ASYMPTOTIC DEPTH OF INVARIANT CHAINS OF EDGE IDEALS

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Abstract. We completely determine the asymptotic depth, equivalently, the asymptotic projective dimension of a chain of edge ideals that is invariant under the action of the monoid Inc of increasing functions on the positive integers. Our results and their proofs also reveal surprising combinatorial and topological properties of corresponding graphs and their independence complexes. In particular, we are able to determine the asymptotic behavior of all reduced homology groups of these independence complexes.

1. Introduction

For $n \geq 1$ let $R_n = \Bbbk[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$. A vibrant area of research at the crossroads of algebraic geometry, combinatorics, commutative algebra, group theory, representation theory, and statistics concerning chains of ideals $\mathcal{I} = (I_n)_{n \geq 1}$ with $I_n \subseteq R_n$ for $n \geq 1$ that are invariant under the actions of the infinite general linear group, the infinite symmetric group, or more generally, the monoid Inc of strictly increasing functions; see, e.g. [1–9, 14, 24, 26, 33, 34, 38, 39]. See also [17, 20, 25, 27, 28] for related directions in discrete geometry, convex optimization, and machine learning.

It follows from a celebrated result of Cohen [3] that any Inc-invariant chain $\mathcal{I} = (I_n)_{n \geq 1}$ *stabilizes* (see Section 3 for more details). That is, from some index r on, any ideal I_n with $n \geq r$ is completely determined by I_r up to the Inc-action. One might therefore hope that invariants related to these ideals are well-behaved. In [21, 22], it is conjectured that the regularity and projective dimension of I_n are eventually linear functions in n. See [32] for extensions of these conjectures to OI-modules. Although significant evidence for the conjectures has been obtained [11, 23, 30, 31, 37, 40], they remain wide open.

Recently, the regularity conjecture [21] has been verified for chains of edge ideals in [15]. It is shown that the regularity of an ideal in such a chain is eventually a constant, that, somewhat surprisingly, can only be 2 or 3. Proving this result requires a deep understanding of the chain of graphs corresponding to the chain of edge ideals. It turns out, for instance, that eventually such a graph can only have an induced matching number of at most 2.

As a continuation of [15], this paper explores further properties of Inc-invariant chains of edge ideals. Our main goal is to verify the projective dimension conjecture [22] for those chains. More specifically, we are interested in the following stronger problem.

Problem 1.1. *Let $\mathcal{I} = (I_n)_{n \geq 1}$ be an* Inc*-invariant chain of edge ideals. Compute explicitly* $\operatorname{depth}(R_n/I_n)$ *for large n, in combinatorial terms.*

2020 *Mathematics Subject Classification.* 13A50, 13C15, 13D02, 13F20, 16P70, 16W22.

Key words and phrases. Depth, edge ideal, independence complex, invariant ideal, reduced homology.

The main result of the paper resolves this problem, providing an explicit and simple formula for $\operatorname{depth}(R_n/I_n)$ when $n \gg 0$. In particular, it shows that $\operatorname{depth}(R_n/I_n)$ is eventually a constant. This implies, via the Auslander–Buchsbaum formula, that the projective dimension of I_n is eventually a linear function in n. Below, the *support* of a monomial ideal is the set of variables that divide at least one minimal monomial generator of that ideal.

Theorem 1.2. *Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an* Inc*-invariant chain of (eventually non-zero) edge ideals with the stability index* $\operatorname{ind}(\mathcal{I}) = r$. *Assume that x_1 and x_r belong to the support of I_r. Denote $j_q = \max\{j \mid 1 \leq j \leq r, x_1x_j \in I_r\}$ and $\operatorname{sp}(\mathcal{I}) = \min\{j-i \mid 1 \leq i \leq j \leq r, x_ix_j \in I_r\}$. Then the following hold for all $n \geq 3r$.*

(i) *If $j_q = r$, then* $\operatorname{depth}(R_n/I_n) = \operatorname{sp}(\mathcal{I})$.
(ii) *If $j_q < r$, then* $\operatorname{depth}(R_n/I_n) = \min\{\operatorname{sp}(\mathcal{I}), 2\}$.

It should be noted that the assumption that x_1 and x_r belong to the support of I_r in the previous theorem is not restrictive, as the general case can always be reduced to this case by simple index shifts (see Lemma 3.2). Note also that the index j_q and the invariant $\operatorname{sp}(\mathcal{I})$, which we call the *sparsity index* of the chain $\mathcal{I}$, play a crucial role in this result.

Theorem 1.2 exhibits a dichotomy for $\operatorname{depth}(R_n/I_n)$ when $n \gg 0$. This is very similar to the dichotomy for $\operatorname{reg}(I_n)$ mentioned above. It would be interesting to have some explanation for this similarity.

Our proof of Theorem 1.2 makes use of Takayama's formula, which allows us to interpret the problem of computing $\operatorname{depth}(R_n/I_n)$ as the problem of computing certain reduced homology groups of the independence complex $\operatorname{IN}(G_n)$ of the graph G_n corresponding to the ideal I_n. This approach, therefore, inspires the following problem.

Problem 1.3. *Determine all reduced homology groups of $\operatorname{IN}(G_n)$ for $n \gg 0$.*

In the present paper, we also obtain a complete solution to this problem. As in Theorem 1.2, the index j_q as well as the sparsity index $\operatorname{sp}(\mathcal{I})$ are essential for the statement of our result.

Theorem 1.4. *Keep the assumptions of Theorem 1.2. Denote by $\widetilde{H}_i(\operatorname{IN}(G_n))$ the i-th reduced homology group of $\operatorname{IN}(G_n)$ over the field $\Bbbk$. Then there exist two nonnegative integers α, β depending only on the chain $\mathcal{I}$ such that the following hold for all $n \gg 0$.*

(i) $\widetilde{H}_i(\operatorname{IN}(G_n)) = 0$ *for* $i \neq 0, 1$.
(ii) $\widetilde{H}_0(\operatorname{IN}(G_n)) \cong \begin{cases} \Bbbk^{n-\alpha} & \text{if } \operatorname{sp}(\mathcal{I}) = 1, \\ 0 & \text{if } \operatorname{sp}(\mathcal{I}) \geq 2. \end{cases}$
(iii) $\widetilde{H}_1(\operatorname{IN}(G_n)) \cong \Bbbk^{\beta}$. *Furthermore, if* $\operatorname{sp}(\mathcal{I}) \geq 2$, *then* $\beta = \begin{cases} 0 & \text{if } j_q = r, \\ 1 & \text{if } j_q < r. \end{cases}$

Theorem 1.4 is an unexpected outcome of our approach to Problem 1.1. On the other hand, the proof of Item (ii) (which is the harder part) of Theorem 1.2 depends crucially on Theorem 1.4(iii) (see Proposition 6.2). It is worth mentioning that in the situation where $\operatorname{sp}(\mathcal{I}) \geq 2$ and $j_q < r$, Theorem 1.4 provides the remarkable information that $\operatorname{IN}(G_n)$ has the same reduced homology as the sphere $\mathbb{S}^1$ for all $n \gg 0$.

This paper marks the first successful application of Takayama's formula to problems on Inc-invariant chains of monomial ideals. We believe that many of our results and proofs, for example those in Section 4 and parts of Sections 5 and 6, are amenable to more general situations beyond edge ideals of graphs. Moreover, as an extention of Problem 1.3, the theme of asymptotic homology stability of simplicial complexes associated to Inc-invariant chains of monomial ideals is worthy of further investigation.

Let us now describe the structure of the paper. Section 2 provides graph terminology and some basic properties of monomial ideals. In Section 3, we review Inc-invariant chains of edge ideals and present some auxiliary results that reveal interesting structures of corresponding graphs. Upper and lower bounds for the asymptotic depth of an Inc-invariant chain of edge ideals are given in Section 4. The next two sections show that the asymptotic depth of any Inc-invariant chain of edge ideals always attains one of the two bounds established in Section 4, thus proving Theorem 1.2. Section 7 is devoted to the proof of Theorem 1.4. Lastly, the Appendix proves a technical result from Section 6 that is decisive to the proof of Theorem 1.2(ii).

2. Preliminaries

We collect here necessary notions and results concerning graphs and monomial ideals. For unexplained terminology, the reader is referred to [13, 43]. Throughout the section, let $S = \Bbbk[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field $\Bbbk$ and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ its graded maximal ideal.

2.1. **Depth, projective dimension, and regularity.**

Let M be a finitely generated graded S-module. The *projective dimension* and the *(Castelnuovo-Mumford) regularity* of M are defined as

$$\begin{aligned} \operatorname{pd}(M) &:= \max\{i \mid \operatorname{Tor}_i^S(M, \Bbbk) \neq 0\}, \\ \operatorname{reg}(M) &:= \max\{j - i \mid \operatorname{Tor}_i^S(M, \Bbbk)_j \neq 0\}. \end{aligned}$$

In particular, for any nonzero homogeneous ideal $I \subseteq S$ we have

$$\operatorname{pd}(S/I) = \operatorname{pd}(I) + 1 \quad \text{and} \quad \operatorname{reg}(S/I) = \operatorname{reg}(I) - 1.$$

The *depth* of M is related to its projective dimension via the Auslander–Buchsbaum formula:

$$\operatorname{pd}(M) + \operatorname{depth}(M) = n.$$

In this paper, we will mainly use the following interpretations of depth and regularity in terms of local cohomology modules

$$\begin{aligned} \operatorname{depth}(M) &= \min\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}, \\ \operatorname{reg}(M) &= \max\{i + j \mid H_{\mathfrak{m}}^i(M)_j \neq 0\}, \end{aligned}$$

where $H_{\mathfrak{m}}^i(M)$ denotes the ith local cohomology module of M with respect to $\mathfrak{m}$.

2.2. **Takayama's formula.** For any monomial ideal $I \subseteq S$, there is a $\mathbb{Z}^n_{\geq 0}$-grading on S/I that is inherited from the natural $\mathbb{Z}^n_{\geq 0}$-grading of S. This induces a $\mathbb{Z}^n$-grading on the local cohomology module $H^i_{\mathfrak{m}}(S/I)$. The dimensions of the $\mathbb{Z}^n$-graded components of $H^i_{\mathfrak{m}}(S/I)$ are described by Takayama's formula, which we now recall. For brevity, we write $[n] = \{1, \dots, n\}$. Let $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ be an integral vector. Set $x^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Also, denote

$$\operatorname{supp}(\boldsymbol{a}) = \{i \in [n] \mid a_i \neq 0\} \quad \text{and} \quad G_{\boldsymbol{a}} = \{i \in [n] \mid a_i < 0\}.$$

For $F \subseteq [n]$, let $S_{(F)} = S[x_i^{-1} \mid i \in F]$. The *degree complex of I with respect to $\boldsymbol{a}$* is defined as

$$\Delta_{\boldsymbol{a}}(I) = \{F \setminus G_{\boldsymbol{a}} \mid G_{\boldsymbol{a}} \subseteq F \subseteq [n], x^{\boldsymbol{a}} \notin IS_{(F)}\}.$$

Then Takayama's formula [41, Theorem 1] (see also [29, Section 1]) states that

$$\dim_{\Bbbk} H^i_{\mathfrak{m}}(S/I)_{\boldsymbol{a}} = \dim_{\Bbbk} \widetilde{H}_{i-|G_{\boldsymbol{a}}|-1}(\Delta_{\boldsymbol{a}}(I)) \quad \text{for all } i \in \mathbb{Z} \text{ and all } \boldsymbol{a} \in \mathbb{Z}^n,$$

where $\widetilde{H}_i(\Delta_{\boldsymbol{a}}(I))$ denotes the i-th reduced homology of $\Delta_{\boldsymbol{a}}(I)$ over $\Bbbk$.

Takayama's formula is useful for studying $\operatorname{depth}(S/I)$. We gather here two simple results in this direction. For $F \subseteq [n]$, let $S_F = \Bbbk[x_i \mid i \notin F]$ and $I_F = IS_{(F)} \cap S_F$. Then I_F is the monomial ideal in S_F obtained from I by setting $x_i = 1$ for all $i \in F$. When $F = \{j\}$ for some $j \in [n]$, we simply write S_j and I_j instead of $S_{\{j\}}$ and $I_{\{j\}}$, respectively.

The following criterion for the vanishing of $H^1_{\mathfrak{m}}(S/I)$ from [42, Proposition 1.6] can be used to give a lower bound for $\operatorname{depth}(S/I)$.

Proposition 2.1. *Let $I \subseteq S$ be a monomial ideal. The following are equivalent:*

(i) $H^1_{\mathfrak{m}}(S/I) = 0$*;*
(ii) $\Delta_{\boldsymbol{a}}(I)$ *is connected for all* $\boldsymbol{a} \in \mathbb{Z}^n_{\geq 0}$ *and* $\operatorname{depth}(S_j/I_j) \geq 1$ *for all* $j \in [n]$.

The next result provides an upper bound for $\operatorname{depth}(S/I)$.

Proposition 2.2. *Let $I \subseteq S$ be a monomial ideal. Then*

$$\operatorname{depth}(S/I) \leq \min\{|F| \mid F \subseteq [n], \operatorname{depth}(S_F/I_F) = 0\}.$$

In order to prove this proposition, we need the following lemma, which is an easy consequence of [42, Corollary 1.4].

Lemma 2.3. *Let $I \subseteq S$ be a monomial ideal and F a subset of $[n]$. For any $\boldsymbol{a} \in \mathbb{Z}^n$, let $\boldsymbol{a}_+$ be the vector obtained from $\boldsymbol{a}$ by replacing each negative entry with zero. Then the following are equivalent:*

(i) $\widetilde{H}_{-1}(\Delta_{\boldsymbol{a}}(I)) = 0$ *for all* $\boldsymbol{a} \in \mathbb{Z}^n$ *with* $G_{\boldsymbol{a}} = F$*;*
(ii) $\Delta_{\boldsymbol{a}}(I) \neq \{\emptyset\}$ *for all* $\boldsymbol{a} \in \mathbb{Z}^n$ *with* $G_{\boldsymbol{a}} = F$*;*
(iii) $\operatorname{depth}(S_F/I_F) \geq 1$.

Proof. (i) $\Leftrightarrow$ (ii) is clear, since $\widetilde{H}_{-1}(\Delta_{\boldsymbol{a}}(I)) = 0$ if and only if $\Delta_{\boldsymbol{a}}(I) \neq \{\emptyset\}$.

(iii) $\Rightarrow$ (ii): Let $\widetilde{I_F} = \bigcup_{k \geq 1}(I_F : \mathfrak{m}_F^k)$ be the *saturation* of I_F, where $\mathfrak{m}_F = \langle x_i \mid i \notin F\rangle$ denotes the graded maximal ideal of S_F. Then it is well-known that $H^0_{\mathfrak{m}}(S_F/I_F) = \widetilde{I_F}/I_F$. Since depth$(S_F/I_F) \geq 1$, we have $H^0_{\mathfrak{m}}(S_F/I_F) = 0$, and hence $\widetilde{I_F} = I_F$. It follows that $x^{\boldsymbol{a}_+} \notin \widetilde{I_F} \setminus I_F$ for all $\boldsymbol{a} \in \mathbb{Z}^n$ with $G_{\boldsymbol{a}} = F$. By [42, Corollary 1.4], this means that $\Delta_{\boldsymbol{a}}(I) \neq \{\emptyset\}$ for all such $\boldsymbol{a}$.

(ii) $\Rightarrow$ (iii): Assume, to the contrary, that depth$(S_F/I_F) = 0$. Then $H^0_{\mathfrak{m}}(S_F/I_F) = \widetilde{I_F}/I_F \neq 0$. So there exists $\boldsymbol{b} \in \mathbb{Z}^n_{\geq 0}$ such that supp$(\boldsymbol{b}) \subseteq [n] \setminus F$ and $x^{\boldsymbol{b}} \in \widetilde{I_F} \setminus I_F$. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be the standard unit vectors of $\mathbb{Z}^n$. Setting $\boldsymbol{a} = \boldsymbol{b} - \sum_{i \in F} \boldsymbol{e}_i$, we get $G_{\boldsymbol{a}} = F$ and $\boldsymbol{a}_+ = \boldsymbol{b}$. Hence $x^{\boldsymbol{a}_+} \in \widetilde{I_F} \setminus I_F$, which by [42, Corollary 1.4] implies that $\Delta_{\boldsymbol{a}}(I) = \{\emptyset\}$. This contradiction shows that depth$(S_F/I_F) \geq 1$, as desired. ☐

Let us now prove Proposition 2.2.

Proof of Proposition 2.2. It suffices to show that if there exists $F \subseteq [n]$ with $|F| = i$ and depth$(S_F/I_F) = 0$, then $H^i_{\mathfrak{m}}(S/I) \neq 0$. Equivalently, this amounts to showing that if $H^i_{\mathfrak{m}}(S/I) = 0$, then for any $F \subseteq [n]$ with $|F| = i$ one has depth$(S_F/I_F) \geq 1$. Indeed, let $\boldsymbol{a} \in \mathbb{Z}^n$ be any vector with $G_{\boldsymbol{a}} = F$. Then by Takayama's formula

$$\dim_{\Bbbk} \widetilde{H}_{-1}(\Delta_{\boldsymbol{a}}(I)) = \dim_{\Bbbk} H^i_{\mathfrak{m}}(S/I)_{\boldsymbol{a}} = 0.$$

Hence $\widetilde{H}_{-1}(\Delta_{\boldsymbol{a}}(I)) = 0$, and therefore depth$(S_F/I_F) \geq 1$ by virtue of Lemma 2.3. ☐

2.3. **Graphs and edge ideals.** Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If, moreover, $E(H) = E(G) \cap \binom{V(H)}{2}$, then H is called an *induced subgraph* of G. The *complement* G^c of G is the graph on $V(G)$ with edge set $\binom{V(G)}{2} \setminus E(G)$. For an integer $m \geq 3$, a *cycle* of length m is a graph C_m with $V(C_m) = \{v_1, \dots, v_m\}$ and $E(C_m) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_m, v_1\}\}$. The graph G is called *weakly chordal* (or *weakly triangulated*) if neither G nor G^c contains an induced cycle of length at least 5.

A subset $U \subseteq V(G)$ is called *independent* if the vertices in U are pairwise non-adjacent. The *independence complex* of G, denote by IN(G), is the simplicial complex whose faces are the independent sets of G. It is evident that the 1-skeleton of IN(G) is exactly G^c. This yields the following fact that should be well-known.

Lemma 2.4. *Let G be a graph with at least one vertex. Then $\widetilde{H}_0(\mathrm{IN}(G)) \cong \Bbbk^{\mathfrak{c}(G^c)-1}$, where $\mathfrak{c}(G^c)$ denotes the number of connected components of G^c. In particular, $\widetilde{H}_0(\mathrm{IN}(G)) = 0$ if and only if G^c is connected.*

For a vertex $v \in V(G)$, its *open neighborhood* $N(v)$ is the set of vertices $u \neq v$ that are adjacent to v, and its *closed neighborhood* is $N[v] \coloneqq N(v) \cup \{v\}$. More generally, for a subset $U \subseteq V(G)$, we define $N[U] \coloneqq \bigcup_{v \in U} N[v]$. Let $G \setminus U$ denote the graph obtained from G by deleting all vertices in U and all edges adjacent to those vertices. Observe that $G \setminus U$ is an induced subgraph of G. The following result, which is essentially a consequence of the Mayer–Vietoris long exact sequence, can be found in [16, Theorem 3.5.1] or [18, Section 2.1].

Lemma 2.5. *Let G be a graph and v a vertex of G. Then there is a long exact sequence*

$$\cdots \to \widetilde{H}_i(\mathrm{IN}(G \setminus N[v])) \to \widetilde{H}_i(\mathrm{IN}(G \setminus v)) \to \widetilde{H}_i(\mathrm{IN}(G)) \to \widetilde{H}_{i-1}(\mathrm{IN}(G \setminus N[v])) \to \cdots.$$

From now on, assume that $V(G) = [n]$. The *edge ideal* of G is defined as

$$I(G) = \langle x_i x_j \mid \{i, j\} \in E(G) \rangle \subseteq S.$$

Evidently, $I(G)$ is the Stanley–Reisner ideal of the independence complex $\mathrm{IN}(G)$. Thus, in particular, $\dim(S/I(G)) = \dim \mathrm{IN}(G) + 1$.

Next, let us recall some notions that are useful for studying the projective dimension of $I(G)$. A *matching* in G is a subset of $E(G)$ that consists of pairwise disjoint edges. If a matching forms the edge set of an induced subgraph of G, it is called an *induced matching*. The *induced matching number* $\mathrm{im}(G)$ of G is the largest cardinality of an induced matching in G. A *strongly disjoint family of complete bipartite subgraphs* of G, as introduced in [19], is a collection $\mathfrak{B}_1, \ldots, \mathfrak{B}_g$ of subgraphs of G satisfying the following conditions:

(i) each $\mathfrak{B}_i$ is a complete bipartite subgraph of G;
(ii) $V(\mathfrak{B}_i) \cap V(\mathfrak{B}_j) = \emptyset$ for all $1 \leq i < j \leq g$;
(iii) there exists an induced matching $\{e_1, \ldots, e_g\}$ in G with $e_i \in E(\mathfrak{B}_i)$ for $i = 1, \ldots, g$.

The following formula for $\mathrm{pd}(S/I(G))$ follows from [36, Theorem 7.7].

Proposition 2.6. *Let G be a weakly chordal graph with at least one edge. Then*

$$\mathrm{pd}(S/I(G)) = \max \left\{ \sum_{i=1}^{g} |V(\mathfrak{B}_i)| - g \right\},$$

where the maximum is taken over all $1 \leq g \leq \mathrm{im}(G)$ and all strongly disjoint family of complete bipartite subgraphs $\mathfrak{B}_1, \ldots, \mathfrak{B}_g$ of G.

We now describe a relationship between degree complexes of $I(G)$ and the independence complex $\mathrm{IN}(G)$, which, together with Takayama's formula, allows us to study $\mathrm{depth}(S/I(G))$ via $\mathrm{IN}(G)$. For a simplicial complex Δ, let $\mathcal{F}(\Delta)$ denote the set of its facets.

Lemma 2.7. *Let G be a graph on $[n]$ with edge ideal $I = I(G)$. Then for any $\boldsymbol{a} \in \mathbb{Z}^n$,*

$$\mathcal{F}(\Delta_{\boldsymbol{a}}(I)) = \{F \setminus G_{\boldsymbol{a}} \mid \mathrm{supp}(\boldsymbol{a}) \subseteq F \subseteq [n], F \in \mathcal{F}(\mathrm{IN}(G))\}.$$

In particular, the following hold.

(i) $\Delta_{\mathbf{0}}(I(G)) = \mathrm{IN}(G)$, *where $\mathbf{0}$ denotes the zero vector of $\mathbb{Z}^n$.*
(ii) *If $\boldsymbol{b} \in \mathbb{Z}^n$ such that $\mathrm{supp}(\boldsymbol{b}) = \mathrm{supp}(\boldsymbol{a})$ and $G_{\boldsymbol{b}} = G_{\boldsymbol{a}}$, then $\Delta_{\boldsymbol{b}}(I) = \Delta_{\boldsymbol{a}}(I)$.*
(iii) *If $\mathrm{supp}(\boldsymbol{a}) \notin \mathrm{IN}(G)$, then $\Delta_{\boldsymbol{a}}(I) = \emptyset$.*
(iv) *If $G_{\boldsymbol{a}} \subsetneq \mathrm{supp}(\boldsymbol{a})$ and $\mathrm{supp}(\boldsymbol{a}) \in \mathrm{IN}(G)$, then $\Delta_{\boldsymbol{a}}(I)$ is a cone.*
(v) $\Delta_{\boldsymbol{a}}(I) = \{\emptyset\}$ *if and only if $\mathrm{supp}(\boldsymbol{a}) = G_{\boldsymbol{a}} \in \mathcal{F}(\mathrm{IN}(G))$.*

The proof of this result requires the following special case of [35, Proposition 1.6].

Proposition 2.8. *Let G be a graph on $[n]$ with edge ideal $I = I(G)$. For $\boldsymbol{a} \in \mathbb{Z}^n$, denote by $\boldsymbol{a}_+$ the vector obtained from $\boldsymbol{a}$ by setting every negative entry to zero. Let $F \subseteq [n]$ be such that $G_{\boldsymbol{a}} \subseteq F$. Then the following are equivalent:*

(i) $F \setminus G_{\boldsymbol{a}} \in \mathcal{F}(\Delta_{\boldsymbol{a}}(I))$;
(ii) $F \in \mathcal{F}(\mathrm{IN}(G))$ *and $x^{\boldsymbol{a}_+} \notin \mathfrak{m}_F$, where $\mathfrak{m}_F = \langle x_i \mid i \notin F \rangle$.*

Proof of Lemma 2.7. From Proposition 2.8 it follows that

$$\begin{aligned} F\setminus G_{\boldsymbol{a}}\in\mathcal{F}(\Delta_{\boldsymbol{a}}(I)) &\iff F\in\mathcal{F}(\mathrm{IN}(G)) \quad\text{and}\quad x^{\boldsymbol{a}_+}\notin\mathfrak{m}_F\\ &\iff F\in\mathcal{F}(\mathrm{IN}(G)) \quad\text{and}\quad \operatorname{supp}(\boldsymbol{a}_+)\cap([n]\setminus F)=\emptyset\\ &\iff F\in\mathcal{F}(\mathrm{IN}(G)) \quad\text{and}\quad \operatorname{supp}(\boldsymbol{a}_+)\subseteq F\\ &\iff F\in\mathcal{F}(\mathrm{IN}(G)) \quad\text{and}\quad \operatorname{supp}(\boldsymbol{a})\subseteq F \quad (\text{as } G_{\boldsymbol{a}}\subseteq F).\end{aligned}$$

This proves the given description of $\mathcal{F}(\Delta_{\boldsymbol{a}}(I))$. The remaining assertions follow readily. ☐

As a consequence of Propositions 2.1 and 2.8, we obtain the following.

Corollary 2.9. *Let G be a graph on $[n]$ with edge ideal $I=I(G)$. Then the following are equivalent:*

(i) $\operatorname{depth}(S/I)\geq 2$;
(ii) $H^1_{\mathfrak{m}}(S/I)=0$;
(iii) *The complement G^c of G is connected.*

Proof. Since I is a squarefree non-maxiamal ideal, $H^0_{\mathfrak{m}}(S/I)=0$, and thus (i) ⇔ (ii).

(ii) ⇒ (iii): By Proposition 2.1 and Lemma 2.7, $\Delta_{\mathbf{0}}(I)=\mathrm{IN}(G)$ is connected. Since the 1-skeleton of $\mathrm{IN}(G)$ is precisely G^c, we deduce that G^c is connected.

(iii) ⇒ (ii): By Proposition 2.1, we have to check the following:

(a) $\Delta_{\boldsymbol{a}}(I)$ is connected for every $\boldsymbol{a}\in\mathbb{Z}^n_{\geq 0}$;
(b) $\operatorname{depth}(S_j/I_j)\geq 1$ for every $j\in[n]$.

For (a), take $\boldsymbol{a}\in\mathbb{Z}^n_{\geq 0}$. If $\boldsymbol{a}=\mathbf{0}$, then by Lemma 2.7, $\Delta_{\boldsymbol{a}}(I)=\mathrm{IN}(G)$, which is connected as its 1-skeleton is nothing but G^c. If $\boldsymbol{a}\in\mathbb{Z}^n_{\geq 0}\setminus\{\mathbf{0}\}$, then $G_{\boldsymbol{a}}=\emptyset\subsetneq\operatorname{supp}(\boldsymbol{a})$. So it follows from Lemma 2.7 that either $\Delta_{\boldsymbol{a}}(I)$ is a cone or $\Delta_{\boldsymbol{a}}(I)=\emptyset$, depending on whether $\operatorname{supp}(\boldsymbol{a})\in\mathrm{IN}(G)$ or not. In any case, $\Delta_{\boldsymbol{a}}(I)$ is connected. Thus (a) is true.

For (b), assume the contrary that $\operatorname{depth}(S_j/I_j)=0$ for some $j\in[n]$. Then being a squarefree monomial ideal, necessarily $I_j=\langle x_i \mid i\in[n]\setminus j\rangle$, so j is an isolated vertex of G^c. This contradicts the connectedness of G^c. Hence (b) is also true, and the proof is complete. ☐

3. Invariant chains of ideals

In this section we fix notation and provide auxiliary results on invariant chains of edge ideals. Let us begin by recalling the notion of invariant chains of ideals.

3.1. **Invariant chains of ideals.** Let $\mathbb{N}$ denote the set of positive integers. As before, for each $n\in\mathbb{N}$, let $R_n=\Bbbk[x_1,\dots,x_n]$ be the polynomial ring in n variables over a field $\Bbbk$. Via the natural embedding we may regard R_n as a subring of R_m when $m\geq n$, and thus obtain a chain of increasing polynomial rings $R_1\subset R_2\subset\cdots$. Let $R:=\bigcup_{n\geq 1}R_n$ denote limit of this chain. Then $R=\Bbbk[x_i\mid i\in\mathbb{N}]$ is a polynomial ring in infinitely many variables. Of interest are ideals in R that are invariant under the action of the *monoid of strictly increasing maps on* $\mathbb{N}$:

$$\mathrm{Inc}=\{\pi\colon\mathbb{N}\to\mathbb{N}\mid \pi(n)<\pi(n+1) \text{ for all } n\geq 1\}.$$

This monoid acts on R by means of ring endomorphisms via

$$\pi \cdot x_i = x_{\pi(i)} \quad \text{for any } \pi \in \operatorname{Inc} \text{ and } i \geq 1.$$

An ideal $I \subseteq R$ is called Inc-*invariant* if $\pi(f) \in I$ for any $\pi \in \operatorname{Inc}$ and $f \in I$. Although the ring R is not Noetherian, a classical result of Cohen [3] (later rediscovered by Aschenbrenner and Hillar [1]; see also Hillar and Sullivant [14]) says that this ring is Inc-*Noetherian*, meaning that any Inc-invariant ideal $I \subseteq R$ is generated by finitely many Inc-orbits of polynomials.

Cohen's result has an interesting implication for chains of ideals $\mathcal{I} = (I_n)_{n\geq 1}$, where I_n is an ideal in R_n for $n \geq 1$. Such a chain is called Inc-*invariant* if

$$\langle \operatorname{Inc}_{m,n}(I_m)\rangle_{R_n} \subseteq I_n \quad \text{for all } n \geq m \geq 1, \tag{3.1}$$

where $\operatorname{Inc}_{m,n}$ denotes the following subset of Inc:

$$\operatorname{Inc}_{m,n} = \{\pi \in \operatorname{Inc} \mid \pi(m) \leq n\}$$

and $\langle \operatorname{Inc}_{m,n}(I_m)\rangle_{R_n}$ is the ideal in R_n generated by $\operatorname{Inc}_{m,n}(I_m)$. When the chain $\mathcal{I}$ is Inc-invariant, we say that it *stabilizes* if there exists $r \geq 1$ such that the inclusion in (3.1) becomes an equality for all $n \geq m \geq r$. The smallest such number r is called the *stability index* of $\mathcal{I}$ and is denoted by $\operatorname{ind}(\mathcal{I})$. A consequence of Cohen's result is that every Inc-invariant chain of ideals $\mathcal{I} = (I_n)_{n\geq 1}$ always stabilizes (see [14, 17]).

The stabilization of the chain $\mathcal{I} = (I_n)_{n\geq 1}$ implies that I_{n+1} can be interpreted in terms of I_n for all $n \geq \operatorname{ind}(\mathcal{I})$. Let us make this interpretation more explicit. For each integer $k \geq 0$, let $\sigma_k : \mathbb{N} \to \mathbb{N}$ be the strictly increasing map given by

$$\sigma_k(i) = \begin{cases} i, & \text{if } 1 \leq i \leq k, \\ i+1, & \text{if } i \geq k+1. \end{cases} \tag{3.2}$$

It is evident that $\sigma_k \in \operatorname{Inc}_{n,n+1}$ for all $k \geq 0$ and $n \geq 1$. Denote $\sigma_k(I_n) = \langle \sigma_k(f) \mid f \in I_n\rangle_{R_{n+1}}$. Then one can easily check that $I_{n+1} = \sum\limits_{k=0}^{n} \sigma_k(I_n)$ for all $n \geq \operatorname{ind}(\mathcal{I})$. The next result provides a more concise representation of I_{n+1}.

Proposition 3.1. *Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an* Inc-*invariant chain of ideals with* $\operatorname{ind}(\mathcal{I}) = r$. *Then for any subset $\Lambda \subseteq \{0, 1, \ldots, n\}$ with $|\Lambda| = r + 1$, it holds that*

$$I_{n+1} = \sum_{k\in\Lambda} \sigma_k(I_n) \quad \textit{for all } n \geq r.$$

Proof. We proceed by induction on n. The case $n = r$ is clearly true since $I_{r+1} = \sum_{k=0}^{r} \sigma_k(I_r)$. Suppose that the assertion has been shown for some $n \geq r$. Let Λ be an arbitrary subset of $\{0, 1, \ldots, n+1\}$ with $|\Lambda| = r + 1$. Since $I_{n+2} = \sum_{k=0}^{n+1} \sigma_k(I_{n+1})$, it suffices to check that

$$\sigma_l(I_{n+1}) \subseteq \sum_{k\in\Lambda} \sigma_k(I_{n+1}) \quad \text{for any } l \in \{0, 1, \ldots, n+1\} \setminus \Lambda.$$

We fix an $l \in \{0, 1, \ldots, n+1\} \setminus \Lambda$. Set

$$\Lambda_1 = \{k \in \Lambda \mid k > l\}, \quad \Lambda_2 = \Lambda \setminus \Lambda_1, \quad \text{and} \quad \Lambda_1' = \{k - 1 \mid k \in \Lambda_1\}.$$

It is evident that $k \geq l$ for all $k \in \Lambda_1'$ and $\Lambda' := \Lambda_1' \cup \Lambda_2$ is a subset of $\{0, 1, \ldots, n\}$ with $|\Lambda'| = r + 1$. Thus $I_{n+1} = \sum_{k \in \Lambda'} \sigma_k(I_n)$ by induction hypothesis. From [33, Corollary 4.2] we know that $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j-1}$ whenever $j > i \geq 0$. This yields

$$\sigma_l \circ \sigma_k = \begin{cases} \sigma_k \circ \sigma_{l-1} & \text{if } k \in \Lambda_2, \\ \sigma_{k+1} \circ \sigma_l & \text{if } k \in \Lambda_1'. \end{cases}$$

Therefore,

$$\begin{aligned} \sigma_l(I_{n+1}) = \sigma_l\Big(\sum_{k \in \Lambda'} \sigma_k(I_n)\Big) &= \sum_{k \in \Lambda_1'} \sigma_l \circ \sigma_k(I_n) + \sum_{k \in \Lambda_2} \sigma_l \circ \sigma_k(I_n) \\ &= \sum_{k \in \Lambda_1'} \sigma_{k+1} \circ \sigma_l(I_n) + \sum_{k \in \Lambda_2} \sigma_k \circ \sigma_{l-1}(I_n) \\ &\subseteq \sum_{k \in \Lambda_1} \sigma_k(I_{n+1}) + \sum_{k \in \Lambda_2} \sigma_k(I_{n+1}) = \sum_{k \in \Lambda} \sigma_k(I_{n+1}). \end{aligned}$$

The proof is complete. □

Given an Inc-invariant chain $\mathcal{I} = (I_n)_{n \geq 1}$, it can happen that the ideal I_n is generated by a proper subset of the set of variables of R_n for $n \gg 0$, and the "superfluous variables" may cause unnecessary complications. One can remove these variables by merely shifting indices. To describe this trick, let us restrict to the case of monomial ideals for simplicity.

Let $\mathcal{I} = (I_n)_{n \geq 1}$ be an Inc-invariant chain of monomial ideals with $\operatorname{ind}(\mathcal{I}) = r$. Denote by $\mathcal{G}(I_r)$ the minimal set of monomial generators of I_r. We define

$$\begin{aligned} \operatorname{msupp}(I_r) &= \min\{i \mid x_i \text{ divides } u \text{ for some } u \in \mathcal{G}(I_r)\}, \\ \operatorname{Msupp}(I_r) &= \max\{i \mid x_i \text{ divides } u \text{ for some } u \in \mathcal{G}(I_r)\}. \end{aligned}$$

Then $1 \leq \operatorname{msupp}(I_r) \leq \operatorname{Msupp}(I_r) \leq r$ and no element of $\mathcal{G}(I_r)$ involves variables with indices in $\{1, \ldots, r\} \setminus \{\operatorname{msupp}(I_r), \ldots, \operatorname{Msupp}(I_r)\}$. As the next lemma indicates, we may always reduce to the case where $\operatorname{msupp}(I_r) = 1$ and $\operatorname{Msupp}(I_r) = r$ by simple index shifts. The proof of the lemma is straightforward and is therefore left to the interested reader.

Lemma 3.2. *Let $\mathcal{I} = (I_n)_{n \geq 1}$ be an* Inc*-invariant chain of monomial ideals with* $\operatorname{ind}(\mathcal{I}) = r$*. Denote $i_1 = \operatorname{msupp}(I_r)$, $p = \operatorname{Msupp}(I_r)$, and $\widetilde{r} = p - i_1 + 1$. Consider the chain $\widetilde{\mathcal{I}} = (\widetilde{I}_n)_{n \geq 1}$ obtained from $\mathcal{I}$ by shifting the variables by $i_1 - 1$ and shifting the index of I_n by $r - \widetilde{r}$, i.e. $\widetilde{I}_n = 0$ for $n < \widetilde{r}$ and*

$$\widetilde{I}_n = \langle \rho(I_{n-\widetilde{r}+r}) \rangle_{R_{n-\widetilde{r}+r}} \cap R_n \quad \text{for } n \geq \widetilde{r},$$

where $\rho : R \to R$ is the $\mathbb{k}$-endomorphism of R induced by $\rho(x_n) = 0$ for $n < i_1$ and $\rho(x_n) = x_{n-i_1+1}$ for $n \geq i_1$. Then the following hold.

(i) *$\widetilde{\mathcal{I}}$ is an* Inc*-invariant chain with* $\operatorname{ind}(\widetilde{\mathcal{I}}) = \widetilde{r}$, $\operatorname{msupp}(\widetilde{I}_{\widetilde{r}}) = 1$ *and* $\operatorname{Msupp}(\widetilde{I}_{\widetilde{r}}) = \widetilde{r}$.

(ii) $\operatorname{depth}(R_n/I_n) = r - \widetilde{r} + \operatorname{depth}(R_{n+\widetilde{r}-r}/\widetilde{I}_{n+\widetilde{r}-r})$ *for all $n \geq r$.*

Example 3.3. Consider the chain $\mathcal{I} = (I_n)_{n \geq 1}$ with $I_n = 0$ for $n < 10$,

$$I_{10} = \langle x_2x_5, x_2x_7, x_3x_5, x_3x_9, x_7x_9 \rangle$$

is the edge ideal of a 5-cycle (see Figure 1), and $I_n = \langle \operatorname{Inc}_{10,n}(I_{10}) \rangle$ for $n \geq 11$.

Then $r = \operatorname{ind}(\mathcal{I}) = 10$, $\operatorname{msupp}(I_{10}) = 2$, $\operatorname{Msupp}(I_{10}) = 9$, and $\widetilde{r} = 8$. So the chain $\widetilde{\mathcal{I}} = (\widetilde{I}_n)_{n\geq 1}$ is given by $\widetilde{I}_n = 0$ for $n < 8$, $\widetilde{I}_8 = \langle x_1x_4, x_1x_6, x_2x_4, x_2x_8, x_6x_8\rangle$, and $\widetilde{I}_n = \langle \operatorname{Inc}_{8,n}(\widetilde{I}_8)\rangle$ for $n \geq 9$. Evidently, $\operatorname{ind}(\widetilde{\mathcal{I}}) = 8 = \operatorname{Msupp}(\widetilde{I}_8)$ and $\operatorname{msupp}(\widetilde{I}_8) = 1$. Moreover, x_1, x_n is a regular sequence on R_n/I_n and one has $\operatorname{depth}(R_n/I_n) = \operatorname{depth}(R_{n-2}/\widetilde{I}_{n-2}) + 2$ for all $n \geq 10$.

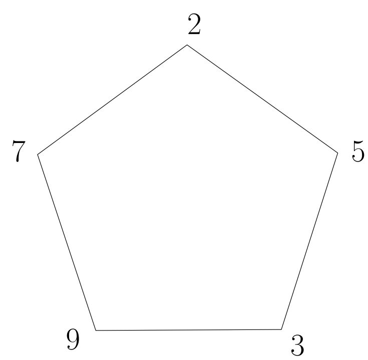

FIGURE 1. An indexed 5-cycle

3.2. **Invariant chains of edge ideals.** From now on, we focus on chains $\mathcal{I} = (I_n)_{n\geq 1}$, where each I_n is an edge ideal. For convenience, the following notation will be fixed throughout the remaining part of the paper.

Notation 3.4.

(i) Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an Inc-invariant chain of eventually nonzero edge ideals with stability index $r = \operatorname{ind}(\mathcal{I})$. For $n \geq 1$ let G_n be the graph corresponding to I_n. We always assume that $E(G_r) = \{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$ with $i_t < j_t$, $i_1 \leq \cdots \leq i_s$, and moreover if $i_t = i_{t+1}$ then $j_t < j_{t+1}$. Thus, in particular, $\operatorname{msupp}(I_r) = i_1$ and $\operatorname{Msupp}(I_r) = \max\{j_1, \dots, j_s\}$. Set

$$\begin{aligned}
j_q &= \max\{j_t \mid i_t = i_1,\ 1 \leq t \leq s\},\\
p &= \operatorname{Msupp}(I_r) = \max\{j_1, \dots, j_s\},\\
b &= \min\{i_t \mid j_t = \operatorname{Msupp}(I_r)\} = \min\{i_t \mid j_t = p\},\\
B &= \max\{i_t \mid j_t = \operatorname{Msupp}(I_r)\} = \max\{i_t \mid j_t = p\},\\
\widetilde{r} &= \operatorname{Msupp}(I_r) - \operatorname{msupp}(I_r) + 1 = p - i_1 + 1.
\end{aligned}$$

Moreover, we write $(i, j) \in E(G_n)$ if $\{i, j\} \in E(G_n)$ and $i < j$.

(ii) For $(i, j) \in \mathbb{N}^2$ and an integer $m \geq 0$, denote by $\Delta((i, j), m)$ the isosceles right triangle with the vertices $(i, j), (i, j + m), (i + m, j + m)$, whose legs are of length m.

Example 3.5. For the chain $\mathcal{I}$ in Example 3.3, it is already known that $r = 10$ and $\widetilde{r} = 8$. As $E(G_{10}) = \{(2, 5), (2, 7), (3, 5), (3, 9), (7, 9)\}$, we see that

$$i_1 = 2,\ j_q = 7,\ p = 9,\ b = 3,\ B = 7.$$

We provide here some useful asymptotic properties of the graphs G_n. Let us first recall a simple observation from [15, Lemma 3.3] that is crucial for testing membership in I_n.

Lemma 3.6. *Let $1 \le i < j \le r$ and $n \ge r$ be positive integers. Then for integers $k < l$, the following are equivalent:*

(i) $x_k x_l \in \mathrm{Inc}_{r,n}(x_i x_j)$;
(ii) *It holds that* $0 \le k - i \le l - j \le n - r$;
(iii) $(k, l) \in \Delta((i, j), n - r)$.

The following result shows a *density* property of G_n for $n \gg 0$: If $(k, l) \in E(G_n)$ is identified with the point $(k, l) \in \mathbb{R}^2$, then moving this point in all four cardinal directions by small integral steps still yields edges of G_n (see Figure 2).

Lemma 3.7. *Let $n \ge r$ be an integer. Using Notation 3.4, the following hold.*

(i) (Short east and short south moves) *Assume that $n \ge 3r$. Let $k \le k' \le r$ and $n - r \le l' \le l$ be integers. If (k, l) is an edge of G_n, then so are (k, l') and (k', l).*
(ii) (Short west moves) *Let $r \le k'' \le k < l$ be integers. If (k, l) is an edge of G_n, then so is (k'', l).*
(iii) (Short north moves) *Let $k < l \le l'' \le n - r$ be integers. If (k, l) is an edge of G_n, then so is (k, l'').*

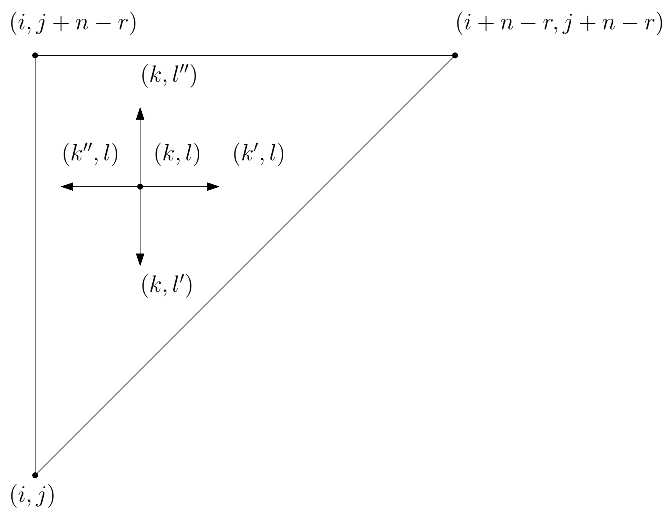

Figure 2. Short moves in the four cardinal directions

Proof. (i) As $n \ge 3r$, we deduce that

$$k \le k' \le r < n - r \le l' \le l.$$

Since $(k, l) \in E(G_n)$, it follows from Lemma 3.6 that $(k, l) \in \Delta((i, j), n - r)$ for some $(i, j) \in E(G_r)$ with $1 \le i < j \le r$. In other words,

$$0 \le k - i \le l - j \le n - r. \tag{3.3}$$

We only prove that $(k, l') \in E(G_n)$; similar arguments work for (k', l) as well. By Lemma 3.6, it suffices to show that $(k, l') \in \Delta((i, j), n - r)$, or equivalently,

$$0 \le k - i \le l' - j \le n - r.$$

The first and third inequalities follow immediately from (3.3) and the fact that $l' \leq l$. Also, the second inequality holds since $n \geq 3r$ and

$$k - i + j \leq r - 1 + r \leq n - r \leq l'.$$

(ii) Similarly to (i), if $(k, l) \in \Delta((i, j), n - r)$ for some $(i, j) \in E(G_r)$, then we also have $(k'', l) \in \Delta((i, j), n - r)$, since $0 \leq r - i \leq k'' - i$.

The proof of (iii) is similar and is left to the attentive reader. $\square$

Our next goal is to show that G_n does not contain long induced cycles for $n \gg 0$. For this, we need the following consequence of [15, Lemma 3.6].

Lemma 3.8. *Use Notation 3.4. Let $n \geq 3r$ and $(u_1, v_1), (u_2, v_2) \in E(G_n)$. If $(u_1, v_1), (u_2, v_2)$ form an induced matching of G_n, then $[u_1, v_1] \cap [u_2, v_2] = \emptyset$.*

Proof. By Lemma 3.6, there exists $(i_k, j_k) \in E(G_r)$ such that $(u_k, v_k) \in \Delta((i_k, j_k), n - r)$ for $k = 1, 2$. Since $n \geq 3r$, we have $n - r \geq 2r \geq 2 \max\{j_1, j_2\}$. Moreover, none of the pairs $\{u_1, u_2\}$, $\{u_1, v_2\}$, $\{u_2, v_1\}$, $\{v_1, v_2\}$ is an edge of G_n since $(u_1, v_1), (u_2, v_2)$ form an induced matching of G_n. So if we assume without loss of generality that $u_1 < u_2$, then it follows from [15, Lemma 3.6] that

$$v_1 < i_2 < n - r + j_1 < u_2.$$

Hence, $[u_1, v_1] \cap [u_2, v_2] = \emptyset$, as desired. $\square$

We are now ready to prove the following.

Lemma 3.9. *Use Notation 3.4. Then for $n \geq 3r$, G_n has no induced cycle C_m with $m \geq 6$.*

Proof. Assume on the contrary that for some $n \geq 3r$, G_n contains an induced cycle C_m with $m \geq 6$. Label the vertices of C_m as $u_1, \ldots, u_m$ such that $u_1 = \min\{u_1, \ldots, u_m\}$. For two real numbers x, y, let $(x, y)^{\leq}$ be the ordered pair $(\min\{x, y\}, \max\{x, y\}) \in \mathbb{R}^2$, and $[x, y]^{\leq}$ be the closed interval $[\min\{x, y\}, \max\{x, y\}] \subseteq \mathbb{R}$. Since $m \geq 6$, $\{(u_1, u_2), (u_{m-2}, u_{m-1})^{\leq}\}$ is an induced matching of G_n. Thus, $[u_1, u_2] \cap [u_{m-2}, u_{m-1}]^{\leq} = \emptyset$ by Lemma 3.8. It follows that

$$u_1 < u_2 < \min\{u_{m-2}, u_{m-1}\}.$$

Analogously, $u_1 < u_m < \min\{u_3, u_4\}$ as $\{(u_1, u_m), (u_3, u_4)^{\leq}\}$ is an induced matching of G_n. So if $u_2 < u_m$, then $u_2 < u_m < u_3$. Otherwise, if $u_m < u_2$, then $u_m < u_2 < u_{m-1}$. In either case, it always holds that $[u_2, u_3]^{\leq} \cap [u_{m-1}, u_m]^{\leq} \neq \emptyset$. Thus, $\{(u_2, u_3)^{\leq}, (u_{m-1}, u_m)^{\leq}\}$ is not an induced matching of G_n by Lemma 3.8. This contradiction concludes the proof. $\square$

For the graph $G_n \setminus N[n]$, which arises from the ideal $I_n : x_n$, a stronger result holds true.

Lemma 3.10. *Using Notation 3.4, assume that $p = r$. Then the following statements hold for all $n \geq 2r + 1$.*

(i) $V(G_n \setminus N[n]) = \{1 \ldots, b - 1\} \cup \{n - r + B + 1, \ldots, n - 1\}$.
(ii) *The graphs $G_n \setminus N[n]$ and $G_{n+1} \setminus N[n + 1]$ are isomorphic.*
(iii) *$G_n \setminus N[n]$ is weakly chordal.*

Proof. (i) Since $2r+1 \geq r+B-b$, it suffices to show that

$$N(n) = \{b, b+1, \ldots, n-r+B\} \quad \text{for } n \geq r+B-b.$$

Let $k \in N(n)$. Then $(k,n) \in E(G_n)$. By Lemma 3.6, there exists $(i,j) \in E(G_r)$ such that $(k,n) \in \Delta((i,j), n-r)$, i.e.

$$0 \leq k-i \leq n-j \leq n-r.$$

This implies $i \leq k \leq n-j+i$ and $j \geq r$. Since $j \leq r$, we get $j = r$ and thus $(i,r) \in E(G_r)$. Note that $p = r$. So the definition of b and B in Notation 3.4 gives $b \leq i \leq B$. Hence, $b \leq i \leq k \leq n-r+i \leq n-r+B$ and therefore $k \in \{b, b+1, \ldots, n-r+B\}$.

Conversely, if $b \leq k \leq n-r+B$, we see that $(k,n) \in \Delta((b,r), n-r)$ if $b \leq k \leq n-r+b$ and $(k,n) \in \Delta((B,r), n-r)$ if $B \leq k \leq n-r+B$. As $n \geq r+B-b$, we deduce $k \in N(n)$, as desired.

(ii) Denote $F_n = G_n \setminus N[n]$. For $n \geq 2r+1$, it follows from (i) that

$$I_n : x_n = \langle x_b, x_{b+1}, \ldots, x_{n-r+B} \rangle + L_n, \tag{3.4}$$

where $L_n \subseteq R_n$ is an edge ideal supported on $V(F_n)$. Moreover, L_n is exactly the edge ideal of F_n if it is viewed as an ideal in the ring $\Bbbk[x_i \mid i \in V(F_n)]$. Recall the map σ_k defined in (3.2). It is apparent that σ_b induces a bijective map from $V(F_n)$ to $V(F_{n+1})$. We show that this map is a graph isomorphism between F_n and F_{n+1} for all $n \geq 2r+1$. In other words, we need to prove that $L_{n+1} = \sigma_b(L_n)$, or equivalently,

$$I_{n+1} : x_{n+1} = \langle x_b, x_{b+1}, \ldots, x_{n-r+B+1} \rangle + \sigma_b(L_n) \quad \text{for all } n \geq 2r+1.$$

Indeed, it follows from [23, Lemma 4.7] that the chain $(I_n : x_n)_{n \geq 1}$ is Inc-invariant with stability index at most $r+1$. So by Proposition 3.1,

$$I_{n+1} : x_{n+1} = \sum_{k \in \Lambda} \sigma_k(I_n : x_n) \quad \text{for all } n \geq r+1, \tag{3.5}$$

where Λ is any subset of $\{0, 1, \ldots, n+1\}$ with $|\Lambda| = r+2$. Choose

$$\Lambda = \{b, b+1, \ldots, b+r+1\}.$$

Then $\Lambda \subseteq \{b, b+1, \ldots, n-r+B\}$ for all $n \geq 2r+1$. Take any $x_u x_v \in L_n$ with $u < v$. Since $u, v \in V(F_n)$, only the following cases can occur.

Case 1: $u < v < b$. In this case, $\sigma_k(x_u x_v) = x_u x_v$ for all $k \in \Lambda$.

Case 2: $u < b < n-r+B < v$. In this case, $\sigma_k(x_u x_v) = x_u x_{v+1}$ for all $k \in \Lambda$.

Case 3: $n-r+B < u < v$. In this case, $\sigma_k(x_u x_v) = x_{u+1} x_{v+1}$ for all $k \in \Lambda$.

It follows that $\sigma_k(L_n) = \sigma_b(L_n)$ for all $k \in \Lambda$. Thus from (3.4) and (3.5) we get

$$\begin{aligned}
I_{n+1} : x_{n+1} &= \sum_{k=b}^{b+r+1} \sigma_k\big(\langle x_b, x_{b+1}, \ldots, x_{n-r+B} \rangle + L_n\big) \\
&= \langle x_b, x_{b+1}, \ldots, x_{n-r+B+1} \rangle + \sum_{i=b}^{b+r+1} \sigma_i(L_n) \\
&= \langle x_b, x_{b+1}, \ldots, x_{n-r+B+1} \rangle + \sigma_b(L_n)
\end{aligned}$$

for all $n \geq 2r+1$, as wanted. The desired assertion follows.

(iii) In view of (ii), it is enough to show that F_n is weakly chordal for some $n \gg 0$. Let $n \geq \max\{5r, r(r-B+b-2)\}$. Since F_n is an induced subgraph of G_n, it follows from Lemma 3.9 that F_n contains no induced cycle C_m for $m \geq 6$. By [15, Proposition 4.1], the complement F_n^c also contains no induced cycle C_m for $5 \leq m \leq n/r$. As

$$\frac{n}{r} \geq r - B + b - 2 = |V(F_n)| = |V(F_n^c)|,$$

F_n^c has no induced cycle C_m for all $m \geq 5$. This implies that F_n also has no induced C_5 since $C_5^c = C_5$. Therefore, F_n is weakly chordal. The proof is complete. □

Remark 3.11. By index shift, Lemma 3.10 is valid even without the assumption that $p = r$. In this case, the graph $G_n \setminus N[n]$ would have to be replaced by $G_n \setminus N[n-r+p]$ and some indices in the result would have to be modified accordingly. The details are left to the reader.

We conclude this section with a strengthened version of [15, Proposition 7.4], in which the index of regularity stability is slightly reduced.

Lemma 3.12. *Use Notation 3.4. If $j_q = p$, i.e. $x_{i_1}x_p \in I_r$, then*

$$\operatorname{reg} I_n = 2 \quad \textit{for all } n \geq 3r-3.$$

Proof. The result actually follows implicitly from the proof of [15, Proposition 7.4]. We just need to tighten an inequality in that proof. Indeed, by Fröberg's theorem [10, Theorem 1], we have to show that G_{n+r} is *cochordal* (i.e. G_{n+r}^c is chordal) for all $n \geq 2r-3$. If G_{n+r} is not cochordal for some $n \geq 2r-3$, then it is shown in the proof of [15, Proposition 7.4] that there exists some $k \in [s]$ such that

$$n < r + j_q - i_q - j_k,$$

which yields $n < 2r-3$ since $j_q \leq r$, $i_q \geq 1$ and $j_k > i_k \geq 1$. This contradiction concludes the proof. □

4. Sparsity index and bounds for the asymptotic depth

This section can be seen as the starting point for the proof of Theorem 1.2, which will be completed in Sections 5 and 6. We provide here upper and lower bounds for $\operatorname{depth}(R_n/I_n)$ when $n \gg 0$. As we will see in the subsequent sections, $\operatorname{depth}(R_n/I_n)$ always attains one of these bounds for $n \gg 0$. Throughout the section, we continue to use Notation 3.4.

Let us first introduce the following crucial notion.

Definition 4.1. We call

$$\operatorname{sp}(I_r) = \min\{j - i \mid (i,j) \in E(G_r)\} = \min\{j_t - i_t \mid 1 \leq t \leq s\}$$

the *sparsity index* of I_r.

The Inc-invariance of the chain $\mathcal{I} = (I_n)_{n \geq 1}$ implies that $\operatorname{sp}(I_n) = \operatorname{sp}(I_r)$ for all $n \geq r$. We can therefore set

$$\operatorname{sp}(\mathcal{I}) \coloneqq \operatorname{sp}(I_r)$$

and call this number the *sparsity index* of $\mathcal{I}$. The nomenclature is justified by the observation that the bigger $\operatorname{sp}(\mathcal{I})$ is, the fewer edges each graph G_n may have:

Remark 4.2. Let $n \geq r$ and $1 \leq i < j \leq n$ be integers. If $(i,j) \in E(G_n)$, then it is evident that $j - i \geq \operatorname{sp}(I_n) = \operatorname{sp}(\mathcal{I})$. Thus, i and j are not adjacent in G_n whenever $j - i < \operatorname{sp}(\mathcal{I})$.

Example 4.3. The chain $\mathcal{I}$ in Example 3.3 has $\operatorname{sp}(\mathcal{I}) = 2$.

The next result gives an upper bound for $\operatorname{depth}(R_n/I_n)$ when $n \gg 0$. Recall that

$$\widetilde{r} = \operatorname{Msupp}(I_r) - \operatorname{msupp}(I_r) + 1 = p - i_1 + 1$$

is the stability index of the chain $\widetilde{\mathcal{I}}$ considered in Lemma 3.2.

Theorem 4.4. *For all $n \geq r + 2\widetilde{r}$, it holds that*

$$\operatorname{depth}(R_n/I_n) \leq r - \widetilde{r} + \operatorname{sp}(\mathcal{I}).$$

Proof. Define the chain $\widetilde{\mathcal{I}}$ as in Lemma 3.2. Observe that $\operatorname{sp}(\mathcal{I}) = \operatorname{sp}(\widetilde{\mathcal{I}})$. So passing to the chain $\widetilde{\mathcal{I}}$ we may assume $i_1 = 1$, $p = r$, and in this case, what we need to show becomes

$$\operatorname{depth}(R_n/I_n) \leq \operatorname{sp}(\mathcal{I}) \quad \text{for all } n \geq 3r.$$

Put $m = \operatorname{sp}(\mathcal{I})$. By Proposition 2.2, it suffices to provide a subset $F \subseteq [n]$ with $|F| = m$ such that $\operatorname{depth}((R_n)_F/(I_n)_F) = 0$. Since $(I_n)_F$ is squarefree, the last condition is equivalent to $(I_n)_F = \mathfrak{m}_F$, where $\mathfrak{m}_F$ is the graded maximal ideal of $(R_n)_F$. From the assumption that $i_1 = 1, p = r$ and $m = \operatorname{sp}(\mathcal{I})$, we deduce that $E(G_r)$ contains (not necessarily distinct) edges of the forms $(1,a), (v-m,v), (b,r)$, where $m+1 \leq a, v \leq r$ and $1 \leq b \leq r - m$. Let

$$F = \{\alpha, \alpha-1, \ldots, \alpha-m+1\} \ \text{ with } \ \alpha = \max\{a+v, b\} \leq 2r.$$

We show that F has the desired property for all $n \geq 3r$. Obviously, $|F| = m$. So it remains to prove that $(I_n)_F = \mathfrak{m}_F$. This is done through the following claims.

Claim 4.4.1. $(I_n)_F \subsetneq (R_n)_F$.

Indeed, if $(I_n)_F = (R_n)_F$, then G_n must have an edge of the form (k,l), where $k < l$ are both in F. But then $l - k \leq m - 1$, contradicting Remark 4.2.

Claim 4.4.2. $(I_n)_F \supseteq \mathfrak{m}_F$*, i.e. each vertex $k \in [n] \setminus F$ is adjacent to a vertex $l \in F$ in G_n.*

In fact, we have $[n] \setminus F = \{1, \ldots, \alpha - m\} \cup \{\alpha+1, \ldots, n\}$. Using Lemma 3.6, it suffices to show that for each $k \in [n] \setminus F$, there exists $l \in F$ such that either (k,l) or (l,k) belongs to one of the triangles $\Delta((1,a), n-r)$, $\Delta((v-m,v), n-r)$ and $\Delta((b,r), n-r)$. We distinguish the following cases.

Case 1: $1 \leq k \leq v - m$. Then it is clear that

$$0 \leq k - 1 \leq \alpha - a \leq n - r$$

since $v \leq \alpha - a$ and $\alpha \leq 2r \leq n - r$. Hence, $(k, \alpha) \in \Delta((1,a), n-r)$.

Case 2: $\alpha + r - b \leq k \leq n$. In this case, $(\alpha, k) \in \Delta((b,r), n-r)$ since

$$0 \leq \alpha - b \leq k - r \leq n - r.$$

Case 3: $v - m < k \leq \alpha - m$ or $\alpha + 1 \leq k < \alpha + r - b$. Set

$$l = \begin{cases} k - m & \text{if } \alpha + 1 \leq k \leq \alpha + m, \\ \alpha & \text{otherwise.} \end{cases}$$

Then $l \in F$. Similarly to the previous cases, one can show that either (k, l) (when $k < l$) or (l, k) (when $k > l$) belongs to $\Delta((v - m, v), n - r)$. The details are left to the reader. ☐

Let us now provide a lower bound for $\operatorname{depth}(R_n/I_n)$ when $n \gg 0$.

Theorem 4.5. *For all $n \geq r$, the following inequality holds*

$$\operatorname{depth}(R_n/I_n) \geq r - \widetilde{r} + \min\{\operatorname{sp}(\mathcal{I}), 2\}.$$

Proof. Using Lemma 3.2, we may assume that $i_1 = 1$ and $p = r$. In this case, we need to show that

$$\operatorname{depth}(R_n/I_n) \geq \min\{\operatorname{sp}(\mathcal{I}), 2\} \quad \text{for all } n \geq r.$$

Since I_n is a squarefree non-maximal ideal of R_n, it always holds that $\operatorname{depth}(R_n/I_n) \geq 1$. Hence, it suffices to prove that $\operatorname{depth}(R_n/I_n) \geq 2$ when $\operatorname{sp}(\mathcal{I}) \geq 2$, which we will assume from now. By Corollary 2.9, we need to show that G_n^c is connected for each $n \geq r$. Indeed, take two arbitrary vertices k, l of G_n^c with $k < l$. These vertices are joined by the edges $(k, k+1), (k+1, k+2), \ldots, (l-1, l)$, which all belong to G_n^c since $\operatorname{sp}(\mathcal{I}) \geq 2$. Hence, G_n^c is connected, as desired. ☐

As a direct consequence of Theorems 4.4 and 4.5 we obtain the following.

Corollary 4.6. *Assume that $\operatorname{sp}(\mathcal{I}) \leq 2$. Then for all $n \geq r + 2\widetilde{r}$, it holds that*

$$\operatorname{depth}(R_n/I_n) = r - \widetilde{r} + \operatorname{sp}(\mathcal{I}).$$

5. Maximal asymptotic depth

In this section we show that the upper bound given in Theorem 4.4 is attained when $j_q = p$, where we use Notation 3.4 throughout as usual. The following result covers Theorem 1.2(i).

Theorem 5.1. *Assume that $j_q = p$. Then for all $n \geq r + 2\widetilde{r}$, one has*

$$\operatorname{depth}(R_n/I_n) = r - \widetilde{r} + \operatorname{sp}(\mathcal{I}).$$

To prove this theorem, we need some auxiliary results. We first give a lower bound for the size of a maximal independent set of the graph G_n when $n \gg 0$.

Lemma 5.2. *Assume that $j_q = p$. Then for all $n \geq r + 2\widetilde{r}$, every maximal independent set of G_n has size at least $\operatorname{sp}(\mathcal{I})$.*

Proof. Set $m = \operatorname{sp}(\mathcal{I})$. Using Lemma 3.2 we may assume that $i_1 = 1$ and $j_q = r$. In this case, it is enough to show that every maximal independent set of G_n has size at least m for all $n \geq 3r$. Suppose to the contrary that G_n has a maximal independent set U of size at most $m - 1$. Denote $\alpha = \min U$ and $\beta = \max U$. To derive a contradiction, let us prove the following claims.

Claim 5.2.1. $\alpha \leq r - 1$.

Indeed, consider the following sets of size m:

$$V_1 = \{1, 2, \dots, m\},$$
$$V_2 = \{\beta - m + 1, \beta - m + 2, \dots, \beta\},$$
$$V_3 = \{\alpha, \alpha + 1, \dots, \alpha + m - 1\}.$$

By Remark 4.2, V_1 is an independent set of G_n. Since $|U| < |V_1|$, it follows from the maximality of U that $U \nsubseteq V_1$. This yields $\beta \geq m + 1$ and thus $V_2 \subseteq [n]$. Again by Remark 4.2, V_2 is an independent set of G_n. Hence, $U \nsubseteq V_2$ due to the maximality of U. It follows that $\alpha \leq \beta - m$, and consequently, $\alpha + m \leq \beta \leq n$. Thus, $V_3 \subseteq [n]$. Since $|U| < |V_3|$, there exists $i \in [m-1]$ such that $\alpha + i \notin U$. The maximality of U implies that $U \cup \{\alpha + i\}$ is a dependent set of G_n. Therefore, $\{\alpha + i, u\} \in E(G_n)$ for some $u \in U$. Note that $|u - (\alpha + i)| \geq m$ by Remark 4.2. Since $\alpha \leq u$ and $i \leq m - 1$, we must have $\alpha + i < u$. Now if $\alpha \geq r$, then moving west from $(\alpha + i, u)$ to (α, u) using Lemma 3.7, we get $(\alpha, u) \in E(G_n)$. This contradicts the independence of U. Hence, $\alpha \leq r - 1$, as claimed.

Claim 5.2.2. *$\beta \geq n - r + 1$.*

We argue similarly as above. Since $U \nsubseteq V_2$, there exists $j \in [m-1]$ such that $U \cup \{\beta - j\}$ is a dependent set of G_n. Thus, $\{\beta - j, v\} \in E(G_n)$ for some $v \in U$. Using Remark 4.2 together with the fact that $v \leq \beta$ and $j \leq m - 1$, we also deduce that $v < \beta - j$. If $\beta \leq n - r$, then moving north from $(v, \beta - j)$ to (v, β) using Lemma 3.7, we get $(v, \beta) \in E(G_n)$. This again contradicts the independence of U. Hence, $\beta \geq n - r + 1$.

Claim 5.2.3. *$(\alpha, \beta) \in E(G_n)$.*

The assumption that $i_1 = 1$ and $j_q = r$ implies $(1, r) \in E(G_r)$. So by Lemma 3.6, it suffices to show that $(\alpha, \beta) \in \Delta((1, r), n - r)$. Indeed, this follows from

$$0 \leq \alpha - 1 \leq r - 2 \leq n - 2r + 1 \leq \beta - r \leq n - r,$$

where the third inequality holds since $n \geq 3r$.

Claim 5.2.3 contradicts the independence of U and thus completes the proof. ☐

The following technical lemma also plays a role in the proof of Theorem 5.1.

Lemma 5.3. *Assume that $j_q = p$ and $\operatorname{sp}(\mathcal{I}) \geq 2$. Let U be an independent set of G_n of size at most $\operatorname{sp}(\mathcal{I}) - 2$. Then the complement of $G_n \setminus N[U]$ is connected for all $n \geq r + 2\widetilde{r}$.*

Proof. Using Lemma 3.2 we may assume that $i_1 = 1$ and $j_q = r$. In this case, we need to show that the graph $F_n := (G_n \setminus N[U])^c = G_n^c \setminus N[U]$ is connected for all $n \geq 3r$. Suppose to the contrary that F_n is not connected for some $n \geq 3r$. Let $i < j$ be vertices of F_n that are not joined by a path in F_n with $j - i$ being minimal. Then in particular, $(i, j) \in E(G_n)$ since $(i, j) \notin E(G_n^c)$. Moreover, any $k \in [n]$ with $i < k < j$ is not a vertex of F_n due to the minimality of $j - i$. Thus,

$$\{i + 1, \dots, j - 1\} \subseteq N[U]. \tag{5.1}$$

Denote $m = \operatorname{sp}(\mathcal{I})$, $d = |U|$, $\alpha = \min U$, and $\beta = \max U$. Then it is clear that $\beta - \alpha \geq d - 1$. Also, $j - i \geq m$ as $(i, j) \in E(G_n)$. We distinguish two cases.

Case 1: $i > \alpha$. A contradiction will be derived from the following claims.

Claim 5.3.1. *$\alpha \le r-1$ and $j \le n-r$.*

If $\alpha \ge r$, then moving west from (i,j) to (α,j) using Lemma 3.7, we get $(\alpha,j) \in E(G_n)$, contradicting the fact that $j \notin N[U]$. Hence, $\alpha \le r-1$. By assumption, $(1,r) \in E(G_r)$. So Lemma 3.6 yields $(\alpha,j) \notin \Delta((1,r),n-r)$, i.e. at least one of the inequalities

$$0 \le \alpha - 1 \le j - r \le n - r$$

is not true. We see that the middle inequality must be false. Thus

$$j < \alpha - 1 + r \le 2r - 2 \le n - r,$$

where the last inequality follows from $n \ge 3r$.

Claim 5.3.2. *$\beta < j$.*

Suppose that $j < \beta$. If $\beta \le n-r$, then moving north from (i,j) to (i,β) using Lemma 3.7, we deduce $(i,\beta) \in E(G_n)$. This contradicts the fact that $i \notin N[U]$. On the other hand, if $\beta \ge n-r+1$, then the hypothesis $n \ge 3r$ and the inequality $\alpha \le r-1$ from Claim 1 give

$$0 \le \alpha - 1 \le r - 2 < n - 2r + 1 \le \beta - r \le n - r.$$

This implies $(\alpha,\beta) \in \Delta((1,r),n-r)$, contradicting the independence of U.

Claim 5.3.3. *$(u, j-h) \in E(G_n)$ for some $u \in U$ and $h \in [d+1]$.*

As $|U| = d$, there exists $h \in [d+1]$ such that $j - h \notin U$. Since $d+1 \le m-1 \le j-i-1$, it is easily seen from (5.1) that $j-h \in N[U]$. Thus, $\{u, j-h\} \in E(G_n)$ for some $u \in U$. This implies $|(j-h)-u| \ge m$. Since $u \le \beta < j$ and $h \le d+1 \le m-1$, it must hold that $u < j-h$. Therefore, $(u, j-h) \in E(G_n)$.

Let us now derive a contradiction. Since $j \le n-r$ and $(u,j-h) \in E(G_n)$, it follows from Lemma 3.7(iii) that $(u,j) \in E(G_n)$. This contradicts the fact that $j \notin N[U]$, as wanted.

Case 2: $i < \alpha$. Let us first prove the following claims.

Claim 5.3.4. *$(i+h, u) \in E(G_n)$ for some $u \in U$ and $h \in [d+1]$.*

We argue analogously to the proof of Claim 5.3.3. By (5.1) and the fact that $d+1 \le j-i-1$, we may choose $h \in [d+1]$ such that $i+h \in N[U] \setminus U$. This implies $\{i+h, u\} \in E(G_n)$ for some $u \in U$. We must have $i+h < u$ since $|(i+h)-u| \ge m$, $i < \alpha \le u$ and $h \le d+1 \le m-1$. Hence, $(i+h, u) \in E(G_n)$.

Claim 5.3.5. *If $\alpha + d + 1 < j$, then $(\alpha + k, v) \in E(G_n)$ for some $v \in U$ and $k \in [d+1]$.*

From the assumption $\alpha + d + 1 < j$ it follows that $\alpha + k \in N[U]$ for all $k \in [d+1]$. The rest of the argument is the same as in the proof of Claim 5.3.4 and is omitted.

Claim 5.3.6. *If $\beta < j$, then $(w, j-l) \in E(G_n)$ for some $w \in U$ and $l \in [d+1]$.*

The argument is the same as in the proof of Claim 5.3.3 and is omitted.

Let us now derive a contradiction. If $i \geq r$, then using Claim 5.3.4 and Lemma 3.7(ii) we infer that $(i,u) \in E(G_n)$. This contradicts the fact that $i \notin N[U]$. Hence, $i \leq r-1$. Notice that $(i,\beta) \notin \Delta((1,r), n-r)$ by Lemma 3.6. So arguing as in the proof of Claim 5.3.1 yields

$$\beta < i + r - 1 \leq 2r - 2 \leq n - r.$$

If $j < \beta$, then moving north from (i,j) to (i,β) using Lemma 3.7, we get the contradiction that $(i,\beta) \in E(G_n)$. Thus, $\beta < j$. If $j \leq n-r$, then Claim 5.3.6 together with Lemma 3.7(iii) implies that $(w,j) \in E(G_n)$, again a contradiction. Hence, $j \geq n-r+1$. As $(\alpha, j) \notin E(G_n)$, also $(\alpha, j) \notin \Delta((1,r), n-r)$, and it follows that $\alpha > j - r + 1 \geq n - 2r + 2 > r$. Moreover, $\alpha + d + 1 < j$ since

$$\alpha + d - 1 \leq \beta < 2r - 2 \leq (n - r + 1) - 3 \leq j - 3.$$

So using Claim 5.3.5 and Lemma 3.7(ii) we deduce that $(\alpha, v) \in E(G_n)$. This contradiction concludes the proof. ☐

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. By Corollary 4.6, it suffices to consider the case $m := \operatorname{sp}(\mathcal{I}) \geq 3$. Moreover, using Lemma 3.2 we may assume that $i_1 = 1$ and $j_q = p = r$. In this case, by Theorem 4.4, it is enough to show that

$$\operatorname{depth}(R_n/I_n) \geq m \quad \text{for all } n \geq 3r.$$

Using Takayama's formula, this is equivalent to proving that

$$\widetilde{H}_{i-|G_{\boldsymbol{a}}|-1}(\Delta_{\boldsymbol{a}}(I_n)) = 0 \quad \text{for all } n \geq 3r, \text{ all } i \leq m-1 \text{ and all } \boldsymbol{a} \in \mathbb{Z}^n.$$

By Lemma 2.7(ii), we may assume that $\boldsymbol{a} \in \{-1,0,1\}^n$. Moreover, it suffices to examine the case $\operatorname{supp}(\boldsymbol{a}) = G_{\boldsymbol{a}} \in \operatorname{IN}(G_n)$ by virtue of Lemma 2.7(iii)–(iv). In this case, $\boldsymbol{a} \in \{-1,0\}^n$. As $j_q = p = r$, we know from Lemma 3.12 that $\operatorname{reg}(R_n/I_n) = 1$ for all $n \geq 3r$. Thus, if $i + \sum_{i=1}^n a_i = i - |G_{\boldsymbol{a}}| \geq 2$, then Takayama's formula gives

$$\widetilde{H}_{i-|G_{\boldsymbol{a}}|-1}(\Delta_{\boldsymbol{a}}(I_n)) \cong H^i_{\mathfrak{m}}(R_n/I_n)_{\boldsymbol{a}} = 0.$$

Therefore, we may assume that $|G_{\boldsymbol{a}}| \geq i-1$. If $|G_{\boldsymbol{a}}| \geq i+1$, then $\widetilde{H}_{i-|G_{\boldsymbol{a}}|-1}(\Delta_{\boldsymbol{a}}(I_n)) = 0$ since $i - |G_{\boldsymbol{a}}| - 1 \leq -2$. So there are only two cases left:

Case 1: $|G_{\boldsymbol{a}}| = i$. Since $i \leq m-1$, it follows from Lemma 5.2 that $G_{\boldsymbol{a}}$ is not a maximal independent set of G_n. Hence, $\Delta_{\boldsymbol{a}}(I_n) \neq \{\emptyset\}$ by Lemma 2.7(v). This implies

$$\widetilde{H}_{i-|G_{\boldsymbol{a}}|-1}(\Delta_{\boldsymbol{a}}(I_n)) = \widetilde{H}_{-1}(\Delta_{\boldsymbol{a}}(I_n)) = 0.$$

Case 2: $|G_{\boldsymbol{a}}| = i-1$. Recall from Lemma 2.7 that

$$\mathcal{F}(\Delta_{\boldsymbol{a}}(I_n)) = \{F \setminus G_{\boldsymbol{a}} \mid G_{\boldsymbol{a}} \subseteq F \subseteq [n], F \in \mathcal{F}(\operatorname{IN}(G_n))\}.$$

Thus, the 1-skeleton of $\Delta_{\boldsymbol{a}}(I_n)$ is exactly the graph $(G_n \setminus N[G_{\boldsymbol{a}}])^c$. Since $|G_{\boldsymbol{a}}| = i-1 \leq m-2$, Lemma 5.3 says that $(G_n \setminus N[G_{\boldsymbol{a}}])^c$ is connected. Hence,

$$\widetilde{H}_{i-|G_{\boldsymbol{a}}|-1}(\Delta_{\boldsymbol{a}}(I_n)) = \widetilde{H}_0(\Delta_{\boldsymbol{a}}(I_n)) = 0,$$

as desired. ☐

We conclude this section with an example illustrating that the lower bound for the stability index of $\operatorname{depth}(R_n/I_n)$ given in Theorem 5.1 could be close to optimal.

Example 5.4. Consider the chain $\mathcal{I} = (I_n)_{n\geq 1}$ with stability index $r \geq 6$ and

$$E(G_r) = \{(1,r), (2,4), (3,5)\}.$$

This chain satisfies $i_1 = 1$ and $j_q = p = r$. Computations with Macaulay2 [12] suggest that

$$\operatorname{depth}(R_n/I_n) = \begin{cases} 2 & \text{if } n = 2r-5, \\ 3 & \text{if } 2r-4 \leq n \leq 3r-10, \\ 2 & \text{if } n \geq 3r-9. \end{cases}$$

Assuming the above result, we see that the lower bound $n \geq 3r$ given in Theorem 5.1 for the stability index of $\operatorname{depth}(R_n/I_n)$ cannot be improved to $n \geq 3r-10$ in general.

6. Minimal asymptotic depth

Our goal in this section is to complete the proof of Theorem 1.2 and thereby provide a comprehensive picture of the asymptotic behavior of $\operatorname{depth}(R_n/I_n)$: we prove the following slight generalization of Theorem 1.2(ii), showing that the lower bound given in Theorem 4.5 is attained when $j_q < p$. As always, Notation 3.4 is used throughout the section.

Theorem 6.1. *Assume that $j_q < p$. Then*

$$\operatorname{depth}(R_n/I_n) = r - \widetilde{r} + \min\{2, \operatorname{sp}(\mathcal{I})\}$$

for all $n \geq r + 2\widetilde{r}$.

The proof of this theorem is mainly based on the following nonvanishing result for the first reduced homology group of the independence complex $\operatorname{IN}(G_n)$ of the graph G_n. A complete description of all reduced homology groups of $\operatorname{IN}(G_n)$ can be found in Section 7.

Proposition 6.2. *Assume that $i_1 = 1$ and $p = r$. If $j_q < r$ and $\operatorname{sp}(\mathcal{I}) \geq 2$, then*

$$\widetilde{H}_1(\operatorname{IN}(G_n)) \cong \Bbbk \quad \textit{for all } n \geq 3r.$$

The main idea to prove Proposition 6.2 is to proceed by induction on $r - j_q$ using the long exact sequence in Lemma 2.5. Let us begin by showing (non)vanishing results for the zeroth and first reduced homology groups of the independence complexes $\operatorname{IN}(G_n \setminus n)$ and $\operatorname{IN}(G_n \setminus N[n])$.

Lemma 6.3. *Assume that $i_1 = 1$ and $p = r$. If $\operatorname{sp}(\mathcal{I}) \geq 2$, then*

$$\widetilde{H}_0(\operatorname{IN}(G_n \setminus n)) = 0 \quad \textit{for all } n \geq r.$$

Proof. Obviously, the graph $G_n \setminus n$ contains at least one vertex for all $n \geq r$. So by Lemma 2.4, it suffices to verify the connectedness of the complementary graph $(G_n \setminus n)^c$. But this is clear, because any two vertices i, j of $(G_n \setminus n)^c$ with $i < j$ are connected by the edges $(i, i+1), (i+1, i+2), \ldots, (j-1, j)$, all of which belong to $(G_n \setminus n)^c$ since $\operatorname{sp}(\mathcal{I}) \geq 2$. ☐

Lemma 6.4. *Assume that $i_1 = 1$ and $p = r$. If $j_q = r - 1$, then*

$$\widetilde{H}_1(\operatorname{IN}(G_n \setminus n)) = 0 \quad \textit{for all } n \geq 3r.$$

Proof. Observe that the edge ideal of $G_n \setminus n$ is the ideal of R_{n-1} generated by monomials in I_n that are not divisible by x_n, i.e. the ideal $\langle I_n \cap R_{n-1}\rangle_{R_{n-1}}$. Thus, if we define the chain $\mathcal{J} = (J_n)_{n\geq 1}$ as follows

$$J_n = \begin{cases} 0 & \text{if} \quad n \leq r-1 \\ \langle I_{n+1} \cap R_n\rangle_{R_n} & \text{if} \quad n \geq r, \end{cases}$$

then J_{n-1} is the edge ideal of $G_n \setminus n$ for $n \geq r+1$. By [23, Lemma 4.7], $\mathcal{J}$ is an Inc-invariant chain with stability index $\operatorname{ind}(\mathcal{J}) = \operatorname{ind}(\mathcal{I}) = r$. (Note that the ideal J_n in the current proof is denoted by J_{n+1} in [23, Lemma 4.7], hence the difference in the stability indices.) A useful property of the chain $\mathcal{J}$ is that the index j_q for this chain increases by 1. Indeed, we have $x_1x_{j_q} \in I_r$ since $i_1 = 1$. It follows that $x_1x_{j_q+1} \in I_{r+1}$, and hence $x_1x_{j_q+1} \in J_r$, as claimed. Now the assumption that $j_q = r-1$ gives $x_1x_r \in J_r$. So by Lemma 3.12, it holds that $\operatorname{reg}(R_{n-1}/J_{n-1}) = 1$ for all $n \geq 3r$ (in fact, it suffices to take $n \geq 3r-2$). This combined with Lemma 2.7(i) and Takayama's formula yields

$$\widetilde{H}_1(\operatorname{IN}(G_n \setminus n)) = \widetilde{H}_1(\Delta_{\mathbf{0}}(J_{n-1})) \cong H^2_{\mathfrak{m}_{n-1}}(R_{n-1}/J_{n-1})_{\mathbf{0}} = 0,$$

where $\mathfrak{m}_{n-1}$ denotes the graded maximal ideal of R_{n-1}. □

Lemma 6.5. *Assume that $i_1 = 1$, $p = r$, $j_q < r$, and $\operatorname{sp}(\mathcal{I}) \geq 2$. Then for $n \geq 2r+1$,*

$$\widetilde{H}_0(\operatorname{IN}(G_n \setminus N[n])) \cong \begin{cases} \Bbbk & \text{if} \;\; j_q = r-1, \\ 0 & \text{if} \;\; j_q \leq r-2. \end{cases}$$

Proof. Set $G \coloneqq G_n \setminus N[n]$. Then for all $n \geq 2r+1$, we know from Lemma 3.10 that G has the vertex set $V(G) = V_1 \cup V_2$, where

$$V_1 = \{1, \dots, b-1\} \quad \text{and} \quad V_2 = \{n-r+B+1, \dots, n-1\}.$$

Denote by Γ_1 and Γ_2 the induced subgraphs of G^c on V_1 and V_2, respectively. We show that Γ_1 and Γ_2 are connected. Indeed, note that $G^c = G_n^c \setminus N[n]$ is the induced subgraph of G_n^c on $V(G) = V_1 \cup V_2$. Hence, Γ_1 and Γ_2 are also the induced subgraphs of G_n^c on V_1 and V_2. From the assumption $\operatorname{sp}(\mathcal{I}) \geq 2$ it follows that $(i, i+1) \in E(G_n^c)$ for all $i \in [n-1]$. Therefore, Γ_1 and Γ_2 are connected, as desired.

Let us first consider the case $j_q = r-1$, i.e. $(1, r-1) \in E(G_r)$. By Lemma 2.4, it suffices to show that G^c has exactly two connected components. We claim that Γ_1 and Γ_2 are the connected components of G^c. Indeed, this means that $(u, v) \notin E(G^c)$, or equivalently, $(u, v) \in E(G)$ for every $u \in V_1$ and $v \in V_2$. As $n \geq 2r+1$, we have

$$v - u \geq (n-r+B+1) - (b-1) = n-r+B-b+2 \geq r-2.$$

Moreover, $v-(r-1) \leq n-r$ since $v \leq n-1$. It follows that

$$0 \leq u-1 \leq v-(r-1) \leq n-r,$$

which yields $(u, v) \in \Delta((1, r-1), n-r)$. Hence, $(u, v) \in E(G_n)$ by Lemma 3.6. Since G is the induced subgraph of G_n on $V(G)$, this implies that $(u, v) \in E(G)$, as claimed.

Now assume that $j_q \leq r-2$. Again by Lemma 2.4, we need to show in this case that G^c is a connected graph. Since Γ_1 and Γ_2 are connected, it suffices to find an edge of G^c connecting them. We claim that $(1, n-1)$ is such an edge. Suppose to the contrary that

$(1, n-1) \notin E(G^c)$, i.e. $(1, n-1) \in E(G)$. This means that $(1, n-1) \in E(G_n)$ since G is an induced subgraph of G_n. Thus, $(1, n-1) \in \Delta((i,j), n-r)$ for some $(i,j) \in E(G_r)$ by Lemma 3.6. It follows that

$$0 \leq 1 - i \leq n - 1 - j \leq n - r.$$

Hence, $i = 1$ and $j \geq r - 1$. But this contradicts the assumption that $j_q \leq r - 2$. Therefore, $(1, n-1) \in E(G^c)$, as desired. □

Lemma 6.6. *Assume that* $i_1 = 1$, $p = r$, *and* $\operatorname{sp}(\mathcal{I}) \geq 2$. *Then*

$$\widetilde{H}_1(\mathrm{IN}(G_n \setminus N[n])) = 0 \quad \text{for all } n \geq 3r.$$

The proof of this lemma is rather technical and lengthy. So we postpone it to the Appendix. Let us present here the proof of Proposition 6.2.

Proof of Proposition 6.2. Let $n \geq 3r$. The long exact sequence in Lemma 2.5, applied to the graph G_n and its vertex n, together with the fact that $\widetilde{H}_0(\mathrm{IN}(G_n \setminus n)) = 0$ (Lemma 6.3) and $\widetilde{H}_1(\mathrm{IN}(G_n \setminus N[n])) = 0$ (Lemma 6.6), yields the following short exact sequence

$$0 \longrightarrow \widetilde{H}_1(\mathrm{IN}(G_n \setminus n)) \longrightarrow \widetilde{H}_1(\mathrm{IN}(G_n)) \longrightarrow \widetilde{H}_0(\mathrm{IN}(G_n \setminus N[n])) \longrightarrow 0. \tag{6.1}$$

Let us show that $\widetilde{H}_1(\mathrm{IN}(G_n)) \cong \Bbbk$ by induction on $r - j_q \geq 1$. If $r - j_q = 1$, then Lemma 6.4 gives $\widetilde{H}_1(\mathrm{IN}(G_n \setminus n)) = 0$. It thus follows from (6.1) and Lemma 6.5 that

$$\widetilde{H}_1(\mathrm{IN}(G_n)) \cong \widetilde{H}_0(\mathrm{IN}(G_n \setminus N[n])) \cong \Bbbk.$$

Now assume that $r - j_q > 1$. In this case, $\widetilde{H}_0(\mathrm{IN}(G_n \setminus N[n])) = 0$ by Lemma 6.5. The short exact sequence (6.1) then implies that $\widetilde{H}_1(\mathrm{IN}(G_n)) \cong \widetilde{H}_1(\mathrm{IN}(G_n \setminus n))$. So it remains to show that $\widetilde{H}_1(\mathrm{IN}(G_n \setminus n)) \cong \Bbbk$. Consider the chain $\mathcal{J} = (J_n)_{n \geq 1}$ as in the proof of Lemma 6.4. It is easy to verify that this chain satisfies all the assumptions of Proposition 6.2. Moreover, we know from the proof of Lemma 6.4 that $x_1 x_{j_q+1} \in J_r$. Thus we may apply the induction hypothesis to the chain $\mathcal{J}$ and obtain $\widetilde{H}_1(\mathrm{IN}(G_n \setminus n)) \cong \Bbbk$. This concludes the proof. □

The following example, which is somewhat similar to Example 5.4, suggests that the lower bound for the index of stability of $\widetilde{H}_1(\mathrm{IN}(G_n))$ in Proposition 6.2 could be close to optimal.

Example 6.7. Consider the chain $\mathcal{I} = (I_n)_{n \geq 1}$ with stability index $r \geq 6$ and

$$E(G_r) = \{(1,3), (2,r), (r-2,r)\},$$

which satisfies all the assumptions of Proposition 6.2. Computations with Macaulay2 [12] suggest that

$$\dim_{\Bbbk} \widetilde{H}_1(\mathrm{IN}(G_n)) = \begin{cases} 0, & \text{if } n \leq 2r - 5, \\ 2, & \text{if } 2r - 4 \leq n \leq 3r - 10, \\ 1, & \text{if } n \geq 3r - 9. \end{cases}$$

That is, $\widetilde{H}_1(\mathrm{IN}(G_n))$ could be stable from $n = 3r - 9$.

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. In view of Corollary 4.6, it suffices to consider the case $\operatorname{sp}(\mathcal{I}) \geq 3$. Moreover, using Lemma 3.2 we may furthermore assume that $i_1 = 1$, $p = r$ and thus reduce the statement we want to prove to

$$\operatorname{depth}(R_n/I_n) = 2 \quad \text{for all } n \geq 3r.$$

Let $n \geq 3r$. Combining Proposition 6.2 with Lemma 2.7(i) and Takayama's formula we get

$$\Bbbk \cong \widetilde{H}_1(\operatorname{IN}(G_n)) = \widetilde{H}_1(\Delta_{\mathbf{0}}(I_n)) \cong H^2_{\mathfrak{m}}(R_n/I_n)_{\mathbf{0}}.$$

It follows that $\operatorname{depth}(R_n/I_n) \leq 2$. Consequently, $\operatorname{depth}(R_n/I_n) = 2$ by Theorem 4.5. The proof is complete. □

The lower bound for the index of depth stability in Theorem 6.1 is also close to optimal, as illustrated by the next example.

Example 6.8. Consider the chain $\mathcal{I} = (I_n)_{n\geq 1}$ with stability index $r \geq 6$ and

$$E(G_r) = \{(1, r-1), (2, 3), (2, r)\},$$

which satisfies all the assumptions of Theorem 6.1. We show that $\operatorname{depth}(R_n/I_n) \geq 2$ for $n = 3r - 8$ and $\operatorname{depth}(R_n/I_n) = 1$ for $n \geq 3r - 7$.

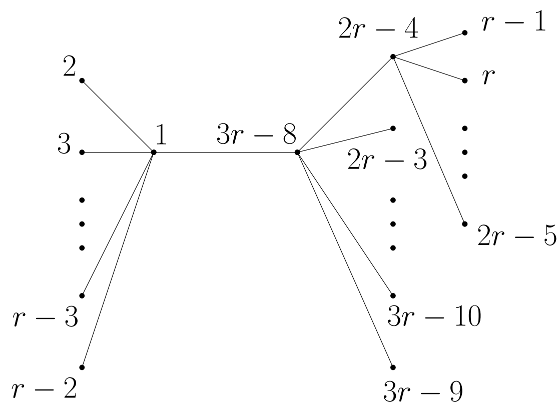

FIGURE 3. A spanning tree of G^c_{3r-8}

To show $\operatorname{depth}(R_{3r-8}/I_{3r-8}) \geq 2$, thanks to Corollary 2.9, is equivalent to proving that G^c_{3r-8} is connected. This holds because G^c_{3r-8} has a spanning tree as in Figure 3.

Next, to show $\operatorname{depth}(R_n/I_n) = 1$ for $n \geq 3r - 7$, we observe that G^c_n is not connected for all such n, as it admits $r - 1$ as an isolated vertex. Therefore, for each $r \geq 6$, the index of depth stability of the chain $\mathcal{I}$ is $3r - 7$.

Remark 6.9. As a summary of Theorems 5.1 and 6.1 and [15, Theorem 7.1], Table 1 provides a complete picture of the asymptotic depth and regularity of Inc-invariant chains of (eventually non-zero) edge ideals. It would be interesting to have similar tables for more general chains, e.g. chains of (squarefree) monomial ideals.

Table 1. Asymptotic depth and regularity of Inc-invariant chains of edge ideals

Invariant	Value	Condition
$\lim_{n\to\infty} \operatorname{depth}(R_n/I_n)$	$r - \widetilde{r} + \operatorname{sp}(\mathcal{I})$	either $j_q = p$, or $\operatorname{sp}(\mathcal{I}) \le 2$
	$r - \widetilde{r} + 2$	$j_q < p$ and $\operatorname{sp}(\mathcal{I}) \ge 2$
$\lim_{n\to\infty} \operatorname{reg}(R_n/I_n)$	1	either $j_q = p$, or ($\operatorname{sp}(\mathcal{I}) = 1$ and $\operatorname{im}(G_{3r}) = 1$)
	2	$j_q < p$ and ($\operatorname{sp}(\mathcal{I}) \ge 2$ or $\operatorname{im}(G_{3r}) = 2$)

7. Asymptotic homology of independence complexes

In view of Proposition 6.2, one may wonder whether all reduced homology groups of the independence complex $\operatorname{IN}(G_n)$ can be determined asymptotically. The main result of the present section provides a complete answer to this question. As before, we keep using Notation 3.4 throughout the section.

Theorem 7.1. *Assume that $i_1 = 1$ and $p = r$. Then there exist two nonnegative integers α, β depending only on the chain $\mathcal{I}$ such that the following hold for all $n \gg 0$:*

(i) $\widetilde{H}_i(\operatorname{IN}(G_n)) = 0$ *for* $i \neq 0, 1$.

(ii) $\widetilde{H}_0(\operatorname{IN}(G_n)) \cong \begin{cases} \Bbbk^{n-\alpha} & \text{if } \operatorname{sp}(\mathcal{I}) = 1, \\ 0 & \text{if } \operatorname{sp}(\mathcal{I}) \ge 2. \end{cases}$

(iii) $\widetilde{H}_1(\operatorname{IN}(G_n)) \cong \Bbbk^{\beta}$. *Furthermore, if* $\operatorname{sp}(\mathcal{I}) \ge 2$, *then* $\beta = \begin{cases} 0 & \text{if } j_q = r, \\ 1 & \text{if } j_q < r. \end{cases}$

Notice that only the case $i_1 = 1$ and $p = r$ considered in the preceding theorem is nontrivial, because otherwise, G_n contains isolated vertices for all $n \ge r$, which means that $\operatorname{IN}(G_n)$ is a cone and thus $\widetilde{H}_i(\operatorname{IN}(G_n)) = 0$ for every i. It is also worth noting that all reduced homology groups of $\operatorname{IN}(G_n)$, except for $\widetilde{H}_0(\operatorname{IN}(G_n))$ in the case $\operatorname{sp}(\mathcal{I}) = 1$, are of bounded dimension.

To prove Theorem 7.1, let us begin with its first statement.

Lemma 7.2. *For all $n \ge 4r$ and all $i \neq 0, 1$, it holds that $\widetilde{H}_i(\operatorname{IN}(G_n)) = 0$.*

Proof. Obviously, $\operatorname{IN}(G_n) \neq \{\emptyset\}$ for all $n \ge r$. Hence, $\widetilde{H}_i(\operatorname{IN}(G_n)) = 0$ for all $i < 0$. Recall from [15, Theorem 6.1] that $\operatorname{reg}(R_n/I_n) \le 2$ for all $n \ge 4r$. Thus combining Lemma 2.7(i) and Takayama's formula we get

$$\widetilde{H}_i(\operatorname{IN}(G_n)) = \widetilde{H}_i(\Delta_{\mathbf{0}}(I_n)) \cong H^{i+1}_{\mathfrak{m}}(R_n/I_n)_{\mathbf{0}} = 0$$

for all $n \ge 4r$ and all $i \ge 2$. The desired conclusion follows. ☐

For the proof of the remaining part of Theorem 7.1, note that the case $\operatorname{sp}(\mathcal{I}) \ge 2$ has essentially been treated in Proposition 6.2 (for $\widetilde{H}_1$) and in the proof of Theorem 4.5 (for $\widetilde{H}_0$).

To deal with the case $\operatorname{sp}(\mathcal{I}) = 1$, we need some more auxiliary results. In what follows, when $n \geq 4r$, we denote

$$U_n = \{2r, 2r+1, \dots, n-2r\}. \tag{7.1}$$

Lemma 7.3. *Assume that $i_1 = 1, p = r$ and $\operatorname{sp}(\mathcal{I}) = 1$. Then U_n consists of isolated vertices of G_n^c for all $n \geq 4r$.*

Proof. It follows from the assumption that $E(G_r)$ contains edges of the forms $(1, a)$, $(v-1, v)$, (b, r), where $2 \leq a, v \leq r$ and $1 \leq b \leq r-1$. Take any $k \in U_n$. We need to prove that $\{k, l\} \in E(G_n)$ for every $l \in [n] \setminus \{k\}$. By Lemma 3.6, it suffices to show that either (k, l) or (l, k) belongs to one of the triangles $\Delta((1, a), n-r)$, $\Delta((v-1, v), n-r)$ and $\Delta((b, r), n-r)$. We distinguish two cases.

Case 1: $l < k$. In this case, one can easily check that

$$(l,k) \in \begin{cases} \Delta((1,a), n-r) & \text{if } \ 1 \leq l \leq k-a+1, \\ \Delta((v-1,v), n-r) & \text{if } \ k-a+2 \leq l \leq k-1. \end{cases}$$

Case 2: $l > k$. It is readily seen that

$$(k,l) \in \begin{cases} \Delta((v-1,v), n-r) & \text{if } \ k+1 \leq l \leq n-r+v, \\ \Delta((b,r), n-r) & \text{if } \ n-r+v < l \leq n. \end{cases}$$

Thus we always have $\{k, l\} \in E(G_n)$, as desired. □

The next result determines the asymptotic dimension of $\widetilde{H}_0(\operatorname{IN}(G_n))$ when $\operatorname{sp}(\mathcal{I}) = 1$. As before, the number of connected components of a graph G is denoted by $\mathfrak{c}(G)$.

Lemma 7.4. *Assume that $i_1 = 1, p = r$ and $\operatorname{sp}(\mathcal{I}) = 1$. For $n \geq 4r$, let $\Gamma_n = G_n^c \setminus U_n$ be the induced subgraph of G_n^c on the vertex set*

$$V(\Gamma_n) = [n] \setminus U_n = \{1, \dots, 2r-1\} \cup \{n-2r+1, \dots, n\}.$$

Then the following hold for all $n \gg 0$.

- (i) $\mathfrak{c}(G_n^c) = \mathfrak{c}(\Gamma_n) + n - 4r + 1$.
- (ii) *The graphs Γ_n and Γ_{n+1} are isomorphic. In particular, $\mathfrak{c}(\Gamma_{n+1}) = \mathfrak{c}(\Gamma_n)$ and one can define $\mathfrak{c}(\mathcal{I}) \coloneqq \mathfrak{c}(\Gamma_n)$ for $n \gg 0$.*
- (iii) $\dim_{\Bbbk} \widetilde{H}_0(\operatorname{IN}(G_n)) = n - \alpha$, *where $\alpha = 4r - \mathfrak{c}(\mathcal{I}) \geq 1$.*

Proof. (i) Since U_n consists of isolated vertices of G_n^c by Lemma 7.3, we have

$$\mathfrak{c}(G_n^c) = \mathfrak{c}(\Gamma_n) + |U_n| = \mathfrak{c}(\Gamma_n) + n - 4r + 1.$$

(ii) It suffices to prove the first assertion. Consider the map σ_{2r} as defined in (3.2). Recall that $\sigma_{2r} \in \operatorname{Inc}_{n,n+1}$ for all $n \geq 1$. Let $\phi_n : V(\Gamma_n) \to V(\Gamma_{n+1})$ be the restriction of σ_{2r} on $V(\Gamma_n)$. We show that ϕ_n is a graph isomorphism between Γ_n and Γ_{n+1} for all $n \gg 0$. Evidently, ϕ_n is a bijection between $V(\Gamma_n)$ and $V(\Gamma_{n+1})$. Denote by $\psi_n : V(\Gamma_{n+1}) \to V(\Gamma_n)$ the inverse map of ϕ_n. We claim that $\psi_n(E(\Gamma_{n+1})) \subseteq E(\Gamma_n)$. Equivalently, we need to show that if $\{i,j\} \notin E(\Gamma_n)$, then $\{\phi_n(i), \phi_n(j)\} \notin E(\Gamma_{n+1})$. Indeed, if $\{i,j\} \notin E(\Gamma_n)$, then $\{i,j\} \in E(G_n)$, i.e. $x_i x_j \in I_n$. Since $\sigma_{2r} \in \mathrm{Inc}_{n,n+1}$, this implies that

$$x_{\phi_n(i)} x_{\phi_n(j)} = x_{\sigma_{2r}(i)} x_{\sigma_{2r}(j)} = \sigma_{2r}(x_i x_j) \in I_{n+1}.$$

Hence, $\{\phi_n(i), \phi_n(j)\} \in E(G_{n+1})$, and thus $\{\phi_n(i), \phi_n(j)\} \notin E(\Gamma_{n+1})$, as desired. So ψ_n is a graph morphism from Γ_{n+1} to Γ_n, which is a bijection on the vertex sets. Consequently, it yields an injective map $E(\Gamma_{n+1}) \to E(\Gamma_n)$. In particular, $|E(\Gamma_n)| \geq |E(\Gamma_{n+1})|$ for all $n \geq 4r$. It follows that $|E(\Gamma_n)| = |E(\Gamma_{n+1})|$ for $n \gg 0$. In other words, for all $n \gg 0$, ψ_n is a graph isomorphism, and hence so is its inverse ϕ_n.

(iii) The formula for the dimension of $\widetilde{H}_0(\mathrm{IN}(G_n))$ results from combining (i) and Lemma 2.4. We have $\alpha \geq 1$ since $\mathfrak{c}(\mathcal{I}) \leq |V(\Gamma_n)| = 4r - 1$. ☐

It remains to determine the asymptotic dimension of $\widetilde{H}_1(\mathrm{IN}(G_n))$ when $\mathrm{sp}(\mathcal{I}) = 1$. Given Lemmas 7.2 and 7.4, this can be done by using the *Euler characteristic* of $\mathrm{IN}(G_n)$:

$$\chi(\mathrm{IN}(G_n)) = \sum_{i=0}^{d_n} (-1)^i f_i(\mathrm{IN}(G_n)) = 1 + \sum_{i=-1}^{d_n} (-1)^i \dim_{\Bbbk} \widetilde{H}_i(\mathrm{IN}(G_n)),$$

where $d_n = \dim \mathrm{IN}(G_n)$ and $(f_i(\mathrm{IN}(G_n)))_{i=0}^{d_n}$ is the f-vector of $\mathrm{IN}(G_n)$. The following result describes the asymptotic behavior of the f-vector and the Euler characteristic of $\mathrm{IN}(G_n)$.

Proposition 7.5. *Assume that $i_1 = 1$, $p = r$ and $\mathrm{sp}(\mathcal{I}) = 1$. Then the following statements hold for all $n \gg 0$.*

(i) *$f_0(\mathrm{IN}(G_{n+1})) = f_0(\mathrm{IN}(G_n)) + 1$ and $f_i(\mathrm{IN}(G_{n+1})) = f_i(\mathrm{IN}(G_n))$ for all $i \geq 1$.*
(ii) *$\chi(\mathrm{IN}(G_{n+1})) = \chi(\mathrm{IN}(G_n)) + 1$.*

Proof. It is apparent that (ii) follows from (i), so we only need to prove (i). Observe that if v is an isolated vertex of G_n^c, then it is also an isolated vertex of $\mathrm{IN}(G_n)$ and this gives

$$f_0(\mathrm{IN}(G_n)) = f_0(\mathrm{IN}(G_n \setminus v)) + 1 \quad \text{and} \quad f_i(\mathrm{IN}(G_n)) = f_i(\mathrm{IN}(G_n \setminus v)) \ \text{ for all } i \geq 1. \tag{7.2}$$

For $n \geq 4r$ consider the set U_n given in (7.1). Recall from Lemma 7.3 that U_n consists of isolated vertices of G_n^c. Hence, applying (7.2) repeatedly we obtain

$$\begin{aligned} f_0(\mathrm{IN}(G_n)) &= f_0(\mathrm{IN}(G_n \setminus U_n)) + |U_n| = f_0(\mathrm{IN}(G_n \setminus U_n)) + n - 4r + 1, \\ f_i(\mathrm{IN}(G_n)) &= f_i(\mathrm{IN}(G_n \setminus U_n)) \ \text{ for all } i \geq 1. \end{aligned} \tag{7.3}$$

Note that $G_n \setminus U_n = \Gamma_n^c$, where $\Gamma_n = G_n^c \setminus U_n$. By Lemma 7.4, the graphs Γ_n and Γ_{n+1} are isomorphic for $n \gg 0$. This implies that the graphs $G_n \setminus U_n$ and $G_{n+1} \setminus U_{n+1}$ are isomorphic for $n \gg 0$. Consequently, the simplicial complexes $\mathrm{IN}(G_n \setminus U_n)$ and $\mathrm{IN}(G_{n+1} \setminus U_{n+1})$ are also isomorphic for $n \gg 0$. The desired conclusion now follows readily from (7.3). ☐

Remark 7.6. Proposition 7.5(i) implies that $\dim \mathrm{IN}(G_n)$ is a constant for $n \gg 0$. This also follows from [21, Theorem 3.8] and the fact that I_n is the Stanley–Reisner ideal of $\mathrm{IN}(G_n)$.

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. The first statement follows from Lemma 7.2. To prove the second and third statements, we distinguish two cases.

Case 1: $\operatorname{sp}(\mathcal{I}) \geq 2$. From the proof of Theorem 4.5 we know that the graph G_n^c is connected for all $n \geq r$. Hence, $\widetilde{H}_0(\mathrm{IN}(G_n)) = 0$ for all $n \geq r$ by Lemma 2.4. Let us now prove the formula for $\widetilde{H}_1(\mathrm{IN}(G_n))$. In view of Proposition 6.2, we only need to consider the case $j_q = r$. In this case, $\operatorname{reg}(R_n/I_n) = 1$ for all $n \geq 3r-3$ by Lemma 3.12. So using Lemma 2.7(i) and Takayama's formula, we get

$$\widetilde{H}_1(\mathrm{IN}(G_n)) = \widetilde{H}_1(\Delta_{\mathbf{0}}(I_n)) \cong H^2_{\mathfrak{m}}(R_n/I_n)_{\mathbf{0}} = 0 \quad \text{for all } n \geq 3r-3.$$

Case 2: $\operatorname{sp}(\mathcal{I}) = 1$. According to Lemma 7.4(iii), there exists a constant $\alpha \geq 1$ such that $\dim_{\Bbbk} \widetilde{H}_0(\mathrm{IN}(G_n)) = n - \alpha$ for $n \gg 0$. It remains to prove that $\dim_{\Bbbk} \widetilde{H}_1(\mathrm{IN}(G_n)) = \beta$ for some constant β when $n \gg 0$. Indeed, it follows from Proposition 7.5(ii) that there exists a constant γ such that

$$\chi(\mathrm{IN}(G_n)) = n - \gamma \quad \text{for } n \gg 0.$$

Note that $\chi(\mathrm{IN}(G_n)) = 1 + \dim_{\Bbbk} \widetilde{H}_0(\mathrm{IN}(G_n)) - \dim_{\Bbbk} \widetilde{H}_1(\mathrm{IN}(G_n))$ for $n \gg 0$ by Lemma 7.2. Therefore,

$$n - \gamma = 1 + (n - \alpha) - \dim_{\Bbbk} \widetilde{H}_1(\mathrm{IN}(G_n)),$$

and hence $\dim_{\Bbbk} \widetilde{H}_1(\mathrm{IN}(G_n)) = \gamma - \alpha + 1$ for $n \gg 0$, as desired. □

Proposition 6.2 and the proof of Theorem 7.1 provide lower bounds for the stability indices of $\widetilde{H}_0(\mathrm{IN}(G_n))$ and $\widetilde{H}_1(\mathrm{IN}(G_n))$ when $\operatorname{sp}(\mathcal{I}) \geq 2$, namely, r and $3r$, respectively. It would therefore be interesting to have similar bounds in the case $\operatorname{sp}(\mathcal{I}) = 1$. In this case, it would also be interesting to determine the constants α and β in Theorem 7.1 explicitly. While α is always positive by Lemma 7.4, the following examples indicate that β can be zero or not.

Example 7.7. Let $r = 2$ and $E(G_2) = \{(1,2)\}$. Then G_n is the complete graph K_n for all $n \geq 2$. Thus, $\widetilde{H}_0(\mathrm{IN}(G_n)) \cong \Bbbk^{n-1}$ and $\widetilde{H}_1(\mathrm{IN}(G_n)) = 0$ for all $n \geq 2$.

Example 7.8. Let $r = 4$ and $E(G_4) = \{(1,2),(3,4)\}$. We claim that for all $n \geq 5$,

$$\widetilde{H}_0(\mathrm{IN}(G_n)) \cong \Bbbk^{n-4} \quad \text{and} \quad \widetilde{H}_1(\mathrm{IN}(G_n)) \cong \Bbbk.$$

In fact, it is not hard to show that for all $n \geq 5$, the facets of $\mathrm{IN}(G_n)$ are precisely

$$\{1, n-1\}, \{2, n-1\}, \{1, n\}, \{2, n\}, \{3\}, \{4\}, \ldots, \{n-2\}.$$

The desired conclusion follows.

8. Appendix

Here, as promised, we provide the proof of Lemma 6.6. It is more convenient to prove the following slightly stronger result, which specializes to Lemma 6.6 when $a = b$ and $A = B$.

Proposition 8.1. *Assume that $i_1 = 1$, $p = r$ and $\operatorname{sp}(\mathcal{I}) \geq 2$. Let a and A be integers with $1 \leq a \leq b$ and $B \leq A \leq r-1$. Denote by $G_n(a, A)$ the induced subgraph of G_n on the vertex set $V(G_n(a, A)) = V_1 \cup V_2$, where $V_1 = \{1, \ldots, a-1\}$ and $V_2 = \{n-r+A+1, \ldots, n-1\}$. Then*

$$\widetilde{H}_1(\mathrm{IN}(G_n(a, A))) = 0 \quad \textit{for all } n \geq 3r.$$

The proof of this result is mainly based on Proposition 2.6. In order to apply Proposition 2.6, some preparations are needed.

Lemma 8.2. *Under the assumptions of Proposition 8.1, the following statements hold.*

(i) $G_n(a, A)$ *is weakly chordal for all* $n \geq 2r + 1$.
(ii) $\operatorname{im}(G_n(a, A)) \leq 2$ *for all* $n \geq 3r$.
(iii) *Let* $n \geq 3r$ *and* $(u_1, v_1), (u_2, v_2) \in E(G_n(a, A))$. *If* $(u_1, v_1), (u_2, v_2)$ *with* $u_1 < u_2$ *form an induced matching of* $G_n(a, A)$, *then*

$$\begin{aligned} &1 \leq u_1 < v_1 \leq a - 1, \\ &n - r + A + 1 \leq u_2 < v_2 \leq n - 1, \\ &a \geq 3 \quad \textit{and} \quad r - A \geq 3. \end{aligned}$$

Proof. (i) By Lemma 3.10, $G_n(b, B) = G_n \setminus N[n]$ is weakly chordal for all $n \geq 2r + 1$. Since $G_n(a, A)$ is an induced subgraph of $G_n(b, B)$, we deduce that $G_n(a, A)$ is also weakly chordal.

(ii) The graph $G_n(a, A)$ is an induced subgraph of G_n. So by [15, Theorem 3.1],

$$\operatorname{im}(G_n(a, A)) \leq \operatorname{im}(G_n) \leq 2 \quad \text{for all } n \geq 3r.$$

(iii) Assume that $(u_i, v_i) \in \Delta((k_i, l_i), n - r)$, where $(k_i, l_i) \in E(G_r)$ for $i = 1, 2$. Since $G_n(a, A)$ is an induced subgraph of G_n, $\{(u_1, v_1), (u_2, v_2)\}$ is also an induced matching of G_n. From the proof of Lemma 3.8 we know that

$$v_1 < k_2 < n - r + l_1 < u_2.$$

As $V(G_n(a, A)) = \{1, \ldots, a-1\} \cup \{n-r+A+1, \ldots, n-1\}$ and $\max\{a-1, k_2\} < r < n-r$, it follows that

$$1 \leq u_1 < v_1 \leq a - 1 \text{ and } n - r + A + 1 \leq u_2 < v_2 \leq n - 1.$$

In particular, these inequalities yield $a \geq 3$ and $r - A \geq 3$. ☐

Lemma 8.3. *Keep the assumptions of Proposition 8.1. Assume further that* $\mathfrak{B}$ *is a complete bipartite subgraph of* $G_n(a, A)$ *with partition* $V(\mathfrak{B}) = W_1 \cup W_2$, *where* $W_1, W_2 \neq \emptyset$. *If* $U \subseteq V(\mathfrak{B})$ *consists of consecutive integers, then either* $U \subseteq W_1$ *or* $U \subseteq W_2$.

Proof. It suffices to prove that $U \subseteq W_1$ if $U \cap W_1 \neq \emptyset$. Indeed, take $k \in U \cap W_1$. Let $l = k-1$ or $l = k+1$. Then $\{k, l\} \notin E(\mathfrak{B})$ since $\operatorname{sp}(\mathcal{I}) \geq 2$. Hence, $l \in W_1$ whenever $l \in V(\mathfrak{B})$. Since U consists of consecutive integers, it follows easily by induction that $U \subseteq W_1$. ☐

Lemma 8.4. *Keep the assumptions of Proposition 8.1. Then for all* $n \geq 3r$, $G_n(a, A)$ *does not contain a strongly disjoint family of two complete bipartite subgraphs* $\mathfrak{B}_1, \mathfrak{B}_2$ *such that* $V(\mathfrak{B}_1) \cup V(\mathfrak{B}_2) = V(G_n(a, A))$.

Proof. Assume the contrary that there exists a strongly disjoint family of two complete bipartite subgraphs $\mathfrak{B}_1, \mathfrak{B}_2$ of $G_n(a, A)$ with $V(\mathfrak{B}_1) \cup V(\mathfrak{B}_2) = V(G_n(a, A))$ for some $n \geq 3r$. Then $G_n(a, A)$ has an induced matching $(u_1, v_1), (u_2, v_2)$, where $(u_i, v_i) \in E(\mathfrak{B}_i)$ for $i = 1, 2$. We may assume that $u_1 < u_2$. Then Lemma 8.2(iii) yields

$$1 \leq u_1 < v_1 \leq a - 1 \text{ and } n - r + A + 1 \leq u_2 < v_2 \leq n - 1.$$

Let $V(\mathfrak{B}_1) = W_1 \cup W_2$ be the vertex partition of $\mathfrak{B}_1$. We first show that $k \notin V(\mathfrak{B}_1)$ for some $k < v_1$. In fact, if $[v_1] \subseteq V(\mathfrak{B}_1)$, then it follows from Lemma 8.3 that either $[v_1] \subseteq W_1$ or $[v_1] \subseteq W_2$. But this contradicts the fact that $(u_1, v_1) \in E(\mathfrak{B}_1)$. Hence, there must exist $k < v_1$ such that $k \notin V(\mathfrak{B}_1)$.

As $k < v_1 \leq a - 1$, we have $k \in V(G_n(a, A)) = V(\mathfrak{B}_1) \cup V(\mathfrak{B}_2)$, and thus $k \in V(\mathfrak{B}_2)$. Since $\mathfrak{B}_2$ is complete bipartite and $(u_2, v_2) \in E(\mathfrak{B}_2)$, we deduce that either (k, u_2) or (k, v_2) belongs to $E(\mathfrak{B}_2) \subseteq E(G_n)$. Consider the case $(k, u_2) \in E(G_n)$; the case $(k, v_2) \in E(G_n)$ being similar. Since $k < v_1 \leq r$ and $n - r < u_2$, moving east from (k, u_2) to (v_1, u_2) using Lemma 3.7(i), we get $(v_1, u_2) \in E(G_n)$. Consequently, $(v_1, u_2) \in E(G_n(a, A))$, as $G_n(a, A)$ is an induced subgraph of G_n. But this is impossible because $(u_1, v_1), (u_2, v_2)$ is an induced matching of $G_n(a, A)$. The desired conclusion follows. ☐

Lemma 8.5. *Keep the assumptions of Proposition 8.1. Assume also that* $\operatorname{im}(G_n(a, A)) = 2$ *and that* $G_n(a, A)$ *has a complete bipartite subgraph* $\mathfrak{B}$ *with* $|V(\mathfrak{B})| \geq |V(G_n(a, A))| - 1$ *and* $E(\mathfrak{B}) \neq \emptyset$ *for some* $n \geq 3r$. *If* $\widetilde{H}_1(\mathrm{IN}(G_n(a-1, A))) = \widetilde{H}_1(\mathrm{IN}(G_n(a, A+1))) = 0$, *then* $\widetilde{H}_1(\mathrm{IN}(G_n(a, A))) = 0$.

Proof. Let $\{(u_1, v_1), (u_2, v_2)\}$ with $u_1 < u_2$ be an induced matching of $G_n(a, A)$. Then we know from Lemma 8.2(iii) that

$$1 \leq u_1 < v_1 \leq a - 1 \text{ and } n - r + A + 1 \leq u_2 < v_2 \leq n - 1.$$

Let $V(\mathfrak{B}) = W_1 \cup W_2$ be the vertex partition of $\mathfrak{B}$. Then $W_1, W_2 \neq \emptyset$ since $E(\mathfrak{B}) \neq \emptyset$. Recall that $V(G_n(a, A)) = V_1 \cup V_2$, where both V_1 and V_2 consist of consecutive integers. Since $|V(\mathfrak{B})| \geq |V(G_n(a, A))| - 1$, either V_1 or V_2 is contained in $V(\mathfrak{B})$. So by reindexing (if needed), it follows from Lemma 8.3 that either $V_1 \subseteq W_2$ or $V_2 \subseteq W_2$. We consider only the case $V_2 \subseteq W_2$; the other case can be treated similarly. In this case, $u_2, v_2 \in V_2 \subseteq W_2$. Since $(u_1, v_1), (u_2, v_2)$ form an induced matching of $G_n(a, A)$, we infer that

$$\text{neither } u_1 \text{ nor } v_1 \text{ belongs to } W_1. \tag{8.1}$$

Hence, either u_1 or v_1 belongs to W_2 because $|V(\mathfrak{B})| \geq |V(G_n(a, A))| - 1$. Thus $V_1 \cap W_2 \neq \emptyset$. This together with Lemma 8.3 and the fact that $W_1 \neq \emptyset$ yields $V_1 \not\subseteq V(\mathfrak{B})$. The assumption $|V(\mathfrak{B})| \geq |V(G_n(a, A))| - 1$ now imples that

$$V(\mathfrak{B}) = (V_1 \setminus \{k\}) \cup V_2$$

for some $k \in V_1$. Note that $k \geq 2$, since otherwise, $V_1 \setminus \{1\} \subseteq V(\mathfrak{B})$. So Lemma 8.3 yields $V_1 \setminus \{1\} \subseteq W_1$ or $V_1 \setminus \{1\} \subseteq W_2$. As $W_1 \neq \emptyset$, we deduce that $V_1 \setminus \{1\} \subseteq W_1$. But then either u_1 or v_1 belongs to W_1, contradicting (8.1).

Claim 8.5.1. *It holds that*

$$W_1 = \{k+1, \ldots, a-1\} \quad \text{and} \quad W_2 = \{1, \ldots, k-1\} \cup V_2.$$

Indeed, we have $1 \in V(\mathfrak{B})$ because $k \geq 2$. If $1 \in W_1$, then $(1, u_2) \in E(\mathfrak{B}) \subseteq E(G_n)$ since $u_2 \in W_2$. Recall that $1 \leq u_1 \leq r$ and $n - r < u_2$. So according to Lemma 3.7(i), it holds that $(u_1, u_2) \in E(G_n)$, whence $(u_1, u_2) \in E(G_n(a, A))$. But this contradicts the fact that $\{(u_1, v_1), (u_2, v_2)\}$ is an induced matching of $G_n(a, A)$. Thus we must have $1 \in W_2$. This together with Lemma 8.3 implies $[k-1] \subseteq W_2$. Hence, $[k-1] \cup V_2 \subseteq W_2$. Now since $W_1 \neq \emptyset$, the desired claim follows easily from Lemma 8.3.

Claim 8.5.2. *The closed neighborhood of* $a-1$ *in* $G_n(a,A)$ *is*

$$N[a-1] = \{1,\dots,a-3,a-1\} \cup V_2 = V(G_n(a,A)) \setminus \{a-2\}.$$

Since $\mathrm{sp}(\mathcal{I}) \geq 2$, we have $a-2 \notin N[a-1]$, hence $N[a-1] \subseteq V(G_n(a,A)) \setminus \{a-2\}$. On the other hand, Claim 8.5.1 gives $W_2 \subseteq N[a-1]$. Thus it remains to show that

$$\{k,k+1,\dots,a-3\} \subseteq N(a-1). \tag{8.2}$$

From Claim 8.5.1 we know that $(k-1,k+1) \in E(\mathfrak{B}) \subseteq E(G_n)$. By Lemma 3.6 and the assumption that $\mathrm{sp}(\mathcal{I}) \geq 2$, this implies that $(k-1,k+1) \in \Delta((l,l+2),n-r)$ for some $(l,l+2) \in E(G_r)$. In particular, one has $l \leq k-1$. Now using Lemma 3.6, it is easy to check that $(i,a-1) \in \Delta((l,l+2),n-r)$ for any $k \leq i \leq a-3$. Hence, (8.2) is true, as required.

Claim 8.5.2 implies that $G_n(a,A) \setminus N[a-1] \cong K_1$, where K_1 is the complete graph on one vertex. Moreover, it is clear that $G_n(a,A) \setminus \{a-1\} = G_n(a-1,A)$. So applying Lemma 2.5 to $G_n(a,A)$ and its vertex $a-1$, and using the fact that $\widetilde{H}_0(\mathrm{IN}(K_1)) = \widetilde{H}_1(\mathrm{IN}(K_1)) = 0$, we obtain

$$\widetilde{H}_1(\mathrm{IN}(G_n(a,A))) \cong \widetilde{H}_1(\mathrm{IN}(G_n(a-1,A))) = 0.$$

The proof is complete. □

We are now prepared to give the proof of Proposition 8.1.

Proof of Proposition 8.1. Fix an $n \geq 3r$. We show that $\widetilde{H}_1(\mathrm{IN}(G_n(a,A))) = 0$ by a double induction on $a \in [b]$ and on $r-A \in [r-B]$. Let $S = \Bbbk[x_i \mid i \in V(G_n(a,A))]$ and denote by $L(a,A) \subseteq S$ the edge ideal of $G_n(a,A)$. Recall that the graph $G_n(a,A)$ is weakly chordal by Lemma 8.2(i). So [44, Theorem 14] yields

$$\mathrm{reg}(S/L(a,A)) = \mathrm{im}(G_n(a,A)).$$

According to Lemma 8.2(ii), we have $\mathrm{im}(G_n(a,A)) \leq 2$. Let us first consider the case $\mathrm{im}(G_n(a,A)) \leq 1$. Notice that this case covers the case that either $a \leq 2$ or $r-A \leq 2$ by virtue of Lemma 8.2(iii). Since $\mathrm{reg}(S/L(a,A)) = \mathrm{im}(G_n(a,A)) \leq 1$, it follows from Takayama's formula and Lemma 2.7(i) that

$$\widetilde{H}_1(\mathrm{IN}(G_n(a,A))) \cong H^2_{\mathfrak{m}}(S/L(a,A))_{\mathbf{0}} = 0,$$

where $\mathfrak{m}$ denotes the graded maximal ideal of S.

Now assume that $\mathrm{im}(G_n(a,A)) = 2$. In this case, $a \geq 3$ and $r-A \geq 3$ by Lemma 8.2(iii). Since $G_n(a,A)$ is weakly chordal, Proposition 2.6 implies that there exists a strongly disjoint family of complete bipartite subgraphs $\mathfrak{B}_1,\dots,\mathfrak{B}_g$ of $G_n(a,A)$ with $1 \leq g \leq \mathrm{im}(G_n(a,A))$ such that

$$\mathrm{pd}(S/L(a,A)) = \sum_{i=1}^{g} |V(\mathfrak{B}_i)| - g.$$

Since $\mathrm{im}(G_n(a,A)) = 2$, there are only two cases to consider.

Case 1: $g=2$. By Lemma 8.4, $V(\mathfrak{B}_1) \cup V(\mathfrak{B}_2) \subsetneq V(G_n(a,A))$. It follows that

$$\mathrm{pd}(S/L(a,A)) = |V(\mathfrak{B}_1)| + |V(\mathfrak{B}_2)| - 2 \leq |V(G_n(a,A))| - 1 - 2 = \dim(S) - 3.$$

Hence, $\operatorname{depth}(S/L(a,A)) \geq 3$ by the Auslander–Buchsbaum formula. This together with Takayama's formula and Lemma 2.7(i) implies that

$$\widetilde{H}_1(\mathrm{IN}(G_n(a,A))) \cong H^2_{\mathfrak{m}}(S/L(a,A))_{\mathbf{0}} = 0.$$

Case 2: $g = 1$. If $|V(\mathfrak{B}_1)| \leq |V(G_n(a,A))| - 2$, then again

$$\operatorname{pd}(S/L(a,A)) = |V(\mathfrak{B}_1)| - 1 \leq |V(G_n(a,A))| - 2 - 1 = \dim(S) - 3,$$

and we reach the desired conclusion with the same argument as in the previous case. Now suppose that $|V(\mathfrak{B}_1)| \geq |V(G_n(a,A))| - 1$. From the induction hypothesis we know that

$$\widetilde{H}_1(\mathrm{IN}(G_n(a-1,A))) = \widetilde{H}_1(\mathrm{IN}(G_n(a,A+1))) = 0.$$

So by Lemma 8.5, $\widetilde{H}_1(\mathrm{IN}(G_n(a,A))) = 0$. This completes the proof. □

Acknowledgments. The first and fifth named authors (TQH and TTN) were supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the grant number 101.04-2023.07. The fourth named author (HDN) was supported by NAFOSTED under the grant number 101.04-2023.30. Part of this work was done while the authors were visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). We would like to thank VIASM for hospitality and financial support.

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