Monotonic complementarity problems

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Dedicated to Professor Le Dung Muu on his 75th birthday

Abstract

A general class of nonlinear complementarity problems is studied that includes polynomial complementarity problems as a subclass. In contrast to most existing methods for nonlinear complementarity problems, our algorithm works under very general conditions. Preliminary computational experiments on polynomial complementarity problems show its practicability for problems with polynomial degree up to 41 and variable number up to 8.

Keywords: Nonlinear complementarity problem; complementarity problem with d.m. functions; polynomial complementarity problem; linear complementarity problem; branch-reduce-and-bound algorithm.

1. Introduction

We are concerned with the following *Monotonic Complementarity Problem*

$$\text{Find } x \in \mathbb{R}^n_+ \text{ satisfying } g(x) \geq 0, h(x) \geq 0, \langle g(x), h(x) \rangle = 0, \tag{MCP}$$

where $g, h : \mathbb{R}^n \to \mathbb{R}^n$, and $g_i, h_i (i = 1, \dots, n)$ are continuous *d.m. functions* such that

$$D = \{x \in \mathbb{R}^n_+ \mid g(x) \geq 0, h(x) \geq 0\} \neq \emptyset. \tag{1}$$

Recall from [24] that a d.m. function on $\mathbb{R}^n_+$ is a function which can be represented as a difference of two increasing functions on $\mathbb{R}^n_+$, where by increasing function on $\mathbb{R}^n_+$ we mean a function $u : \mathbb{R}^n \to \mathbb{R}$ such that $x' \geq x \geq 0$ implies $u(x') \geq u(x)$. Outstanding examples of d.m. functions on $\mathbb{R}^n_+$ are polynomials (generalized polynomials, resp.) of the form $\sum_\alpha c_\alpha x^\alpha$ with α being natural numbers ($\alpha \in \mathbb{R}^n_+$, resp.), $c_\alpha \in \mathbb{R}_+$, and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. To represent a polynomial (or generalized polynomial) as a difference of two increasing functions on $\mathbb{R}^n_+$, it suffices to group separately all terms with positive coefficients c_α and all terms with negative coefficients c_α.

Polynomial and generalized polynomial complementarity problems, i.e. (MCP) involving polynomials or generalized polynomials, are encountered in diverse fields (see e.g. [4]). When $g(x)$ is an affine mapping and $h(x) \equiv x$, (MCP) is the classical linear complementarity problem [2] which has been extensively studied during the last five decades.

So far most numerical studies of nonlinear complementarity problems have been based on nonlinear programming (e.g., [9], [14]), using sometimes merit functions satisfying certain conditions (e.g. [18], [28]). A nonlinear complementarity problem more general than (MCP) has been considered in [20], where g, h are just assumed to be continuous and the constraints are $g(x) \in \mathcal{K}$, $h(x) \in \mathcal{K}^\circ$ with $\mathcal{K} \subset \mathbb{R}^n$ being a closed convex cone and $\mathcal{K}^\circ$ its polar. In [20] it is shown that this general problem can be reduced to a differentiable minimization problem, but no numerical result is provided.

The best known numerical algorithms for solving polynomial complementarity problems are the homotopy methods [6, 19], the interior-point methods [10], the nonsmooth-equations approach [14], and the basic Newton method [17]. For a discussion of these methods we refer the reader to [15] where a comprehensive treatment of the general complementarity problem can also be found.

More recently, in [16] the reformulation-linearization technique (RLT) was used to formulate the linear complementarity problem as an equivalent mixed integer linear program which was solved by an implicit enumeration method that utilized Langrangian relaxation, surrogate constraints, cutting planes and a heuristic to successfully exploit the resulting linearization. The reported computational results were the best to date for unstructured linear complementarity problems. The same approach can readily be extended to handle polynomial complementarity problems.

However, to the best of our knowledge, so far no numerical algorithm has been developed for nonlinear complementarity problems of the general class (MCP). The purpose of the present paper is to suggest an approach to (MCP) based on exploiting the monotonic structure by using recently developed techniques of monotonic optimization [22, 24].

In contrast to the above mentioned methods, each of which can be applied to a restricted subclass of (MCP), the algorithm to be proposed works under very general conditions. All the development below is valid, in particular, when the functions $g_i(x), h_i(x)$ are arbitrary generalized polynomials. Also note that neither g nor h is assumed to be a monotone mapping as defined in monotone complementarity problems in [13]. In addition, an important feature of the method to be proposed is that it actually applies to complementarity problems with an additional box constraint of the form $c \leq x \leq d$, where $0 \leq c_i \leq d_i \leq +\infty$ for $i = 1, \ldots, n$.

The paper is divided into several sections. First, in Section 2, we discuss a reformulation of (MCP) as a monotonic optimization problem, in a format most suitable for numerical study purpose. Based on this monotonic reformulation, a new branch-reduce-and-bound algorithm for solving (MCP) is developed in Sections 3 through 6. Details on the three basic operations involved in this algorithm: reducing, bounding, and branching are discussed in Sections 3 and 4, then a formal description of the algorithm is given in Section 5. In Section 6, an improved bounding method is described, while Section 7 discusses some specializations of (MCP). Finally, Section 8 closes the paper with some illustrative numerical examples and preliminary computational results.

2. Reduction to monotonic optimization

Since $g_i, h_i (i = 1, \ldots, n)$ are assumed to be nonnegative d.m. functions on $\mathbb{R}^n_+$, it can easily be checked that the function

$$\langle g(x), h(x) \rangle = \sum_{i=1}^{n} g_i(x) h_i(x)$$

is also a d.m. function on $\mathbb{R}^n_+$. Let us then associate with (MCP) the following d.m. optimization problem

$$\min\{\langle g(x), h(x) \rangle \mid x \in \mathbb{R}^n_+, g(x) \geq 0, h(x) \geq 0\}. \tag{P}$$

The following proposition is obvious, where we write min (P) to denote the optimal value of problem (P).

Proposition 1. (MCP) *has a solution if and only if* min (P) $= 0$.

Thus, solving (MCP) reduces to solving the *d.m. programming problem* (P), which can be done by methods of monotonic optimization developed in [22, 23, 24] and especially, by the SIT Algorithm discussed in [25]. Furthermore, when $g(x), h(x)$ are affine, the problem (P) belongs to a class of nonconvex quadratic optimization problems studied earlier in [12] via multiplicative programming or in [8] and more recently in [27] via a monotonic approach. As it turns out, without much effort the method in [8] can be extended to solve (P) in the general case.

A peculiar feature of problem (P) which may require special treatment is that its feasible set may be unbounded. Therefore, we first convert the problem to a form directly amenable to standard techniques of monotonic optimization. For this we define

$$\Phi(y,z) = \min\{\langle g(x), h(x)\rangle \mid y \le h(x) \le z, x \in D\} \tag{2}$$

with the usual convention $\min \emptyset = +\infty$, and D is defined in (1). Since $g(x) \ge 0, h(x) \ge 0$ for all $x \in D$, we clearly have

$$\Phi(y,z) \ge 0 \quad \forall y, z \ge 0. \tag{3}$$

As can easily be checked, we have

(i) $0 \le y \le y'$ implies $\Phi(y,z) \le \Phi(y',z)$, i.e., with fixed z the function $\Phi(y,z)$ is increasing with respect to y in $\mathbb{R}^n_+$;

(ii) $0 \le z' \le z$ implies $\Phi(y,z) \le \Phi(y,z')$, i.e., with fixed y the function $\Phi(y,z)$ is decreasing with respect to z in $\mathbb{R}^n_+$;

(iii) $[y', z'] \subset [y,z] \subset \mathbb{R}^n_+$ implies $\Phi(y',z') \ge \Phi(y,z)$, since $\Phi(y',z') \ge \Phi(y',z) \ge \Phi(y,z)$.

Setting $h(D) := \{y \in \mathbb{R}^n_+ \mid y = h(x), x \in D\}$. Obviously from the definition of the function $\Phi(y,z)$, if $\overline{x} \in D$ is a solution of (MCP), then $\Phi(y,z) = 0$ for all $y, z \in \mathbb{R}^n_+$ such that $y \le h(\overline{x}) \le z$. Conversely, if $\Phi(y,z) = 0$ for some $0 \le y \le z$ and $h(D) \cap [y,z] \ne \emptyset$, then (MCP) has a solution $\overline{x} \in D$ such that $h(\overline{x}) \in [y,z]$.

Employing these properties of the function $\Phi(y,z)$, to find solution $\overline{x} \in D$ of (MCP) we construct a sequence of boxes $[y,z] \subset \mathbb{R}^n_+$ converging to $h(\overline{x})$. We first find a lower bound for the lower vertex of the initial box of such a sequence. To do that, let z be sufficiently large and fixed. Consider the following problem:

$$\min\{\Phi(y,z) \mid y \in h(D)\}. \tag{Q}$$

Proposition 2. *If $\overline{x}$ is a solution of* (MCP)*, then $\overline{y} = h(\overline{x})$ is an optimal solution of the problem* (Q) *with $\Phi(\overline{y}, z) = 0$. Conversely, if $\overline{y}$ is an optimal solution of* (Q) *with $\Phi(\overline{y}, z) = 0$, then every solution $\overline{x} \in D$, if exists, of (2) corresponding to $\overline{y}$ and z is a solution of* (MCP) *with $\overline{y} \le h(\overline{x}) \le z$.*

Proof. If $\overline{x}$ is a solution of (MCP), i.e., $\overline{x} \in D$ and $\langle g(\overline{x}), h(\overline{x})\rangle = 0$, $g(\overline{x}) \ge 0$, $h(\overline{x}) \ge 0$, then $\overline{y} = h(\overline{x}) \in h(D)$ and by (2) we have

$$0 \le \min\{\Phi(y,z) \mid y \in h(D)\} \le \Phi(\overline{y}, z) \le \langle g(\overline{x}), h(\overline{x})\rangle = 0.$$

So $\Phi(\overline{y}, z) = 0$ and $\overline{y}$ solves (Q).

Conversely, if $\overline{y}$ is an optimal solution of (Q) with $\Phi(\overline{y}, z) = 0$, while $\overline{x} \in D$ is a solution of

$$\Phi(\overline{y}, z) = \min\{\langle g(x), h(x)\rangle \mid \overline{y} \leq h(x) \leq z, x \in D\} = 0,$$

then $g(\overline{x}) \geq 0, h(\overline{x}) \geq 0$, and $\langle g(\overline{x}), h(\overline{x})\rangle = 0$, so $\overline{x}$ is a solution of (MCP), moreover, $\overline{y} \leq h(\overline{x}) \leq z$. □

Note that (Q) is equivalent to a monotonic optimization problem. To prove this fact, define $H = h(D) + \mathbb{R}^n_+ = \{y \in \mathbb{R}^n_+ \mid y \geq h(x) \text{ for some } x \in D\}$. Clearly $y' \geq y \in H$ implies $y' \in H$, so H is a *conormal set* in $\mathbb{R}^n_+$. Actually H is the conormal hull of $h(D)$, i.e. the smallest conormal set containing $h(D)$. It can be easily seen that the problem (Q) is equivalent to the following monotonic optimization problem

$$\min\{\Phi(u, z) \mid u \in H\}. \tag{$\tilde{\mathrm{Q}}$}$$

Indeed, if $\overline{u}$ solves ($\tilde{\mathrm{Q}}$), then $\overline{u} \geq \overline{y} = h(\overline{x})$ for some $\overline{x} \in D$ and $\Phi(\overline{y}, z) \leq \Phi(\overline{u}, z) \leq \Phi(y, z)$ for all $y \in H$, hence $\overline{y}$ solves (Q). Conversely, if $\overline{y}$ solves (Q), then $\overline{y} \in h(D) \subset H$. Furthermore, for any $u \in H$ there exists $\tilde{y} \in h(D)$ such that $u \geq \tilde{y}$. We therefore have

$$\Phi(u, z) \geq \Phi(\tilde{y}, z) \geq \Phi(\overline{y}, z) = \min\{\Phi(y, z) \mid y \in h(D)\},$$

so $\overline{y}$ solves ($\tilde{\mathrm{Q}}$).

We have thus reduced (MCP) to solving the problem

$$\min\{\Phi(y, z) \mid y \in h(D)\} = 0. \tag{Q_0}$$

For any $y \in \mathbb{R}^n_+$ let $E(y) := \{x \in D \mid \langle g(x), h(x)\rangle = \Phi(y, z)\}$. If $\overline{y}$ solves (Q_0) and $E(\overline{y}) \neq \emptyset$, then any $\overline{x} \in E(\overline{y})$ solves (MCP). Conversely, any solution $\overline{x}$ to (MCP) satisfies $\overline{x} \in E(\overline{y})$ for some solution $\overline{y}$ of (Q_0).

Problem (Q_0) consists in checking whether the monotonic optimization problem (P) has an optimal solution $\overline{y}$ with objective function value $\Phi(\overline{y}, z) = 0$. This can be done by applying the SIT algorithm (Successive Incumbent Transcending Algorithm) developed in [25, 26, 27] for transcending the value 0 in the monotonic optimization problem (P). The following sections describe this algorithm adapted to solve (MCP).

3. Bounding operation

Given a nonempty closed set $G \subset \mathbb{R}^n_+$, a point $y \in G$ is called a *lower boundary point* of G if there is no $z \in G$ such that $z \leq y$ and $z \neq y$, i.e., if $\{z \in \mathbb{R}^n \mid G \cap (z - \mathbb{R}^n_+)\} = \{z\}$. Denote the set of all lower boundary points of G by $\partial_- G$.

Proposition 3. *(i) For any $x \in G$ there exists $\sigma(x) \in \partial_- G$ such that $\sigma(x) \leq x$.*

(ii) If $\overline{y} \in h(D)$ is a solution of the problem (Q)*, then so is every $u \in h(D) \cap (\overline{y} - \mathbb{R}^n_+)$. In particular, $\sigma(\overline{y})$ is also a solution of* (Q).

Proof. (i) Given any $x \in G$, let $\sigma(x) := z^n$, where z^n is defined inductively by

$$z^1 = \arg\min\{y_1 \mid y \in G, y \leq x\},$$
$$z^i = \arg\min\{y_i \mid y \in G, y \leq z^{i-1}\} \quad \text{for } i = 2, \ldots, n.$$

Then $z^n \in \{x\} - \mathbb{R}^n_+$, and there is no $y \in G$ such that $y \neq z^n$, $y \leq z^n$, i.e., $G \cap (z^n - \mathbb{R}^n_+) = \{z^n\}$. Therefore $\sigma(x) \in \partial_- G$.

(ii) If $u \in h(D) \cap (\overline{y} - \mathbb{R}^n_+)$, then $u \in h(D)$ and $u \leq \overline{y}$. The latter inequality implies $\Phi(u, z) \leq \Phi(\overline{y}, z) = \min\{\Phi(y, z) \mid y \in h(D)\}$, because the function $\Phi(\cdot, z)$ is increasing. This fact, together with $u \in h(D)$, implies that u is also a solution of the problem (Q). $\square$

As a corollary of Proposition 3(i), we have

$$\arg\min\{\Phi(y, z) \mid y \in \partial_- G\} \subset \arg\min\{\Phi(y, z) \mid y \in G\}. \tag{4}$$

As a consequence of (4) and Proposition 3(ii), to find a solution of (Q) we can restrict the set of interest points to $\partial_- h(D)$.

Proposition 4. *For each $i = 1, \ldots, n$ let*

$$a_i = \inf\{y_i \mid y = h(x) \text{ for some } x \in D\},$$
$$\theta_i = \inf\{\theta \mid \theta \geq 0, a + \theta e^i = h(x) \text{ for some } x \in D\},$$

in which e^i is the i-th unit vector of $\mathbb{R}^n_+$. Define

$$b = a + \sum_{i=1}^{n} \theta_i e^i.$$

Then $\partial_- h(D) \subset [a, b]$.

Proof. Let $y \in \partial_- h(D)$. If $y \ngeq a$, i.e., $y_i < a_i$ for some $i \in \{1, \ldots, n\}$, then $y_i < z_i$ for all $z \in h(D)$, hence $y \notin h(D)$. This contradicts our setting that $y \in \partial_- h(D)$. Therefore $y \geq a$. On the other hand, if $y \nleq b$, i.e., $y_i > b_i = a_i + \theta_i e^i$ for some $i \in \{1, \ldots, n\}$, then $y \neq a + \theta_i e^i \in h(D)$, while $a + \theta_i e^i \leq y$. Thus $y \notin \partial_- h(D)$, which again contradicts our choice of y. Therefore $y \leq b$. So every lower boundary point of $h(D)$ must be contained in $[a, b]$. $\square$

Note that $a \geq 0$, i.e., $[a, b] \subset \mathbb{R}^n_+$. By Proposition 3(ii) and Proposition 4, if (MCP) has a solution $\overline{x} \in D$, the function $\Phi(\cdot, z)$ must have a zero in $[a, b]$ and $h(\overline{x}) \geq a$.

Now we come back to problem (2). To find a solution $\overline{x} \in D$ of (MCP) we can start searching in an initial box with a defined in Proposition 4 being the lower vertex. Let c be the upper vertex of the initial box, i.e. such that $a \leq h(\overline{x}) \leq c$. The algorithm to be proposed for solving (MCP) proceeds by successive partitioning of $[a, c]$ according to a branch and bound scheme. The partition sets in this algorithm are boxes of the form $M = [p, q] \subset \mathbb{R}^n_+$. At each iteration, for each box $M = [p, q]$ a lower bound $\beta(M)$ is computed for $\Phi(p, q)$ and if $\beta(M) > 0$ then M is deleted, so that only partition sets M with $\beta(M) = 0$ remain for consideration. A partition set M with $\beta(M) = 0$ is then selected and further partitioned, generating a new collection of boxes for exploration at the next iteration. The algorithm continues until occurrence of either of the following events:

1) no partition set remains for exploration (then the problem is infeasible);

2) a solution of (MCP) is obtained.

VALID REDUCTION

A basic operation in this algorithm is *bounding*: for a given box $M := [p, q] \subset \mathbb{R}^n_+$, compute a lower bound $\beta(M)$ for the optimal value of the subproblem

$$\Phi(p, q) = \min\{\langle g(x), h(x)\rangle \mid p \leq h(x) \leq q, x \in D\}. \tag{Q_M}$$

Since, naturally, we want a bound as tight as possible, before computing $\beta(M)$ we should try to replace the box M by a smaller one $M' = [p', q'] \subset M$ such that

$$\Phi(p, q) = \Phi(p', q'). \tag{5}$$

A box $M' \subset M$ satisfying (5) will be referred as a *valid reduction* of M, written $M' = \text{red}M$. In that spirit, let $p' \in \mathbb{R}^n_+$ be such that

$$p'_i = \min\{y_i \mid y = h(x) \text{ for some } x \in D, p \leq h(x) \leq q\} \quad (i = 1, \ldots, n).$$

Proposition 5. *$M' = [p', q]$ is a valid reduction of $M = [p, q]$.*

Proof. Let x^M be a solution of (Q_M). Then $y^M := h(x^M) \in h(D) \cap [p, q]$. We need to show that $p' \leq y^M$. Indeed, if $y^M \not\geq p'$, i.e. $y^M_i < p'_i$ for some $i \in \{1, \ldots, n\}$, then $y^M_i < z_i$ for all $z \in h(D) \cap [p, q]$. Hence $y^M \notin h(D) \cap [p, q]$, contradicts the construction of y^M. Therefore $y^M \geq p'$. □

COMPUTATION OF BOUND

Proposition 6. *Let* $\text{red}M = [p', q']$.
(i) A lower bound $\beta(M)$ for the optimal value of (Q_M) is

$$\beta(M) := \min\{\langle g(x), p' \rangle \mid p' \leq h(x) \leq q', x \in D\}. \tag{LQ_M}$$

(ii) If an optimal solution x^M of (LQ_M) satisfies $h(x^M) = p'$ and $\beta(M) = 0$, then x^M is a solution to (MCP).

Proof. (i) Since $\langle g(x), h(x) - p' \rangle \geq 0$ with $h(x) \geq p', g(x) \geq 0$, we have

$$\begin{aligned}
\Phi(p, q) &= \Phi(p', q') \\
&= \min\{\langle g(x), h(x) \rangle \mid x \in D, p' \leq h(x) \leq q'\} \\
&\geq \min\{\langle g(x), p' \rangle \mid x \in D, p' \leq h(x) \leq q'\} \\
&= \beta(M).
\end{aligned}$$

(ii) Obviously from $x^M \in D$, $\beta(M) = \langle g(x^M), h(x^M) \rangle = 0$. □

Despite its simplicity, the above bounding method can be combined with an adaptive branching process to produce a convergent branch and bound algorithm for solving (MCP). In fact, by using an adaptive rectangular subdivision rule (see [21]) starting from the initial box M_0 one can generate a nested sequence of boxes $M_\nu = [p^\nu, q^\nu]$ such that $\beta(M_\nu) = 0$, $x^{M_\nu} \to \overline{x} \in D$, and $p^\nu - h(x^{M_\nu}) \to 0$. Then $g(x^{M_\nu}) \to g(\overline{x})$, $h(x^{M_\nu}) \to h(\overline{x})$, $\beta(M_\nu) \to \langle g(\overline{x}), h(\overline{x}) \rangle = 0$, so $\overline{x}$ is a solution to (MCP).

4. Branching process

Let $M = [p, q]$ be a box with $\text{red}M = [p', q']$ and such that $\beta(M) = 0$. If x^M is an optimal solution of the problem (LQ_M), then $p' \leq h(x^M) \leq q'$. If it so happens that $h(x^M) = p'$, then by Proposition 6 we have

$$\langle g(x^M), p' \rangle = 0, x^M \in D,$$

i.e., x^M solves (MCP).

Proposition 7. *Let $M_\nu = [p^\nu, q^\nu]$ with $\nu \in \mathbb{Z}_+$ be an infinite nested sequence of subboxes of the initial box $[a, c]$ such that for every ν we have*

(i) $\beta(M_\nu) = 0$;

(ii) $M_{\nu+1}$ is a child of M_ν in the subdivision via the hyperplane $y_{i_\nu} = \eta_\nu$, where $\eta_\nu := (p^\nu_{i_\nu} + h_{i_\nu}(x^\nu))/2$;

(iii) $x^\nu := x^{M_\nu}$ is an optimal solution of (LQ_{M_ν}) *while $i_\nu \in \arg\max_i(h_i(x^\nu) - p^\nu_i)$.*

Then any cluster point $\overline{x}$, if exists, of the sequence $\{x^\nu\}$ yields a solution of (MCP).

Proof. Since $i_\nu \in \{1, \ldots, n\}$, by passing to subsequences if necessary, we can assume that $i_\nu = i_0$ for some $i_0 \in \{1, \ldots, n\}$ and for all $\nu \in \mathbb{Z}_+$. Then, condition (ii) gives us $\eta_\nu = (p^\nu_{i_0} + h_{i_0}(x^\nu))/2$ for all $\nu \in \mathbb{Z}_+$. Furthermore, since $(M_\nu)_{\nu\in\mathbb{Z}_+}$ is an infinite nested sequence of subboxes of $[a, c]$, by (iii) we have $x^\nu \to \overline{x}$, $h_{i_0}(x^\nu) \to h_{i_0}(\overline{x})$, $p^\nu_{i_0} \to \overline{p}_{i_0}$. Then we have $\eta_\nu = (p^\nu_{i_0} + h_{i_0}(x^\nu))/2 \to (\overline{p}_{i_0} + h_{i_0}(\overline{x}))/2 =: \eta_0$. By the subdivision rule, $\eta_\nu \in \{p^{\nu+1}_{i_0}, q^{\nu+1}_{i_0}\}$ and we have $p^{\nu+1}_{i_0} \to \overline{p}_{i_0}$, $q^{\nu+1}_{i_0} \to \overline{q}_{i_0}$ (possibly $\overline{q}_{i_0} = +\infty$). Therefore, $\eta_0 \in \{\overline{p}_{i_0}, \overline{q}_{i_0}\}$ if $\overline{q}_{i_0} < +\infty$, or $\eta_0 = \overline{p}_{i_0}$ otherwise. Since $\overline{p}_{i_0} \le h_{i_0}(\overline{x}) \le \overline{q}_{i_0}$ while $\eta_0 = (\overline{p}_{i_0} + h_{i_0}(\overline{x}))/2$, this implies that $h_{i_0}(\overline{x}) = \overline{p}_{i_0}$, hence $h_{i_0}(x^\nu) - p^\nu_{i_0} \to h_{i_0}(\overline{x}) - \overline{p}_{i_0} = 0$. From the definition of i_0, we then deduce that $h(x^\nu) - p^\nu \to 0$, and consequently, $h(x^\nu) \to \overline{p}$ as $\nu \to +\infty$, i.e., $h(\overline{x}) = \overline{p}$. Since x^ν is an optimal solution of (LQ_{M_ν}), it follows that $\overline{x}$ is an optimal solution of $(\mathrm{LQ}_{[\overline{p},\overline{q}]})$, and finally, as noted above, the fact $h(\overline{x}) = \overline{p}$ implies that $\overline{x}$ solves (MCP). □

5. Branch and bound method

The above development leads to the following branch and bound method for solving (MCP), in which branching is performed by rectangular subdivision of the initial box $[a, c]$.

Algorithm 1

Step 0. Let $M_0 = [a, c]$ in which

$$a_i = \inf\{y_i \mid y = h(x) \text{ for some } x \in D\} \quad \text{for } i = 1, \ldots, n$$

and c be sufficiently large. Start with $\mathcal{R}_0 = \mathcal{P}_1 = \{M_0\}$. Set $k = 1$.

Step 1. Replace each box $M \in \mathcal{P}_k$ with its valid reduction $\mathrm{red}M = [p^M, q^M]$, then compute the lower bound

$$\beta(M) = \min\{\langle g(x), p^M\rangle \mid x \in D, p^M \le x \le q^M\} \tag{6}$$

together with an optimal solution x^M of (6). Let $\mathcal{P}'_k = \{M \in \mathcal{P}_k \mid \beta(M) = 0\}$.

Step 2. If $p^M = h(x^M)$ for some $M \in \mathcal{P}'_k$, then terminate: x^M is a solution of (MCP). Otherwise continue.

Step 3. Let $\mathcal{R}_k = (\mathcal{R}_{k-1}\setminus\{M_{k-1}\}) \cup \mathcal{P}'_k$. If $\mathcal{R}_k = \emptyset$, then terminate: (MCP) is infeasible. Otherwise, select $M_k \in \mathcal{R}_k$ (the most recent partition set) and go to Step 4.

Step 4. Let $x^k = x^{M_k}$, $y^k = h(x^k)$, $p^k = p^{M_k}$. Choose $i_k \in \arg\max_i\{y^k_i - p^k_i\}$ and divide M_k into two subboxes via the hyperplane $y_{i_k} = (p^k_{i_k} + y^k_{i_k})/2$. Reduce each subbox in the partition of M_k and let $\mathcal{P}_{k+1}$ be the collection of these reduced subboxes.

Step 5. Increment k and go back to Step 1.

Theorem 1. *Either the above algorithm terminates after finitely many iterations, yielding a solution to* (MCP) *or an evidence that* (MCP) *is infeasible - or it generates an infinite sequence* $\{x^k\}$ *every cluster point* $\overline{x}$ *of which is a solution to* (MCP)*.*

Proof. When finite, the algorithm terminates at either Step 2 or Step 3 of some iteration. In the former case, the point x^M at that step is a solution to (MCP). In the latter case, $\mathcal{R}_k = \emptyset$ is an evidence for the infeasibility of (MCP). When infinite, the algorithm generates a nested sequence of boxes $M_{k_\nu} = [p^{k_\nu}, q^{k_\nu}]$ $(\nu = 1, 2, \ldots)$ satisfying the conditions of Proposition 7, hence the conclusion. □

Remark 1. The type of subdivision described in Proposition 7 and used in Algorithm 1 is often referred to as an *adaptive subdivision*. The rationale behinds it is that it tends to bring the distance $\|h(x^k) - p^k\|$ to zero, thus forcing $\{x^k\}$ to converge to a solution of (MCP).

Remark 2. Each subproblem (LQ_M) for computing the lower bound $\beta(M)$ is a d.m. optimization problem and, in the most general case, can be solved by suitable methods of d.m. optimization as developed in [27]. However, instead of solving (LQ_M) for obtaining a lower bound $\beta(M)$ as indicated in Proposition 6 one may also compute a number

$$\gamma(M) \leq \min\{\langle g(x), h(x)\rangle \mid p' \leq h(x) \leq q', g(x) \geq 0\},$$

i.e., a lower bound for the optimal value of the problem (2). Convergence will be ensured, provided that for any nested sequence $M_\nu = [p^\nu, q^\nu]$ satisfying $\cap_\nu M_\nu = \{\overline{x}\}$ we have $\gamma(M_\nu) \to \langle g(\overline{x}), h(\overline{x})\rangle$. For example, if $g(x) = u(x) - v(x)$ with $u(x), v(x)$ increasing in $\mathbb{R}^n_+$, then this requirement is satisfied by the number $\gamma(M) = \langle p', u(p') - v(q')$ which is the most easily computable underestimate of the optimal value of (2). When $g(x), h(x)$ are polynomials, it may also be advantageous to take $\gamma(M)$ as the optimal value of a RLT relaxation of the problem (2) (see e.g. [21]).

6. Specializations

6.1. Linear complementarity problems

In this subsection we specialize the above method to the linear complementarity problem:

$$\text{Find } x \in \mathbb{R}^n \text{ satisfying } x \geq 0,\ C(x) \geq 0,\ \langle C(x), x\rangle = 0, \qquad \text{(LCP)}$$

where $C : \mathbb{R}^n \to \mathbb{R}^n$ is an affine mapping. Here $g(x) = C(x)$, $h(x) \equiv x$, and the set $D = \{x \in \mathbb{R}^n \mid C(x) \geq 0, x \geq 0\}$ is a polyhedron. This linear structure implies some important properties. Since the polyhedron D is line free, it has at least one extreme point, and at least one such extreme point is a solution of (LCP), if the latter is solvable. This follows from the well known fact that (LCP) has a solution if and only if the concave minimization problem

$$\min\{\sum_{i=1}^{n} \min\{x_i, C_i x \mid x \geq 0, C(x) \geq 0\}\}$$

has an optimal solution with objective value 0, and at least one optimal solution of this concave minimization problem is an extreme point of D. Note that the convex hull of the extreme points of D is a compact set, so we can concentrate the search for a solution of (LCP) on a bounded initial box.

6.2. Increasing complementarity problems

In this subsection we specialize the above method to the increasing complementarity problem, i.e. (MCP) in which $g_i(x), h_i(x)$ are increasing functions on $\mathbb{R}^n_+$.

7. Numerical experiments

In this section we present some computational experiments to illustrate the behavior of the proposed algorithm. The algorithm has been coded in MATLAB R2020a and run on a PC Intel(R) Core(TM) i7-6700HQ CPU 2*2.60 GHz, RAM 16.0 GB.

Experiment 1. (Test problem 1 in Section 10.2 [5], taken from MCPLIB [3]).
$g(x) = Ax + q, h(x) = x$ in which $q \in \mathbb{R}^{16}$ with $q_i = -1$ for all $i = 1, \ldots, 16$ and

$$A = \begin{bmatrix} 1 & 2 & 2 & \ldots & 2 & 2 \\ 0 & 1 & 2 & \ldots & 2 & 2 \\ 0 & 0 & 1 & \ldots & 2 & 2 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & 2 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

is a matrix in $\mathbb{R}^{16\times16}$ whose diagonal entries equal 1, lower-diagonal entries equal 0, upper-diagonal entries equal 2.

Experiment 2. (Test problem 2 in Section 10.2 [5], taken from [2]).
$g(x) = Ax + q, h(x) = x$ in which $q = (-1, -1)^t$ and

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Experiment 3. (Test problem 3 in Section 10.2 [5], taken from [2]).
$g(x) = Ax + q, h(x) = x$ in which $q = (-3, 6, -1)^t$ and

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}.$$

Experiment 4. (Test problem 4 in Section 10.2 [5], taken from [2]).
$g(x) = Ax + q, h(x) = x$ in which $q = (-1, -1, -1, -1)^t$ and

$$A = \begin{bmatrix} 0 & 0 & 10 & 20 \\ 0 & 0 & 30 & 15 \\ 10 & 20 & 0 & 0 \\ 30 & 15 & 0 & 0 \end{bmatrix}.$$

Experiment 5. (Test problem 5 in Section 10.2 [5]).

Experiment 6. (Test problem 6 in Section 10.2 [5], taken from [2]).
$g(x) = Ax + q, h(x) = x$ in which $q = (50, 50, \lambda, -6)^t$ with $\lambda \geq 0$ and

$$A = \begin{bmatrix} 11 & 0 & 10 & -1 \\ 0 & 11 & 10 & -1 \\ 10 & 10 & 21 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Experiment 7. (Test problem 1 in Section 10.3 [5], taken from [2]).
$g(x) = \arctan(x - 10), h(x) = x$.
Experiment 8. (Test problem 2 in Section 10.3 [5], taken from [2]).

$$g(x) = \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_2^2 + x_1 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}$$

and $h(x) = x \in \mathbb{R}^4$.
Experiment 9. (Test problem 3 in Section 10.3 [5]).

$$g(x) = \begin{bmatrix} -x_1 + x_2 + x_3 \\ x_4 - 0.75(x_2 + \beta x_3) \\ 1 - x_4 - 0.25(x_2 + \beta x_3)/x_2 \\ \beta - x_4 \end{bmatrix}$$

and $h(x) = x \in \mathbb{R}^4$.
Experiment 10. (Test problem 4 in Section 10.3 [5]).
Experiment 11. (Test problem 5 in Section 10.3 [5]).
Experiment 12. (Test problem 6 in Section 10.3 [5]).

Acknowledgments

The authors are indebted to the late Professor Hoang Tuy for inspiring this work and for his detailed comments on earlier drafts of this paper. The authors would like to thank Nguyen Thi Hoai Phuong for her helpful discussions. Part of this work was done during a research stay of the first author (Xuan Thanh Le) at Vietnam Institute for Advanced Study in Mathematics.

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