm{St}_G] = [R_{\mathbf{T}}^{\mathbf{G}}(\theta), D_{\mathbf{G}}(1_G)] &= [D_{\mathbf{G}}R_{\mathbf{T}}^{\mathbf{G}}(\theta), 1_G] \\ &\overset{6.6}{=} [\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R_{\mathbf{T}}^{\mathbf{G}}(\theta), 1_G] = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}[\theta, {}^*R_{\mathbf{T}}^{\mathbf{G}}(1_G)] \\ &\overset{5.11}{=} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}[\theta, 1_T] = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}\delta_{1_T,\theta}.\end{aligned}$$

Hence $[\sum_{\theta\in\mathrm{Irr}(T)} R_{\mathbf{T}}^{\mathbf{G}}(\theta), \mathrm{St}_G] = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}$.

Next, $\sum_{\theta\in\mathrm{Irr}(T)} R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is the G-character afforded by $H_c^*(\mathcal{L}^{-1}(\mathbf{U}))$.

Now, if $1 \neq s \in G$ is semisimple, then s acts on $\mathcal{L}^{-1}(\mathbf{U})$ by left multiplication, whence $\mathcal{L}^{-1}(\mathbf{U})^s = \emptyset$.

Hence, by 5.6, $\mathfrak{L}(s, \mathcal{L}^{-1}(\mathbf{U})) = \mathfrak{L}(1, \mathcal{L}^{-1}(\mathbf{U})^s) = 0$.

Thus $\sum_{\theta\in\mathrm{Irr}(T)} R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = 0$ if $g \neq 1$ is semisimple.

By 6.9, $\mathrm{St}(g) = 0$ unless g is semisimple, and $\mathrm{St}_G(1) = |G|_p$.

By 6.2, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1)$ does not depend on θ.

Putting altogether:

$$\begin{aligned}
\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} &= [\sum_{\theta\in\mathrm{Irr}(T)} R_{\mathbf{T}}^{\mathbf{G}}(\theta), \mathrm{St}_G] \\
&= \sum_{\theta\in\mathrm{Irr}(T)} R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1)\,\mathrm{St}_G(1)/|G| \\
&= |T| \cdot R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1)|G|_p/|G| \\
&= R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1)|T|/|G|_{p'}.\square
\end{aligned}$$

An analogue of 6.11 also holds for Deligne-Lusztig induction:

Proposition 6.12

For any L-stable Levi $\mathbf{L}$ *and* $\varphi \in \mathrm{Irr}(\mathbf{L}^F)$,

$$R_{\mathbf{L}}^{\mathbf{G}}(\varphi)(1) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}\varphi(1)|\mathbf{G}^F|_{p'}/|\mathbf{L}^F|_{p'}.$$

This is obvious for Harish-Chandra induction, but not for Deligne-Lusztig induction!

Example 6.13 (Example 5.21 continued)

Consider $\mathbf{G} = GL_2$ and $G = \mathbf{G}^F = GL_2(q)$.
Since $W = C_2$, there are two G-classes of F-stable maximal tori in $\mathbf{G}$.
The maximally split ones give $\mathbf{T}^F = T_1 \cong C_{q-1} \times C_{q-1}$; the other ones lead to $T_2 \cong C_{q^2-1}$, as in §2. Note that $\varepsilon_G = 2 = \varepsilon_{T_1},\ \ \varepsilon_{T_2} = 1$.
We already computed $R_{T_1}^G(\theta)$ in Example 5.21.

Consider $\mu := R_{T_2}^G(1_{T_2})$. Then

$[1_G, \mu] = [1_G, R_{T_2}^G(1_{T_2})] = [{}^*R_{T_2}^G(1_G), 1_{T_2}] \overset{5.11}{=} [1_{T_2}, 1_{T_2}] = 1,$

$[\mathrm{St}_G, \mu] = [D_G(1_G), R_{T_2}^G(1_{T_2})] = [1_G, D_G(R_{T_2}^G(1_{T_2}))] \overset{6.6}{=} [1_G, -\mu] = -1,$
and $[\mu, \mu] = 2$ by 5.16.
Hence, $\mu = 1_G - \mathrm{St}_G$.

This yields the values of the Green function
$Q_{T_2}^G(1) = \mu(1) = 1 - q, \ \ Q_{T_2}^G(u) = \mu(u) = 1$ for $1 \neq u \in G$ unipotent.

Now using Theorem 6.2 one can compute all $R_{T_2}^G(\theta)$.

Outline of Section 7

7 Lusztig's classification of characters

- $\mathrm{Irr}(\mathbf{G}^F)$ and $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$'s
- Duality of reductive groups
- Geometric conjugacy
- Lusztig series
- Jordan decomposition of $\chi \in \mathrm{Irr}(\mathbf{G}^F)$

Let $\mathcal{T}$ be the set of all F-stable maximal tori of $\mathbf{G}$, and let $G = \mathbf{G}^F$.

Proposition 7.1

The principal character 1_G *is*

$$1_G = |G|^{-1} \sum_{\mathbf{T} \in \mathcal{T}} |\mathbf{T}^F| R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F}).$$

Proof. Let σ be the RHS.

By 5.11, ${}^*R_{\mathbf{T}}^{\mathbf{G}}(1_G) = 1_{\mathbf{T}^F}$. Hence

$$[1_G, \sigma] = |G|^{-1} \sum_{\mathbf{T} \in \mathcal{T}} |\mathbf{T}^F| [{}^*R_{\mathbf{T}}^{\mathbf{G}}(1_G), 1_{\mathbf{T}^F}] = \sum_{\mathbf{T} \in \mathcal{T}} |\mathbf{T}^F| / |G|$$

which can be shown to be 1.

By the proof of 5.16, $[R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F}), R_{\mathbf{T}'}^{\mathbf{G}}(1_{\mathbf{T}'^F})]$ is 0 unless $\mathbf{T}, \mathbf{T}' \in \mathcal{T}$ are G-conjugate, in which case it is $|\mathrm{N}_G(\mathbf{T})|/|\mathbf{T}^F|$.
The number of $\mathbf{T}' \in \mathcal{T}$ that are G-conjugate to $\mathbf{T}$ is $|G|/|\mathrm{N}_G(\mathbf{T})|$.
Hence,

$$\begin{aligned}[\sigma, \sigma] &= \frac{1}{|G|^2} \sum_{\mathbf{T}, \mathbf{T}' \in \mathcal{T}} |\mathbf{T}^F| \cdot |\mathbf{T}'^F| \cdot [R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F}), R_{\mathbf{T}'}^{\mathbf{G}}(1_{\mathbf{T}'^F})] \\ &= \frac{1}{|G|^2} \sum_{\mathbf{T} \in \mathcal{T}} |\mathbf{T}^F|^2 \cdot (|\mathrm{N}_G(\mathbf{T})|/|\mathbf{T}^F|) \cdot (|G|/|\mathrm{N}_G(\mathbf{T})|) \\ &= \sum_{\mathbf{T} \in \mathcal{T}} |\mathbf{T}^F|/|G| = 1.\end{aligned}$$

It follows that

$$[\sigma - 1_G, \sigma - 1_G] = [\sigma, \sigma] - 2[\sigma, 1_G] + 1 = 1 - 2 + 1 = 0,$$

whence $\sigma = 1_G$. □

Theorem 7.2

The character reg_G *of the regular representation of* $G = \mathbf{G}^F$ *is*

$$\mathrm{reg}_G = \frac{1}{|G|_p} \sum_{\mathbf{T}\in\mathcal{T}} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{reg}_{\mathbf{T}^F}) = \frac{1}{|G|_p} \sum_{\mathbf{T}\in\mathcal{T},\ \theta\in\mathrm{Irr}(\mathbf{T}^F)} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta).$$

Proof. The two formulae are equivalent. Let's prove the 1st.
Let $\gamma : G \to \mathbb{Z}$ with $\gamma(g) = |G|_{p'}$ if $g \in G_u$ and 0 otherwise.
Write $T := \mathbf{T}^F$ and compute $R_{\mathbf{T}}^{\mathbf{G}}(\gamma|_T)(g)$.
By Theorem 6.2, it is 0 unless g is unipotent, in which case it is $\left(R_{\mathbf{T}}^{\mathbf{G}}(1_T)\gamma\right)(g)$. Hence

$$R_{\mathbf{T}}^{\mathbf{G}}(1_T)\gamma = R_{\mathbf{T}}^{\mathbf{G}}(\gamma|_T).$$

By the definition of γ, $\gamma|_T = (\gamma(1)/|T|)\mathrm{reg}_T = (|G|_{p'}/|T|)\mathrm{reg}_T$.

Hence, $R_{\mathbf{T}}^{\mathbf{G}}(1_T)\gamma = (|G|_{p'}/|T|)R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{reg}_T)$.
Applying 6.6 and 7.1, we get

$$\mathrm{St}_G = D_{\mathbf{G}}(1_G) = |G|^{-1}\sum_{\mathbf{T}\in\mathcal{T}} |\mathbf{T}^F|\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F}).$$

The formula 6.9 for St_G implies $\mathrm{reg}_G = \gamma\cdot\mathrm{St}_G$. Consequently,

$$\begin{aligned}\mathrm{reg}_G &= |G|^{-1}\sum_{\mathbf{T}\in\mathcal{T}} |\mathbf{T}^F|\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})\gamma \\ &= |G|^{-1}\sum_{\mathbf{T}\in\mathcal{T}} |\mathbf{T}^F|\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}(|G|_{p'}/|\mathbf{T}^F|) R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{reg}_{\mathbf{T}^F}) \\ &= |G|_p^{-1}\sum_{\mathbf{T}\in\mathcal{T}} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{reg}_{\mathbf{T}^F}).\square\end{aligned}$$

Theorem 7.2 shows: *Each irreducible character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ is an irreducible constituent of some $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.*

So, to find $\mathrm{Irr}(\mathbf{G}^F)$, "just" have to decompose all Deligne-Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ ...

Hence, $R_{\mathbf{T}}^{\mathbf{G}}(1_T)\gamma = (|G|_{p'}/|T|)R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{reg}_T)$.
Applying 6.6 and 7.1, we get

$$\mathrm{St}_G = D_{\mathbf{G}}(1_G) = |G|^{-1}\sum_{\mathbf{T}\in\mathcal{T}}|\mathbf{T}^F|\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F}).$$

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Definition 7.3

Let $\mathbf{G}$ and $\mathbf{G}^*$ be two connected reductive groups over k.
(i) Say $\mathbf{G}$ and $\mathbf{G}^*$ are dual to each other, if there are maximal tori $\mathbf{T} \subset \mathbf{G}$ and $\mathbf{T}^* \subset \mathbf{G}$ and an isomorphism $\pi : X(\mathbf{T}) \cong Y(\mathbf{T}^*)$ that sends roots of $\mathbf{G}$ to coroots of $\mathbf{G}^*$.

(ii) Suppose in addition that $F : \mathbf{G} \to \mathbf{G}$, $F^* : \mathbf{G}^* \to \mathbf{G}^*$ are Frobenii, $\mathbf{T}$ is F-stable, $\mathbf{T}^*$ is F^*-stable, and π is compatible with the actions of F and F^*. Then we say the pair $(\mathbf{G}, F)$ is dual to the pair $(\mathbf{G}^*, F^*)$.
In this case we also say that the dual of $\mathbf{G}^F$ is ${\mathbf{G}^*}^{F^*}$.

Fact 7.4

If $(\mathbf{G}, F)$ is dual to $(\mathbf{G}^*, F^*)$ with corresponding dual tori $\mathbf{T}$, $\mathbf{T}^*$, then $|\mathbf{G}^F| = |{\mathbf{G}^*}^{F^*}|$, and $\mathrm{Irr}(\mathbf{T}^F) \cong {\mathbf{T}^*}^{F^*}$.

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Example 7.5 (Some dual pairs)

- (GL_n, GL_n), (Sp_{2n}, SO_{2n+1}), (SO_{2n}, SO_{2n}).
- (SL_n, PSL_n), where $PSL_n = SL_n/\mathrm{Z}(SL_n)$ is the projective special linear group over k.
- $(GL_n(q), GL_n(q))$, $(GU_n(q), GU_n(q))$
- $(SL_n(q), PGL_n(q))$, where $PGL_n(q) = GL_n(q)/\mathrm{Z}(GL_n(q))$ is the projective general linear group over $\mathbb{F}_q$.
- $(Sp_{2n}(q), SO_{2n+1}(q))$.
- $(SO^\epsilon_{2n}(q), SO^\epsilon_{2n}(q))$, $\epsilon = \pm$. Here,

$SO^\epsilon_{2n}(q) = \{f \in SL(V) \mid f \text{ fixes } Q^\epsilon\}$ for $V = \langle e_1, \ldots, e_{2n}\rangle_{\mathbb{F}_q}$

$Q^+(\sum_{i=1}^{2n} x_i e_i) = \sum_{i=1}^{n} x_{2i-1}x_{2i}$

$Q^-(\sum_{i=1}^{2n} x_i e_i) = x_1^2 + x_1x_2 + ax_2^2 + \sum_{i=2}^{n} x_{2i-1}x_{2i}$, where $t^2 + t + a \in \mathbb{F}_q[t]$ is irreducible.

Let $\mathbf{T}$ be an F-stable maximal torus, $Y := Y(\mathbf{T})$, and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. Since $\mathbf{T}^F \cong Y/(F-1)Y$, θ can be viewed as a character of Y.

There is a norm map:

$$\mathrm{Norm}_n : \mathbf{T}^{F^n} \to \mathbf{T}^F, \quad t \mapsto t \cdot F(t) \cdot F^2(t) \cdot \ldots \cdot F^{n-1}(t).$$

Definition 7.6

Let $\mathbf{T}$, $\mathbf{T}'$ be F-stable maximal tori of $\mathbf{G}$, $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, $\theta' \in \mathrm{Irr}(\mathbf{T}'^F)$. $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are geometrically conjugate if they satisfy one of the two equivalent conditions :

(i) There is a $g \in \mathbf{G}$ that conjugates $\mathbf{T}$ to $\mathbf{T}'$ and the $Y(\mathbf{T})$-character θ to the $Y(\mathbf{T}')$-character θ'.

(ii) For some $n \in \mathbb{N}$, there is $g \in \mathbf{G}^{F^n}$ that conjugates $\mathbf{T}$ to $\mathbf{T}'$ and the $\mathbf{T}^{F^n}$-character $\theta \circ \mathrm{Norm}_n$ to the $\mathbf{T}'^{F^n}$-character $\theta' \circ \mathrm{Norm}_n$.

Recall 5.15: If $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are not $\mathbf{G}^F$-conjugate, then

$$[R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')]_{\mathbf{G}^F} = 0.$$

Note: virtual characters α, β with $[\alpha, \beta]_G = 0$ may still share common irreducible constituents.

But here we have a much stronger result:

Theorem 7.7

Let $\mathbf{T}$, $\mathbf{T}'$ be F-stable maximal tori of $\mathbf{G}$, $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, $\theta' \in \mathrm{Irr}({\mathbf{T}'}^F)$. Suppose $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are not geometrically conjugate. Then the virtual characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ have no irreducible constituents in common.

Geometric conjugacy is best understood in terms of dual group.

Proposition 7.8

Suppose that $(\mathbf{G}, F)$ *is dual to* $(\mathbf{G}^*, F^*)$. *Then*

(i) *The geometric conjugacy classes of* $(\mathbf{T}, \theta)$, *where* $\mathbf{T}$ *is* F*-stable and* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, *are in bijective correspondence with* $\mathbf{G}^*$*-conjugacy classes of semisimple elements in* $\mathbf{G}^*$ *that meet* ${\mathbf{G}^*}^{F^*}$.

(ii) *The* $\mathbf{G}^F$*-conjugacy classes of* $(\mathbf{T}, \theta)$, *where* $\mathbf{T}$ *is* F*-stable and* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, *are in bijective correspondence with* ${\mathbf{G}^*}^{F^*}$*-conjugacy classes of pairs* $(\mathbf{T}^*, s)$, *where* $\mathbf{T}^*$ *is an* F^**-stable maximal torus of* $\mathbf{G}^*$ *and* $s \in {\mathbf{T}^*}^{F^*}$.

In this case, instead of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ we also write $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$.

Definition 7.9

Suppose that $G = \mathbf{G}^F$ is dual to $G^* := \mathbf{G}^{*F^*}$, and fix a semisimple element $s \in G^*$.

(i) The Lusztig series $\mathcal{E}(G, (s)_{\mathbf{G}^*})$ is the set of all irreducible characters of $\mathrm{Irr}(G)$ that occur in some $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, where the geometric conjugacy class of $(\mathbf{T}, \theta)$ corresponds to the $\mathbf{G}^*$-conjugacy class of s in 7.8(i).

(ii) The rational Lusztig series $\mathcal{E}(G, (s))$ is the set of all irreducible characters of $\mathrm{Irr}(G)$ that occur in some $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ as in 7.8(ii).

Note: If $C_{\mathbf{G}^*}(s)$ is connected (eg. when $Z(\mathbf{G})$ is connected), then the Lusztig series and the rational Lusztig series corresponding to s coincide.

GL_n has connected center, but SL_n does not.
Hence $GL_n(q)$ behaves better than $SL_n(q)$!

Theorems 7.2, 7.7, and Proposition 7.8 imply:

Corollary 7.10

$\mathrm{Irr}(\mathbf{G}^F)$ *is the disjoint union of distinct Lusztig series* $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^*})$.

A passage to groups with connected center is needed to prove

Proposition 7.11

$\mathrm{Irr}(\mathbf{G}^F)$ *is the disjoint union of distinct rational Lusztig series* $\mathcal{E}(\mathbf{G}^F, (s))$.

Definition 7.12

The characters in $\mathcal{E}(\mathbf{G}^F, (1))$, *i.e. the irreducible constituents of* $R^{\mathbf{G}}_{\mathbf{T}}(1_{\mathbf{T}^F})$, *where* $\mathbf{T}$ *is any* F*-stable maximal torus of* $\mathbf{G}$, *are called unipotent characters of* $\mathbf{G}^F$.

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Example 7.13 (Example 6.13 continued.)

Again consider $G = GL_2(q)$. Keep the notation of §2.
As shown in Example 6.13, $\mathcal{E}(G,(1)) = \{1_G, \mathrm{St}_G\} = \{U_1, V_1\}$.
Next, G^* can be identified with G. Can also identify

$$\mu_1 : \mathbb{F}_q^\times \longleftrightarrow \mathrm{Irr}(\mathbb{F}_q^\times), \quad \mu_2 = \mu_1 \times \mu_1 : T_1 \longleftrightarrow \mathrm{Irr}(T_1).$$
$$\nu : T_2 \longleftrightarrow \mathrm{Irr}(T_2).$$

If $s \in \mathrm{Z}(G) \cong \mathrm{Irr}(\mathbb{F}_q^\times)$ corresponds to $\alpha \in \mathrm{Irr}(\mathbb{F}_q^\times)$ under μ_1, then $\mathcal{E}(G,(s)) = \{U_\alpha, V_\alpha\}$.

All other semisimple elements are either $c_{x,y}$ or d_z.

If $s = c_{x,y}$ corresponds to $L_{\alpha,\beta}$ under μ_2, then $\mathcal{E}(G,(s)) = W_{\alpha,\beta}$.

If $s = d_z$ corresponds to γ under ν, then $\mathcal{E}(G,(s)) = T_\gamma$.

This explains the symmetry of the character table of $GL_2(q)$.

Example 7.14 (Example 7.13 continued.)

Again consider $G = GL_2(q)$. Example 6.13 implies:

$$1_G = (R^G_{T_1}(1_{T_1}) + R^G_{T_2}(1_{T_2}))/2, \quad \mathrm{St} = (R^G_{T_1}(1_{T_1}) - R^G_{T_2}(1_{T_2}))/2.$$

It follows that any $\chi \in \mathrm{Irr}(G)$ is a $\mathbb{Q}$-linear combinations of $R^G_T(\theta)$'s, i.e. G is uniform.

Example 7.15

However, the Weil representations $W_\pm$, $T_\pm$ of $G = SL_2(q)$ show: the $R^G_T(\theta)$'s do not span $\mathbb{C}[\mathrm{Irr}(G)]$ (as they are the same on unipotent elements b_1 and b_1'). Thus $SL_2(q)$ is not uniform.

They also illustrate the issues with disconnected centralizers of semisimple elements in $G^* = PGL_2(q)$:
Eg. $s = \mathrm{diag}(1, -1)$ has disconnected centralizer in PGL_2.

Example 7.14 (Example 7.13 continued.)

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A highlight of Deligne-Lusztig theory is

Theorem 7.16 (Lusztig)

Let $\mathbf{G}$ *be connected reductive with Frobenius* F*, and let* $G = \mathbf{G}^F$ *be dual to* $G^* = \mathbf{G}^{*F^*}$*. For any semisimple element* $s \in G^*$*, there is a bijection* $J_s : \mathcal{E}(G, (s)) \to \mathcal{E}(\mathrm{C}_{G^*}(s), (1))$*, such that*

$$\chi(1) = [G : \mathrm{C}_{G^*}(s)]_{p'} \cdot J_s(\chi)(1)$$

for all $\chi \in \mathcal{E}(G, (s))$*, and* $J_s(R^{\mathbf{G}}_{\mathbf{T}^*}(s)) = \varepsilon_{\mathbf{G}} \varepsilon_{\mathrm{C}^0_{\mathbf{G}^*}(s)} R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(1_{\mathbf{T}^{*F^*}})$*.*

Lusztig first proved Theorem 7.16 for $\mathbf{G}$ with connected center. To handle the remaining groups, he used regular embeddings $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ (i.e. such that $\mathrm{Z}(\tilde{\mathbf{G}})$ is connected and $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = [\mathbf{G}, \mathbf{G}]$). One also needs to extend the notion of $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ to finite groups arising from disconnected reductive groups $\mathbf{G}$: $R^{\mathbf{G}}_{\mathbf{T}}(\theta) = \mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{0F}}(R^{\mathbf{G}^0}_{\mathbf{T}}(\theta))$.

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Regular embeddings

Let $\mathbf{G}$ be connected reductive, with possibly disconnected $Z(\mathbf{G})$, and a Frobenius F.
Let r be the rank of the abelian group $Z(\mathbf{G})$. Then one can embed $Z(\mathbf{G})$ in $\mathbf{S} \cong \mathbb{G}_m^r$ with Frobenius $F : \mathbf{S} \to \mathbf{S}$ extending the action of F on $Z(\mathbf{G})$. Define

$$\tilde{\mathbf{G}} = \mathbf{G} \times_{Z(\mathbf{G})} \mathbf{S} := (\mathbf{G} \times \mathbf{S})/Z, \text{ with } Z := \{(z, z^{-1}) \mid z \in Z(\mathbf{G})\}.$$

Then $g \mapsto (g, 1)Z$ gives a regular embedding $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$.

Lusztig: *If $\chi \in \mathrm{Irr}(\tilde{\mathbf{G}}^F)$ then $\chi|_{\mathbf{G}^F}$ is multiplicity-free.*

Example 7.17

Let $G = SL_n$. Then $Z(\mathbf{G})$ is finite of order dividing n, and so usually disconnected.
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Weil representations

Let $\mathbf{G} = Sp_{2n}$, $G = \mathbf{G}^F = Sp_{2n}(q)$, with odd q.

Then $\mathbf{G}^* = SO_{2n+1}$, and $G^* = \mathbf{G}^{*F^*} = SO_{2n+1}(q) = SO(V)$, the group of linear transformations of det. 1 that preserve the quadratic form $Q(\sum_{i=0}^{2n} x_i e_i) = x_0^2 + \sum_{i=1}^{2n} x_i x_{n+i}$ on $V = \mathbb{F}_q^{2n+1}$.

Consider a "minus-reflection" $s^+ = \mathrm{diag}(1, -I_{2n}) \in G^*$, so that

$$\mathrm{Ker}(s+1) = W := e_0^{\perp} = \langle e_1, \ldots, e_{2n} \rangle_{\mathbb{F}_q}.$$

Then $\mathbf{C} := \mathrm{C}_{\mathbf{G}^*}(s^+) \cong GO(W \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q) = GO_{2n}$ is disconnected, and $\mathbf{C}/\mathbf{C}^0 \cong C_2$.

Hence $s^{\mathbf{G}^*} \cap G^*$ breaks into 2 G^*-conjugacy classes, one of s^+, and another of a "minus-reflection" s^-, which fixes a vector $e' \in V$ with $Q(e') =$non-square in $\mathbb{F}_q$. Note

$$\mathrm{C}_{G^*}(s^+) \cong GO(W) \cong GO_{2n}^+(q), \ \ \mathrm{C}_{G^*}(s^-) \cong GO((e')^{\perp}) \cong GO_{2n}^-(q).$$

For $\epsilon = \pm$: induce the principal (unipotent) character of $SO^{\epsilon}_{2n}(q)$ to $C_{G^*}(s^{\epsilon})$ and decompose to get two unipotent characters of $C_{G^*}(s^{\epsilon})$ of degree 1. Also,

$$[G^* : C_{G^*}(s^{\epsilon})]_{p'} = (q^n + \epsilon)/2.$$

Theorem 7.16 gives

- two irreducible chars $W_{\pm}$ of degree $(q^n + 1)/2$ in $\mathcal{E}(G, (s^+))$;
- two irreducible chars $T_{\pm}$ of degree $(q^n - 1)/2$ in $\mathcal{E}(G, (s^-))$.

These are Weil characters of $G = Sp_{2n}(q)$.

Taking $n = 1$, we obtain the four Weil characters, and therefore all irreducible characters of $SL_2(q)$, as described in §3.

Weil representations are remarkable in many respects. Eg. they lead to dense sphere packings (**Elkies, Shioda, Gross, Dummigan, T.**), by considering Mordell-Weil groups of certain elliptic curves over some function fields.

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